Homeomorphisms of the circle

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August 2018

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$$
\begin{tikzcd}
S^1 \arrow[r, "f"] \arrow[d, "\phi"'] & S^1 \arrow[d, "\phi"] \\
S^1 \arrow[r, "R_\alpha"] & S^1
\end{tikzcd}
$$

Figure 1: Poincaré classification theorem

ABSTRACT

This project tries to answer the basic question: If f, g are homeomorphisms of the circle, when does there exist another homeomorphism τ such that $f = \tau^{-1} \circ g \circ \tau$? We focus on homeomorphisms with irrational rotation number. Poincaré's fundamental result solves the question for transitive homeomorphisms, but in this project we provide an answer for the case of homeomorphisms which are not transitive. We show that every non-transitive homeomorphism with irrational rotation number is related to Denjoy's surgery construction and give necessary and sufficient conditions for topological conjugacy using this model.

This projects consists of two parts. The first part of the project is the introduction, where we develop the necessary tools and concepts used in studying circle homeomorphisms from a dynamical point of view. The second part contains the classification of homeomorphisms with irrational rotation number, with emphasis on non-transitive homeomorphisms.

Our theory of classification of non-transitive homeomorphisms with irrational rotation number is original and we are not aware of any such classification. All the pictures (except one) were done by us.

We would like to thank our supervisor, Dr. Davoud Cheraghi, for his help and support during the project.

Contents

1 Introduction and general theory

1.1 Basic Definitions and Examples

We will use two ways to formally define the circle.

Definition 1.1. Define the circle S^1 to be the set

$$S^1 = \{e^{2\pi ix} : x \in [0,1)\} \subset \mathbb{C}$$

with the metric inherited by $\mathbb{C}$

This definition gives rise to an identification of the circle with $\frac{\mathbb{R}}{\mathbb{Z}}$, where we denote $\frac{\mathbb{R}}{\mathbb{Z}} = [0,1)$ with $0 = 1$. The identification is via the map

$$\frac{\mathbb{R}}{\mathbb{Z}} \to S^1 \qquad x \mapsto e^{2\pi ix}$$

This is a group isomorphism from $(\frac{\mathbb{R}}{\mathbb{Z}}, +)$ to $(S^1, \times)$. Moreover, we will use the following metric on this set:

Definition 1.2. $d_{\text{arc}} : [0,1) \to [0,1)$ is defined as

$$\forall x, y \in [0,1) \qquad d_{\text{arc}}(x,y) := \min(|x-y|, |x-(y+1)|, |x-(y-1)|)$$

It can be checked that d_{arc} satisfies the axioms of a metric. Hence $([0,1), d_{\text{arc}})$ forms a metric space and we identify it with the circle $S^1 \subset \mathbb{C}$. In fact d_{arc} represents the arc length of any 2 points on a circle.

For presentation purposes and simplicity, it is sometimes more convenient to use $[0,1)$, therefore we will use S^1 and $[0,1)$ interchangeably.

$f : S^1 \to S^1$ is said to be a circle homeomorphism if it is bijective, continuous and its inverse is continuous. Note that as we are on S^1, f is a homeomorphism if and only if it is bijective and continuous. We denote the set of all circle homeomorphisms as $\text{Hom}(S^1)$. Some simple examples are listed below:

Example 1.3. *Rotation by an angle α*
The rotations of the circle are to most fundamental and simple homeomorphisms of the circle. Ther are defined as follows. For all $\alpha \in [0,1)$, define

$$\widetilde{R_\alpha} : S^1 \to S^1 \qquad \widetilde{R_\alpha} : z \mapsto e^{2\pi i\alpha} z$$

Which is the classical rotation by angle α on $S^1 \subset \mathbb{C}$. Note that this corresponds to the map $R_\alpha : [0,1) \to [0,1)$ defined by

$$\forall x \in [0,1) \qquad R_\alpha(x) := x + \alpha \mod 1 = x + \alpha - \lfloor x + \alpha \rfloor$$

Example 1.4. *An orientation-reversing homeomorphism*
$f : S^1 \to S^1$ is defined as

$$\forall x \in [0,1),\ f(x) = \begin{cases} p(-\frac{2x}{3}) & x \in [0, \frac{1}{2}) \\ p(-\frac{4x}{3}) & [\frac{1}{2}, 1) \end{cases}$$

where $p(x)$ is the fractional part of x, i.e. the natural projection

$$p : \mathbb{R} \to [0,1) \qquad p(x) = x \mod 1$$

When we study a circle homeomorphism, in many cases it is convenient to consider an associated function from $\mathbb{R}$ to $\mathbb{R}$:

Definition 1.5. If $f \in \mathrm{Hom}(S^1)$ then a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ is said to be lift of f if it satisfies

$$\forall x \in \mathbb{R} \qquad f \circ p(x) = p \circ F(x)$$

Example 1.6. If $R_\alpha : [0, 1) \to [0, 1)$ is the rotation then the lifts of R_α are the functions $F_k : \mathbb{R} \to \mathbb{R}$ defined by $\forall k \in \mathbb{Z}$

$$\forall x \in \mathbb{R} \qquad F_k(x) = x + \alpha + k$$

Proposition 1.7. *For any $f \in \mathrm{Hom}(S^1)$, there exists a lift $F : \mathbb{R} \to \mathbb{R}$ of f. Moreover if F, G are lifts of f then $\exists k \in \mathbb{Z}$ such that*

$$\forall x \in \mathbb{R} \qquad G(x) = F(x) + k$$

Proof. Let $f(0) = y_0 \in [0, 1)$. Then define $F(0) = y_0$. Then the equation

$$\forall x \in \mathbb{R} \qquad f \circ p(x) = p \circ F(x)$$

uniquely extends F to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$.

Suppose F, G are lifts of f. Then

$$\forall x \in [0, 1) \qquad p \circ F(x) = f \circ p(x) = p \circ G(x)$$

Therefore $\forall x \in [0, 1)$, $\exists k_x \in \mathbb{Z}$ such that $G(x) = F(x) + k_x$. Therefore $G - F : \mathbb{R} \to \mathbb{Z}$. But, by assumption, F, G are continuous. Therefore $G - F$ is continuous and hence $\exists k \in \mathbb{Z}$ such that

$$\forall x \in \mathbb{R} \qquad G(x) = F(x) + k$$

□

Definition 1.8. A homeomorphism $f : S^1 \to S^1$ is said to be orientation preserving if its lift, F, is strictly increasing. Otherwise, we say that f is orientation-reversing.

From this point onwards, we will assume that all circle homeomorphisms considered are orientation-preserving, unless otherwise specified. The set of all orientation-preserving homeomorphisms is denoted as $\mathrm{Hom}^+(S^1)$.

Proposition 1.9. *An increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$ is a lift of some orientation preserving circle homeomorphism $f : S^1 \to S^1$ if and only if*

$$\forall x \in \mathbb{R} \qquad F(x + 1) = F(x) + 1$$

Proof. Suppose that F is the lift of some circle homeomorphism f so that

$$f \circ p = p \circ F$$

holds on $\mathbb{R}$. Therefore $\forall x \in \mathbb{R}$

$$p \circ F(x + 1) = f \circ p(x + 1) = f \circ p(x) = p \circ F(x)$$

Therefore $\exists k \in \mathbb{Z}$ such that

$$F(x + 1) = F(x) + k$$

Now, as F is strictly increasing, we have that $k \geq 1$. Suppose, for a contradiction, that $k \geq 2$. Then, as F is continuous, by the intermediate value theorem, there is $y \in (x, x+1)$ such that

$$F(y) = F(x) + 1$$

Therefore

$$f(p(x)) = p(F(x))$$
$$f(p(y)) = p(F(y)) = p(F(x) + 1) = p(F(x))$$

But note that as $y \in (x, x+1)$ we have that $p(x) \neq p(y)$. Therefore the above equations are a contradiction to injectivity of f.

Conversely suppose that $F : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism which satisfies

$$\forall x \in \mathbb{R} \qquad F(x+1) = F(x) + 1$$

Then define $f : S^1 \to S^1$ by

$$\forall x \in [0,1) \qquad f(x) = p \circ F(x)$$

To check that F is indeed a lift of f it is sufficient to note that, by our assumption

$$\forall x \in \mathbb{R} \qquad F(x) = F(x-1) = \cdots = F(p(x))$$

Therefore

$$\forall x \in \mathbb{R} \qquad p \circ F(x) = p \circ F(p(x) = f \circ p(x)$$

and this, by definition of a lift, proves that F is a lift of f.

□

Corollary 1.10. *Any lift $F : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\forall x \in \mathbb{R},\ \forall n \in \mathbb{Z} \qquad F(x+n) = F(x) + n$$

It is also convenient to have the following lemma:

Lemma 1.11. *For circle homeomorphisms f and g, if F, G are their respective lifts, then:*
(a) $F \circ G$ is a lift of $f \circ g$.
(b) f^{-1} is also orientation-preserving and F^{-1} is a lift of f^{-1}.

Definition 1.12. We define a circle homeomorphism $f : S^1 \to S^1$ to be transitive if $\exists x \in [0,1)$ such that $\{f^n(x) : n \in \mathbb{Z}\} \subset [0,1)$ is dense in $[0,1)$.

Definition 1.13. We define a circle homeomorphism $f : S^1 \to S^1$ to be minimal if $\forall x \in [0,1)$, $\{f^n(x) : n \in \mathbb{Z}\} \subset [0,1)$ is dense in $[0,1)$.

Now we will show that the rotations of the circle are an example of minimal homeomorphisms. For this we will need to show that the orbit, under the rotation, of any point x (the set $\{R_\alpha^n(x) : n \in \mathbb{Z}\}$) is infinite.

Lemma 1.14. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $x \in [0,1)$. Then if $m \neq n \in \mathbb{Z}$ then $R_\alpha^n(x) \neq R_\alpha^m(x)$*

Proof. Suppose not. Then there are $m \neq n \in \mathbb{Z}$ such that

$$R_\alpha^n(x) = R_\alpha^m(x)$$

Therefore $\exists k \in \mathbb{Z}$ such that

$$x + n\alpha = x + m\alpha + k$$
$$\alpha = \frac{k}{n - m}$$

This is a contradiction to α being irrational

$\square$

Proposition 1.15. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $R_\alpha : [0, 1) \to [0, 1)$ is minimal.*

Proof. Fix any $x, y \in [0, 1)$ and $\epsilon > 0$. It is sufficient to prove that there exists $M \in \mathbb{Z}$ such that

$$R_\alpha^M(x) \in (y - \epsilon, y + \epsilon)$$

Let $N \in \mathbb{N}$ be large enough so that $\frac{1}{N} < \epsilon$. Consider

$$\{R_\alpha^n(x) : n \in \{1, 2, \dots, N + 1\}\}$$

Then by the previous lemma this set contains $N + 1$ elements and therefore there must be some $n_1, n_2 \in \mathbb{N}$ such that

$$|R_\alpha^{n_1}(x) - R_\alpha^{n_2}(x)| \leq \frac{1}{N + 1} < \epsilon$$

But note that R_α is an isometry (R_α preserves distances), therefore

$$|R_\alpha^{n_1 - n_2}(x) - x| < \epsilon$$

and therefore $\forall k \in \mathbb{Z}$

$$|R_\alpha^{(k+1)(n_1 - n_2)}(x) - R_\alpha^{k(n_1 - n_2)}(x)| < \epsilon$$

Let $K \in \mathbb{Z}$ be the smallest integer such that

$$y \in [R_\alpha^{K(n_1 - n_2)}(x), R_\alpha^{(K+1)(n_1 - n_2)}(x)] \qquad \text{or}$$
$$y \in [R_\alpha^{(K+1)(n_1 - n_2)}(x), R_\alpha^{K(n_1 - n_2)}(x)]$$

Such an integer clearly exists. Then we have that

$$|R_\alpha^{K(n_1 - n_2)}(x) - y| < \epsilon$$

and the proposition is proved.

$\square$

1.2 Rotation number

Now we will introduce the concept of rotation number of a circle homeomorphisms. It turns out, rotation number is a very useful property in determining behavior of circle homeomorphism and is a fundamental concept in classifying circle homeomorphisms.

