State estimation

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Control y Sistemas

Ingeniería Mecatrónica,
Facultad de Ingeniería,
Universidad Nacional de Cuyo

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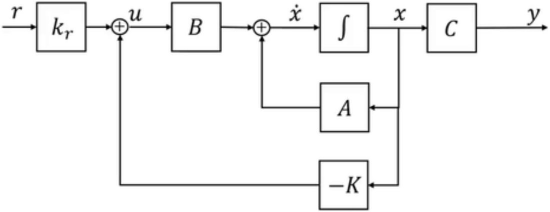
Podemos modificar el desempeño de un sistema lineal modificiando los polos del sistema a LC, es decir sus autovalores.
Esto se hace agregando una matriz de realimentacion K y una de referencia Kr. Así creamos nueva acción de control

Este enfoque requiere acceso completo al vector de estados... Desventaja

State feedback control

Fuerte limtiacion

Idea with state feedback control design: Modify the eigenvalues of the system by using the input, $u = -Kx + k_r r$.

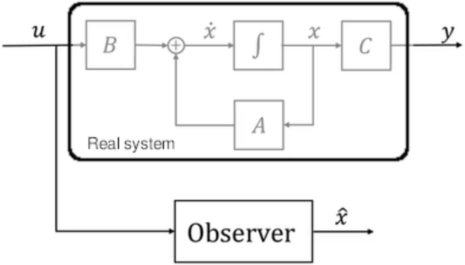


Problem: Requires full access to the state vector, $u = -Kx$

Se estiman los estados para sortear la limitación previa.

State estimation

Idea of state estimation: Develop an observer of the dynamic system that provides an estimate, $\hat{x}$, of the system's states.



El observador tiene un modelo matemático del sistema, y lo compara respecto al real

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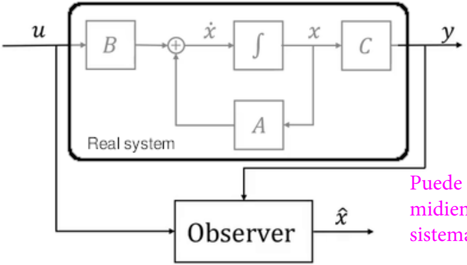


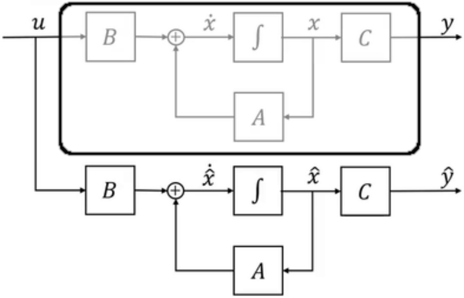
Puede ser también a LC midiendo las salidas del sistema real

State estimation – Open loop estimator

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$

Realistic?

Idea: Use a ***copy*** of the model description in the observer: $\dot{\hat{x}} = A\hat{x} + Bu$



Analyze the error dynamics:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu$$
$$= A(x - \hat{x}) = A\tilde{x}$$

which has the solution:

$$\tilde{x}(t) = e^{At}\tilde{x}(0) = e^{At}(x(0) - \hat{x}(0))$$

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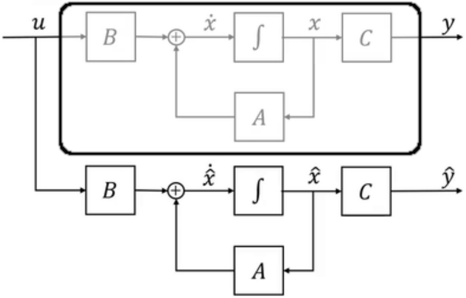

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Anal

$\dot{\tilde{x}} =$

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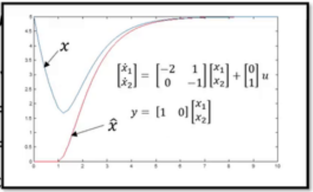

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Para sistema estable tienden a ser iguales y el error tiende a cero.

