`Ryacas` – an `R` interface to the `yacas` computer algebra system

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1   Introduction

`Ryacas` makes the `yacas` computer algebra system available from within `R`. The name `yacas` is short for "Yet Another Computer Algebra System". The `yacas` program is developed by Ayal Pinkhuis (who is also the maintainer) and others, and is available at http://yacas.sourceforge.net for various platforms. There is a comprehensive documentation (300+ pages) of `yacas` (also available at http://yacas.sourceforge.net) and the documentation contains many examples.

2   `R` expressions, `yacas` expressions and `Sym` objects

The `Ryacas` package works by sending "commands" to `yacas` which makes the calculations and returns the result to `R`. There are various different formats of the return value as well

2.1   `R` expressions

A call to `yacas` may be in the form of an `R` expression which involves valid R calls, symbols or constants (though not all valid `R` expressions are valid). For example:

```
> exp1 <- yacas(expression(Factor(x^2-1)))

expression((x + 1) * (x - 1))
```

The result `exp1` is not an expression in the `R` sense but an object of class `"yacas"`. To evaluate the resulting expression numerically, we can do

```
> Eval(exp1, list(x=4))

[1] 15
```

2.2   `yacas` expressions

Some commands are not proper `R` expressions. For example, typing

```
  yacas(expression(D(x)Sin(x)))
```

produces an error. For such cases we can make a specification using the `yacas` syntax:

```
> yacas("D(x)Sin(x)")

expression(cos(x))
```

2.3   `Sym` objects

Probably the most elegant way of working with `yacas` is by using `Sym` objects. A `Sym` object is a `yacas` character string that has the "Sym" class. One can combine `Sym` objects with other `Sym` objects as well as to other `R` objects using `+`, `-` and other similar `R` operators.

The function `Sym(x)` coerces an object `x` to a `Sym` object by first coercing it to character and then changing its class to "Sym":

```
> x <- Sym("x")
expression(x)
```

Operations on `Sym` objects lead to new `Sym` objects:

```
> x+4
expression(x + 4)
```

One can apply `sin`, `cos`, `tan`, `deriv`, `Integrate` and other provided functions to `Sym` objects. For example:

```
> Integrate(sin(x), x)
expression(-cos(x))
```

In this way the communication with `yacas` is "tacit".

It is important to note the difference between the `R` name `x` and the symbol `"x"` as illustrated below:

```
> x <- Sym("xs")
expression(xs)
> x
expression(xs)
> x+4
expression(xs + 4)
> Eval(x+4, list(xs=5))
[1] 9
```

The convention in the following is 1) that `Sym` objects match with their names that they end with an 's', e.g.

```
> xs <- Sym('xs')
```

3   A sample session

Algebraic calculations:

```
> yacas(expression((10 + 2) * 5 + 7^7))
expression(823603)
> yacas(expression(1/14+5/21* (30- 1+ 1/2)))
expression(149/21)
```

```
> #(Sym(10) + Sym(2)) * Sym(5) + Sym(7) ^ Sym(7)
> Sym("10 * 2") * 5 + Sym(7) ^ 7

expression(823643)

> #(Sym(10) + 2) * 5 + Sym(7) ^ 7
> #Sym("(10+2)*5 + 7^7")
> Sym("1/14 + 5/21 * (30 - 1+1/2)")

expression(149/21)
```

Numerical evaluations:

```
> yacas(expression(N(-12/2)))

expression(-6)
```

```
> Sym("-12/2")

expression(-6)

> #Eval(Sym("-12/2"))
```

Symbolic expressions:

```
> yacas(expression(Factor(x^2-1)))

expression((x + 1) * (x - 1))

> exp1 <- expression(x^2 + 2 * x^2)
> exp2 <- expression(2 * exp0)
> exp3 <- expression(6 * pi * x)
> exp4 <- expression((exp1 * (1 - sin(exp3))) / exp2)
> yacas(exp4)

expression(3 * (x^2 * (1 - sin(6 * (x * pi)))))/(2 * exp0))
```

```
> Factor(xs^2-1)

expression((xs + 1) * (xs - 1))

> exp1 <- xs^2 + 2 * xs^2
> exp0 <- Sym("exp0")
> exp2 <- 2 * Sym(exp0)
> exp3 <- 6 * Pi * xs
> exp4 <- exp1 * (1 - sin(exp3)) / exp2
> exp4

expression(3 * (xs^2 * (1 - sin(6 * (xs * pi)))))/(2 * exp0))
```