Proposition 1.16. *Fix any $f \in \mathrm{Hom}^+(S^1)$ and any lift F of f. Then $\left(\frac{F^n(0)}{n}\right)_{n=1}^{\infty}$ is convergent.*

Proof. Fix any $n \in \mathbb{N}$. Define $p_n := \lfloor F^{\circ n}(0) \rfloor$. Note that $p_n \leq F^{\circ n}(0) < p_n + 1$. It can be proved inductively that

$$\forall m \in \mathbb{N} \qquad mp_n \leq F^{\circ(nm)}(0) < m(p_n + 1)$$

Hence

$$\left|\frac{F^{\circ(nm)}(0)}{nm} - \frac{p_n}{n}\right| \leq \frac{1}{n}$$

Then

$$\left|\frac{F^{\circ m}(0)}{m} - \frac{F^{\circ n}(0)}{n}\right|$$

$$\leq \left|\frac{F^{\circ(m)}(0)}{m} - \frac{p_m}{m}\right| + \left|\frac{F^{\circ(nm)}(0)}{nm} - \frac{p_m}{m}\right| + \left|\frac{F^{\circ(nm)}(0)}{nm} - \frac{p_n}{n}\right| + \left|\frac{F^{\circ(n)}(0)}{n} - \frac{p_n}{n}\right| \leq \frac{2}{m} + \frac{2}{n}$$

By making m, n arbitrarily large, we can see that $\left(\frac{F^n(0)}{n}\right)_{n=1}^{\infty}$ is Cauchy. Hence the required result follows. □

Proposition 1.17. *Fix any $f \in \mathrm{Hom}^+(S^1)$ and any lift F of f. Then for any $x, y \in \mathbb{R}$*

$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(y)}{n}$$

Proof. Define $\lim_{n\to\infty} \frac{F^n(0)}{n} =: a$. Fix any integer k. Note that

$$\frac{F^n(k)}{n} = \frac{F^n(0) + k}{n} \to a \qquad \text{as} \qquad n \to \infty$$

Fix any $b \in \mathbb{R}$. Note that

$$\forall n \in \mathbb{Z} \qquad \frac{F^{\circ n}\lfloor b \rfloor}{n} \leq \frac{F^{\circ n}(b)}{n} \leq \frac{F^{\circ n}\lceil b \rceil}{n}$$

But as $\lfloor b \rfloor, \lceil b \rceil \in \mathbb{Z}$ the left hand side and the right hand side converge to a. Therefore

$$\frac{F^n(b)}{n} \to a \qquad \text{as} \qquad n \to \infty$$

□

Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism and let $F : \mathbb{R} \to \mathbb{R}$ be its lift. Define, for any $x \in \mathbb{R}$

$$\psi(F) := \lim_{n\to\infty} \frac{F^n(x)}{n}$$

Proposition 1.18. *Fix any $f \in \mathrm{Hom}^+(S^1)$. Then for any lifts F, G of f*

$$\psi(F) \; mod \; 1 = \psi(G) \; \; mod \; 1$$

Proof. Suppose F, G are lifts of $f \in \mathrm{Hom}^+(S^1)$. Note that by Proposition 1.7, there exists $k \in \mathbb{Z}$ such that

$$\forall x \in \mathbb{R} \qquad F(x) - G(x) = k$$

. Then $\forall n \in \mathbb{Z}$

$$\frac{F^{\circ n}(x)}{n} = \frac{G^{\circ n}(x) + kn}{n} = \frac{G^{\circ n}(x)}{n} + k$$

This implies $\psi(F) = \psi(G) + k$ and hence

$$\psi(F) \bmod 1 = \psi(G) \bmod 1$$

□

The three propositions above allow us to uniquely define the concept of rotation number of a circle homeomorphism.

Definition 1.19. Let $f \in \mathrm{Hom}^+(S^1)$, let F be its lift and let $x \in \mathbb{R}$. Then the rotation number $\rho(f) \in [0, 1)$ is defined to be

$$\rho(f) := \lim_{n \to \infty} \frac{F^n(x)}{n} \bmod 1 = \psi(F) \bmod 1$$

Note that the above propositions show that this is well-defined and moreover the propositions 1.18 and 1.7 showa that we can always choose a lift such that

$$\psi(F) = \lim_{n \to \infty} \frac{F^n(x)}{n} \in [0, 1)$$

so that $\psi(F) = \rho(f)$.

Example 1.20. The rotation number of the rotation $R_\alpha : [0, 1) \to [0, 1)$ is $\alpha \in [0, 1)$

$$\rho(f) := \lim_{n \to \infty} \frac{R_\alpha^n(0)}{n} \bmod 1 = \lim_{n \to \infty} \frac{n\alpha}{n} \bmod 1 = \alpha$$

Definition 1.21. We say that homeomorphisms $f_1, f_2 \in \mathrm{Hom}^+(S^1)$ are topologically conjugate if $\exists \tau \in \mathrm{Hom}^+(S^1)$ such that

$$f_1 = \tau^{-1} \circ f_2 \circ \tau$$

It can be easily shown that topological conjugacy is an equivalence relation, from now onwards we denote $f_1 \sim f_2$ if and only if f_1, f_2 are topologically conjugate. Moreover, from the definition of a topological conjugacy we see that the function τ above is a function which sends orbits to orbits. More accurately, if τ is a conjugacy of f_1, f_2 then

$$\forall x \in [0, 1) \qquad \tau : \{f_1^n(x) : n \in \mathbb{Z}\} \to \{f_2^n(\tau(x)) : n \in \mathbb{Z}\}$$

We will show that functions which are topologically conjugate share some properties and exhibit similar behavior. Later on, we will be classifying homeomorphisms up to topological conjugacy. The first property that is an invariant under conjugacy is the rotation number.

Proposition 1.22. *If $f_1, f_2 \in \mathrm{Hom}^+(S^1)$ are topologically conjugate then they have the same rotation number*

Proof. Let $\phi \in \mathrm{Hom}^+(S^1)$ satisfy $f_1 = \phi^{-1} \circ f_2 \circ \phi$. Fix any lift F_2 of f_2 and consider the lift Φ of ϕ such that $0 \leq \Phi(0) < 1$. Then, by lemma 1.11, $\Phi^{-1} \circ F_2 \circ \Phi$ is a lift of $\phi^{-1} \circ f_2 \circ \phi = f_1$. Name it F_1.

Note that $\forall x \in [0, 1)$

$$\Phi(x) - x < \Phi(1) - 0 < 2 \qquad \text{and} \qquad 0 - 1 < \Phi(x) - x$$

Since $\Phi(x) - x$ is periodic, this holds on the whole of $\mathbb{R}$ and we have that

$$\forall x \in \mathbb{R} \qquad |\Phi(x) - x| < 2 \tag{1}$$

By substituting x with $\Phi^{-1}(y)$, we get

$$\forall y \in \mathbb{R} \qquad |\Phi^{-1}(y) - y| < 2 \tag{2}$$

Consider the expression $|F_1^{\circ n}(x) - F_2^{\circ n}(x)|$. For all $x \in \mathbb{R}$,

$$\begin{aligned} |F_1^{\circ n}(x) - F_2^{\circ n}(x)| &= |\Phi^{-1} \circ F_2^{\circ n} \circ \Phi(x) - F_2^{\circ n}(x)| \\ &\leq |\Phi^{-1} \circ F_2^{\circ n} \circ \Phi(x) - F_2^{\circ n} \circ \Phi(x)| + |F_2^{\circ n} \circ \Phi(x) - F_2^{\circ n}(x)| \\ &< 2 + 2 = 4 \end{aligned}$$

Where the last inequality follows from equations (1),(2). Hence

$$\left|\frac{F_1^{\circ n}(x)}{n} - \frac{F_2^{\circ n}(x)}{n}\right| < \frac{4}{n} \to 0 \qquad \text{as} \qquad n \to \infty$$

The required result follows.

□

Now we will present a result which shows that rotation number is, indeed, quite informative about the given homeomorphism. We will show that a homeomorphism with rational rotation number is characterized by existence of periodic points.

Definition 1.23. If $f \in \mathrm{Hom}^+(S^1)$ we say that $x \in [0, 1)$ is a periodic point of f if $\exists k \in \mathbb{Z}$ such that

$$f^k(x) = x$$

Lemma 1.24. *Suppose F is a lift of some circle homeomorphism and $\delta \in \mathbb{R}$ is such that*

$$\forall x \in \mathbb{R} \qquad F(x) - x > \delta$$

then

$$\psi(F) > \delta$$

Proof. Note that, as the restriction $F - id : [0, 1] \to \mathbb{R}$ defined by $F - id : x \mapsto F(x) - x$ is a continuous function on a compact set, $\exists \widetilde{\delta} > 0$ such that

$$\forall x \in [0, 1] \qquad F(x) - x > \delta + \widetilde{\delta} := \epsilon$$

As the function $F - id$ is periodic with period 1 we have that

$$\forall x \in \mathbb{R} \qquad F(x) - x > \epsilon$$

Now we prove by induction that the above implies that $\forall k \in \mathbb{N}$

$$F^k(x) - x > k\epsilon$$

For $k = 1$ we already have the statement. Assume the statement to be true for some $k-1 \in \mathbb{N} \cup \{0\}$. Then

$$F^k(x) - x = F^k(x) - F^{k-1}(x) + F^{k-1}(x) - x$$

if we set $y = F^{k-1}(x)$ then we have that

$$F^k(x) - F^{k-1}(x) = F(y) - y > \epsilon$$

Therefore, using the induction assumption we have

$$F^k(x) - x > \epsilon + (k-1)\epsilon = k\epsilon$$

Which proves the statement. Therefore we have that

$$\psi(F) = \lim_{k\to\infty} \frac{F^k(x)}{k} = \lim_{k\to\infty} \frac{F^k(x) - x}{k} \geq \lim_{k\to\infty} \frac{k\epsilon}{k} = \epsilon = \delta + \widetilde{\delta} > \delta$$

□

Note that the above lemma is, of course, also true with $>$ replaced by $<$.

Proposition 1.25. *Let $\frac{p}{q} \in \mathbb{Q}$ be in lowest terms. Then $\psi(F) = \frac{p}{q}$ if and only if $\exists x \in \mathbb{R}$ such that $F^q(x) = x + p$*

Proof. Suppose that $\psi(F) = \frac{p}{q}$. Then

$$\psi(F^q) = \lim_{n\to\infty} \frac{F^{nq}(x)}{n} = q \lim_{n\to\infty} \frac{F^{nq}(x)}{nq} = q\psi(F) = p$$

Now suppose, for a contradiction, that $\nexists x \in \mathbb{R}$ such that $F^q(x) = x + p$. Therefore, as $F^q - x : \mathbb{R} \to \mathbb{R}$ is continuous, we must have that

$$\forall x \in \mathbb{R} \qquad F^q(x) - x > p$$

or

$$\forall x \in \mathbb{R} \qquad F^q(x) - x < p$$

Otherwise the intermediate value theorem would give us a contradiction. Without loss of generality we can take $>$, because the same proof follows for $<$.
Therefore, by lemma 1.24, we have that

$$p = \psi(F^q) > p$$

Which is a contradiction. So we have proved that $\exists x \in \mathbb{R}$ such that $F^q(x) = x + p$.

Conversely, suppose that $\exists x \in \mathbb{R}$ such that $F^q(x) = x + p$. Then $\forall k \in \mathbb{Z}$

$$F^{qk}(x) = x + kp$$

and hence

$$\psi(F) = \lim_{k\to\infty} \frac{F^k(x)}{k} = \lim_{k\to\infty} \frac{x + kp}{k} = p$$

□

Corollary 1.26. *Let $f \in \mathrm{Hom}^+(S^1)$. Then $\rho(f) \in \mathbb{Q}$ if and only if f has a periodic point*

Now we will show that the function $f \mapsto \rho(f)$ is continuous and use this to prove existence of non-rotation homeomorphisms with irrational rotation number α. In the appendix we sketch a method which allows us to produce any transitive homeomorphism with irrational rotation number α.

