Space feedback control

Pole placement

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Pole placement

So far we have learnt how a state feedback looks like and when it is possible to design a state feedback controller to stabilize a system:

$$\dot{x} = Ax + Bu$$
$$u = -Kx + k_r r$$

$$\dot{x} = (A - BK)x + Bk_r r$$

K ubica los polos de LC y kr se encarga de eliminar el error de estado estacionario

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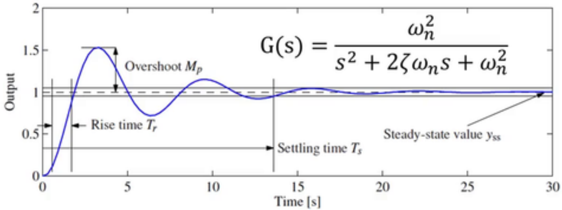
The questions that remain are: *How do we design a state feedback controller* and *Where do we place the closed loop system's poles?*

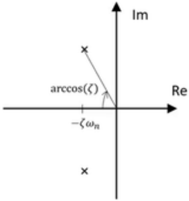
rise time: 10 a 90% rta

settling time: +-1%

Specifications and pole placement

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$





Pole placement

Where do we place the closed loop system's poles?

Nos enfocamos en los polos dominantes que dominan la rta en el transitorio

Idea:

- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.

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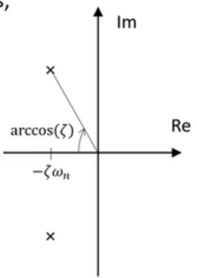

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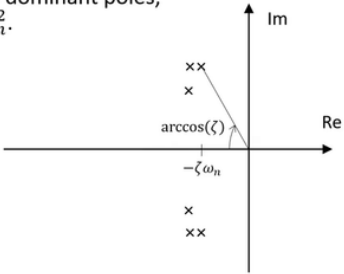
- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.
- Place the rest of the poles so they become faster than the dominant poles.



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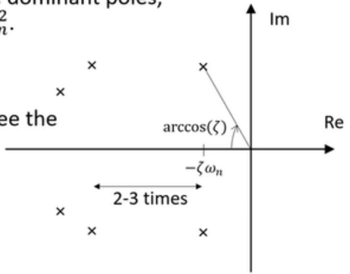

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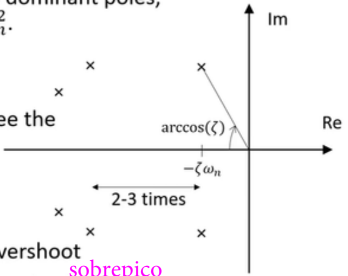

Usually you end up with some zeros as well:

- Zeros in the left half plane give additional overshoot
- Zeros in the right half plane give a negative undershoot

sobrepico

Pole placement (Ackermann's formula)

Pole placement is performed by matching the desired characteristic polynomial with the closed loop system's characteristic polynomial.

From earlier example (vehicle steering) we have seen:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

$$k_1 = \frac{b\omega_n^2}{v_0^2} \qquad k_2 = \frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}$$

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For low order systems it is ok, but for larger systems this is boring work.

Ackermann's formula offers us a method to do this in one computational step.

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n.$$

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$$a(s) = s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n.$$

If the system is reachable, then there exist a control law, $u = -Kx$, that gives a closed loop system with the charateristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots p_{n-1} s + p_n.$$

acá p no son los polos. Ojo

Pole placement (Ackermann's formula)

The feedback gain is given by p no son polos

$$K = [p_1 - a_1 \quad p_2 - a_2 \quad \dots \quad p_n - a_n]\widetilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB \quad \dots \quad A^{n-1}B]$$

Alcanzable cuando Wr tiene inversa

and

$$\widetilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}^{-1}$$

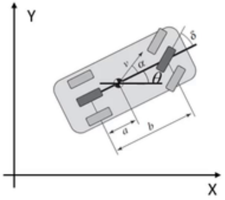
This is called ***Ackermann's formula***.

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



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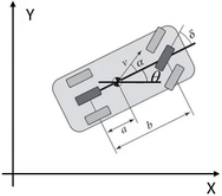

Determine the characteristic polynomial for the system:

$$\det(sI - A) = \begin{vmatrix}s & -v_0 \\ 0 & s\end{vmatrix} = s^2 = s^2 + 0s + 0$$

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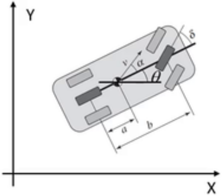

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Desired characteristic polynomial for the closed loop system:

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\widetilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\widetilde{W}_r W_r^{-1}$$

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Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 7.