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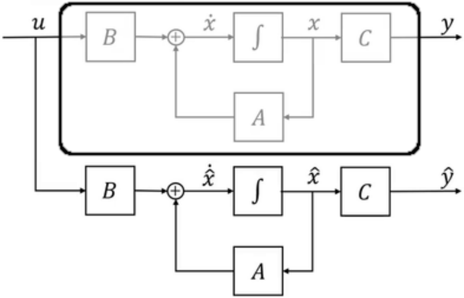

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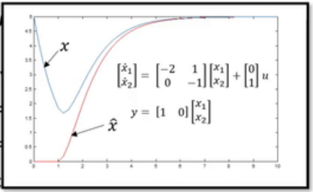
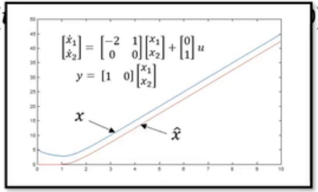
$\tilde{x}($ ... $))$





Para sistema inestable hay error de estado estacionario

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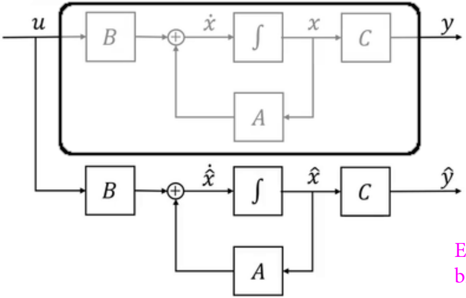

Analyze the error dynamics:

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Problem, A needs to be stable, then $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$.

Es decir que observador a LA solo funciona bien para sistemas estables

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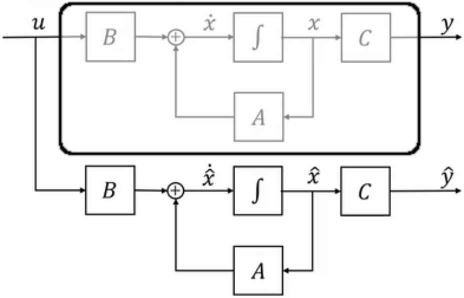

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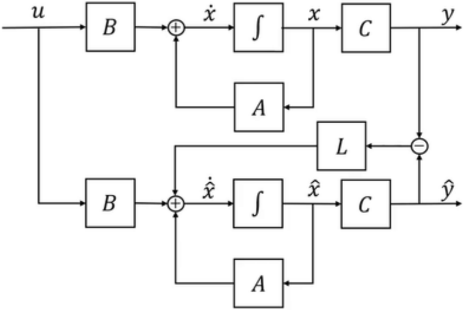
Open loop estimation does not seem to be a good idea!

State estimation – Closed loop estimator

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$

Idea: Use feedback from y in the observer to improve the estimates.



Compare the estimated output with measured:

$$\tilde{y} = y - \hat{y} = Cx - C\hat{x} = C\tilde{x}$$

Feed back the error to the open loop estimator via a feedback gain L:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$$

$$\hat{y} = C\hat{x}$$

L: Ganancia de observación

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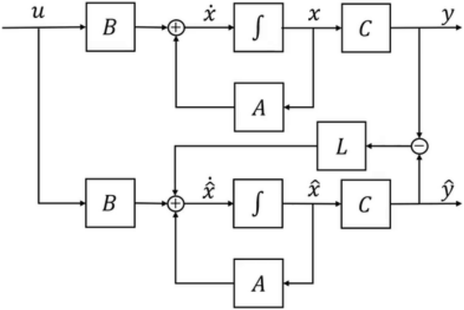

Analyze the error dynamics:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$
$$= Ax + Bu - A\hat{x} - Bu - L\tilde{y}$$
$$= A(x - \hat{x}) - L(y - \hat{y})$$
$$= A(x - \hat{x}) - L(Cx - C\hat{x})$$
$$= A\tilde{x} - LC\tilde{x}$$
$$= (A - LC)\tilde{x}$$

which has the solution:

$$\tilde{x}(t) = e^{(A-LC)t}\tilde{x}(0)$$

L can be chosen such that the error dynamics converges, $\tilde{x} \to 0$ as $t \to \infty$, (if *observable*).