Lemma 1.27. *Let $\frac{p}{q} \in \mathbb{Q}$. Then $\psi(F) < \frac{p}{q}$ if and only if $\forall x \in \mathbb{R}\ \ F^q(x) < x + p$*

Proof. Assume that $\psi(F) < \frac{p}{q}$ and that $\exists y \in \mathbb{R}$ such that $F^q(y) \geq y + p$ then, as $p \in \mathbb{Z}$, we have that $F(y+p) = F(y) + p$ and hence by an inductive argument $\forall n \in \mathbb{Z}$

$$F^{nq}(y) \geq y + np$$

so that

$$\psi(F) = \lim_{n\to\infty} \frac{F^{qn}(y)}{qn} \geq \lim_{n\to\infty} \frac{y + pn}{qn} = \frac{p}{q}$$

This contradicts the assumption.

The converse is proved by lemma 1.24. $\square$

Proposition 1.28. *The function $\psi : F \mapsto \lim_{n\to\infty} \frac{F^n(x)}{n}$ is continuous with respect to the C^0 topology induced by the supremum norm*

Proof. Let F be some lift of an orientation preserving homeomorphism of the circle, let $\psi(F) \in \mathbb{R}$ and let $\epsilon > 0$. Then $\exists \frac{p'}{q'}, \frac{p}{q} \in \mathbb{Q}$ such that

$$\frac{p'}{q'} < \psi(F) < \frac{p}{q} \qquad \text{and}$$
$$\frac{p'}{q'} < \frac{p}{q} < \frac{p'}{q'} + \epsilon$$

Then by the previous lemma $\forall x \in \mathbb{R}$

$$F^q(x) < x + p \qquad \text{and}$$
$$F^{q'}(x) > x + p'$$

Now let

$$\epsilon_2 = \frac{1}{2}\min\left(p - \sup_{\mathbb{R}}(F^q(x) - x), \inf_{\mathbb{R}}(F^{q'}(x) - x) - p'\right) > 0$$

Where this is greater than zero as F is periodic and continuous. Moreover, it can be shown that $\forall \widetilde{\epsilon} > 0\ \ \exists \delta > 0$ such that if $||F - G|| < \delta$ then $||F^q - G^q|| < \widetilde{\epsilon}$. Now applying this statement to $\epsilon_2 > 0$, gives that $\exists \delta_2 > 0$ such that if G is a lift of an orientation preserving homeomorphism and $||F - G|| < \delta_2$ then $\forall x \in \mathbb{R}$

$$G^q(x) < x + p \qquad \text{and}$$
$$G^{q'}(x) > x + p'$$

Hence, by the previous lemma $\frac{p'}{q'} < \psi(G) < \frac{p}{q}$ and hence $|\psi(G) - \psi(F)| < \epsilon$. $\square$

Now using the previous proposition we can prove the existence of a non-rotation homeomorphism of the circle with irrational rotation number.

Consider the family of functions $f_w : [0, 1) \to [0, 1)$ for $w \in [0, \frac{1}{2}]$, defined by

$$\forall x \in [0,1) \qquad f_w(x) = w + x + \frac{1}{100}\sin(2\pi x) \mod 1$$

It can be shown that these functions are orientation preserving homeomorphisms. Now define the function $\varphi : [0, \frac{1}{2}] \to \mathbb{R}$ by

$$\varphi(w) = \rho(f_w)$$

Using the previous proposition it can be shown that φ is a continuous function. Now as $f_0(0) = 0$, f_0 has a fixed point and hence $\rho(f_0) = 0$. Moreover

$$f_{\frac{1}{2}}(0) = 1/2 \qquad \text{and}$$
$$f_{\frac{1}{2}}^{\circ 2}(0) = f_{\frac{1}{2}}(1/2) = 1/2 + 1/2 + \frac{1}{100}\sin(\pi) \mod 1 = 0$$

This implies that that $\rho(f_{\frac{1}{2}}) = \frac{1}{2}$. Therefore

$$\varphi(0) = \rho(f_0) = 0 \qquad \text{and} \qquad \varphi(\frac{1}{2}) = \rho(f_{\frac{1}{2}}) = \frac{1}{2}$$

hence by the intermediate value theorem $[0, \frac{1}{2}] \subseteq \varphi([0, \frac{1}{2}])$. This shows that $\exists w \in [0, \frac{1}{2}]$ such that $\rho(f_w) \in [0, \frac{1}{2}]$ is irrational.

2 Classification of transitive homeomorphisms

The question that we are trying to answer in this section is when two general homomorphisms f_1, f_2 with $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ are topologically conjugate.

2.1 Poincaré Classification theorem

Theorem 2.1. *Poincaré Classification Theorem. If $f : S^1 \to S^1$ is a homeomorphism that is transitive and $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ then there is a homeomorphism $\pi : S^1 \to S^1$ such that*

$$f = \pi^{-1} \circ R_\alpha \circ \pi$$

The above theorem can be illustrated by the commutative diagram

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\pi"'] & S^1 \arrow[d, "\pi"] \\ S^1 \arrow[r, "R_\alpha"] & S^1 \end{tikzcd}$$

Proof. Suppose that $f : S^1 \to S^1$ is a transitive homeomorphism with $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $F : \mathbb{R} \to \mathbb{R}$ and $R : \mathbb{R} \to \mathbb{R}$ be the lifts of f, R_α which satisfy $R(0) = \alpha$ and

$$\lim_{n\to\infty} \frac{F^n(x)}{n} \in [0, 1)$$

By propositions 1.7, 1.18 we can choose such a lift. As R_α is minimal and as f is transitive there is $x \in [0, 1)$ such that

$$\{f^n(x) : n \in \mathbb{Z}\} \subset [0, 1)$$
$$\{R_\alpha^n(0) : n \in \mathbb{Z}\} \subset [0, 1)$$

are dense in $[0, 1)$. This implies that

$$\Omega := \{F^n(x) + m : n, m \in \mathbb{Z}\} \subset \mathbb{R}$$
$$\Lambda := \{R^n(0) + m : n, m \in \mathbb{Z}\} \subset \mathbb{R}$$

are dense in $\mathbb{R}$. Now we define a function $\widetilde{\pi} : \Omega \to \Lambda$ by

$$\widetilde{\pi}(F^n(x) + m) = R^n(0) + m = n\alpha + m$$

First, we prove that π is strictly increasing.

Lemma 2.2. *If $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and are such that*

$$F^{n_1-n_2}(x) - x < m_2 - m_1$$

then $\forall y \in \mathbb{R}$

$$F^{n_1-n_2}(y) - y < m_2 - m_1$$

Proof. Suppose for a contradiction that $\exists y \in \mathbb{R}$ such that the equation does not hold. Then, as F is continuous, by the intermediate value theorem $\exists z \in \mathbb{R}$ such that

$$F^{n_1-n_2}(z) = z + m_2 - m_1$$

Hence $p(z)$ (the fractional part of z) is a periodic point of f. Therefore f must have rational rotation number, which is a contradiction to $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

□

Now we can prove that $\widetilde{\pi}$ is strictly increasing. Suppose $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ are such that

$$F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$$
$$F^{n_1-n_2}(x) - x < m_2 - m_1$$

Therefore, by the previous lemma $\forall y \in \mathbb{R}$

$$F^{n_1-n_2}(y) - y < m_2 - m_1$$

Therefore by lemma 1.24, we have

$$\lim_{k\to\infty} \frac{F^{k(n_1-n_2)}(x)}{k} = \psi(F^{n_1-n_2}) < m_2 - m_1$$

$$\alpha = \rho(f) = \psi(F) = \frac{\psi(F^{n_1-n_2})}{n_1 - n_2} < \frac{m_2 - m_1}{n_1 - n_2}$$

Therefore

$$n_1\alpha + m_1 < n_2\alpha + m_2$$
$$\widetilde{\pi}(F^{n_1}(x) + m_1) < \widetilde{\pi}(F^{n_2}(x) + m_2)$$

So $\widetilde{\pi}$ is strictly increasing. Moreover we have that on Ω

$$\widetilde{\pi} \circ F = R \circ \widetilde{\pi}$$

Because $\forall n, m \in \mathbb{Z}$

$$\widetilde{\pi} \circ F(F^n(x)+m) = \widetilde{\pi}(F^{n+1}(x)+m) = R^{n+1}(0)+m = R(R^n(0)+m) = R \circ \widetilde{\pi}(F^n(x)+m)$$

Now, as $\widetilde{\pi}$ is strictly increasing from a dense set in $\mathbb{R}$ onto a dense set in $\mathbb{R}$, it can be uniquely extended to a strictly increasing function $\widetilde{\pi} : \mathbb{R} \to \mathbb{R}$, hence a homeomorphism. Moreover, by continuity it can be easily shown that the extension also satisfies the property

$$\widetilde{\pi} \circ F = R \circ \widetilde{\pi}$$

Finally, we need to prove that $\widetilde{\pi}$ is the lift of some homeomorphism of the circle. We have that for all $F^n(x) + m \in \Omega$

$$\widetilde{\pi}(F^n(x) + m + 1) = n\alpha + m + 1 = \widetilde{\pi}(F^n(x) + m) + 1$$

By continuity, this property is also inherited by the extension of $\widetilde{\pi}$, so that $\forall y \in \mathbb{R}$

$$\widetilde{\pi}(y + 1) = \widetilde{\pi}(y) + 1$$

Therefore by proposition 1.9, there is a homeomorphism $\pi : [0, 1) \to [0, 1)$ defined by the relation

$$\forall x \in \mathbb{R} \qquad p \circ \widetilde{\pi}(x) = \pi \circ p(x)$$

where $\widetilde{\pi}$ is a lift of π. The relation

$$\pi \circ f = R_\alpha \circ \pi$$

is also inherited by the homeomorphisms and hence the theorem is proved.

$\square$

Lemma 2.3. *If $f : S^1 \to S^1$ is a homeomorphism that is transitive and $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ then by Poincaré's theorem there exists $\pi \in \mathrm{Hom}^+(S^1)$ such that*

$$f = \pi^{-1} \circ R_\alpha \circ \pi$$

then

$$\{\phi \in \mathrm{Hom}^+(S^1) : f = \phi^{-1} \circ R_\alpha \circ \phi\} = \{R_\theta \circ \pi : \theta \in [0,1)\}$$

Proof. Suppose ϕ is a homeomorphism which satisfies $f = \phi^{-1} \circ R_\alpha \circ \phi$ and Let $\widetilde{\pi}, F, \Phi$ be lifts of π, f, ϕ, respectively. As f is transitive $\exists x \in [0,1)$ such that $\{f^n(x) : n \in \mathbb{Z}\} \subset [0,1)$ is dense in $[0,1)$. Then we have

$$\widetilde{\pi} \circ F = R_\alpha \circ \widetilde{\pi}(x) = \widetilde{\pi}(x) + \alpha$$
$$\Phi \circ F = R_\alpha \circ \Phi = \Phi(x) + \alpha$$

Hence

$$(\widetilde{\pi} - \Phi)(F(x)) = (\widetilde{\pi} - \Phi)(x)$$

This, by an inductive argument, means that $(\widetilde{\pi} - \Phi)$ is constant on $\{F^n(x) : n \in \mathbb{Z}\} \subset \mathbb{R}$. Moreover, as $\widetilde{\pi}, \Phi$ are lifts of homeomorphisms, $\forall m \in \mathbb{Z}, x \in \mathbb{R}$

$$(\widetilde{\pi} - \Phi)(x+m) = \widetilde{\pi}(x+m) - \Phi(x+m) = \widetilde{\pi}(x) + m - (\Phi(x) + m) = (\widetilde{\pi} - \Phi)(x)$$

Hence $(\widetilde{\pi} - \Phi)$ is constant on $\{F^n(x) + m : n, m \in \mathbb{Z}\} \subset \mathbb{R}$, which is dense in $\mathbb{R}$. As $(\widetilde{\pi} - \Phi)$ is continuous, this implies that $(\widetilde{\pi} - \Phi)$ is constant on $\mathbb{R}$. Therefore $\exists \widetilde{\theta} \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} \qquad \Phi(x) = \widetilde{\pi}(x) + \widetilde{\theta}$$

Hence we have that the following holds for the homeomorphisms

$$\phi = R_\theta \circ \pi$$

where $\theta = p(\widetilde{\theta})$ (fractional part).

