The Z-transform

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncu.edu.ar

Control y Sistemas

Facultad de Ingeniería,
Universidad Nacional de Cuyo

UNCUYO
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NACIONAL DE CUYO

Summary

Laplace transfrom

- The Z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- Like the Laplace transform for continuous systems, but for discrete ones.

The Laplace transform is defined as:

$$\mathcal{L}\{x(t)\} = F(s) = \int_0^\infty x(t)e^{-st}\mathrm{dt} \tag{1}$$

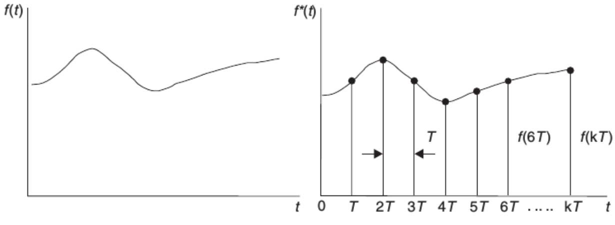
$$\mathcal{L}\{\dot{x}(t)\} = s\,F(s) \quad \text{Asumiendo CI nulas} \tag{2}$$

Eq. 2 enables us to find the transfer function of a linear continuous time system, given the differential equation description of that system.

tiempo continuo-->Laplace-->frecuencia continua

Firstly, we define:

$x[n]$ is the sampled version of $x(t)$, n is an integer and T is the sampling time.



(a) Continuous Signal (b) Sampled Signal

z is a complex number,

$$z = r\, e^{j\omega} = r \cdot (\cos\omega + j \sin\omega)\,. \tag{3}$$

The bilateral or two-sided Z-transform of a discrete-time signal $x[n]$ is the formal power series $X(z)$ defined as, Los negativos serían valores que ya sucedieron. Es teórica, no usamos esta transformada.

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}\,. \tag{4}$$

Unilateral Z-transform

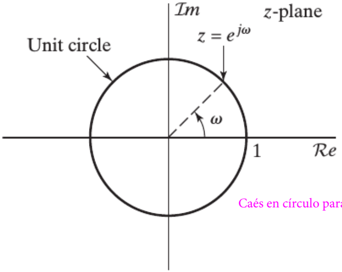
Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$ the single-sided or unilateral Z-transform is defined as,

Esta es la que usamos

Podés agregarle atrasos

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \, . \tag{5}$$

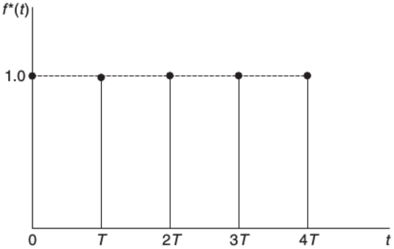
$$z = r\, e^{j\omega} = r \cdot (\cos\omega + j \sin\omega) \, .$$



Caés en círculo para r=1

Example

Find the Z-transform of the unit step function $u(t) = 1$.



$$\mathcal{Z}\{u[n]\} = U(z) = \sum_{n=0}^{\infty} u(kT)z^{-n}, \tag{6}$$

$$U(z) = z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-n} + \cdots \tag{7}$$

Equation 7 can be written in closed-form as,

$$\mathcal{Z}\{u[n]\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}. \tag{8}$$

Only for |z|>1

The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$ROC = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\}. \tag{9}$$

Example (no ROC)

Let $x[n] = 0.5^n = \{\cdots, 0.5^{-3}, 0.5^{-2}, 0.5^{-1}, 1, 0.5, 0.5^2, \cdots\} \Rightarrow$

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} \to \infty$$

Therefore, there are no values of z that satisfy this condition.

La podés separar en la parte de 0 a inf y de 0 a -inf y ambas convergen en zonas distintas.

Causal ROC

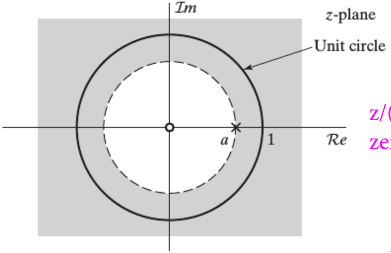
Sistema es causal cuando arranca en 0

Let $x[n] = 0.5^n u[n] = \{\cdots, 0, 0, 0, 1, 0.5, 0.5^2, 0.5^3, \cdots\}$.

inequality holds for
1 != 0.5z^-1

$$\sum_{n=-\infty}^{\infty} (0.5^n u[n])\, z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n = \frac{1}{1-0.5z^{-1}} < \infty$$

equality holds for condition A



z/(z-a)
zero at z=0 and pole at z=a

condition A

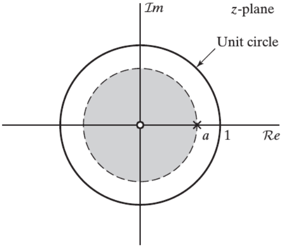
Using infinite geometric series, the equality only holds if $|0.5\, z^{-1}| < 1$, which can be rewritten in terms of z as $|z| > 0.5$. Thus, the ROC is $|z| > 0.5$, the region **outside** a circle of radius 0.5. Generally, there will be one zero at $z = 0$, and one pole at $z = a$.

