Infinite impulse response filters

Bilinear z-transform

Dr. Ing. Rodrigo Gonzalez

rodralez@frm.utn.edu.ar
rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Summary

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2. IIR filtering in frequency domain
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4. IIR structures
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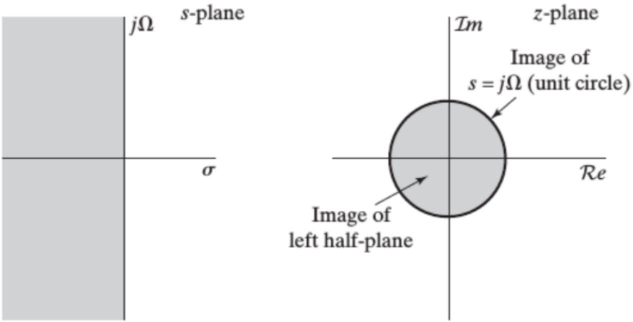
Classification of discrete filters

Table: Classification of discrete filters

	Finite impulse response (FIR)	Infinite impulse response (IIR)
Filtering in time domain	Moving average	Leaky Integrator
Filtering in frequency domain	Windowed Filters Equiripple Minimax	Bilinear z-transform ZOH method

- The main idea is to transform an analog filter to the discrete domain.
- From *s* domain to *z* domain.
- This way, all the theory behind analog filter can be reused to implement a filter in a computer (Butterworth filter, Chebyshev filters, Elliptic filter).

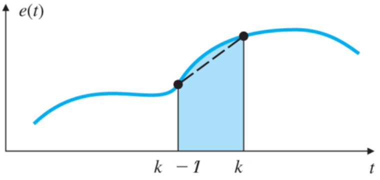


Suppose the following integrator,

$$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}\,. \tag{1}$$

The area under $e(t)$ over $k \times T$ periods is,

$$u(k) = \int_0^{k-1} e(t)dt + \int_{k-1}^{k} e(t)dt\,. \tag{2}$$



Tustin's method uses the trapezoidal integration, to approximate $e(t)$ by a straight line between two samples. The technique is an algebraic transformation between variables s and z.

$$u(k) = u(k-1) + \frac{T}{2}\left[e(k-1) + e(k)\right], \tag{3}$$

$$U(z) = z^{-1}U(z) + \frac{T}{2}\left[z^{-1}E(z) + E(z)\right], \tag{4}$$

$$U(z)(1 - z^{-1}) = \frac{T}{2}\left[E(z)(1 + z^{-1})\right], \tag{5}$$

$$\Longrightarrow \frac{U(z)}{E(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}\,. \tag{6}$$

Comparing Eq. 1 and 6,

$$\boxed{s \approx \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}\,. \tag{7}$$

Reemplazando s por esta expresión podemos digitalizar rápidamente una señal

Relationship between analog and digital frequencies

- Ω is the analog frequency, $-\infty, < \Omega < \infty$.
- ω, the "digital" frequency, $-\pi, < \omega < \pi$, i.e., $-2\pi f_s/2, < \omega < 2\pi f_s/2$.
- What is the relationship between Ω and ω.

Doing $s = j\Omega$, z should be evaluated in the unity circle, so, $z = r \cdot e^{j\omega} = \cdot e^{j\omega}$, with $r = 1$.

$$s = \frac{2}{T}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \frac{2}{T}\left[\frac{2e^{-j\omega/2}(j \sin \omega/2)}{2e^{-j\omega/2}(\cos \omega/2)}\right] = j\frac{2}{T}\tan(\omega/2)\,. \tag{8}$$

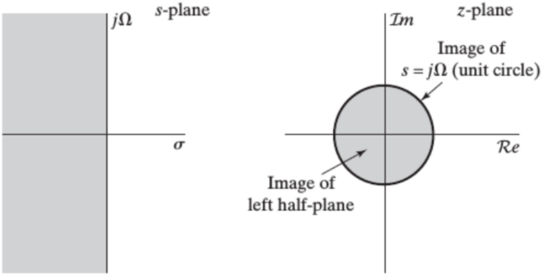
Real and imaginary parts on both sides of Eq. 8 are,