Conversely, we also have that if $\exists \theta \in [0,1)$ such that $\phi = R_\theta \circ \pi$ then

$$\phi^{-1} \circ R_\alpha \circ \phi = \pi^{-1} \circ R_\theta^{-1} \circ R_\alpha \circ R_\theta \circ \pi = \pi^{-1} \circ R_\alpha \circ \pi = f$$

Therefore the lemma is proved.

□

As topological conjugacy is an equivalence relation, Poincaré's theorem shows that if $f_1, f_2 : S^1 \to S^1$ are transitive and $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ then they are topologically conjugate. Therefore the question we wanted to tackle is solved for transitive homeomorphisms. However, we will investigate transitive homeomorphisms further as the following results will be useful when classifying non-transitive homoemorphisms.

Lemma 2.4. *If $f_1, f_2 : S^1 \to S^1$ are transitive homeomorphisms and $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ then by Poincaré's theorem there exists $\pi_1, \pi_2 \in \mathrm{Hom}^+(S^1)$ such that*

$$f_1 = \pi_1^{-1} \circ R_\alpha \circ \pi_1$$
$$f_2 = \pi_2^{-1} \circ R_\alpha \circ \pi_2$$

then

$$\{\tau \in \mathrm{Hom}^+(S^1) : f_1 = \tau^{-1} \circ f_2 \circ \tau\} = \{\pi_2^{-1} \circ R_\theta \circ \pi_1 : \theta \in [0,1)\}$$

The functions τ above are functions which send orbits to orbits, where the image of an orbit is determined by θ. More accurately, if τ_θ is a conjugacy of f_1, f_2 then $\forall x \in [0,1), \exists y \in [0,1)$ such that

$$\tau_\theta : \{f_1^n(x) : n \in \mathbb{Z}\} \to \{f_2^n(y) : n \in \mathbb{Z}\}$$

Where the choice of θ determines a unique y for every x.

Proof. Suppose that $\tau \in \mathrm{Hom}^+(S^1)$ is such that $f_1 = \tau^{-1} \circ f_2 \circ \tau$. Then

$$\tau^{-1} \circ f_2 \circ \tau = f_1 = \pi_1^{-1} \circ R_\alpha \circ \pi_1$$
$$f_2 = \tau \circ \pi_1^{-1} \circ R_\alpha \circ \pi_1 \circ \tau^{-1}$$

Therefore, by previous lemma

$$\pi_1 \circ \tau^{-1} \in \{R_\theta \circ \pi_2 : \theta \in [0,1)\}$$
$$\tau^{-1} \in \{\pi_1^{-1} \circ R_\theta \circ \pi_2 : \theta \in [0,1)\}$$
$$\tau \in \{\pi_2^{-1} \circ R_\theta \circ \pi_1 : \theta \in [0,1)\}$$

On the other hand if $\phi = \pi_2^{-1} \circ R_\theta \circ \pi_1$ for some $\theta \in [0,1)$ then

$$\phi^{-1} \circ f_2 \circ \phi = \pi_1^{-1} \circ R_{-\theta} \circ \pi_2 \circ f_2 \circ \pi_2^{-1} \circ R_\theta \circ \pi_1$$
$$\phi^{-1} \circ f_2 \circ \phi = \pi_1^{-1} \circ R_{-\theta} \circ R_\alpha \circ R_\theta \circ \pi_1$$
$$\phi^{-1} \circ f_2 \circ \phi = \pi_1^{-1} \circ R_\alpha \circ \pi_1 = f_1$$

Therefore $\phi \in \{\tau \in \mathrm{Hom}^+(S^1) : f_1 = \tau^{-1} \circ f_2 \circ \tau\}$ and this finishes the proof.

□

Lemma 2.5. *If $f_1, f_2 : S^1 \to S^1$ are transitive homeomorphisms with $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $\forall x, y \in S^1$, $\exists! \tau \in \mathrm{Hom}^+(S^1)$ such that*

$$f_1 = \tau^{-1} \circ f_2 \circ \tau$$

and

$$\tau(x) = y$$

Proof. Suppose that $x, y \in [0,1)$ are arbitrary, now define $\theta \in [0,1)$ by

$$\theta = \begin{cases} \pi_2(y) - \pi_1(x) & \text{if } \pi_2(y) \geq \pi_1(x) \\ (\pi_2(y) + 1) - \pi_1(x) & \text{if } \pi_2(y) < \pi_1(x) \end{cases}$$

This implies that

$$(\pi_1(x) + \theta) \mod 1 = \pi_2(y)$$
$$R_\theta(\pi_1(x)) = \pi_2(y)$$
$$\pi_2^{-1} \circ R_\theta \circ \pi_1(x) = y$$

Now, for this θ, we define τ to be the homeomorphism $\tau = \pi_2^{-1} \circ R_\theta \circ \pi_1$. Then, using the previous lemma, we have that τ satisfies

$$\tau(x) = y$$
$$f_1 = \tau^{-1} \circ f_2 \circ \tau$$

□

Now we will prove two very useful results about homeomorphisms with irrational rotation number, namely that the omega limit set is the same for every point on S^1, and that the omega limit set is, in a sense, invariant under topological conjugacy.

Lemma 2.6. *Suppose that $f : S^1 \to S^1$ is a homeomorphism (not assuming transitivity) with $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $n \neq m \in \mathbb{Z}$ and $x \in S^1$ are such that $I = [f^m(x), f^n(x)]$ is a closed interval in S^1. Then $\forall y \in S^1, \exists N \in \mathbb{N}$ such that $f^N(y) \in I$*

Proof. Suppose that $n > m$. We will prove that

$$\bigcup_{k \in \mathbb{Z}_{\leq -1}} f^{k(n-m)}(I) = S^1$$

The lemma will easily follow from this. As f is a homeomorphism

$$f^{n-m}(I) = [f^n(x), f^{2n-m}(x)]$$

is an interval adjacent to $I = [f^m(x), f^n(x)]$ (by adjacent we mean that they have a common endpoint). Therefore, by an inductive argument, $\forall k \in \mathbb{Z}_{\leq -1}$, $f^{k(n-m)}(I)$ and $f^{(k-1)(n-m)}(I)$ are adjacent intervals. Hence, if we define

$$U = \bigcup_{k \in \mathbb{Z}_{\leq -1}} f^{k(n-m)}(I)$$

then this is an interval in S^1. Moreover,

$$f^{n-m}(U) = \bigcup_{k \in \mathbb{Z}_{\leq -1}} f^{(k+1)(n-m)}(I) = U$$

So that U is f^{n-m} invariant. Now suppose that $U \neq S^1$ and consider two cases:

1. The endpoints of U are the same point
2. The endpoints of U are distinct

For example the interval $(0, 1)$ in S^1 has the same endpoints. Suppose that $x \in S^1$ is the unique endpoint of U. Then, as U is an interval, $S^1 \setminus U = \{x\}$. But as U is f^{n-m} invariant we also must have that $S^1 \setminus U = \{x\}$ is f^{n-m} invariant. Hence $f^{n-m}(x) = x$ and f has a periodic point. But this is a contradiction to corollary 1.26 as f has irrational rotation number.

Now supose that U has distinct endpoints. As U is f^{n-m} invariant and as f^{n-m} is continuous we have

$$f^{n-m}(\overline{U}) = \overline{U}$$

But, importantly, as the endpoints of U are distinct we have $\overline{U} \neq S^1$ and hence $\overline{U}$ is a closed interval. But, by a standard result from analysis, a homeomorphism from a closed interval to itself must have a fixed point. hence

$$f^{n-m} : \overline{U} \to \overline{U}$$

has a fixed point. But therefore, f has a periodic point which is, again, a contradiction.

Therefore we have proved that $U = S^1$. Now to prove the lemma, suppose that $y \in S^1$ is arbitrary. Then

$$y \in U = \bigcup_{k \in \mathbb{Z}_{\leq -1}} f^{k(n-m)}(I)$$

and hence there is a negative integer $-K \in \mathbb{Z}_{\leq -1}$ and a point $x \in I$ such that

$$f^{-K(n-m)}(x) = y$$

Hence $f^{K(n-m)}(y) = x \in I$ and $K(n-m) \in \mathbb{N}$ and we are done.
If we assumed $m > n$ then the proof would be identical, only we would be proving

$$\bigcup_{k \in \mathbb{N}} f^{k(n-m)}(I) = S^1$$

$\square$

Proposition 2.7. *If f is any homeomorphism (not assuming transitivity) with $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and we define, for $x \in S^1$*

$$\omega(x) = \{y \in S^1 : \exists \textit{ sequence } (n_k)_{k=1}^{\infty} \to \infty \textit{ such that } \lim_{k\to\infty} f^{n_k}(x) = y\}$$

then

$$\forall x, y \in S^1 \qquad \omega(x) = \omega(y)$$

Proof. Suppose $x, y \in S^1$ and $z \in \omega(x)$ are arbitrary. Then there is a sequence $(n_k)_{k=1}^{\infty} \to \infty$ such that

$$\lim_{k\to\infty} f_1^{n_k}(x) = z$$

But note that, by the previous lemma, $\forall k \in \mathbb{N},\ \exists N_k \in \mathbb{N}$ such that

$$f^{N_k}(y) \in [f^{n_k}(x), f^{n_{k+1}}(x)]$$

Hence

$$\lim_{k\to\infty} f^{N_k}(y) = z \in \omega(y)$$

and $N_k \to \infty$ as $k \to \infty$. As $z \in \omega(x)$ was arbitrary this implies $\omega(x) \subseteq \omega(y)$. By symmetry we have the opposite inclusion and therefore $\omega(x) = \omega(y)$

$\square$

Hence it makes sense to define the ω-limit set of any homeomorphism f with irrational rotation number. Define it by

$$\omega(f) := \{y \in S^1 : \exists x \in S^1 \text{ and } \exists \text{ sequence } (n_k)_{k=1}^{\infty} \to \infty \text{ such that } \lim_{k\to\infty} f^{n_k}(x) = y\}$$

By the previous proposition

$$\forall x \in S^1 \qquad \omega(x) = \omega(f)$$

Corollary 2.8. *If $f : S^1 \to S^1$ is a homeomorphism with $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and f is transitive then it is minimal*

Lemma 2.9. *If f_1, f_2 are any homeomorphisms (not assuming transitivity) with $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\exists \tau \in \mathrm{Hom}^+(S^1)$ such that*

$$f_1 = \tau^{-1} \circ f_2 \circ \tau$$

then

$$\tau(\omega(f_1)) = \omega(f_2)$$

Proof. If $y \in \tau(\omega(f_1))$ then $\exists x \in S^1$ and a sequence $(n_k)_{k=1}^{\infty} \to \infty$ such that

$$\lim_{k\to\infty} f_1^{n_k}(x) = \tau^{-1}(y)$$

so

$$\lim_{k\to\infty} f_2^{n_k}(\tau(x)) = \lim_{k\to\infty} \tau \circ f_1^{n_k} \circ \tau^{-1}(\tau(x)) = \tau(\lim_{k\to\infty} f_1^{n_k}(x)) = y$$

so $y \in \omega(f_2)$. Therefore $\tau(\omega(f_1)) \subseteq \omega(f_2)$. Now as τ^{-1} is also a homeomorphism which satisfies

$$f_2 = \left(\tau^{-1}\right)^{-1} \circ f_1 \circ \tau^{-1}$$

by symmetry we have $\tau^{-1}(\omega(f_2)) \subseteq \omega(f_1)$ and hence we are done.

$\square$

3 Classification of non-transitive homeomorphisms

In this section we will give a method how to answer when any two non-transitive homeomorphisms f_1, f_2 with irrational rotation number α are conjugate.

3.1 Denjoy example and surgery

We start with introducing the most simple non-transitive homeomorphisms. The following example, also known as Denjoy's example or surgery, provides a way of constructing a homeomorphism with irrational rotation number which is not transitive.

