Dynamic Macroeconomics using Matlab Lecture 3

University of Callao
Edinson Tolentino
email: et396@kent.ac.uk

Twitter: @edutoleraymondi

Content

Stochastic growth model

- Social planner maximizes expected intertemporal utility

$$U\left([c_t]_{t=0}^{\infty}\right) = E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{1}$$

  with strictly concave period utility $u'(c_t) > 0$ and $u''(c_t) < 0$
- Future is discounted by constant factor β

$$1, \beta, \beta^2, \cdots$$

- $U(\cdot)$ is `time-separable` marginal utility of date-t consumption

$$\frac{\partial U(\cdot)}{\partial c_t} = \beta^t u'(c_t)$$

  depends only on c_t , not consumption on other dates
- Infinite horizon keeps model `stationary`, no life-cycle effects
- Initial $k_0 > 0$ and stochastic process for productivity z_t given

Stochastic growth model

- In per worker units

$$y \equiv \frac{Y}{L} \; k \equiv \frac{K}{L}$$

- Aggregate production function

$$y = f(k)$$

- Resource constraints

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t \tag{2}$$

Social planner's problem

- Using Eq (1) and Eq (2):

$$\max_{c_t, k_{t+1}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$\text{s.t:} \;\; c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t$$

- There are two ways to solve this problem
  - Find to F.O.C
  - Using dynamic solution (programacion dinamica)

Social planner's problem: F.O.C

- Lagrangian with multiplier $\lambda_t > 0$ for each resource constraint

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) - c_t + (1-\delta)k_t - k_{t+1} \right] \right\}$$

- Some key first order conditions

$$\begin{aligned} c_t: &\quad \beta^t u'(c_t) - \lambda_t = 0 \\ k_{t+1}: &\quad -\lambda_t + E_t \left\{ \lambda_{t+1} \left[z_{t+1} f'(k_{t+1}) + (1-\delta) \right] \right\} = 0 \\ \lambda_t: &\quad z_t f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0 \end{aligned}$$

- Using first and second equation

$$\lambda_t = \beta^t u'(c_t) \qquad \lambda_{t+1} = \beta^{t+1} u'(c_{t+1})$$

- Eliminating the Lagrange multipliers (`Euler equation`)

$$u'(c_t) = \beta u'(c_{t+1}) \left[z_{t+1} f'(k_{t+1}) + (1-\delta) \right]$$

Social planner's problem: F.O.C

- The `Euler equation`:

$$u'(c_t) = \beta u'(c_{t+1}) \left[z_{t+1} f'(k_{t+1}) + (1-\delta) \right]$$

MRS between t and t + 1

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

MRT between t and t + 1

$$z_{t+1} f'(k_{t+1}) + (1-\delta)$$

Planner equates MRS and MRT

Dynamical system

- Gives a system of two nolinear difference equations in c_t, k_t

$$u'(c_t) = \beta u'(c_{t+1})\left[z_{t+1}f'(k_{t+1}) + (1-\delta)\right]$$
$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t$$

- Two boundary conditions: (i) initial $k_0 > 0$ given, and (ii) `transversality condition`

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0$$

- Maps exogenous stochastic process z_t into endogenous stochastic c_t, k_t

Steady state

- Steady state where $\triangle c_t = 0$ and $\triangle k_t = 0$, Let c_{ss} k_{ss} denote steady state values. These are determined by

$$u'(c_t) = \beta u'(c_{t+1})\left[z_{t+1}f'(k_{t+1}) + (1-\delta)\right]$$
$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t$$

$$u'(c_{ss}) = \beta u'(c_{ss})\left[z_{ss}f'(k_{ss}) + (1-\delta)\right]$$
$$1 = \beta\left[z_{ss}f'(k_{ss}) + (1-\delta)\right]$$

$$c_{ss} + k_{ss} = z_{ss}f(k_{ss}) + (1-\delta)k_{ss}$$
$$c_{ss} = z_{ss}f(k_{ss}) - \delta k_{ss}$$

Log-linearization

- One method to solve and analyze nonlinear dynamic stochastic models is to approximate the nonlinear equations characterizing the equilibrium with loglinear ones.
- The strategy is to use a first order Taylor approximation around the steady state
- Taylor's theorem tells us that this can be expressed as a power series about a particular point x^*

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \cdots$$

  - Here $f'(x^*)$ is the first derivative of f with respect to x evaluated at the point x^*
  - Here $f''(x^*)$ is the second derivative of f with respect to x evaluated at the point x^*
- the function can be well approximated