Combining symbolic and numerical expressions:

```
> yacas(expression(N(Sin(1)^2 + Cos(x)^2)))

expression(cos(x)^2 + 0.7080734182)
```

```
> N(sin(1)^2 + cos(xs)^2)

expression(cos(xs)^2 + 0.708073418273571)
```

Differentiation:

```
> yacas("D(x)Sin(x)")
expression(cos(x))
```

```
> deriv(sin(xs), xs)
expression(cos(xs))
```

Integration:

```
> yacas("Integrate(x,a,b)Sin(x)")
expression(cos(a) - cos(b))
```

```
> as <- Sym("as")
> bs <- Sym("bs")
> Integrate(sin(xs), xs, as, bs)
expression(cos(as) - cos(bs))
```

Expanding polynomials:

```
> yacas("Expand((1+x)^3)")
expression(x^3 + 3 * x^2 + 3 * x + 1)
```

```
> Expand((1+xs)^3)
expression(xs^3 + 3 * xs^2 + 3 * xs + 1)
```

Taylor expansion:

```
> yacas("texp := Taylor(x,0,3) Exp(x)")
expression(x + x^2/2 + x^3/6 + 1)
```

```
> texp <- Taylor(exp(xs), xs, 0, 3)
expression(xs + xs^2/2 + xs^3/6 + 1)
```

Printing the result in nice forms:

```
> yacas("PrettyForm(texp)")

     2    3
    x    x
x + -- + -- + 1
    2    6

> yacas("TeXForm(texp)", retclass = "unquote")
$x + \frac{x ^{2}}{2}  + \frac{x ^{3}}{6}  + 1$
```

```
> PrettyForm(texp)

      2     3
    xs    xs
xs + --- + --- + 1
     2     6

> TeXForm(texp)
expression("$xs + \frac{xs ^{2}}{2}  + \frac{xs ^{3}}{6}  + 1$")
```

4 Simple Yacas calculations

4.1 Setting and clearing a variable

The function Set() and the operator := can both be used to assign values to global variables.

```
> yacas("n := (10 + 2) * 5")

expression(60)

> yacas("n := n+n")

expression(120)

> #yacas("Set(z, Cos(a))")
> #yacas("z+z")
```

The same can be achieved with Sym objects: Consider:

```
> Set(ns, (10 + 2) * 5)

expression(60)
```

Now ns exists as a variable in yacas(and we can make computations on this variable as above). However we have no handle on this variable in R. Such a handle is obtained with

```
> ns <- Sym("ns")
```

Now the R variable ns refers to the yacas variable ns and we can make calculations directly from R, e.g:

```
> Set(ns,123)

expression(123)

> ns

expression(123)

> ns^2

expression(15129)
```

Likewise:

```
> as <- Sym("as")
> zs <- Sym("zs")
> Set(zs, cos(as))

expression(cos(as))

> zs + zs

expression(2 * cos(as))
```

o clear a variable binding execute Clear():

```
> yacas(expression(n))
expression(120)
> yacas("Clear(n)")
expression(TRUE)
> yacas(expression(n))
expression(n)
```

```
> Set(ns, 1)
expression(1)
> ns <- Sym("ns")
> ns
expression(1)
> Clear(ns)
expression(TRUE)
> ns
expression(ns)
```

4.2 Symbolic and numerical evaluations, precision

Evaluations are generally exact:

```
> yacas("Exp(0)")
expression(1)
> yacas("Exp(1)")
expression(exp(1))
> yacas("Sin(Pi/4)")
expression(root(1/2, 2))
> yacas("355/113")
expression(355/113)
```

```
> exp(Sym(0))
expression(1)
> exp(Sym(1))
expression(exp(1))
> sin(Pi/4)
expression(root(1/2, 2))
> Sym("355/113")
expression(355/113)
```