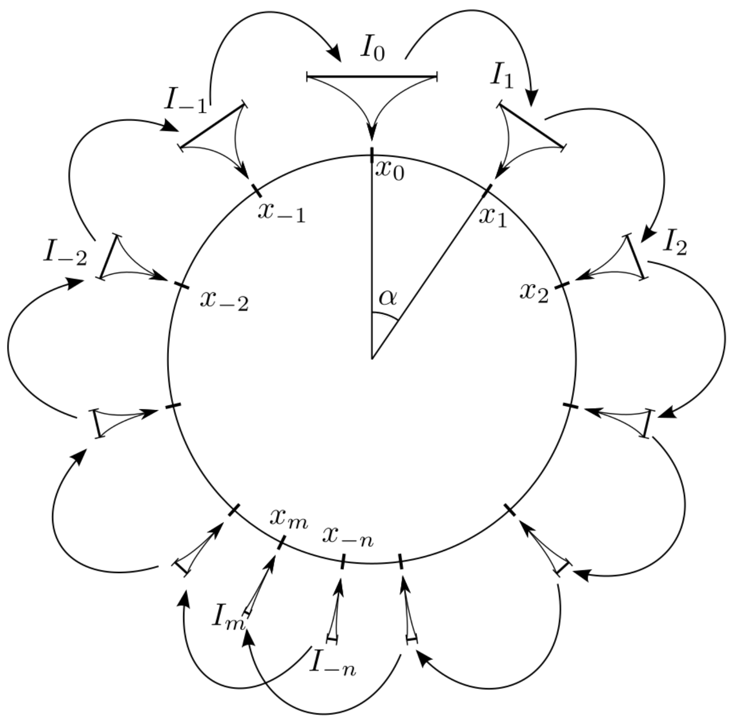
Suppose $\widetilde{f}$ is a transitive, orientation preserving homeomorphism with irrational rotation number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let $\forall n \in \mathbb{Z},\ x_n = \widetilde{f}^n(0)$ and let $(l_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that

$$\sum_{n\in\mathbb{Z}} l_n = d < 1.$$

We will try to make a new homeomorphism f by "replacing" the points x_n by intervals I_n in S^1 while trying to keep f to be "similar" to $\widetilde{f}$ on $S^1 \setminus \bigcup_{n\in\mathbb{Z}} I_n$. Define φ by, $\forall x \in [0,1)$

$$\varphi(x) = x(1-d) + \sum_{n:x_n\in[0,x)} l_n$$

Moreover define $\forall n \in \mathbb{Z},\ a_n := \varphi(x_n),\ b_n := a_n + l_n$ and $I_n := (a_n, b_n)$.



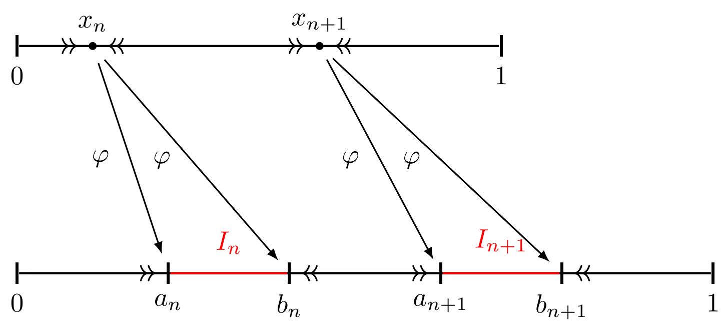
(this picture is from wikipedia)

Lemma 3.1. *φ is a strictly increasing bijection*

$$\varphi : [0,1) \to [0,1) \setminus \bigcup_{n\in\mathbb{Z}} (a_n, b_n]$$

with inverse

$$\varphi^{-1}(y) = \left(y - \sum_{n : a_n \in [0,y)} l_n \right) \frac{1}{1-d}$$



Proof. First, φ is clearly strictly increasing by its definition. To prove surjectivity of φ, and that its inverse is φ^{-1}, it is sufficient to show that

$$\forall y \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} (a_n, b_n] \qquad \varphi(\varphi^{-1}(y)) = y$$

For the proof of this, we will first need to prove that φ^{-1} is strictly increasing. Suppose $y_1 > y_2 \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} (a_n, b_n]$, then

$$\left(\varphi^{-1}(y_1) - \varphi^{-1}(y_2)\right)(1-d) = (y_1 - y_2) - \sum_{n : a_n \in [y_2, y_1)} l_n$$

As $y_1, y_2 \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} (a_n, a_n + l_n]$ we have that

$$a_n \in [y_2, y_1) \iff [a_n, a_n + l_n] \subset [y_2, y_1)$$

Therefore

$$\bigcup_{n : a_n \in [y_2, y_1)} [a_n, a_n + l_n] \subset [y_2, y_1)$$

Therefore, as the intervals $[a_n, a_n + l_n]$ are disjoint we have

$$(y_1 - y_2) > \sum_{n : a_n \in [y_2, y-1)} l_n$$

Therefore

$$\varphi^{-1}(y_1) - \varphi^{-1}(y_2) > 0$$

and so φ^{-1} is strictly increasing. Now, to finish the proof, $\forall y \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} (a_n, b_n]$

$$\varphi(\varphi^{-1}(y)) = \varphi^{-1}(y)(1-d) + \sum_{n : x_n \in [0, \varphi^{-1}(y))} l_n$$

$$\varphi(\varphi^{-1}(y)) = y - \sum_{n : a_n \in [0,y)} l_n + \sum_{n : x_n \in [0, \varphi^{-1}(y))} l_n$$

But as φ^{-1} is strictly increasing

$$a_n \in [0, y) \iff x_n = \varphi^{-1}(a_n) \in [0, \varphi^{-1}(y))$$

Therefore the above equation simplifies to

$$\varphi(\varphi^{-1}(y)) = y$$

$\square$

Lemma 3.2. *The restriction of* φ

$$\varphi : [0, 1) \setminus \{x_n : n \in \mathbb{Z}\} \to [0, 1) \setminus \bigcup_{n \in \mathbb{Z}} [a_n, b_n]$$

is a homeomorphism. Moreover, the following hold

$$\lim_{x \uparrow x_n} \varphi(x) = a_n \qquad \textit{and} \qquad \lim_{y \uparrow a_n} \varphi^{-1}(y) = x_n$$
$$\lim_{x \downarrow x_n} \varphi(x) = b_n \qquad \textit{and} \qquad \lim_{y \downarrow b_n} \varphi^{-1}(y) = x_n$$

Proof. To prove continuity of φ assume that $x \in [0, 1) \setminus \{x_n : n \in \mathbb{Z}\}$ and that $\epsilon > 0$ is arbitrary. As $\sum_{n \in \mathbb{Z}} l_n = d$ is convergent, $\exists N \in \mathbb{N}$ such that

$$\sum_{\substack{n: \in \mathbb{Z}, \\ |n| > N}} l_n < \frac{\epsilon}{2}$$

Now as $x \notin \{x_n : n \in \mathbb{Z}\}$ there exists $\delta' > 0$ such that

$$(x - \delta', x + \delta') \cap \{x_n : |n| \leq N\} = \emptyset$$

Let $\delta = \min(\delta', \frac{\epsilon}{2})$. Then, if $y \in [0, 1)$ is such that $|x - y| < \delta$ (assume w.l.o.g $y \geq x$) then

$$\varphi(y) - \varphi(x) = (y - x)(1 - d) + \sum_{n: x_n \in [x, y)} l_n < (y - x) + \sum_{\substack{n: \in \mathbb{Z}, \\ |n| > N}} l_n < \delta + \frac{\epsilon}{2} < \epsilon$$

As φ^{-1} has similar form to φ, by an analogous proof, if $y \notin \{a_n : n \in \mathbb{Z}\}$ then

$$\varphi^{-1}(y) = \left(y - \sum_{n: a_n \in [0, y)} l_n \right) \frac{1}{1 - d}$$

is continuous at y. This proves that φ is a homeomorphism on the restriction given in the lemma. Moreover, note that even if $x = x_n$ for some $n \in \mathbb{Z}$ then there exists $\delta' > 0$ such that

$$(x - \delta', x) \cap \{x_n : |n| \leq N\} = \emptyset$$

Therefore if $y \in [0, 1)$ and $x - \delta' < y < x$ then, analogously,

$$\varphi(x) - \varphi(y) = (x - y)(1 - d) + \sum_{n: x_n \in [y, x)} l_n < (x - y) + \sum_{\substack{n: \in \mathbb{Z}, \\ |n| > N}} l_n < \epsilon$$

which proves that φ is continuous from the left on $[0, 1)$. Analogously φ^{-1} is continuous from the left on its domain $[0, 1) \setminus \bigcup_{n \in \mathbb{Z}} (a_n, b_n]$ (which, in fact, shows that φ^{-1} is continuous). This shows

$$\lim_{x \uparrow x_n} \varphi(x) = \varphi(x_n) = a_n \qquad \text{and} \qquad \lim_{y \uparrow a_n} \varphi^{-1}(y) = \varphi^{-1}(a_n) = x_n$$

To prove the second set of equations, note that if $x > x_n$ then

$$\varphi(x) - \varphi(x_n) = \sum_{k:x_k \in [x_n, x)} l_k = l_n + \sum_{k:x_k \in (x_n, x)} l_k$$

If x is chosen sufficiently close to x_n then $\sum_{k:x_k \in (x_n,x)} l_k$ is arbitrarily small. Therefore

$$\lim_{x \downarrow x_n} \varphi(x) - \varphi(x_n) = l_n$$
$$\lim_{x \downarrow x_n} \varphi(x) = \varphi(x_n) + l_n = b_n$$

Finally, suppose that $y > b_n = a_n + l_n$, then

$$\varphi^{-1}(y) - x_n = \varphi^{-1}(y) - \varphi^{-1}(a_n) = y - a_n - \sum_{k:a_k \in [a_n, y)} l_k$$
$$\varphi^{-1}(y) - x_n = y - a_n - l_n - \sum_{k:a_k \in (a_n, y)} l_k = (y - b_n) - \sum_{k:a_k \in (a_n, y)} l_k$$
$$\varphi^{-1}(y) - x_n = (y - b_n) - \sum_{k:a_k \in (b_n, y)} l_k$$

But again, if y is chosen sufficiently close to b_n then $\sum_{k:a_k \in (b_n,y)} l_k$ and $(y - b_n)$ are arbitrarily small. Therefore

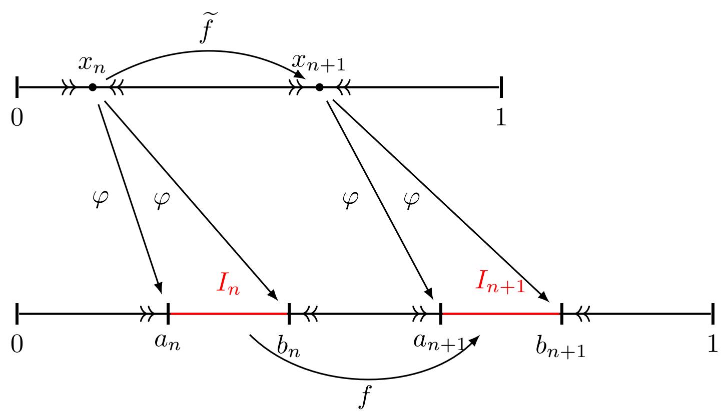
$$\lim_{y \downarrow b_n} \varphi^{-1}(y) = x_n$$

□

Now using φ we can define the new homeomorphism $f : [0, 1) \to [0, 1)$ (with x_n replaced by I_n) as following:

$$f(x) = \begin{cases} \varphi \circ \widetilde{f} \circ \varphi^{-1}(x) & x \in [0,1) \setminus \bigcup_{n \in \mathbb{Z}} \overline{I_n} \\ G_n(x) & x \in \overline{I_n} \end{cases}$$

Where $G_n : \overline{I_n} \to \overline{I_{n+1}}$ can be any increasing homeomorphism from $\overline{I_n}$ to $\overline{I_{n+1}}$, for example the linear map between the intervals.



Noting that $\widetilde{f}([0,1) \setminus \{x_n : n \in \mathbb{Z}\}) = ([0,1) \setminus \{x_n : n \in \mathbb{Z}\})$ and by the last lemma we have that $f : [0,1) \to [0,1)$ is a bijection. To check that it is a homeomorphism, it is enough to check that it is continuous, because continuity of the inverse follows as we are on S^1. We have that

$$f = \varphi \circ \widetilde{f} \circ \varphi^{-1}$$

is a composition of homeomorphisms on $[0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n}$ and

$$f = G_n : \overline{I_n} \to \overline{I_{n+1}}$$

is a homeomorphism. So to prove continuity we just need to check if the values at the endpoints of the intervals match. We need that $\forall n \in \mathbb{Z}$

$$\lim_{x \downarrow b_n} f(x) = b_{n+1} \qquad \text{and} \qquad \lim_{x \uparrow a_n} f(x) = a_{n+1}$$

But, this follows from lemma 3.2 above.