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*)$$

Log-linearization

- Taylor's theorem also applies equally well to multivariate functions.

$$f(x,y) = f(x^*,y^*) + f_x(x^*,y^*)(x-x^*) + f_y(x^*,y^*)(y-y^*)$$

  - Here f_x is the first derivative of f with respect to x evaluated at the point x^*
  - Here f_y is the first derivative of f with respect to y evaluated at the point y^*

Log-linearization

- Given next example:

$$f(x) = \frac{g(x)}{h(x)}$$

To log-linearize it, first take natural logs of both sides:

$$\ln f(x) = \ln g(x) - \ln h(x)$$

Now use the first order Taylor series expansions:

$$\begin{aligned}
\ln f(x) &\approx \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) \\
\ln g(x) &\approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) \\
\ln h(x) &\approx \ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)
\end{aligned}$$

Now put these all together:

$$\begin{aligned}
\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) &= \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) \\
- \left[\ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)\right]
\end{aligned}$$

Log-linearization

- But since $\ln f(x^*) = \ln g(x^*) - \ln h(x^*$

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) = \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) - \left[\ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)\right]$$

$$\frac{f'(x^*)}{f(x^*)}(x - x^*) = \frac{g'(x^*)}{g(x^*)}(x - x^*) - \left[\frac{h'(x^*)}{h(x^*)}(x - x^*)\right]$$

- To put everything in percentage terms, multiply and divide each term by x^*

$$\frac{x^* f'(x^*)}{f(x^*)x^*}(x - x^*) = \frac{x^* g'(x^*)}{g(x^*)x^*}(x - x^*) - \left[x^* \frac{h'(x^*)}{h(x^*)x^*}(x - x^*)\right]$$

- For notational ease, define $\frac{(x-x^*)}{x^*} \approx \widehat{x}$

$$\frac{x^* f'(x^*)}{f(x^*)}\widehat{x} = \frac{x^* g'(x^*)}{g(x^*)}\widehat{x} - \left[x^* \frac{h'(x^*)}{h(x^*)}\widehat{x}\right]$$

Log-linearization

- Given example

$$Y_t = C_t + I_t$$

- But since $\ln Y_t = \ln(C_t + I_t)$

$$\ln Y_{ss} + \frac{1}{Y_{ss}}(Y_t - Y_{ss}) = \ln(C_{ss} + I_{ss}) + \frac{1}{(C_{ss} + I_{ss})}(C_t - C_{ss}) + \frac{1}{(C_{ss} + I_{ss})}(I_t - I_{ss})$$

- Given $\ln Y_{ss} = \ln(C_{ss} + I_{ss})$

$$\frac{1}{Y_{ss}}(Y_t - Y_{ss}) = \frac{1}{(C_{ss} + I_{ss})}(C_t - C_{ss})\frac{C_{ss}}{C_{ss}} + \frac{1}{(C_{ss} + I_{ss})}(I_t - I_{ss})\frac{I_{ss}}{I_{ss}}$$

$$\widehat{Y} = \frac{C_{ss}}{Y_{ss}}\widehat{C}_t + \frac{I_{ss}}{Y_{ss}}\widehat{I}_t$$

Uhlig (1999)

- Replacing a variable X_t with $X_{ss}e^{\widehat{X}_t}$, where $\widehat{X}_t = \log X_t - \log X_{ss}$
- Taking a first order Taylor approximation around the steady state yields

$$X_{ss}e^{\widehat{X}_t} = X(1+\widehat{X}_t)$$

$$e^{\widehat{X}_t + a\widehat{Y}_t} = (1+\widehat{X}_t + a\widehat{Y}_t)$$

- Given example

$$Y_t = C_t + I_t$$

Replace $e^{\widehat{X}_t} = 1+\widehat{X}_t$

$$\begin{aligned} Y_{ss}e^{\widehat{Y}_t} &= C_{ss}e^{\widehat{C}_t} + I_{ss}e^{\widehat{I}_t} \\ Y_{ss}(1+\widehat{Y}_t) &= C_{ss}(1+\widehat{C}_t) + I_{ss}(1+\widehat{I}_t) \end{aligned}$$

Replace $Y_{ss} = C_{ss} + I_{ss}$

$$Y_{ss}\widehat{Y}_t = C_{ss}\widehat{C}_t + I_{ss}\widehat{I}_t$$

$$\widehat{Y}_t = \frac{C_{ss}}{Y_{ss}}\widehat{C}_t + \frac{I_{ss}}{Y_{ss}}\widehat{I}_t$$