Space feedback control

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Control y Sistemas

Ingeniería Mecatrónica,
Facultad de Ingeniería,
Universidad Nacional de Cuyo

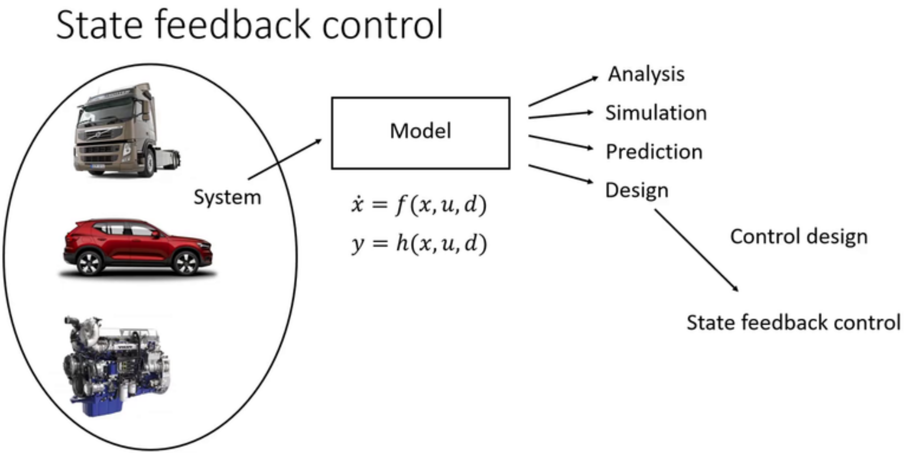
June 2020

Summary

1. State feedback control
   - Feedback gain
   - Reference tracking
   - Integral action
   - Example

2. Reachability
   - Definition
   - Revisit Example

State feedback control



State feedback control

Consider a linear time-invariant state-space model given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$ Tiene que ser LTI

$$y(t) = Cx(t) + Du(t)$$ D típicamente es nula

where $x(t) \in \mathbb{R}^n$ is the state (vector), $u(t) \in \mathbb{R}^p$ is the input or control signal and $y(t) \in \mathbb{R}^q$ is the output signal. (For SISO case, $p = 1, q = 1$)

The system poles are given by the eigenvalues of the system matrix $A \in \mathbb{R}^{n \times n}$.

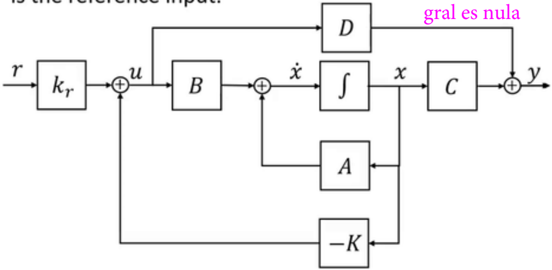
State feedback control

Idea with control design: Modify the eigenvalues of A by using the input $u(t)$

State feedback controller: $u(t) = -Kx(t) + k_r r(t)$

where $K \in \mathbb{R}^{p\times n}$ is the feedback gain, $k_r \in \mathbb{R}^{p\times r}$ is the steady-state reference gain and $r(t) \in \mathbb{R}^r$ is the reference input.

D se suele quitar porque por lo gral es nula

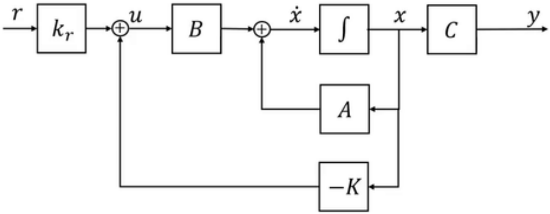


State feedback control

Idea with control design: Modify the eigenvalues of A by using the input $u(t)$

State feedback controller: $u(t) = -Kx(t) + k_r r(t)$

where $K \in \mathbb{R}^{p\times n}$ is the feedback gain, $k_r \in \mathbb{R}^{p\times r}$ is the steady-state reference gain and $r(t) \in \mathbb{R}^r$ is the reference input.