Now if we consider the lifts $F, \widetilde{F}$ of $f, \widetilde{f}$ then by construction

$$\forall x \in [0,1),\ \forall n \in \mathbb{Z} \qquad \lfloor F^n(x) \rfloor = \lfloor \widetilde{F}^n(x) \rfloor$$

and therefore

$$\begin{aligned}
\rho(f) &= \left(\lim_{n\to\infty} \frac{F^n(0)}{n} \right) \text{mod } 1 = \left(\lim_{n\to\infty} \frac{\lfloor F^n(0) \rfloor}{n} \right) \text{mod } 1 \\
\rho(\widetilde{f}) &= \left(\lim_{n\to\infty} \frac{\widetilde{F}^n(0)}{n} \right) \text{mod } 1 = \left(\lim_{n\to\infty} \frac{\lfloor \widetilde{F}^n(0) \rfloor}{n} \right) \text{mod } 1 \\
\rho(f) &= \rho(\widetilde{f})
\end{aligned}$$

and so the rotation number of f is the same as the rotation number of $\widetilde{f}$.

One can similarly define N surgeries, for any $N \in \mathbb{N} \cup \{\infty\}$ (by $N = \infty$ we mean, in the subsequent discussion, that $k \in \{1, 2, \dots, N\} \iff k \in \mathbb{N}$). Suppose $\widetilde{f}$ is a transitive, homeomorphism with rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define for any $x_k \in S^1$

$$Orb_{\widetilde{f}}(x_k) := \{x_{k,n} : n \in \mathbb{Z}\} := \{\widetilde{f}^n(x_k) : n \in \mathbb{Z}\}$$

If the sets $Orb_{\widetilde{f}}(x_1), Orb_{\widetilde{f}}(x_2), \dots, Orb_{\widetilde{f}}(x_N)$ are disjoint and if $(l_{1,n})_{n\in\mathbb{Z}}, (l_{2,n})_{n\in\mathbb{Z}}, \dots (l_{N,n})_{n\in\mathbb{Z}}$ are sequences of positive numbers such that

$$\sum_{k=1}^{N} \sum_{n\in\mathbb{Z}} l_{k,n} = d < 1.$$

Then we define φ by, $\forall x \in [0,1)$

$$\varphi(x) = x(1-d) + \sum_{k=1}^{N} \left(\sum_{n : x_{k,n} \in [0,x)} l_{k,n} \right)$$

Moreover define $\forall k \in \{1, 2 \dots, N\},\ \forall n \in \mathbb{Z}$

$$a_{k,n} := \varphi(x_{k,n}) \qquad b_{k,n} := a_{k,n} + l_{k,n} \qquad I_{k,n} := (a_{k,n}, b_{k,n})$$

Then φ can be shown to be an increasing, left continuous bijection

$$\varphi : [0,1) \to [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N\}}} (a_{k,n}, b_{k,n}]$$

with inverse

$$\varphi^{-1}(y) = \left(y - \sum_{k=1}^{M} \left(\sum_{n:a_{k,n}\in[0,x)} l_{k,n} \right) \right) \frac{1}{1-d}$$

and as before

Lemma 3.3. *The restriction of φ*

$$\varphi : [0,1) \setminus \bigcup_{k=1}^{N} Orb_{\widetilde{f}}(x_k) \to [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N\}}} [a_{k,n}, b_{k,n}]$$

is a homeomorphism. Moreover, the following hold

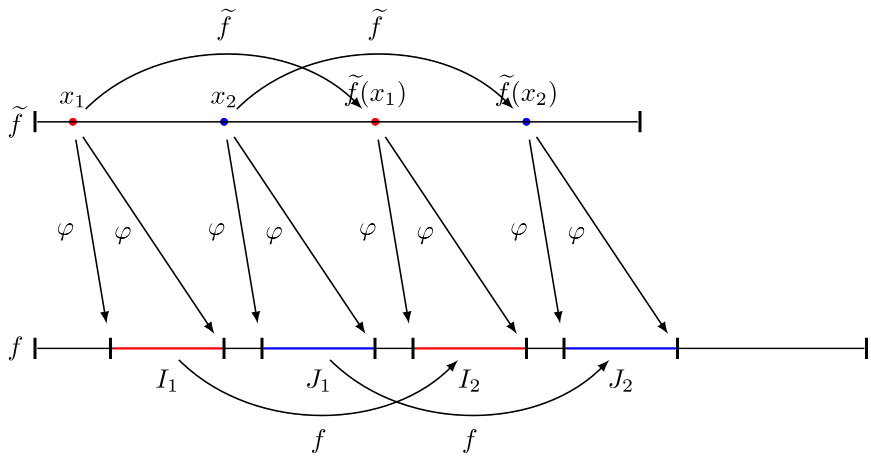
$$\lim_{x\uparrow x_{k,n}} \varphi(x) = a_{k,n} \qquad \text{and} \qquad \lim_{y\uparrow a_{k,n}} \varphi^{-1}(y) = x_{k,n}$$
$$\lim_{x\downarrow x_{k,n}} \varphi(x) = b_{k,n} \qquad \text{and} \qquad \lim_{y\downarrow b_{k,n}} \varphi^{-1}(y) = x_{k,n}$$

Now we can define $f : [0,1) \to [0,1)$

$$f(x) = \begin{cases} \varphi \circ \widetilde{f} \circ \varphi^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2,\dots,N\}} \overline{I_{k,n}} \\ G_{k,n}(x) & x \in \overline{I_{k,n}} \end{cases}$$

Where $\forall n \in \mathbb{Z}, k \in \{1,2,\dots,N\}$, $G_{k,n} : \overline{I_{k,n}} \to \overline{I_{k,n+1}}$ is some increasing homeomorphism. As in the previous case, this function f can be shown to be a homeomorphism with the same rotation number α as $\widetilde{f}$.

The picture illustrates the surgery for $N = 2$;



Definition 3.4. If $\widetilde{f}$ is transitive with $\rho(\widetilde{f}) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and f is defined from $\widetilde{f}$ by surgery on disjoint orbits $Orb_f(x_1), Orb_f(x_2), \dots, Orb_f(x_N)$, then define $\chi(f) = N \in \mathbb{N} \cup \{\infty\}$ to be Denjoy number of f.

Lemma 3.5. $\omega(f) = [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2,\ldots,N\}} I_{k,n}$

Proof. On $[0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2,\ldots,N\}} Orb_{\widetilde{f}}(x_{k,n})$ we have that φ is a homeomorphism and $\widetilde{f} = \varphi^{-1} \circ f \circ \varphi$. Moreover $\widetilde{f}$ is tranitive, i.e. $\omega(\widetilde{f}) = S^1$. Therefore by an argument identical to that of lemma 3.7. we have

$$\varphi\left([0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\ldots,N\}}} Orb_{\widetilde{f}}(x_{k,n})\right) \subseteq \omega(f)$$

$$[0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\ldots,N\}}} \overline{I_{k,n}} \subseteq \omega(f)$$

So to complete the proof we need to show that

$$\forall n \in \mathbb{Z}, k \in \{1,2,\ldots,N\} \qquad I_{k,n} = (a_{k,n}, b_{k,n}) \not\subset \omega(f) \qquad \text{and} \qquad a_{k,n}, b_{k,n} \in \omega(f)$$

Suppose $x \in I_{k,n}$ and $y \in [0,1)$ and $n_j \in \mathbb{Z}$ are such that

$$f^{n_j}(y) \in (x-\epsilon, x+\epsilon)$$

Where $\epsilon > 0$ is small enough so that $(x-\epsilon, y-\epsilon) \subset I_{k,n}$. Then

$$\forall z \in \mathbb{N},\ f^{n_j+z}(y) \in I_{k,n+z}$$

but the intervals $I_{k,n}$ are disjoint. Therefore

$$\forall z \in \mathbb{N},\ f^{n_j+z}(y) \notin I_{k,n}$$

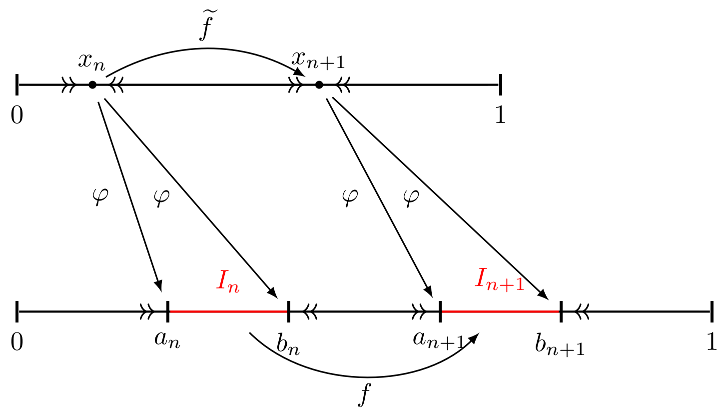
hence it cannot be the case that there is a subsequence converging to x. So $x \notin \omega(f)$, and as $x \in I_{k,n}$ was arbitrary $I_{k,n} \not\subset \omega(f)$.

Now we show that $a_{k,n}, b_{k,n} \in \omega(f)$. As $\widetilde{f}$ is conjugate to the rotation, there exist $x \in [0,1) \setminus \left(\bigcup_{n\in\mathbb{Z},k\in\{1,2,\ldots,N\}} Orb_{\widetilde{f}}(x_{k,n})\right)$ and sequences $(n_j)_{j=1}^{\infty} \to \infty, (m_j)_{j=1}^{\infty} \to \infty$ such that

$$\widetilde{f}^{n_j}(x) \downarrow x_{k,n}$$
$$\widetilde{f}^{m_j}(x) \uparrow x_{k,n}$$

Using lemma 3.3 we have

$$\lim_{x\downarrow x_{k,n}} \varphi(x) = b_{k,n}$$
$$\lim_{x\uparrow x_{k,n}} \varphi(x) = a_{k,n}$$



So that

$$\lim_{j\to\infty} f^{n_j}(\varphi(x)) = \lim_{j\to\infty} \varphi \circ \widetilde{f}^{n_j} \circ \varphi^{-1}(\varphi(x)) = \lim_{j\to\infty} \varphi(\widetilde{f}^{n_j}(x)) = b_{k,n}$$
$$\lim_{j\to\infty} f^{m_j}(\varphi(x)) = \lim_{j\to\infty} \varphi \circ \widetilde{f}^{n_j} \circ \varphi^{-1}(\varphi(x)) = \lim_{j\to\infty} \varphi(\widetilde{f}^{n_j}(x)) = a_{k,n}$$

So $a_{k,n}, b_{k,n} \in \omega(f)$, and we have finished proving

$$\omega(f) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\ldots,N\}}} I_{k,n}.$$

□

3.2 Topological conjugacy under surgery

First, we start with a proposition about non-transitive homeomorphisms with Denjoy number $\chi(f) = 1$, i.e. homeomorphisms produced from a transitive homeomorphism by one surgery.

Proposition 3.6. *If f_1, f_2 are homeomorphisms with $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $\chi(f_1) = \chi(f_2) = 1$ then f_1 and f_2 are topologically conjugate.*

Proof. As $\chi(f_1) = \chi(f_2) = 1$, then there exist transitive homeomorphisms $\widetilde{f_1}, \widetilde{f_2}$ with rotation number α and $x, x' \in [0,1)$ such that f_1, f_2 are produced from $\widetilde{f_1}, \widetilde{f_2}$ via surgery on

$$Orb_{\widetilde{f_1}}(x) := \{\widetilde{f_1}^{\,n}(x) : n \in \mathbb{Z}\}$$
$$Orb_{\widetilde{f_2}}(x') := \{\widetilde{f_2}^{\,n}(x') : n \in \mathbb{Z}\}$$

to get intervals $I_n, n \in \mathbb{Z}$ and $J_n, n \in \mathbb{Z}$ corresponding to f_1 and f_2 respectively, which are defined by:

$$f_1(x) = \begin{cases} \varphi_1 \circ \widetilde{f_1} \circ \varphi_1^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n} \\ G_n(x) \quad x \in \overline{I_n} \end{cases}$$
$$f_2(x) = \begin{cases} \varphi_2 \circ \widetilde{f_2} \circ \varphi_2^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{J_n} \\ H_n(x) \quad x \in \overline{J_n} \end{cases}$$

Where φ_i are as before, $G_n : \overline{I_n} \to \overline{I_{n+1}}$ and $H_n : \overline{J_n} \to \overline{J_{n+1}}$ are some increasing homeomorphisms.

