Spectrum of the random matrix

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Consider a square random matrix $\mathbf{A} = (\mathbf{a_{ij}})$ size of N-by-N. All its elements are independent and equally distributed with some distribution f. Let's denote mat. expectation and variance as

$$E(a_{ij}) = \mu, D(a_{ij}) = \sigma^2 \quad (1)$$

If $\mu = 0$, then as is known from [1], singular values $\{s_i\}_{i=1}^N$ of matrix $\mathbf{A}$ in the limit $N \gg 1$ obey distribution

$$\frac{dw}{ds} = \frac{4}{\pi a}\sqrt{1 - \frac{s^2}{a^2}}, s \in [0, a], \quad (2)$$

where the distribution boundary is $a = 2\sigma\sqrt{N}$. This doesn't depend on distribution f.

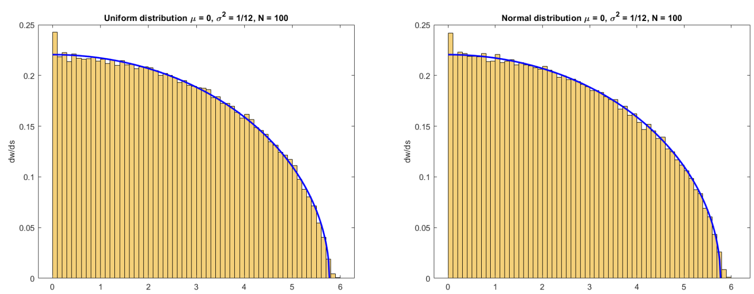

Figure 1: Spectrum of the singular values for two distributions of the matrix elements

If $\mu \neq 0$, a phase transition occurs [2], and for sufficiently large μ the maximum singular value s_n leaves the region $[0, a]$ and becomes isolated.

$$\frac{< s_N >}{N} = \begin{cases} \frac{2\sigma}{\sqrt{N}}, & \text{if } \mu < \frac{\sigma}{\sqrt{N}} \\ \mu + \frac{\sigma^2}{\mu N}, & \text{if } \mu > \frac{\sigma}{\sqrt{N}} \end{cases}$$

The remaining singular values have the same distribution.

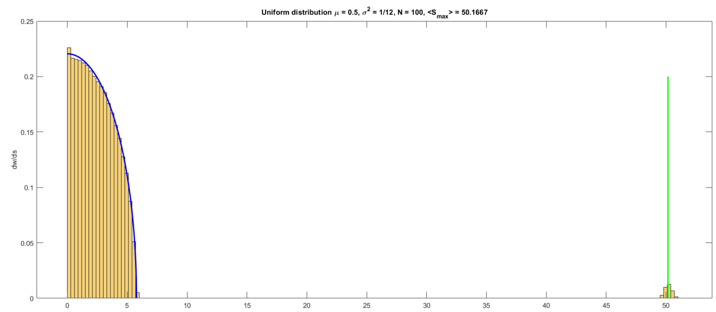

Figure 2: Spectrum of the singular values after phase transition

References

[1] Wikipedia. *Marchenko-Pastur distribution.* URL: `https://en.wikipedia.org/wiki/Marchenko%E2%80%93Pastur_distribution`.

[2] Nadakuditi R. R. Benaych-Georges F. "The singular values and vectors of low rank perturbations of large rectangular random matrices." In: *Journal of Multivariate Analysis* (). DOI: `https://doi.org/10.1016/j.jmva.2012.04.019`.