A Note on the Tight Simplification of Mechanisms

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Abstract

Milgrom (2008) proposes to simplify mechanisms by restricting their message space. When doing so, it is important not to create new equilibria. A tight simplification is one that does not create new Nash equilibria, a strongly tight simplification is one that does not create new ε-Nash equilibria. This note offers characterizations of tightness and strong tightness. When the preference domain is that of continuous utility functions on the outcome space, the two notions are equivalent, and are also equivalent to the outcome closure property of Milgrom (2008).

1 Introduction

The theory of mechanism design often relies on direct mechanisms where the message space corresponds to the type space of the agents, which includes preferences and any private knowledge that they may have. In practice, these message spaces are often much richer than what can be reasonably implemented with a real world design. It is therefore important to know how to design simple mechanisms without losing desirable theoretical properties. To address this question, Milgrom (2008) introduces a notion of simplification that consists in restricting the space of messages available to the players. Simplification has the additional advantage that it can eliminate undesirable equilibria. However, a simplification can also create new equilibria by eliminating profitable deviations. To avoid this, it is important to use tight simplifications. Milgrom (2008) defines a strong notion of tightness, according to which no new ε-Nash equilibrium must be introduced by the simplification. A weaker alternative is to merely rule out the creation of new Nash equilibria.

This note proposes characterizations of the two properties. These characterizations are valid for any preference domain of the players over the outcome space. By specifying the preference domain, it is possible to produce characterizations of tightness that bear on the outcome function of the mechanism. For example, Milgrom (2008) and Milgrom (2009) use the space of continuous utility functions as their preference domain. Under this condition, I show that the outcome closure property of Milgrom (2008) is not only a sufficient condition for strong tightness, but also necessary. I do this by proving that tightness (which is obviously implied by strong tightness) implies the outcome closure property. A byproduct of this demonstration is that there is no difference between tightness and strong tightness when considering the domain of all continuous utility functions.

2 Setup and Definitions

Let $N = \{1, \ldots, N\}$ be a set of players, and $\Omega \subseteq \Omega_1 \times \cdots \times \Omega_N$ denote the set of outcome profiles where Ω_n is the set of possible outcomes for player n. Together, they define an *environment*. A mechanism $\mu = (X, \omega)$ for the environment (Ω, N) specifies a message space $X = X_1 \times \cdots \times X_N$ that defines the strategies available to the players, and an outcome function $\omega : X \to \Omega$. Taken together, a profile of utility functions over outcomes $u = (u_1, \ldots, u_N)$ where $u_n : \Omega_n \to \mathbb{R}$ and a mechanism $\mu(\Omega, N)$ characterize a game (μ, u). Denote by $\mathcal{U}$ a set of acceptable preference profiles over outcomes, and define a *complete environment* as a triple $\mathcal{E} = (\Omega, N, \mathcal{U})$.

Definition 1 (Simplification)**.** *For a given environment* (Ω, N)*, the mechanism* $\mu' = (X', \omega')$ *is a simplification of* $\mu = (X, \omega)$ *(and* μ *is an extension of* μ'*) if for all* $n \in N$*,* $X'_n \subseteq X_n$*, and* ω' *is the restriction of* ω *to* X'*:* $\omega' = \omega_{|X'}$*.*

A simplification is a mechanism that restrains the strategy space of the initial mechanism. By so doing it makes the expression of preferences less complicated, and it can eliminate undesirable equilibria. However, it can also create new Nash equilibria by eliminating the profitable deviations of the players. This is an undesirable property. A tight simplification is one that does not create such problems.

Definition 2 (Tightness)**.** *The simplification μ' of μ is tight in the complete environment $\mathcal{E} = (\Omega, N, \mathcal{U})$ if for every preference profile $u \in \mathcal{U}$, every pure strategy Nash equilibrium of (μ', u) is a pure strategy Nash equilibrium of (μ, u).*

For applications, it is sometimes useful to consider a stronger property.

Definition 3 (Strong Tightness)**.** *The simplification μ' of a mechanism μ is strongly tight for the complete environment $\mathcal{E} = (\Omega, N, \mathcal{U})$ if for every profile $u \in \mathcal{U}$ and every $\varepsilon \geq 0$, every pure strategy profile $x \in X$ that is an ε-Nash equilibrium of (μ', u) is also an ε-Nash equilibrium of (μ, u).*

Where, for reminders, an ε-Nash equilibrium is defined as follows.

