

# Observability

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## Observability

**Definition** (Observability): A linear system is **observable** if for every  $T > 0$  it is possible to determine the state of the system  $x(T)$  through the measurements of  $y(t)$  and  $u(t)$  on the interval  $[0, T]$ .

Recall the solution to the differential equation:

$$y(t) = Ce^{At}x(0) \quad A \text{ es una matriz. Oho}$$

Since we know  $u(t)$ , we only need to consider zero input case.  $D^*u=0$

So, if  $x(0)$  can be determined, then we can reconstruct  $x(t)$  exactly.

## Observability

**Definition:** A **state**  $x^* \neq 0$  is said to be **unobservable** if the zero-input solution  $y(t) = Ce^{At}x(0)$ , with  $x(0) = x^*$ , is zero for all  $t \geq 0$ .

So, if we can find a state  $x^*$  for which  $Ce^{At}x(0) = 0$  for all  $t \geq 0$ , then the system is unobservable. For this to hold, all derivatives must be zero at  $t = 0$ .

$$\begin{aligned}
 Ce^{At}x^*|_{t=0} = 0 &\Rightarrow Cx^* = 0 \\
 \frac{d}{dt}Ce^{At}x^*|_{t=0} = 0 &\Rightarrow CAe^{At}x^*|_{t=0} = CAx^* = 0 \\
 \frac{d^2}{dt^2}Ce^{At}x^*|_{t=0} = 0 &\Rightarrow CA^2e^{At}x^*|_{t=0} = CA^2x^* = 0 \\
 &\vdots \\
 \frac{d^{n-1}}{dt^{n-1}}Ce^{At}x^*|_{t=0} = 0 &\Rightarrow CA^{n-1}e^{At}x^*|_{t=0} = CA^{n-1}x^* = 0
 \end{aligned}$$

Como  $x^* \neq 0$ , entonces la matriz debe tener rango menor a  $n \rightarrow$  SE sin solución

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x^* = 0$$

Son filas

$\underbrace{\hspace{10em}}_{W_o}$

Matriz de observabilidad

Si tiene rango completo decimos que el sistema es observable

## Observability

**Theorem** (Observability rank condition): *A linear system is observable if and only if the observability matrix reachability  $W_o$*

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

*has full row rank.*

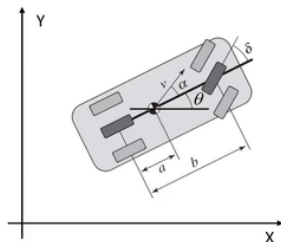
## Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

where  $x_1$  is the lateral position  $Y$ ,  $x_2$  is the heading orientation  $\theta$  and  $u$  is the steering angle  $\delta$ .



Vehicle data:  $v_0 = 12 \text{ m/s}$

$a = 2 \text{ m}$

$b = 4 \text{ m}$

## Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

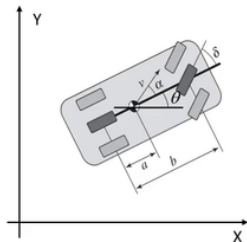
Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 12 \end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 1 & 0 \\ 0 & 12 \end{vmatrix} = 1 \cdot 12 - 0 \cdot 0 = 12$$

*The system is observable!*



Note: A square matrix  $M$  ( $n \times n$ ) has full rank  $n$  iff the  $\det(M) \neq 0$

## Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with heading angle as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

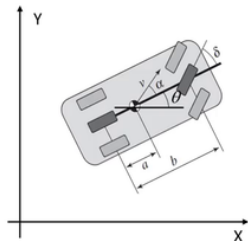
Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 0 = 0$$

*The system is not observable!*



Esto nos dice qué tipo de sensor usar en el sistema. Sería conveniente un gps y no un giróscopo o una brújula.

# Observability

**Definition:** A **state**  $x^* \neq 0$  is said to be **unobservable** if the zero-input solution  $y(t) = Ce^{At}x(0)$ , with  $x(0) = x^*$ , is zero for all  $t \geq 0$ .

So, look at our vehicle steering example with heading angle as output:

$$\begin{bmatrix} C \\ CA \end{bmatrix} x^* = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^* = 0 \quad \Rightarrow \quad x^* = \begin{bmatrix} x_1(0) \neq 0 \\ 0 \end{bmatrix}$$

Este es el estado inicial que hace que la salida del sistema sea siempre nula. Tiene sentido al tener angulo de orientacion 0. La salida va a seguir siendo 0.



- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.