State feedback control

Idea with control design: Modify the eigenvalues of A by using the input $u(t)$

State feedback controller: $u(t) = -Kx(t) + k_r r(t)$

where $K \in \mathbb{R}^{p\times n}$ is the feedback gain, $k_r \in \mathbb{R}^{p\times r}$ is the steady-state reference gain and $r(t) \in \mathbb{R}^r$ is the reference input.

Using the state feedback controller the closed loop dynamics becomes:

$$\dot{x}(t) = Ax(t) + B(-Kx(t) + k_r r(t))$$
$$= (A - BK)x(t) + Bk_r r(t)$$

Control objective: Choose K such that the closed loop dynamics $A - BK$ get desired properties, i.e fulfill the specifications or stabilize the system.

SISO case: *n parameters in K and n eigenvalues in A, so it might be possible!*

Reference tracking

The steady-state reference gain, k_r, does not affect the stability, but it does affect the steady-state solution.

The steady-state gain is usually chosen such that:

$$y(t) \approx r(t) \;\; as\; t \rightarrow \infty$$

At steady-state the time derivative of the state variable is $\dot{x}\,(t) \equiv 0$, so

$$0 = (A - BK)x(t) + Bk_r r(t)$$

$$y(t) = Cx(t) + Du(t)$$

Reference tracking

The steady-state reference gain, k_r, does not affect the stability, but it does affect the steady-state solution.

The steady-state gain is usually chosen such that:

$$y(t) \approx r(t) \;\; as\; t \to \infty$$

At steady-state the time derivative of the state variable is $\dot{x}(t) \equiv 0$, so

$$\left.\begin{aligned} 0 &= (A - BK)x(t) + Bk_r r(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \quad y = -C(A - BK)^{-1}Bk_r r$$

If $y(t) \approx r(t) \;\; as\; t \to \infty$, then k_r should be chosen as

$$k_r = -(C(A - BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A - BK)^{-1}B$$

kr debe eliminar error de estado estacionario, es decir hacer y=r en t infinito

Integral action

Using the steady-state feedback gain, k_r, can achieve zero steady-state error, but it does depend on the model parameters, as

$$k_r = -(C(A-BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A-BK)^{-1}B$$

Si lidiamos con un sistema real, las matrices C,A,B no representan el sistema de forma exacta.
Esto hace que Kr no "elimine" el error de estado estacionario.

Así surge señal de control:

Integral action

Using the steady-state feedback gain, k_r, can achieve zero steady-state error, but it does depend on the model parameters, as

$$k_r = -(C(A-BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A-BK)^{-1}B$$

Introduce ***integral action*** to remove the steady-state error. Approach: introduce an additional state variable in our system which computes the integral of the error

$$\dot{z}(t) = y(t) - r(t)$$

The new state-space model becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ Cx - r \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

Given the new state-space model, we design a controller in the usual fashion and the resulting controller becomes:

$$u(t) = -Kx(t) - K_I z(t) + k_r r(t)$$

Volante es la acción de control en este ejemplo.

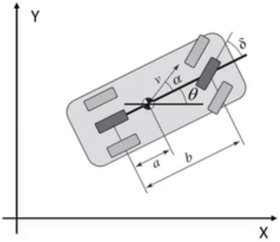
Example 1 - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



The idea is to design a controller that ***stabilizes*** the dynamics and ***tracks*** a given lateral position of the vehicle.

Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
$b = 4\ m$

Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

Example 1 - Vehicle steering (Ex 7.4)

State feedback control:

$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$$

Example 1 - Vehicle steering (Ex 7.4)

State feedback control:

$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$$

The closed loop system dynamics becomes

$$\dot{x} = (A - BK)x + Bk_r r = \left(\begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_1 a v_0 / b & k_2 a v_0 / b \\ k_1 v_0 / b & k_2 v_0 / b \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_r a v_0 / b \\ k_r v_0 / b \end{bmatrix} r$$

Example 1 - Vehicle steering (Ex 7.4)

State feedback control:

$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$$

The closed loop system dynamics becomes

$$\dot{x} = (A - BK)x + Bk_r r = \begin{bmatrix} -k_1 a v_0 / b & v_0 - k_2 a v_0 / b \\ -k_1 v_0 / b & -k_2 v_0 / b \end{bmatrix} x + \begin{bmatrix} k_r a v_0 / b \\ k_r v_0 / b \end{bmatrix} r$$

Example 1 - Vehicle steering (Ex 7.4)

State feedback control:

$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$$

The closed loop system dynamics becomes

$$\dot{x} = (A - BK)x + Bk_r r = \begin{bmatrix} -k_1 a v_0/b & v_0 - k_2 a v_0/b \\ -k_1 v_0/b & -k_2 v_0/b \end{bmatrix} x + \begin{bmatrix} k_r a v_0/b \\ k_r v_0/b \end{bmatrix} r$$

$$y = Cx + Du = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + BK) = \cdots = \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

Matching with desired characteristic polynomial gives:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

Example 1 - Vehicle steering (Ex 7.4)

The steady-state gain can be determined:

$$k_r = -1/C(A - BK)^{-1}B = \cdots = k_1 = \frac{b\omega_n^2}{v_0^2}$$

Example 1 - Vehicle steering (Ex 7.4)

The steady-state gain can be determined:

$$k_r = -1/C(A-BK)^{-1}B = \cdots = k_1 = \frac{b\omega_n^2}{v_0^2}$$

Inserting these control design parameters into the feedback controller gives:

$$u = -k_1 x_1 - k_2 x_2 + k_r r = -\frac{b\omega_n^2}{v_0^2} x_1 - \left(\frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}\right) x_2 + \frac{b\omega_n^2}{v_0^2} r$$

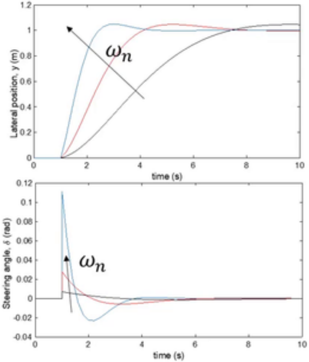
acción de control

Aumentar ohmn hace al sistema más rápido, pero crece drásticamente la acción de control

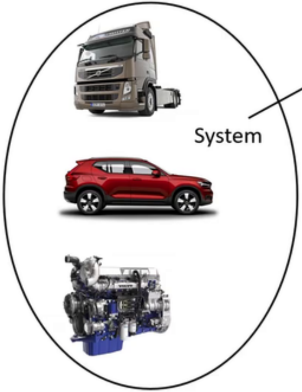
aumentar zita reduce el sobrepico sin influir en la accion de control

Example 1 - Vehicle steering (Ex 7.4)

Simulations with different values of ζ and ω_n:



Reachability



Model

$$\dot{x} = f(x,u,d)$$
$$y = h(x,u,d)$$

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Analysis → ***Reachability***

Simulation

Prediction

Design

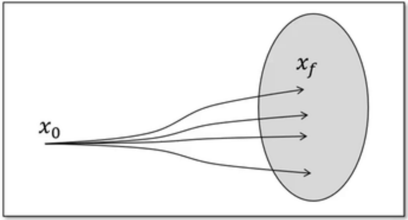
Control design

State feedback control

$$u = -Kx + k_r r$$

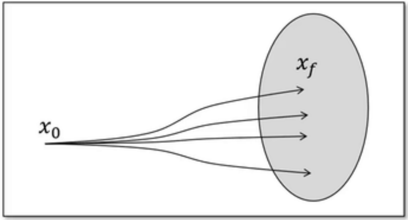
Reachability

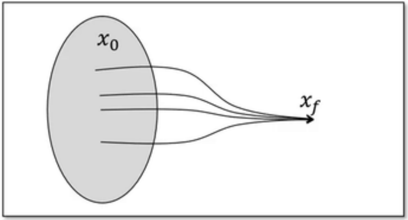
Definition (Reachability): A linear system is ***reachable*** if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and u: $[0, T] \rightarrow \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.