Definition 4 (ε-Nash equilibrium)**.** *For $\varepsilon \geq 0$, $x \in X$ is an ε-Nash equilibrium of a game (μ, u) if for each player n, and every strategy $x'_n \in X_n$,*

$$u_n\left(\omega_n(x'_n, x_{-n})\right) \leq u_n\left(\omega_n(x_n, x_{-n})\right) + \varepsilon.$$

A Nash equilibrium being a particular sort of ε-Nash equilibrium, it is clear that strong tightness implies tightness.

3 Necessary and Sufficient Conditions for Tightness

In this section I characterize tightness by proposing equivalent properties. A simplification satisfies the *deviation conservation property* if for every strategy profile in the restricted set that is not a Nash equilibrium of the extended game, there exists one player with a profitable deviation in her restricted strategy set. Denoting by $NE(\Gamma)$ the set of pure strategy Nash equilibria of a complete information game Γ, the definition can be formally stated as follows.

Definition 5 (Deviation Conservation Property)**.** *Given a complete environment $\mathcal{E} = (\Omega, N, \mathcal{P})$,* *the mechanism μ' has the deviation conservation property with respect to μ if*

$$\big(\forall u \in \mathcal{U}\big)\big(\forall x' \in X' \setminus NE(\mu, \succ)\big)\big(\exists n \in N\big)\big(\exists \tilde{x}_n \in X'_n\big)\ u_n(\omega_n(\tilde{x}_n, x'_{-n})) > u_n(\omega_n(x')).$$

This is an obvious characterization of tightness. Indeed, a pure strategy Nash equilibrium of a game is a strategy profile that no player wants to deviate from. The deviation conservation property is very close to the definition of tightness itself. It appeals to game theoretic and equilibrium concepts, and it would be more attractive to have a condition that appeals only to preference related concepts. In order to do that, and for a given preference profile $u \in \mathcal{U}$ define the *upper-contour for a player* n of a strategy profile x as

$$U_n(x) = \left\{(\tilde{x}_n, x_{-n}) \middle| \Big(\tilde{x}_n \in X_n\Big) \text{ and } \Big(u_n(\omega_n(\tilde{x}_n, x_{-n})) > u_n(\omega_n(x))\Big)\right\} \subseteq X.$$

It is the set of strategy profiles of the extended game that are strictly preferred to x by player n and that she can reach by a unilateral deviation from x. Then the *upper-contour set* of x defined as

$$U(x) = \bigcup_{n \in N} U_n(x)$$

is the set of strategy profiles in X that are preferred to x by any of the players and that can be reached by a unilateral deviation of the same player from x.

Definition 6 (Upper-Contour Closure Property)**.** *Given a complete environment $\mathcal{E} = (X, N, \mathcal{U})$, the mechanism μ' has the upper-contour closure property with respect to μ if*

$$(\forall u \in \mathcal{U})(\forall x' \in X')\ \Big(U(x') = \emptyset \textit{ or } U(x') \cap X' \neq \emptyset\Big).$$

The following theorem shows the desired characterization.

Theorem 1. *The following statements are equivalent:*

(i) The simplification μ' of μ is tight for $\mathcal{E}$.

(ii) μ' *satisfies the deviation conservation property with respect to* μ.

(iii) μ' *satisfies the upper-contour closure property with respect to* μ.

Proof. $(i) \Leftrightarrow (ii)$. The property says that for any strategy profile of the restricted game from which a player would have a profitable deviation in the extended game, there is a player with a profitable deviation in the restricted game. Therefore, it is clear that no new pure strategy Nash equilibrium can be created by the restriction, which proves the sufficiency of the property. Necessity is also true for if the property did not hold, there would be a restricted strategy profile that is not a pure strategy Nash equilibrium of the extended game and from which no player would be willing to deviate in the restricted game, ie. a new pure strategy Nash equilibrium of the restricted game.