Now by lemma 2.5 $\exists \tau \in Hom(S^1)$ such that $\tau(x) = x'$ and $\widetilde{f}_1, \widetilde{f}_2$ are conjugate via τ. So that we have

$$\tau(Orb_{\widetilde{f}_1}(x)) = Orb_{\widetilde{f}_2}(x')$$
$$\tau\left([0,1) \setminus Orb_{\widetilde{f}_1}(x)\right) = \left([0,1) \setminus Orb_{\widetilde{f}_2}(x')\right)$$

Now define $\sigma : [0,1) \setminus \bigcup_{n\in\mathbb{Z},} \overline{I_n} \to [0,1) \setminus \bigcup_{n\in\mathbb{Z},} \overline{J_n}$ by

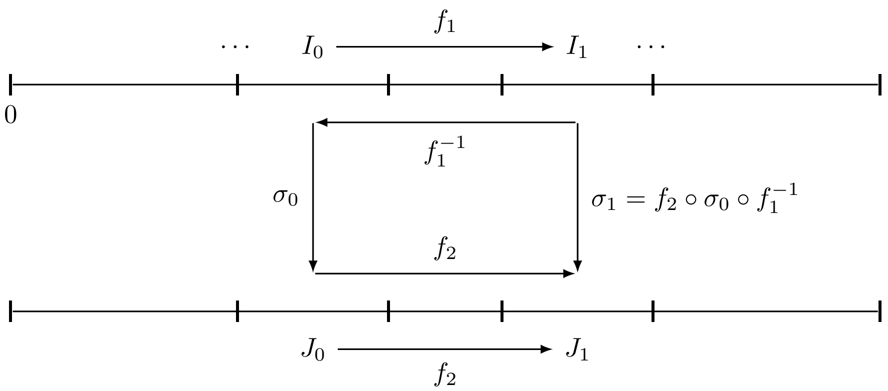
$$\sigma(x) = \varphi_2 \circ \tau \circ \varphi_1^{-1}(x)$$

Then on this domain σ is a composition of homeomorphisms and satisfies:

$$\begin{aligned}
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \tau^{-1} \circ \varphi_2^{-1} \circ f_2 \circ \varphi_2 \circ \tau \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \tau^{-1} \circ \widetilde{f}_2 \circ \tau \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \widetilde{f}_1 \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= f_1 \qquad (3)
\end{aligned}$$

Moreover, define $\sigma_0 : \overline{I_0} \to \overline{J_0}$ to be any increasing homeomorphism, for example the linear rescaling of the intervals and inductively define

$$\forall n \in \mathbb{N} \qquad \sigma_n = f_2 \circ \sigma_{n-1} \circ f_1^{-1} \qquad \overline{I_n} \to \overline{J_n} \qquad (4)$$
$$\forall n \in \mathbb{Z}_{\leq -1} \qquad \sigma_n = f_2^{-1} \circ \sigma_{n-1} \circ f_1 \qquad \overline{I_n} \to \overline{J_n} \qquad (5)$$



Now if we extend $\sigma : [0,1) \to [0,1)$ by

$$\sigma(x) = \begin{cases} \varphi_2 \circ \tau^{-1} \circ \varphi_1^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n} \\ \sigma_n(x) & x \in \overline{I_n} \end{cases}$$

Then σ is a bijection on $[0,1)$ and equations $(3), (4), (5)$ imply that

$$\forall x \in S^1 \qquad f_1 = \sigma^{-1} \circ f_2 \circ \sigma$$

To conclude that σ is a conjugacy we have to show that it is a homeomorphism. For this it is sufficient to check that it is continuous, because continuity of the inverse follows as we are on S^1. We have that

$$\sigma = \varphi_2 \circ \tau \circ \varphi_1^{-1}$$

is a composition of homeomorphisms on $[0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n}$ and

$$\sigma = \sigma_n : \overline{I_n} \to \overline{J_n}$$

is a homeomorphism. So to prove continuity we just need to check if the values at the endpoints of the intervals match. So we need to show that $\forall n \in \mathbb{Z}$

$$\lim_{x\downarrow b_n} \sigma(x) = d_n \qquad \text{and} \qquad \lim_{x\uparrow a_n} \sigma(x) = c_n$$

(where $I_n = (a_n, b_n)$ and $J_n = (c_n, d_n)$). To show $\lim_{x\uparrow a_n} \sigma(x) = c_n$ it is enough to quote lemma 3.3, that the following hold:

$$\begin{aligned}
\lim_{t\uparrow a_n} \varphi_1^{-1}(t) &= \widetilde{f}_1^n(x) \\
\lim_{s\uparrow \widetilde{f}_1^n(x)} \tau(s) &= \tau(\widetilde{f}_1^n(x)) = \widetilde{f}_2^n(x') \\
\lim_{z\uparrow \widetilde{f}_2^n(x')} \varphi_2(z) &= c_n
\end{aligned}$$

where $x \uparrow a_n$ over $[0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n}$. It is an exercise in analysis to show that the limit can be taken over $[0,1)$. The second equation can be shown similarly. This finishes the proof of the proposition. □

Now that we showed that Denjoy number 1 homeomorphisms are topologically conjugate, we can start discussing homeomorphisms with higher Denjoy numbers. We will shortly illustrate that equality of Denjoy number is a neccessary condition for topological conjugacy.

Suppose f_1 is produced by performing surgery (as defined in previous section) on $Orb_{R_\alpha}(0)$ of the rotation R_α. Let $y = \frac{p}{q} \in \mathbb{Q} \cap (0, \alpha)$ so that the orbit of y under R_α is disjoint from the orbit of 0 under R_α. Now define f_2 by surgery on $Orb_{R_\alpha}(0)$ and $Orb_{R_\alpha}(y)$. Hence $f_1, f_2 : [0,1) \to [0,1)$ are defined as following:

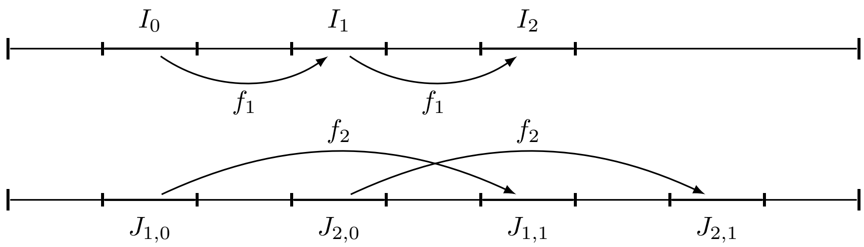
$$f_1(x) = \begin{cases} \varphi_1 \circ R_\alpha \circ \varphi_1^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z},} \overline{I_n} \\ G_n(x) & x \in \overline{I_n} \end{cases}$$

$$f_2(x) = \begin{cases} \varphi_2 \circ R_\alpha \circ \varphi_2^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{J_{k,n}} \\ H_{1,n}(x) & x \in \overline{J_{1,n}} \\ H_{2,n}(x) & x \in \overline{J_{2,n}} \end{cases}$$

Where

$$\begin{aligned}
\varphi_1 &: [0,1) \setminus Orb_{R_\alpha}(0) \to [0,1) \setminus \bigcup_{n\in\mathbb{Z}} \overline{I_n} \\
\varphi_2 &: [0,1) \setminus Orb_{R_\alpha}(0) \cup Orb_{R_\alpha}(y) \to [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z}, \\ k\in\{1,2\}}} \overline{J_{k,n}}
\end{aligned}$$

are homeomorphisms as before and $G_n : \overline{I_n} \to \overline{I_{n+1}}$, $H_{k,n} : \overline{J_{k,n}} \to \overline{J_{k,n+1}}$ are some increasing homeomorphisms.



Now we are going to prove that f_1 and f_2 are not topologically conjugate. Suppose for a contradiction that there is a homeomorphism $\pi : S^1 \to S^1$ such that $\pi \circ f_1 \circ \pi^{-1} = f_2$. Then by lemma 3.5, we have

$$\omega(f_1) = [0,1) \setminus \bigcup_{n\in\mathbb{Z}} I_n$$
$$\omega(f_2) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$

And by lemma 2.9 we have

$$\pi(\omega(f_1)) = \pi(f_2)$$
$$\pi\left([0,1) \setminus \bigcup_{n\in\mathbb{Z}} I_n\right) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$
$$\pi\left(\bigcup_{n\in\mathbb{Z}} I_n\right) = \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$

Now as π is continuous $\pi(I_0)$ is an interval and as $J_{k,n} : n \in \mathbb{Z}$ are disjoint intervals we must have that $\exists m \in \mathbb{Z}$ and $\exists j \in \{1,2\}$ such that

$$\pi(I_0) \subseteq J_{j,m}$$

Let without loss of generality $j = 1$ and $m = 0$. By symmetry $\exists l \in \mathbb{Z}$ such that

$$\pi^{-1}(J_{1,0}) \subseteq I_l$$

Hence as $I_n : n \in \mathbb{Z}$ are disjoint we have $l = 0$ and

$$\pi(I_0) = J_{1,0}$$

Moreover, as $\pi \circ f_1 = f_2 \circ \pi$, by an inductive argument we must have

$$\begin{aligned}
\pi(I_1) &= \pi \circ f_1(I_0) = f_2 \circ \pi(I_0) = f_2(J_{1,0}) = J_{1,1} \\
\pi(I_1) &= J_{1,1} \\
&\vdots \\
\pi(I_n) &= \pi \circ f_1(I_{n-1}) = f_2 \circ \pi(I_{n-1}) = f_2(J_{1,n-1}) = J_{1,n} \\
\pi(I_n) &= J_{1,n}
\end{aligned}$$

So $\forall n \in \mathbb{Z},\ \pi(I_n) = J_{1,n}$ and hence

$$\pi\left(\bigcup_{n\in\mathbb{Z}} I_n\right) = \bigcup_{n\in\mathbb{Z}} J_{1,n}$$
$$\pi\left(\bigcup_{n\in\mathbb{Z}} I_n\right) \neq \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$

Which is a contradiction to surjectivity of π. So f_1 and f_2 are not topologically conjugate.