To obtain a numerical evaluation (approximation), the N() function can be used:

```
> yacas("N(Exp(1))")
expression(2.7182818284)
> yacas("N(Sin(Pi/4))")
expression(0.7071067811)
> yacas("N(355/113)")
expression(3.1415929203)
```

```
> N(exp(1))
expression(2.71828182845905)
> N(sin(Pi/4))
expression(0.7071067811)
> N(355/113)
expression(3.14159292035398)
```

The `N()` function has an optional second argument, the required precision:

```
> yacas("N(355/133,20)")
expression(2.66917293233083)
```

```
> N("355/113",20)
expression(3.14159292035398)
```

The command `Precision(n)` can be used to specify that all floating point numbers should have a fixed precision of n digits:

```
> yacas("Precision(5)")
expression(Precision(5))
> yacas("N(355/113)")
expression(3.1415929203)
```

```
> Precision(5)
expression(Precision(5))
> N("355/113")
expression(3.1415929203)
```

4.3   Rational numbers

Rational numbers will stay rational as long as the numerator and denominator are integers:

```
> yacas(expression(55/10))
expression(11/2)
```

```
> Sym("55 / 10")
expression(11/2)
```

4.4 Symbolic calculation

Some exact manipulations :

```
> yacas("1/14+5/21*(30-(1+1/2)*5^2)")
expression(-12/7)
> yacas("0+x")
expression(x)
> yacas("x+1*y")
expression(x + y)
> yacas("Sin(ArcSin(alpha))+Tan(ArcTan(beta))")
expression(alpha + beta)
```

```
> Sym("1/14+5/21*(1*30-(1+1/2)*5^2)")
expression(-12/7)
> xs <- Sym("xs")
> ys <- Sym("ys")
> 0+xs
expression(xs)
> xs+1*ys
expression(xs + ys)
> sin(asin(xs))+tan(atan(ys))
expression(xs + ys)
```

4.5 Complex numbers and the imaginary unit

The imaginary unit i is denoted I and complex numbers can be entered as either expressions involving I or explicitly Complex(a,b) for a+ib.

```
> yacas("I^2")
expression(-1)
> yacas("7+3*I")
expression(complex_cartesian(7, 3))
> yacas("Conjugate(%)")
expression(complex_cartesian(7, -3))
> yacas("Exp(3*I)")
expression(complex_cartesian(cos(3), sin(3)))
```

```
> I^2
expression(-1)
> 7+3*I
expression(complex_cartesian(7, 3))
> Conjugate(7+3*I)
expression(complex_cartesian(7, -3))
> exp(3*I)
expression(complex_cartesian(cos(3), sin(3)))
```

4.6 Recall the most recent line – the % operator

The operator % automatically recalls the result from the previous line.

```
> yacas("(1+x)^3")
expression((x + 1)^3)
> yacas("%")
expression((x + 1)^3)
> yacas("z:= %")
expression((x + 1)^3)
```

```
> (1+x)^3
expression((xs + 1)^3)
> zs <- Sym("%")
> zs
expression((xs + 1)^3)
```

4.7 Printing with PrettyForm, PrettyPrint, TexForm and TeX-Form

There are different ways of displaying the output.

4.7.1 Standard form

The (standard) yacas form is:

```
> yacas("A:={{a,b},{c,d}}")
expression(list(list(a, b), list(c, d)))
> yacas("B:= (1+x)^2+k^3")
expression((x + 1)^2 + k^3)
> yacas("A")
expression(list(list(a, b), list(c, d)))
> yacas("B")
expression((x + 1)^2 + k^3)
```

```
> as <- Sym("as"); bs <- Sym("bs"); cs <- Sym("cs"); ds <- Sym("ds")
> A <- List(List(as,bs), List(cs,ds))
> ks <- Sym("ks")
> B <- (1+xs)^2+ks^3
> A

expression(list(list(as, bs), list(cs, ds)))

> B

expression((xs + 1)^2 + ks^3)
```

4.7.2 Pretty form

The Pretty form is:

```
> yacas("PrettyForm(A)")

/             \
| ( a ) ( b ) |
|             |
| ( c ) ( d ) |
\             /
> yacas("PrettyForm(B)")

        2    3
( x + 1 )  + k
```

```
> PrettyForm(A)

/               \
| ( as ) ( bs ) |
|               |
| ( cs ) ( ds ) |
\               /
> PrettyForm(B)

         2     3
( xs + 1 )  + ks
```