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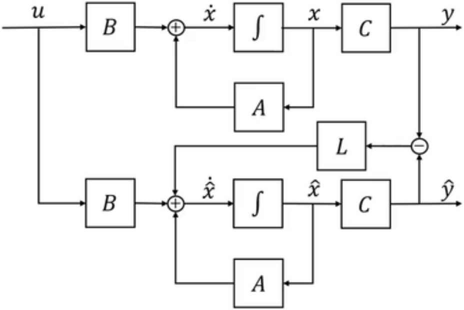
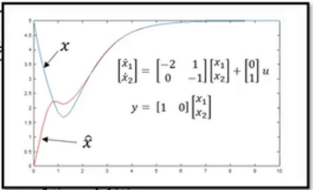


Ar

$\dot{\tilde{x}}$



$= A\tilde{x} - LC\tilde{x}$

Error converge a 0

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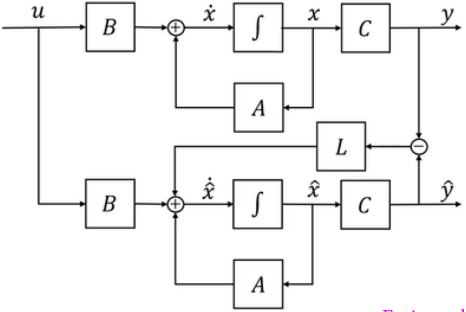

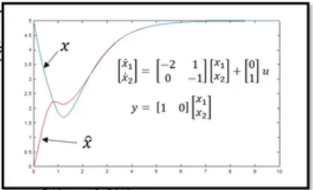

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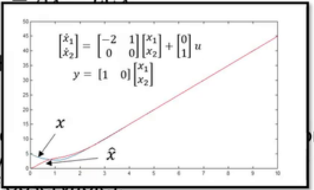


En inestable ya no hay error de estado estacionario

State estimation gain

The way to choose estimator gain is similar to that used for control design. Using the closed loop estimator, the error dynamics becomes:

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

Objective: Choose L such that the closed loop error dynamics $A - LC$ get desired properties, i.e a suitable convergence rate.

Queremos que el observador siga fiel los cambios del sistema real, es decir que sea rápido

State estimation gain

The way to choose estimator gain is similar to that used for control design. Using the closed loop estimator, the error dynamics becomes:

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

Objective: Choose L such that the closed loop error dynamics $A - LC$ get desired properties, i.e a suitable convergence rate.

The closed loop poles of the estimator are the roots to the characteristic polynomial:

$$\det(sI - A + LC) = 0$$

Use pole placement with a desired characteristic polynomial to choose the estimator gain, L.

Elegimos L tal que estos polos sean rápidos

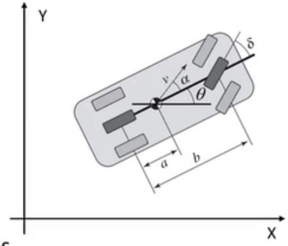
Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



Design a state estimator to estimate the vehicle's states, from measurement of the lateral position.

Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
$b = 4\ m$

Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

Revisit Example - Vehicle steering (Ex 7.4)

State estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \left(\begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

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$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} -l_1 & 12 \\ -l_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda + l_1 & -12 \\ l_2 & \lambda \end{vmatrix} = \lambda^2 + l_1\lambda + 12 l_2$$

Matching with desired characteristic polynomial gives:

$$\lambda^2 + 10\lambda + 24 \equiv \lambda^2 + l_1\lambda + 12 l_2 \quad \Longrightarrow \quad l_1 = 10 \qquad l_2 = 2$$

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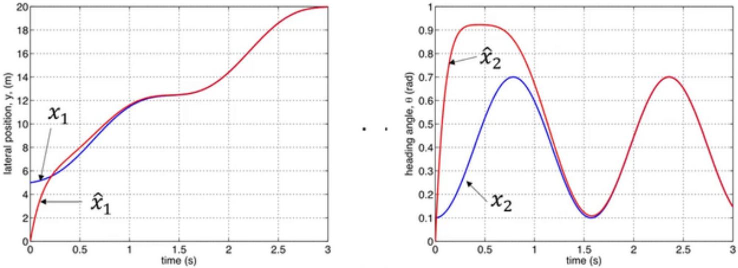
State estimator: $$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u + \begin{bmatrix} 10 \\ 2 \end{bmatrix} (y - \hat{x}_1)$$