Since $s = \sigma + j\Omega$,

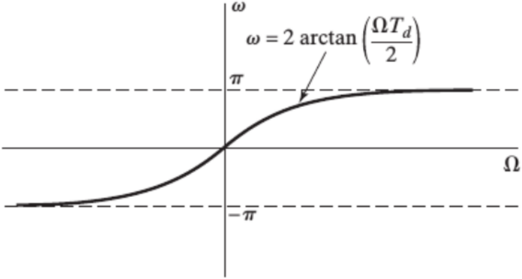
$$\sigma = 0\,, \tag{9}$$

$$\Omega = \frac{2}{T}\tan(\omega/2)\,, \tag{10}$$

$$\Longrightarrow \omega = \arctan(\Omega T/2)\,. \tag{11}$$

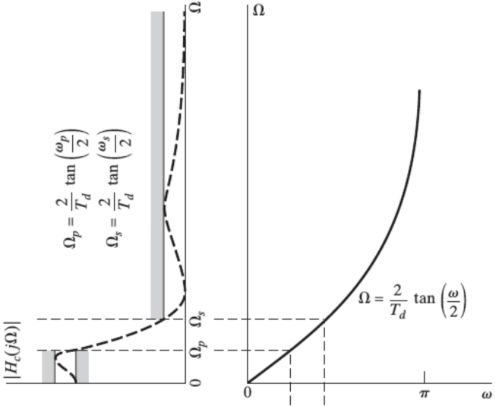


jΩ
s-plane
σ
Im
z-plane
Image of
s = jΩ (unit circle)
Re
Image of
left half-plane



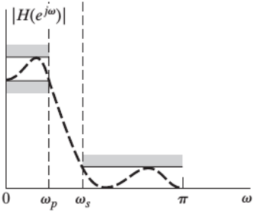
ω
ω = 2 arctan (ΩT_d / 2)
π
Ω
−π

- Non-linear relationship between Ω and ω.
- Analog frequencies has to be adjust **before** analog filter design.



En el dominio analógico no tenemos el mismo valor de frecuencia de corte que en el digital. Pero a nosotros nos interesa el corte digital, por lo que hay que elegir una para el analógico que luego nos de la que queremos en el dominio digital. Esto se llama Pre-warping

Pre-combado



Example of IIR design using bilinear transform

1. Choose the analog filter that complains with the desired performance.

   For example, second-order Butterworth low-pass filter.

$$G(s) = \frac{\Omega_c^2}{s^2 + s\sqrt{2}\Omega_c + \Omega_c^2}$$

2. Cut-off digital frequency is normalized.

$$f_{dc} = 100 \text{ Hz}, \ f_s = 1000 \text{ Hz}, \ T = 0.001 \text{ s}.$$

$$\omega_c = 2\,\pi\,100/1000 = 0.628 \text{ rad/s}.$$ Normalizado por f de muestreo

3. Pre-warp the analog frequencies.

$$\Omega_c = \frac{2}{T}\tan(\omega_c/2) = \frac{2}{0.001}\tan\left(\frac{0.628}{2}\right) = 649.839 \text{ rad/s}.$$

$$f_{ac} = 103.42 \text{ Hz}.$$

4 Replace s by the bilinear transform, $s \approx \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$.

$$H(s) = \frac{\Omega_c^2}{s^2 + s\sqrt{2}\Omega_c + \Omega_c^2}$$

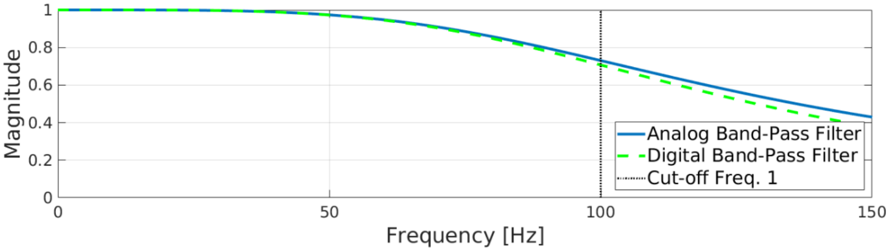
$$H(z) = \frac{(649.84)^2}{\left(\frac{2}{T}\right)^2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\sqrt{2}\,(649.84) + (649.84)^2}$$