4.7.3 TeX form

The output can be displayed in TeX form:

```
> #FIXME
> #yacas("TeXForm(B)")
```

```
> #FIXME
> #TeXForm(B)
```

5 Commands

5.1 Factorial

```
> yacas("40!")

expression(8.15915283247898e+47)
```

```
> Factorial(40)

expression(Factorial(40))
```

5.2   Taylor expansions

Expand $\exp(x)$ in three terms around 0 and a:

```
> yacas("Taylor(x,0,3) Exp(x)")

expression(x + x^2/2 + x^3/6 + 1)

> yacas("Taylor(x,a,3) Exp(x)")

expression(exp(a) + exp(a) * (x - a) + (x - a)^2 * exp(a)/2 +
    (x - a)^3 * exp(a)/6)
```

```
> xs <- Sym("xs")
> Taylor(exp(xs),xs,0,3)

expression(xs + xs^2/2 + xs^3/6 + 1)

> as <- Sym("as")
> Taylor(exp(x),x,as,3)

expression(exp(as) + exp(as) * (xs - as) + (xs - as)^2 * exp(as)/2 +
    (xs - as)^3 * exp(as)/6)
```

The `InverseTaylor()` function builds the Taylor series expansion of the inverse of an expression. For example, the Taylor expansion in two terms of the inverse of $\exp(x)$ around $x = 0$ (which is the Taylor expansion of $Ln(y)$ around $y = 1$):

```
> yacas("InverseTaylor(x,0,2)Exp(x)")

expression(x - 1 - (x - 1)^2/2)

> yacas("Taylor(y,1,2)Ln(y)")

expression(y - 1 - (y - 1)^2/2)
```

```
> InverseTaylor(exp(xs),xs,0,2)

expression(xs + xs^2/2 + 1)

> Taylor(log(ys),ys,1,2)

expression(ys - 1 - (ys - 1)^2/2)
```

5.3   Solving equations

5.3.1   Solving equations symbolically

Solve equations symbolically with the `Solve()` function:

```
> yacas("Solve(x/(1+x) == a, x)")

expression(list(x == a/(1 - a)))

> yacas("Solve(x^2+x == 0, x)")

expression(list(x == 0, x == -1))
```

```
> Solve(xs/(1+xs) == as, xs)

expression(list(xs == as/(1 - as)))

> Solve(xs^2+xs == 0, xs)

expression(list(xs == 0, xs == -1))
```

```
> Solve(List(xs^2+ys^2==6, xs-ys==3), List(xs,ys))

expression(list(list(xs == root(6 - ys^2, 2), ys == ys)))
```

```
> # FIXME
> #Solve(List(mean==(xs/(xs+ys)), variance==((xs*ys)/(((xs+ys)^2) * (xs+ys+1)))),
> #    List(xs,ys))
```

(Note the use of the == operator, which does not evaluate to anything, to denote an
"equation" object.)

5.3.2   Solving equations numerically

To solve an equation (in one variable) like $sin(x) - exp(x) = 0$ numerically taking 0.5 as
initial guess and an accuracy of 0.0001 do:

```
> yacas("Newton(Sin(x)-Exp(x),x, 0.5, 0.0001)")

expression(-3.1830630118)
```

```
> Newton(sin(xs)-exp(xs),xs, 0.5, 0.0001)

expression(-3.1830630118)
```

5.4   Expanding polynomials

```
> yacas(expression(Expand((1+x)^3)))

expression(x^3 + 3 * x^2 + 3 * x + 1)
```

```
> Expand((x+1)^3)

expression(xs^3 + 3 * xs^2 + 3 * xs + 1)
```

5.5   Simplifying an expression

The function Simplify() attempts to reduce an expression to a simpler form.

```
> y <- Sym("y")
> yacas("(x+y)^3-(x-y)^3")

expression((x + y)^3 - (x - y)^3)

> yacas("Simplify(%)")

expression(6 * (x^2 * y) + 2 * y^3)

>
```

```
> (x+y)^3-(x-y)^3
expression((xs + y)^3 - (xs - y)^3)
> Simplify("%")
expression(6 * (xs^2 * y) + 2 * y^3)
```