Anti causal ROC

Let $x[n] = -0.5^n u[-n-1] = \{\cdots, -0.5^{-3}, -0.5^{-2}, -0.5^{-1}, 0, 0, 0, \cdots\}$.

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} 0.5^n z^{-n} = -\sum_{m=1}^{\infty} \left(0.5^{-1}z\right)^m =$$

$$= -\frac{0.5^{-1}z}{1-0.5^{-1}z} = -\frac{1}{0.5z^{-1}-1} = \frac{1}{1-0.5z^{-1}} < \infty$$



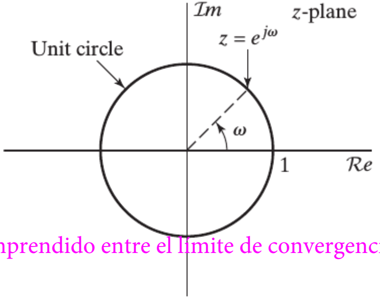
No usamos sistemas anti causales, pero los entendemos

Es inestable porque la ROC no incluye circunferencia de radio 1

Using infinite geometric series, the equality only holds if $|0.5\,z^{-1}| > 1$, which can be rewritten in terms of z as $|z| < 0.5$. Thus, the ROC is $|z| < 0.5$, the region **inside** a circle of radius 0.5. This is intentional to demonstrate that the transform result alone is insufficient. We take into account the ROC

Stability, Causality, and the ROC

- Z-transform X(z) of x[n] is unique when and only when specifying the ROC.
- The stability of a system can also be determined by knowing the ROC alone.
- If the ROC contains the unit circle (i.e., $|z| = 1$) then the system is stable.
- If a system is causal, then the ROC must contain infinity and the system function will be a right-sided sequence.
- If both stability and causality are needed, all the poles of the system function must be inside the unit circle. We usuallly work with this for stability (i think)



La ROC suele ser el anillo comprendido entre el límite de convergencia del sistema causal y el anticausal.

Common Z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m} Atraso en gral	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Se separan sistemas en su parte causal y anti causal y se aplican estas propiedades

Properties: Linearity

The Z-transform of $x[n] = a_1 x_1[n] + a_2 x_2[n]$ is,

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} (a_1\, x_1[n] + a_2\, x_2[n]) z^{-n}\,, && (10)\\
&= a_1 \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n}\,, && (11)\\
&= a_1 X_1(z) + a_2 X_2(z)\,. && (12)
\end{aligned}
$$

Properties: Shift in time

The Z-transform of $x[n-n_0]$ is,

$$X(x[n-n_0]) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}\,. \tag{13}$$

Let $m = n - n_0$,

$$X(x[n-n_0]) = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}\,, \tag{14}$$

$$= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m}\,, \tag{15}$$

$$= z^{-n_0}X(z)\,. \tag{16}$$

This leads directly to a property analogous to Eq. 2,

Mepa que no es a la eq 2 a lo que se refiere porque esa era derivada y esto es atraso.

$$\mathcal{Z}\{x[n-1]\} = z^{-1}X(z)\,. \tag{17}$$

Ojo, capaz sí es. Atrasos discretos son derivadas?

The Z-transform of

La TZ de la convolución es el producto de las TZ de sus funciones. DEMOSTRACION

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \,, \tag{18}$$

is,

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right] z^{-n} \,, \tag{19}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \,, \tag{20}$$

Let $m = n - k$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{m=-\infty}^{\infty} x_2[m] z^{-(m+k)} \,, \tag{21}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \sum_{m=-\infty}^{\infty} x_2[m] z^{-m} \,, \tag{22}$$

$$= X_1(z)\, X_2(z) \,. \tag{23}$$

Consider a system described by the linear constant-coefficient difference equation,

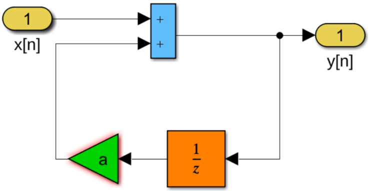
$$y[n] = x[n] + a\,y[n-1]\,, \qquad (24)$$

$$y[n] - a\,y[n-1] = x[n]\,, \qquad (25)$$

$$Y(z) - a\,Y(z)\,z^{-1} = X(z)\,, \qquad (26)$$

$$Y(z)(1 - a\,z^{-1}) = X(z)\,, \qquad (27)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - a\,z^{-1})} \qquad (28)$$



In general,

$$\sum_{k=0}^{N} a_k\,y[n-k] = \sum_{k=0}^{M} b_k\,x[n-k]\,, \qquad (29)$$

$$y[n] = \sum_{k=0}^{M} b_k\,x[n-k] - \sum_{k=1}^{N} a_k\,y[n-k]\,, \qquad (30)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1\,z^{-1} + b_2\,z^{-2} + \ldots + b_M\,z^{-(M)}}{1 + a_1\,z^{-1} + a_2\,z^{-2} + \ldots + a_N\,z^{-(N)}}\,. \qquad (31)$$

Ec típica de un filtro digital

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