$(ii) \Leftrightarrow (iii)$. Suppose that μ' satisfies the deviation conservation property and let $x' \in X'$. If x' is a pure strategy Nash equilibrium of the extended game, then for every player n, $U_n(x') = \emptyset$ and $U(x') = \emptyset$. If x' is not a pure strategy Nash equilibrium of the extended game, by the deviation conservation property there exists a player n with a deviation $x_n \in X'_n$ from x'. But then $(x_n, x'_{-n)} \in \big(U_n(x') \cap X'\big) \subseteq \big(U(x') \cap X'\big)$. This shows the necessity.

Suppose now that μ' satisfies the upper-contour closure property and consider a strategy profile $x' \in X'$ that is not a pure strategy Nash equilibrium of the extended game. Then $U(x') \neq \emptyset$. By the upper-contour closure property it is possible to pick a strategy profile $x \in U(x') \cap X'$, implying that there is some n such that $x \in U_n(x') \cap X'$. Then $x = (x_n, x'_{-n})$ where $x_n \in X'_n$ is a profitable deviation from x'_n for player n, and this concludes the proof. $\square$

It is easy to offer a similar characterization of strong tightness. I only write down the analog of the upper-contour closure property. For this purpose, I define for any $\varepsilon \geq 0$, $n \in N$ and $x \in X$,

$$U_n^{\varepsilon}(x) = \left\{ (\tilde{x}_n, x_{-n}) \middle| \Big(\tilde{x}_n \in X_n\Big) \text{ and } \Big(u_n(\omega_n(\tilde{x}_n, x_{-n})) > u_n(\omega_n(x)) + \varepsilon\Big) \right\} \subseteq X,$$

and $U^{\varepsilon}(x) = \bigcup_{n\in N} U_n^{\varepsilon}(x)$. The proof of the following theorem can be deduced from the former results.

Theorem 2. *μ' is a strongly tight simplification of μ if and only if for every $\varepsilon \geq 0$, every $u \in \mathcal{U}$, and every $x' \in X'$, $U^{\varepsilon}(x') = \emptyset$ or $U(x') \cap X' \neq \emptyset$.*

4 A sufficient condition that is not necessary

An earlier version of Milgrom (2008) defined the *best-reply* closure property, and proved it to be a sufficient condition for tightness. A simplification satisfies the best-reply closure property if, to any strategy profile of her competitors that lies in the restricted set, a player can best-respond with a strategy that lies in her restricted strategy set.

Definition 7 (Best-Reply Closure Property)**.** *Given a complete environment $\mathcal{E} = (\Omega, N, \mathcal{U})$, the mechanism μ' has the best-reply closure property with respect to μ if*

$$(\forall u \in \mathcal{U})(\forall n \in N)(\forall x'_{-n} \in X'_{-n}) \left(\arg\max_{x_n \in X_n} u_n(\omega_n(x_n, x'_{-n}))\right) \cap X'_n \neq \emptyset.$$

Note that one drawback of the property is that best replies do not always exist. The definition is always correct because when a maximum does not exist the maximizing set is $\emptyset$. But this problem implies that it is possible for a strategy profile not to be a Nash equilibrium in a game even though no player has a best reply to this profile, and this justifies the use of deviations rather than best replies. However, this is not the main reason why the best-reply closure property is only a sufficient condition. Indeed, the counter example that follows uses finite strategy sets in which best replies always exist. In fact, the property is too strong because it assumes that every player conserves a best reply in the simplification. Before showing the counter example, I reproduce the simplification theorem with a proof of my own since it has disappeared from the final version of Milgrom (2008).

Theorem 3 (Milgrom)**.** *If the simplification μ' of μ has the best-reply closure property given $\mathcal{E}$, then the simplification is tight.*

Proof. Suppose that the best-reply closure is satisfied. Suppose that $x' \in X'$ is not a Nash equilibrium of the extended game. The best-reply closure property implies that every player has a best-reply to x' that lies in X', and the fact that x' is not a Nash equilibrium of the extended game implies that for at least one player, this best-reply constitutes a strict improvement over x'. This implies that the deviation conservation property is satisfied. $\square$

Example 1. Consider the following complete information game in normal form.

	L	C	R
U	$(0,2)$	$(2,0)$	$(1,1)$
D	$(1,-1)$	$(1,0)$	$(2,1)$

Its unique pure strategy Nash equilibrium is at the strategy profile (D,R). Now consider the simplification that consists in leaving only C and R available for player 2 without changing the strategy set of player 1. This simplification does not satisfy the best-reply closure property since the best-reply of player 2 to U in the extended game is to play L, and is no longer available in the restricted game, even though U lies in the restricted strategy set of player 1. It is however easy to check that no new pure strategy equilibrium is created by the simplification.