5.6   Analytical derivatives

Analytical derivatives of functions can be evaluated with the D() and deriv() functions:

```
> yacas("D(x) Sin(x)")
expression(cos(x))
```

```
> deriv(sin(xs), xs)
expression(cos(xs))
```

These functions also accepts an argument specifying how often the derivative has to be taken, e.g:

```
> yacas("D(x,4)Sin(x)")
expression(sin(x))
```

```
> deriv(sin(xs),xs,4)
expression(sin(xs))
```

5.7   Integration

```
> #yacas("Integrate(x,a,b)Sin(x)")
> #yacas("Integrate(x,a,b)Ln(x)+x")
> #yacas("Integrate(x)1/(x^2-1)")
> yacas("Integrate(x)Sin(a*x)^2*Cos(b*x)")

expression((2 * sin(b * x)/b - (sin(-2 * (x * a) - b * x)/(-2 *
    a - b) + sin(-2 * (x * a) + b * x)/(-2 * a + b)))/4)
```

```
> #Integrate(sin(x),x,a,b)
> #Integrate(log(x),x,a,b)
> #Integrate(1/(x^2-1),x)
> a <- Sym("a")
> b <- Sym("b")
> Integrate(sin(a*x)^2*cos(b*x),x)

expression((2 * sin(b * xs)/b - (sin(-2 * (xs * a) - b * xs)/(-2 *
    a - b) + sin(-2 * (xs * a) + b * xs)/(-2 * a + b)))/4)
```

5.8 Limits

```
> #yacas("Limit(x,0)Sin(x)/x")
> yacas("Limit(n,Infinity)(1+(1/n))^n")

expression(exp(1))

> yacas("Limit(h,0) (Sin(x+h)-Sin(x))/h")

expression(cos(x))
```

```
> #Limit(sin(x)/x,x,0)
> ns <- Sym("ns")
> Limit((1+(1/ns))^ns,ns,Infinity)

expression(exp(1))

> hs <- Sym("hs")
> Limit((sin(xs+hs)-sin(xs))/hs,hs,0)

expression(cos(xs))
```

5.9 Variable substitution

```
> yacas("Subst(x,Cos(a))x+x")

expression(2 * cos(a))
```

```
> Subst(xs+xs,xs,cos(as))

expression(2 * cos(as))
```

5.10 Solving ordinary differential equations

```
> yacas("OdeSolve(y''==4*y)")

expression(C7010 * exp(2 * x) + C7014 * exp(-2 * x))

> yacas("OdeSolve(y'==8*y)")

expression(C7044 * exp(8 * x))
```

6 Matrices

```
> yacas("E4:={ {u1,u1,0},{u1,0,u2},{0,u2,0} }")

expression(list(list(u1, u1, 0), list(u1, 0, u2), list(0, u2,
    0)))

> yacas("PrettyForm(E4)")

/                  \
| ( u1 ) ( u1 ) ( 0 )  |
|                  |
| ( u1 ) ( 0 )  ( u2 ) |
|                  |
| ( 0 )  ( u2 ) ( 0 )  |
\                  /
```

```
> u1 <- Sym("u1")
> u2 <- Sym("u2")
> E4 <- List(List(u1, u1, 0), List(u1, 0, u2), List(0, u2, 0))
> PrettyForm(E4)

/                       \
| ( u1 ) ( u1 ) ( 0 )   |
|                       |
| ( u1 ) ( 0 )  ( u2 )  |
|                       |
| ( 0 )  ( u2 ) ( 0 )   |
\                       /
```

