

# The Z-transform

Dr. Ing. Rodrigo Gonzalez

`rodrigo.gonzalez@ingenieria.uncu.edu.ar`

Control y Sistemas

Facultad de Ingeniería,  
Universidad Nacional de Cuyo



**UNCUYO**  
UNIVERSIDAD  
NACIONAL DE CUYO

# Summary

- 1 Introduction
- 2 Definition
- 3 Region of Convergence
- 4 Z-Transform Properties
- 5 Constant-Coefficient Difference Equations

- The Z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- It is analogous to the Laplace transform for continuous systems.

The Laplace transform is defined as:

$$\mathcal{L}\{x(t)\} = F(s) = \int_0^{\infty} x(t)e^{-st} dt, \quad (1)$$

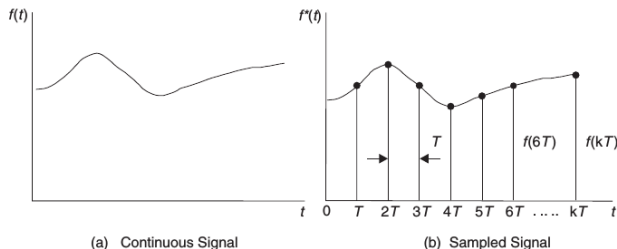
$$\mathcal{L}\{\dot{x}(t)\} = sF(s) \quad (2)$$

Eq. 2 enables us to find the transfer function of a linear continuous time system, given the differential equation description of that system.

# Bilateral Z-transform

Firstly, we define:

$x[n]$  is the sampled version of  $x(t)$ ,  $n$  is an integer and  $T$  is the sampling time.



$z$  is a complex number,

$$z = r e^{j\omega} = r \cdot (\cos \omega + j \sin \omega). \quad (3)$$

The bilateral or two-sided Z-transform of a discrete-time signal  $x[n]$  is the formal power series  $X(z)$  defined as,

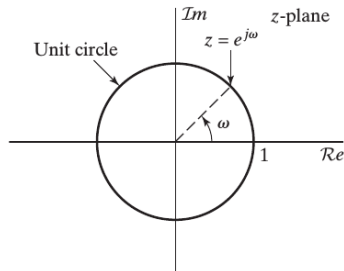
$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \quad (4)$$

# Unilateral Z-transform

Alternatively, in cases where  $x[n]$  is defined only for  $n \geq 0$  the single-sided or unilateral Z-transform is defined as,

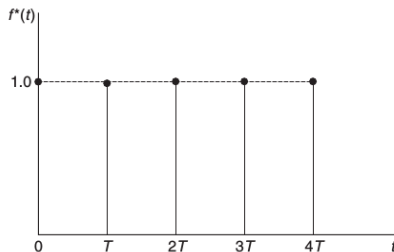
$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \quad (5)$$

$$z = r e^{j\omega} = r \cdot (\cos \omega + j \sin \omega).$$



## Example

Find the Z-transform of the unit step function  $u(t) = 1$ .



$$\mathcal{Z}\{u[n]\} = U(z) = \sum_{n=0}^{\infty} u(kT)z^{-n}, \quad (6)$$

$$U(z) = z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-n} + \cdots \quad (7)$$

Equation 7 can be written in closed-form as,

$$\mathcal{Z}\{u[n]\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}. \quad (8)$$

The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$ROC = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty \right\}. \quad (9)$$

## Example (no ROC)

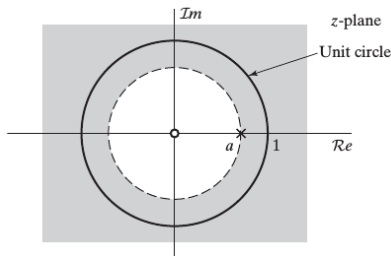
Let  $x[n] = 0.5^n = \{\dots, 0.5^{-3}, 0.5^{-2}, 0.5^{-1}, 1, 0.5, 0.5^2, \dots\} \Rightarrow$

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} \rightarrow \infty$$

Therefore, there are no values of  $z$  that satisfy this condition.

Let  $x[n] = 0.5^n u[n] = \{\dots, 0, 0, 0, 1, 0.5, 0.5^2, 0.5^3, \dots\}$ .

$$\sum_{n=-\infty}^{\infty} (0.5^n u[n]) z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n = \frac{1}{1 - 0.5z^{-1}} < \infty$$



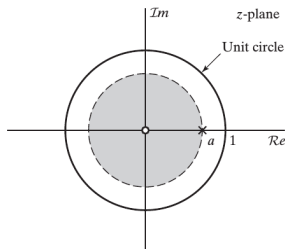
Using infinite geometric series, the equality only holds if  $|0.5 z^{-1}| < 1$ , which can be rewritten in terms of  $z$  as  $|z| > 0.5$ . Thus, the ROC is  $|z| > 0.5$ , the region **outside** a circle of radius 0.5. Generally, there will be one zero at  $z = 0$ , and one pole at  $z = a$ .