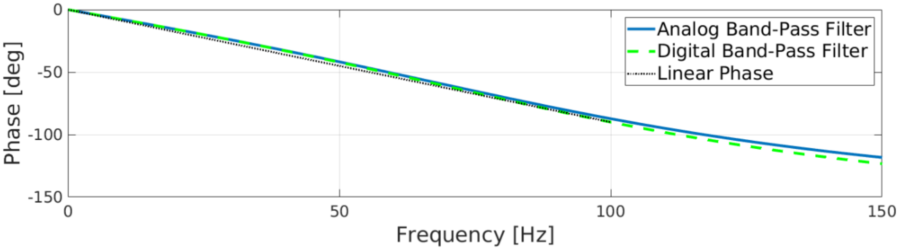
$$H(z) = \frac{0.067 + 0.135z^{-1} + 0.067z^{-2}}{1 - 1.143z^{-1} + 0.413z^{-2}}$$

5 Invert the Z-transform to find the difference equation.

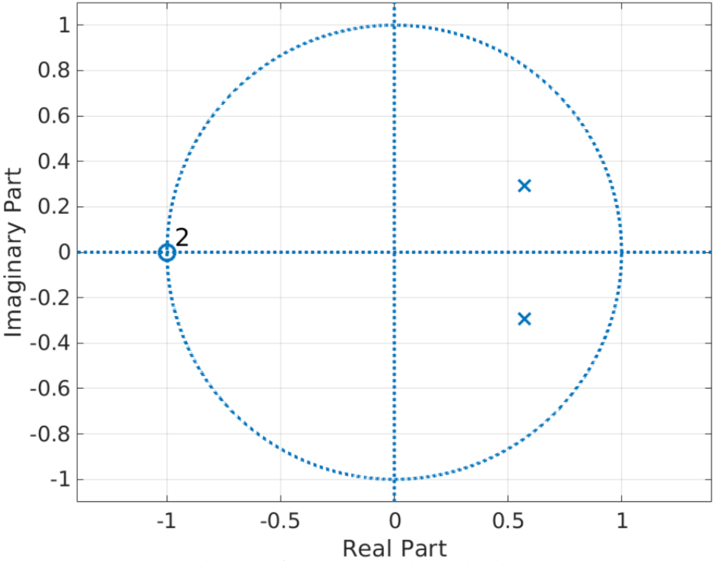
$$\begin{aligned} y[n] = {} & 0.067\,x[n] + 0.135\,x[n-1] + 0.067\,x[n-2] + \\ & + 1.143\,y[n-1] - 0.413\,y[n-2] \end{aligned} \tag{12}$$

Frequency and phase responses





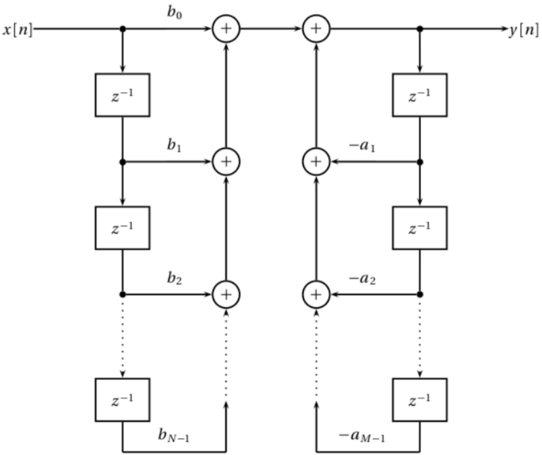
Z plane



Si los polos están muy cerca de la circunferencia un ruido en el cálculo o de cuantización puede inestabilizar el sistema

Direct form I IIR implementation

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{N-1} z^{N-1}}{1 + a_1 z^{-1} + \dots + a_{M-1} z^{M-1}}$$