Revisit Example - Vehicle steering (Ex 7.4)

Simulations using a sinusiodal input, with $x(0)$ = (5, 0.1) and $\hat{x}(0)$ = (0, 0):



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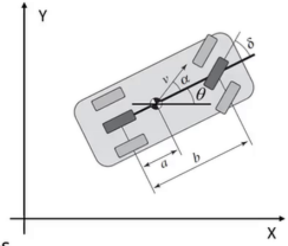

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State estimator:

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The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} 0 & 12 - l_1 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda & -12 + l_1 \\ 0 & \lambda + l_2 \end{vmatrix} = \lambda(\lambda + l_2)$$

Hemos perdido un gdl y por ende no podemos determinar el valor de l1

Esto sucede porque el sistema no es observable

Es un requisito que el sistema sea observable para implementar un observador

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It is not possible shape the error dynamics. We say that the system is not ***observable***.

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The closed loop system has the characteristic polynomial

unobservable

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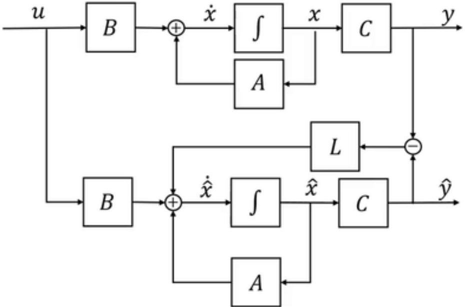
Control using estimated states

Use the estimated states for feedback, $u = -K\hat{x} + k_r r$.



Una vez diseñado el observador que siga rápidamente los cambios del sistema real, resta ver cómo controlamos al sistema. Dónde cerramos el lazo con nuestro observador.

Control using estimated states

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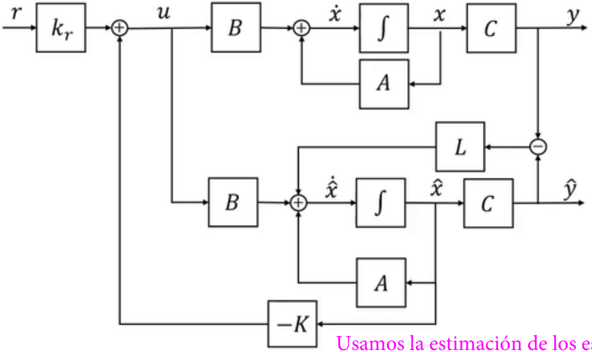
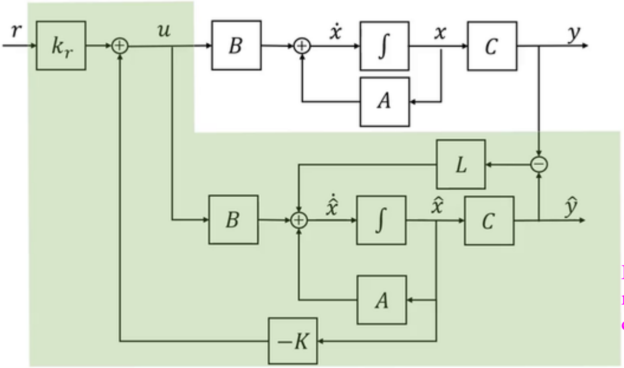


Usamos la estimación de los estados del sistema real.

Control using estimated states

Use the estimated states for feedback, $u = -K\hat{x} + k_r r$.



Esto será parte de nuestro algoritmo de control.

Queda plantear el nuevo sistema que contemple tanto el cálculo de la matriz K como de la matriz L

Será un sistema aumentado que contendrá también los estados del error de estimación(segunda fila)

Control using estimated states

Given the system, $\dot{x} = Ax + Bu$, $y = Cx$, the controller, $u = -K\hat{x} + k_r r$, and the state estimator, $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, the closed loop system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - CL \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

Control using estimated states

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The closed loop system has the characteristic polynomial

$$\lambda(s) = det(sI - A + Bk)det(sI - A + LC)$$

This polynomial can be assigned arbitrary roots if the system is reachable and observable.