## Anti causal ROC

Let  $x[n] = -0.5^n u[-n-1] = \{\dots, -0.5^{-3}, -0.5^{-2}, -0.5^{-1}, 0, 0, 0, \dots\}$ .

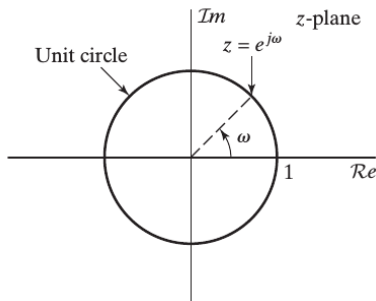
$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]z^{-n} &= - \sum_{n=-\infty}^{-1} 0.5^n z^{-n} = - \sum_{m=1}^{\infty} (0.5^{-1} z)^m = \\ &= - \frac{0.5^{-1} z}{1 - 0.5^{-1} z} = - \frac{1}{0.5z^{-1} - 1} = \frac{1}{1 - 0.5z^{-1}} < \infty\end{aligned}$$



Using infinite geometric series, the equality only holds if  $|0.5 z^{-1}| > 1$ , which can be rewritten in terms of  $z$  as  $|z| < 0.5$ . Thus, the ROC is  $|z| < 0.5$ , the region **inside** a circle of radius 0.5. This is intentional to demonstrate that the transform result alone is insufficient.

## Stability, Causality, and the ROC

- Z-transform  $X(z)$  of  $x[n]$  is unique when and only when specifying the ROC.
- The stability of a system can also be determined by knowing the ROC alone.
- If the ROC contains the unit circle (i.e.,  $|z| = 1$ ) then the system is stable.
- If a system is causal, then the ROC must contain infinity and the system function will be a right-sided sequence.
- If both stability and causality are needed, all the poles of the system function must be inside the unit circle.



# Common Z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All $z$
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z  > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z  < 1$
4. $\delta[n - m]$	$z^{-m}$	All $z$ except 0 (if $m > 0$ ) or $\infty$ (if $m < 0$ )
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
9. $\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z  > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z  > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$

The Z-transform of  $x[n] = a_1 x_1[n] + a_2 x_2[n]$  is,

$$X(z) = \sum_{n=-\infty}^{\infty} (a_1 x_1[n] + a_2 x_2[n]) z^{-n}, \quad (10)$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}, \quad (11)$$

$$= a_1 X_1(z) + a_2 X_2(z). \quad (12)$$

## Properties: Shift in time

The Z-transform of  $x[n - n_0]$  is,

$$X(x[n - n_0]) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}. \quad (13)$$

Let  $m = n - n_0$ ,

$$X(x[n - n_0]) = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}, \quad (14)$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m}, \quad (15)$$

$$= z^{-n_0} X(z). \quad (16)$$

This leads directly to a property analogous to Eq. 2,

$$\mathcal{Z}\{x[n - 1]\} = z^{-1} X(z). \quad (17)$$

The Z-transform of

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k], \quad (18)$$

is,

$$X(z) = \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] z^{-n}, \quad (19)$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}, \quad (20)$$

Let  $m = n - k$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{m=-\infty}^{\infty} x_2[m]z^{-(m+k)}, \quad (21)$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{m=-\infty}^{\infty} x_2[m]z^{-m}, \quad (22)$$

$$= X_1(z) X_2(z). \quad (23)$$

# Constant-Coefficient Difference Equations

Consider a system described by the linear constant-coefficient difference equation,

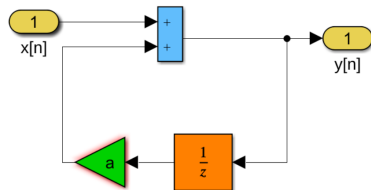
$$y[n] = x[n] + a y[n - 1], \quad (24)$$

$$y[n] - a y[n - 1] = x[n], \quad (25)$$

$$Y(z) - a Y(z) z^{-1} = X(z), \quad (26)$$

$$Y(z)(1 - a z^{-1}) = X(z), \quad (27)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - a z^{-1})} \quad (28)$$



In general,

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k], \quad (29)$$

$$y[n] = \sum_{k=0}^M b_k x[n - k] - \sum_{k=1}^N a_k y[n - k], \quad (30)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-(M)}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-(N)}}. \quad (31)$$

- 1 Wikipedia. Z-transform.
- 2 Alan V. Oppenheim and Ronald W. Schaffer. *Discrete-time signal processing, 3rd Ed.* Prentice Hall. 2010. Chapter 3.
- 3 Paolo Prandoni and Martin Vetterli. *Signal processing for communications.* Taylor and Francis Group, LLC. 2008. Chapter 6.