Reachability

Definition (Reachability): A linear system is ***reachable*** if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and u: $[0,T] \rightarrow \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.

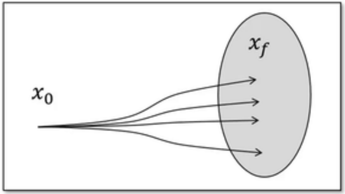




Sometimes the definition of ***controllable*** and ***controllability*** is used, and that is similar.

Para cq x0 tenemos accion de control que nos lleva al resultado deseado xf.
Los resultados para ambas definiciones son los mismos.

Reachability



To see that an arbitrary point can be reached, we can use the convolution equation.

Assume that the system starts from zero, the state of a linear system is given by:

$$x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau) d\tau = \int_0^t e^{A\tau} B\, u(t-\tau) d\tau$$

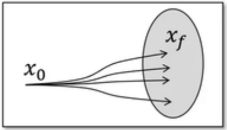
From linear theory it can be shown that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau)$$

where $\alpha_i(t)$ are scalar functions, so that

$$x(t) = B\int_0^t \alpha_0(\tau)\, u(t-\tau) d\tau + AB\int_0^t \alpha_1(\tau)\, u(t-\tau) d\tau + \cdots$$
$$+ A^{n-1}B\int_0^t \alpha_{n-1}(\tau)\, u(t-\tau) d\tau$$

Reachability



By writing it in "vector form", we get:

$$x(t) = \underbrace{[B \quad AB \quad \dots \quad A^{n-1}B]}_{W_r} \begin{bmatrix} \int_0^t \alpha_0(\tau)\, u(t-\tau)d\tau \\ \int_0^t \alpha_1(\tau)\, u(t-\tau)d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau)\, u(t-\tau)d\tau \end{bmatrix}$$

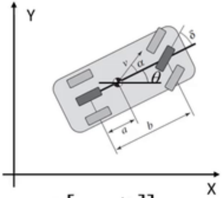
To reach an arbitrary point is state-space, we require that W_r is nonsingular. The matrix W_r is called the ***reachability matrix***.

Theorem (Reachability rank condition): *A linear system is reachable if and only if the reachability matrix is invertable (has full rank).*

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Reachability matrix:

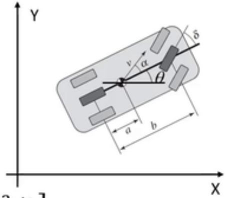
$$W_r = [B \quad AB] = \begin{bmatrix}\begin{matrix}av_0/b\\ v_0/b\end{matrix} & \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}\end{bmatrix}$$

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} av_0/b \\ v_0/b \end{bmatrix} u$$

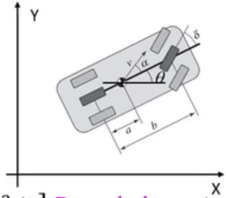
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$



Reachability matrix:

$$W_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

Revisit Example - Vehicle steering (Ex 7.4)



Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

Reachability matrix:

$$W_r = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}av_0/b & v_0^2/b\\ v_0/b & 0\end{bmatrix}$$

Depende de matrices del sistema. Si no es alcanzable se pueden alterar estas matrices para llegar a un sistema alcanzable

Compute the determinant:

$$\det(W_r) = \begin{vmatrix}av_0/b & v_0^2/b\\ v_0/b & 0\end{vmatrix} = av_0/b \cdot 0 - v_0/b \cdot v_0^2/b = -v_0^3/b^2 \neq 0$$

The system is reachable, as long as $v_0 \neq 0$.

Revisit - Example

Return to our example, with the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

Note: A square matrix M (n×n) has full rank n iff the det(M)≠0

Reachability matrix:

$$W_r = [B \quad AB] = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

Compute the determinant:

$$\det(W_r) = \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = 1 \cdot 0 - 0 \cdot 0 = 0$$

The system is not reachable!

->Proponer otra matriz B

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 7.