Rule of thumb: Make the estimator poles 4-5 times faster then the "feedback" poles.
4 o 5 veces más hacia la izquierda

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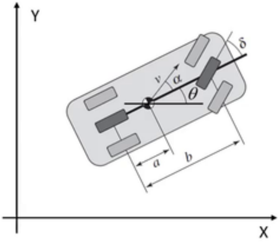


State feedback control (poles in -1 (double pole)):

$$u = -0.0278x_1 - 0.6111x_2 + 0.0278r$$

Estos definen K

Este valor define Kr

Vehicle data: $v_0 = 12\ m/s$

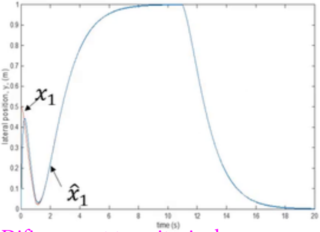
$a = 2\ m$

$b = 4\ m$

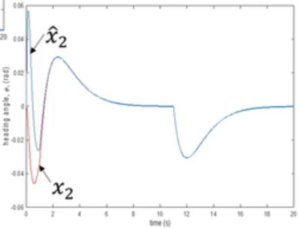
State estimator (poles in -4 and -6):

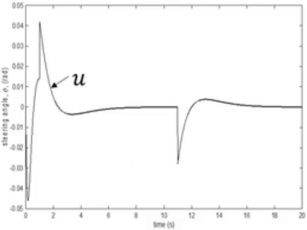
$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u + \begin{bmatrix}10\\ 2\end{bmatrix}(y - \hat{x}_1)$$

Revisit Example - Vehicle steering (Ex 7.4)

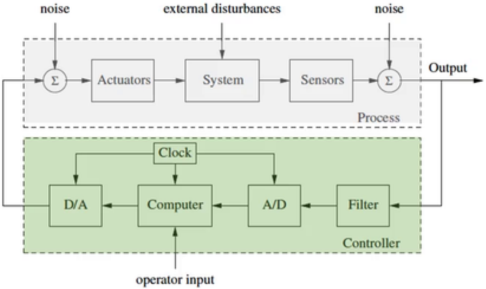


Difieren en t transitorio, luego son indistinguibles





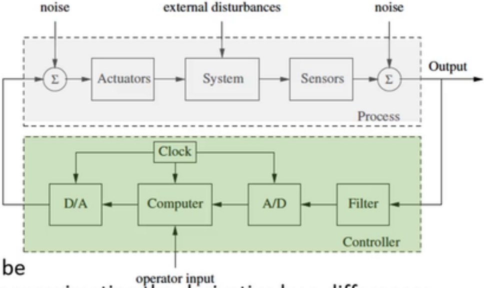
Implementation



Gris: Sistema físico real

Verde: Implementación digital (hoy en dia son algoritmos)

Implementation



Our controller consists of the state feedback controller,

$$u = -K\hat{x} + k_r r,$$

and the state estimator,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

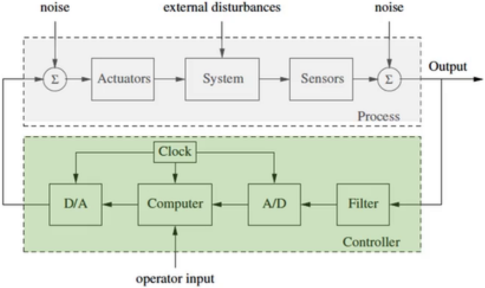
We need to discretize the controller to be able to implement it in a computer, by approximating the derivative by a difference:

$$\dot{\hat{x}} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{t_{k+1} - t_k} = A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k)),$$

Rewriting it as a difference equation:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + \underbrace{(t_{k+1} - t_k)}_{h\text{ - sampling time}}(A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k)),$$

Implementation



In pseudocode:

Ley de control con observador del sistema

```
% Control algorithm - main loop
r = adin(ch1)                          % read reference
y = adin(ch2)                          % get process output
u = -K*xhat + kr*r                     % compute control variable
daout(ch1, u)                          % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x))    % update state estimate
```

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.