5 Strong Tightness

Milgrom (2008) works with the notion of strong tightness and uses restrictions on the set $\mathcal{U}$ to define conditions on the outcome function that ensure tightness. For this purpose, I endow each Ω_n with a topology $\mathcal{T}_n$ and define $\mathcal{C}_n$ to be the set of continuous functions from Ω_n to $\mathbb{R}$ endowed with the usual topology. Let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$. Consider the following property of the simplified outcome function.

Definition 8 (Outcome Closure). *A simplification $\mu' = (X', \omega')$ of the mechanism $\mu = (X, \omega)$ satisfies the outcome closure property if for every player n, every profile $x'_{-n} \in X'_{-n}$, every $x_n \in X_n$, and every open neighborhood $\mathcal{O}$ of $\omega_n(x'_{-n}, x_n)$, there exists $x'_n \in X'_n$ such that $\omega_n(x') \in \mathcal{O}$.*

This property means that if a given outcome is reachable by a player when other players play according to strategies in the simplified set, then she can approach it as closely as desired by picking strategies in her restricted set. Using the language of topology, the outcome closure property says that for every n, the space $\omega_n(X')$ is dense in the space $\omega_n\left(X_n, X'_{-n}\right)$. Milgrom (2008) proves that if μ' satisfies the outcome closure property, then it is strongly tight. I show a more general result under the slight restriction that each $(\Omega_n, \mathcal{T}_n)$ is metrizable[1] with a distance d_n.

Theorem 4. *If each $(\Omega_n, \mathcal{T}_n)$ is metrizable with a distance d_n, the following statements are equivalent in the complete environment $\mathcal{E} = (\Omega, N, \mathcal{C})$*

(i) μ' has the outcome closure property.

(ii) μ' is a strongly tight simplification of μ.

(iii) μ' is a tight simplification of μ.

Proof. Milgrom (2008) proves that (i) implies (ii), and it is obvious that (ii) implies (iii). Therefore I need only show that (iii) implies (i). To do that, I show that if μ' does not have the outcome closure property, then it is not a tight simplification. Suppose indeed that (i) is not true. Then there exists some n, some profile $x'_{-n} \in X'_{-n}$, and some strategy $x_n \in X_n$ such that for a certain neighborhood $\mathcal{O}$ of $\tilde{\omega}_n = \omega_n(x_n, x'_{-n})$, and every $x'_n \in X'_n$, $\omega_n(x'_n, x'_{-n}) \notin \mathcal{O}$. Since $\mathcal{O}$ is an open neighborhood, there must exist some $r > 0$ such that the bowl $\mathcal{B}(\tilde{\omega}_n, r) \subsetneq \mathcal{O}$. Then for every $\hat{\omega}_n \in \Omega_n$, let $u_n(\hat{\omega}_n) = r - d_n(\tilde{\omega}, \hat{\omega}_n)$ if $\hat{\omega}_n \in \mathcal{B}(\tilde{\omega}_n, r)$, and otherwise $u_n(\hat{\omega}_n) = 0$.

[1] A metrizable space is a topological space that is homeomorphic to a metric space. This includes countable spaces and vector spaces with their usual topology, as well as their products. This covers all the usual applications of the theory.

u_n is continuous by construction since the distance function is continuous. Let the other utility functions be uniformly equal to 0, so that they are continuous as well. In the simplified game associated with this utility profile, player n cannot get a utility higher than 0 when the other players use the profile x'_{-n}. Therefore, for any $x'_n \in X'_n$, the profile (x'_n, x'_{-n}) is an equilibrium of the simplified game. However, it is clear that none of these strategy profiles is a Nash equilibrium of the initial game as player n would be better off by playing $x_n \notin X'_n$, which is a violation of tightness. $\square$

6 Final Remarks

The results in this note clarify some aspects of the tight simplification of mechanisms. It highlights the correspondence between the domain of preferences considered and the characterizations of different notions of tightness, and it may be interesting to pursue such characterizations with different preference domains.

References

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