6.1 Inverse

```
> yacas("E4i:=Inverse(E4)")

expression(list(list(u1 * u2^2/(u1^2 * u2^2), 1/u1 - u1 * u2^2/(u2^2 *
    u1^2), -(u1 * u2/u1)/u2^2), list(1/u1 - u1 * u2^2/(u2^2 *
    u1^2), u1 * u2^2/(u2^2 * u1^2) - 1/u1, u2 * u1/(u2^2 * u1)),
    list(-(u1 * u2/u1)/u2^2, u1 * u2/(u1 * u2^2), u1/u2^2)))

> yacas("Simplify(E4i)")

expression(list(list(1/u1, 0, -1/u2), list(0, 0, 1/u2), list(-1/u2,
    1/u2, u1/u2^2)))

> yacas("PrettyForm(Simplify(E4i))")

/                           \
| / 1  \ ( 0 )  / -1 \      |
| | -- |        | -- |      |
| \ u1 /        \ u2 /      |
|                           |
| ( 0 )  ( 0 )  / 1  \      |
|               | -- |      |
|               \ u2 /      |
|                           |
| / -1 \ / 1  \ / u1  \     |
| | -- | | -- | | --- |     |
| \ u2 / \ u2 / |   2 |     |
|               \ u2  /     |
\                           /
```

```
> E4i <- Inverse(E4)
> Simplify(E4i)

expression(list(list(1/u1, 0, -1/u2), list(0, 0, 1/u2), list(-1/u2,
    1/u2, u1/u2^2)))

> PrettyForm(Simplify(E4i))

/                          \
| / 1  \ ( 0 )  / -1 \   |
| | -- |        | -- |   |
| \ u1 /        \ u2 /   |
|                          |
| ( 0 )  ( 0 )  / 1  \   |
|               | -- |   |
|               \ u2 /   |
|                          |
| / -1 \ / 1  \ / u1  \  |
| | -- | | -- | | --- |  |
| \ u2 / \ u2 / |   2 |  |
|               \ u2 /   |
\                          /
```

6.2   Determinant

```
> yacas("Determinant(E4)")

expression(-(u1 * u2^2))

> yacas("Determinant(E4i)")

expression((u1 * u2^2/(u2^2 * u1^2) - 1/u1) * (u1 * u2^2) * u1/(u1^2 *
    u2^4) - u1 * u2^2 * (u2 * u1) * (u1 * u2)/(u1 * u2^2 * (u2^2 *
    u1 * (u1^2 * u2^2))) - (1/u1 - u1 * u2^2/(u2^2 * u1^2)) *
    (u1 * u2)^2/(u2^4 * u1^2) - (1/u1 - u1 * u2^2/(u2^2 * u1^2))^2 *
    u1/u2^2 - (1/u1 - u1 * u2^2/(u2^2 * u1^2)) * (u2 * u1) *
    (u1 * u2)/(u2^4 * u1^2) - (u1 * u2^2/(u2^2 * u1^2) - 1/u1) *
    (u1 * u2)^2/(u1^2 * u2^4))

> yacas("Simplify(E4i)")

expression(list(list(1/u1, 0, -1/u2), list(0, 0, 1/u2), list(-1/u2,
    1/u2, u1/u2^2)))

> yacas("Simplify(Determinant(E4i))")

expression(-1/(u1 * u2^2))
```

```
> determinant(E4)

expression(-(u1 * u2^2))

> determinant(E4i)

expression((u1 * u2^2/(u2^2 * u1^2) - 1/u1) * (u1 * u2^2) * u1/(u1^2 *
    u2^4) - u1 * u2^2 * (u2 * u1) * (u1 * u2)/(u1 * u2^2 * (u2^2 *
    u1 * (u1^2 * u2^2))) - (1/u1 - u1 * u2^2/(u2^2 * u1^2)) *
    (u1 * u2)^2/(u2^4 * u1^2) - (1/u1 - u1 * u2^2/(u2^2 * u1^2))^2 *
    u1/u2^2 - (1/u1 - u1 * u2^2/(u2^2 * u1^2)) * (u2 * u1) *
    (u1 * u2)/(u2^4 * u1^2) - (u1 * u2^2/(u2^2 * u1^2) - 1/u1) *
    (u1 * u2)^2/(u1^2 * u2^4))

> Simplify(E4i)

expression(list(list(1/u1, 0, -1/u2), list(0, 0, 1/u2), list(-1/u2,
    1/u2, u1/u2^2)))

> Simplify(determinant(E4i))

expression(-1/(u1 * u2^2))
```

7   Miscellaneous

Note that the value returned by `yacas` can be of different types:

```
> yacas(expression(Factor(x^2-1)),retclass='unquote')

*" ("+" (x ,1 ),"- (x ,1 ))

> yacas(expression(Factor(x^2-1)),retclass='character')

"*" ("+" (x ,1 ),"-" (x ,1 ))
```