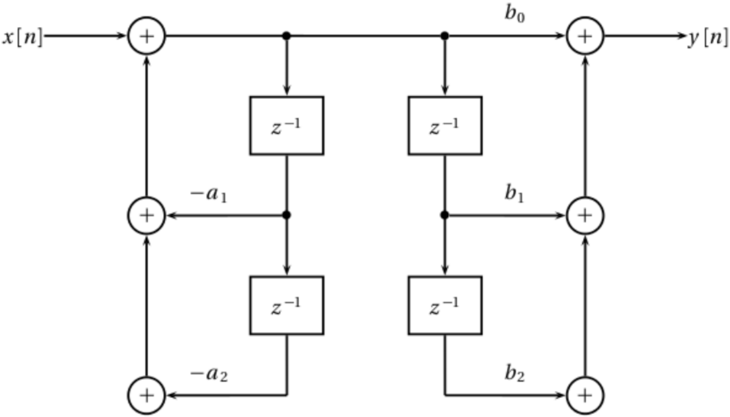


Por cada z^-1 necesito una variable que me guarde un valor pasado

Direct form I IIR implementation inverted

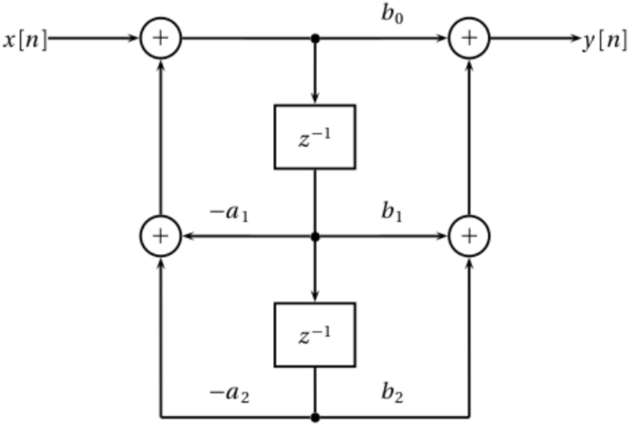
$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

By the commutative properties of the z-transform, we can invert the order of computation to turn the Direct Form I structure into a new structure.



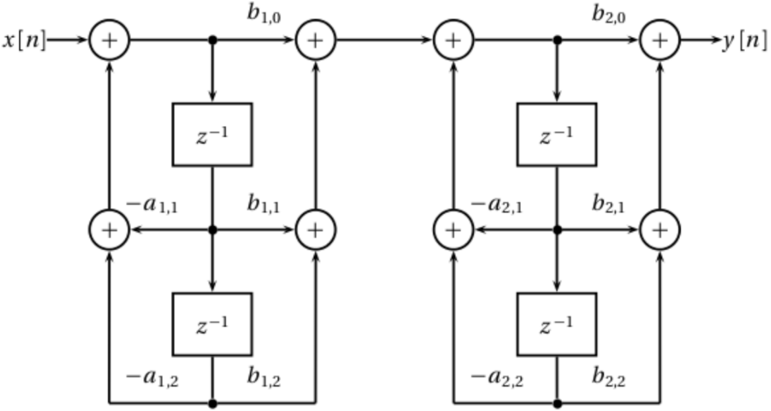
$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

We can then combine the parallel delays together. This implementation is called Direct Form II; its obvious advantage is the reduced number of the required delay elements (hence of memory storage).



The cascade structure of N second-order sections is much less sensitive to quantization errors than the previous Direct form II of order $2 \cdot N$.

$$H(z) = \prod_{k=1}^{N} G_k \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$



FIR vs IIR

FIR, pros:

- Unconditional stability (no poles).
- Precise control of the phase response and, in particular, exact linear phase.
- Robustness with respect to finite numerical precision hardware.

FIR, cons:

- Longer input-output delay.
- Higher computational cost with respect to IIR solutions.

IIR, pros:

- Lower computational cost with respect to an FIR with similar behavior.
- Shorter input-output delay.
- Compact representation.

IIR, cons:

- Stability is not guaranteed.
- Phase response is difficult to control.
- Design is complex in the general case.
- Sensitive to numerical precision.

Bibliography

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3 Gene F. Franklin, J. David Powell and Abbas Emami-Naeini. *Feedback Control of Dynamic Systems*. 7th Edition. 2014. Section 8.3.