Proposition 3.7. *If $f_1 \sim f_2$ then $\chi(f_1) = \chi(f_2)$*

Proof. Suppose, for a contradiction that,

$$\chi(f_1) = N_1 < N_2 = \chi(f_2) \qquad N_1 \in \mathbb{N}, N_2 \in \mathbb{N} \cup \{\infty\}$$

and $\exists \pi \in Hom(S^1)$ such that

$$f_1 = \pi^{-1} \circ f_2 \circ \pi$$

By previous discussion f_1 and f_2 are defined by

$$f_1(x) = \begin{cases} \varphi_1 \circ \widetilde{f} \circ \varphi_1^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}, k\in\{1,2,\dots,N_1\}} \overline{I_{k,n}} \\ G_{k,n}(x) & x \in \overline{I_{k,n}} \end{cases}$$
$$f_2(x) = \begin{cases} \varphi_2 \circ \widetilde{f} \circ \varphi_2^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z}, k\in\{1,2,\dots,N_2\}} \overline{J_{k,n}} \\ H_{k,n}(x) & x \in \overline{J_{k,n}} \end{cases}$$

Where φ_i are as before, $G_{k,n} : \overline{I_{k,n}} \to \overline{I_{k,n+1}}$ and $H_{k,n} : \overline{J_{k,n}} \to \overline{J_{k,n+1}}$ are some increasing homeomorphisms. Then, again, by lemma 3.5, we have

$$\omega(f_1) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N_1\}}} I_{k,n}$$
$$\omega(f_2) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N_2\}}} J_{k,n}$$

And by lemma 2.9 we have

$$\pi(\omega(f_1)) = \pi(f_2)$$
$$\pi\left(\bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N_1\}}} I_{k,n}\right) = \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N_2\}}} J_{k,n} \tag{6}$$

This implies, analogously by performing induction on $k \in \{1,2,\dots,N_1\}$ to the argument presented in the previous example, that there exists an injective function $i : \{1,2,\dots,N_1\} \to \{1,2,\dots,N_2\}$ such that

$$\forall k \in \{1,2,\dots,N_1\} \qquad \pi\left(\bigcup_{n\in\mathbb{Z},} I_{k,n}\right) = \bigcup_{n\in\mathbb{Z}} J_{i(k),n}$$

Hence

$$\pi\left(\bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N_1\}}} I_{k,n}\right)=\bigcup_{\substack{n\in\mathbb{Z},\\ k\in i(\{1,2,\dots,N_1\})}} J_{k,n}$$

But as $N_1 < N_2$ and as i is injective

$$i\left(\{1,2,\dots,N_1\}\right)\neq\{1,2,\dots,N_2\}$$

and hence this is a contradiction to equation 6, i.e. a contradction to surejectivity of π. $\square$

So we have showed that if $\chi(f_1)\neq\chi(f_2)$ then f_1 and f_2 are not topologically conjugate. On the other hand lemma 3.6 Shows that if $\chi(f_1)=\chi(f_2)=1$ then f_1 and f_2 are topologically conjugate. Is this true if $\chi(f_1)=\chi(f_2)=2$? Is this true if $\chi(f_1)=\chi(f_2)=N\in\mathbb{N}\cup\{\infty\}$? We start with answering the question with $N=2$ and then extend to an arbitrary integer.

Proposition 3.8. *If $\chi(f_1)=\chi(f_2)=2$ are produced from transitive $\widetilde{f}_1,\widetilde{f}_2$ via surgery on*

$$Orb_{\widetilde{f}_1}(x_1), Orb_{\widetilde{f}_1}(x_2)$$

and

$$Orb_{\widetilde{f}_2}(x'_1), Orb_{\widetilde{f}_2}(x'_2)$$

respectively then $f_1\sim f_2$ if and only if $\exists\tau\in Hom(S^1)$ such that $\widetilde{f}_1=\tau^{-1}\circ\widetilde{f}_2\circ\tau$ and one of the following holds:

$$\begin{array}{lcl} \tau(x_1)\in Orb_{\widetilde{f}_2}(x'_1) & \textit{and} & \tau(x_2)\in Orb_{\widetilde{f}_2}(x'_2)\\ \tau(x_2)\in Orb_{\widetilde{f}_2}(x'_1) & \textit{and} & \tau(x_1)\in Orb_{\widetilde{f}_2}(x'_2)\end{array}$$

Proof:

By previous discussion f_1 and f_2 are defined by

$$f_1(x)=\begin{cases}\varphi_1\circ\widetilde{f}\circ\varphi_1^{-1}(x) & x\in[0,1)\setminus\bigcup_{n\in\mathbb{Z},k\in\{1,2\}}\overline{I_{k,n}}\\ G_{1,n}(x) & x\in\overline{I_{1,n}}\\ G_{2,n}(x) & x\in\overline{I_{2,n}}\end{cases}$$

$$f_2(x)=\begin{cases}\varphi_2\circ\widetilde{f}\circ\varphi_2^{-1}(x) & x\in[0,1)\setminus\bigcup_{n\in\mathbb{Z},k\in\{1,2\}}\overline{J_{k,n}}\\ H_{1,n}(x) & x\in\overline{J_{1,n}}\\ H_{2,n}(x) & x\in\overline{J_{2,n}}\end{cases}$$

Where φ_i are as before, $G_{k,n}:\overline{I_{k,n}}\to\overline{I_{k,n+1}}$ and $H_{k,n}:\overline{J_{k,n}}\to\overline{J_{k,n+1}}$ are some increasing homeomorphisms. Suppose $f_1\sim f_2$. Then $\exists\sigma\in Hom(S^1)$ such that

$$f_1=\sigma^{-1}\circ f_2\circ\sigma$$

By lemma 3.5, we have

$$\omega(f_1) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} I_{k,n}$$

$$\omega(f_2) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$

And by lemma 2.9 we have

$$\sigma(\omega(f_1)) = \omega(f_2)$$

$$\sigma\left([0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} I_{k,n}\right) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} J_{k,n}$$

$$\sigma\left(\bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} \overline{I_{k,n}}\right) = \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} \overline{J_{k,n}}$$

Therefore $\exists z_1 \in \mathbb{Z}$ such that

$$\sigma(\overline{I_{1,0}}) = \overline{J_{1,z_1}} \qquad \text{or}$$

$$\sigma(\overline{I_{1,0}}) = \overline{J_{2,z_1}}$$

Let $\sigma(\overline{I_{1,0}}) = \overline{J_{1,z_1}}$, say. As $f_1 = \sigma^{-1} \circ f_2 \circ \sigma$ therefore, by an inductive argument, we have

$$\sigma(\overline{I_{1,1}}) = \sigma \circ f_1(\overline{I_{1,0}}) = f_2 \circ \sigma(\overline{I_{1,0}}) = f_2(\overline{J_{1,z_1}}) = \overline{J_{1,z_1+1}}$$

$$\sigma(\overline{I_{1,1}}) = \overline{J_{1,z_1+1}}$$

$$\vdots$$

$$\sigma(\overline{I_{1,n}}) = \sigma \circ f_1(\overline{I_{1,n-1}}) = f_2 \circ \sigma(\overline{I_{1,n-1}}) = f_2(\overline{J_{1,z_1+n-1}}) = \overline{J_{1,z_1+n}}$$

$$\sigma(\overline{I_{1,n}}) = \overline{J_{1,z_1+n}}$$

This gives

$$\sigma\left(\bigcup_{n\in\mathbb{Z}} \overline{I_{1,n}}\right) = \bigcup_{n\in\mathbb{Z}} \overline{J_{1,n}}$$

Therefore by injectivity of σ there exists $z_2 \in \mathbb{Z}$ such that $\sigma(\overline{I_{2,0}}) = \overline{J_{2,z_2}}$ and by a similar argument

$$\sigma(\overline{I_{2,n}}) = \overline{J_{2,z_2+n}} \qquad \forall n \in \mathbb{Z}$$

$$\sigma\left(\bigcup_{n\in\mathbb{Z}} \overline{I_{2,n}}\right) = \bigcup_{n\in\mathbb{Z}} \overline{J_{2,n}}$$

Now denote $I_{k,n} = [a_{k,n}, b_{k,n}]$ and $J_{k,n} = [c_{k,n}, d_{k,n}]$ and define $\tau : [0,1) \to [0,1)$ by

$$\tau = \varphi_2^{-1} \circ \sigma \circ \varphi_1$$

note that $\varphi_1, \varphi_2, \sigma$ are bijections and

$$\varphi_1([0,1)) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} (a_{k,n}, b_{k,n}]$$

$$\sigma\left([0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} (a_{k,n}, b_{k,n}]\right) = [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} (c_{k,n}, d_{k,n}]$$

$$\varphi_2^{-1}\left([0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2\}}} (c_{k,n}, d_{k,n}]\right) = [0,1)$$

So τ is indeed a well defined $[0,1) \to [0,1)$ bijection. Moreover, τ satisfies the property:

$$\begin{aligned}
\tau^{-1} \circ \widetilde{f}_2 \circ \tau &= \varphi_1^{-1} \circ \sigma^{-1} \circ \varphi_2 \circ \widetilde{f}_2 \circ \varphi_2^{-1} \circ \sigma \circ \varphi_1 \\
\tau^{-1} \circ \widetilde{f}_2 \circ \tau &= \varphi_1^{-1} \circ \sigma^{-1} \circ f_2 \circ \sigma \circ \varphi_1 \\
\tau^{-1} \circ \widetilde{f}_2 \circ \tau &= \varphi_1^{-1} \circ f_1 \circ \varphi_1 \\
\tau^{-1} \circ \widetilde{f}_2 \circ \tau &= \widetilde{f}_1
\end{aligned}$$

To conclude that τ is a conjugacy we need to prove that it is a homeomorphism. We have that $\tau = \varphi_2^{-1} \circ \sigma \circ \varphi_1$ is a composition of homeomorphisms on the domain

$$\tau : [0,1) \setminus Orb_{\widetilde{f}_1}(x_1) \cup Orb_{\widetilde{f}_1}(x_2) \to [0,1) \setminus Orb_{\widetilde{f}_2}(x_1') \cup Orb_{\widetilde{f}_2}(x_2')$$

but both the preimage and the image are dense in $[0,1)$. Therefore τ must be a homeomorphism on $[0,1)$. This finally shows that τ is a topological conjugacy of $\widetilde{f}_1, \widetilde{f}_2$ which satisfies

$$\begin{aligned}
\tau(x_1) &= \widetilde{f}_2^{\,z_1}(x_1') \in Orb_{\widetilde{f}_2}(x_1') \\
\tau(x_2) &= \widetilde{f}_2^{\,z_2}(x_2') \in Orb_{\widetilde{f}_2}(x_2')
\end{aligned}$$

If we had instead chosen $\sigma(\overline{I_{1,0}}) = \overline{J_{2,z_1}}$ in the middle of the proof, then by following the same arguments we would have gotten:

$$\begin{aligned}
\tau(x_1) &= \widetilde{f}_2^{\,z_1}(x_2') \in Orb_{\widetilde{f}_2}(x_2') \\
\tau(x_2) &= \widetilde{f}_2^{\,z_2}(x_1') \in Orb_{\widetilde{f}_2}(x_1')
\end{aligned}$$

So the first part of the proposition is proved.

For the second part of the proposition, assume that $\widetilde{f}_1 = \tau^{-1} \circ \widetilde{f}_2 \circ \tau$ where τ is a homeomorphism and w.l.o.g $\exists z_1, z_2 \in \mathbb{Z}$ such that

$$\begin{aligned}
\tau(x_1) &= \widetilde{f}_2^{\,z_1}(x_1') \in Orb_{\widetilde{f}_2}(x_1') \\
\tau(x_2) &= \widetilde{f}_2^{\,z_2}(x_2') \in Orb_{\widetilde{f}_2}(x_2')
\end{aligned}$$

This inductively implies that $\forall n \in \mathbb{Z}$

$$\begin{aligned}
\tau(\widetilde{f}_1^{\,n}(x_1)) &= \widetilde{f}_2^{\,n+z_1}(x_1') \in Orb_{\widetilde{f}_2}(x_1') \\
\tau(\widetilde{f}_1^{\,n}(x_2)) &= \widetilde{f}_2^{\,n+z_2}(x_2') \in Orb_{\widetilde{f}_2}(x_2')
\end{aligned}$$

Now define, as before, $\sigma : [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{I_{k,n}} \to [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{J_{k,n}}$ by

$$\sigma = \varphi_2 \circ \tau \circ \varphi_1^{-1}$$

Then on $[0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{I_{k,n}}$ we have that σ is a homeomorphism and satisfies

$$\begin{aligned}
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \tau^{-1} \circ \varphi_2^{-1} \circ f_2 \circ \varphi_2 \circ \tau \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \tau^{-1} \circ \widetilde{f_2} \circ \tau \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= \varphi_1 \circ \widetilde{f_1} \circ \varphi_1^{-1} \\
\sigma^{-1} \circ f_2 \circ \sigma &= f_1 \qquad (7)
\end{aligned}$$

Moreover, define $\forall k \in \{1,2\},\ \sigma_{k,1} : \overline{I_{k,1}} \to \overline{J_{k,1+z_k}}$ to be any increasing homeomorphism. Then we inductively define for $k \in \{1,2\}$

$$\begin{aligned}
&\forall n \in \mathbb{N}_{\geq 2} && \sigma_{k,n} = f_2 \circ \sigma_{k,n-1} \circ f_1^{-1} && \overline{I_{k,n}} \to \overline{J_{k,n}} \\
&\forall n \in \mathbb{Z}_{\leq 0} && \sigma_{k,n} = f_2^{-1} \circ \sigma_{k,n-1} \circ f_1 && \overline{I_{k,n}} \to \overline{J_{k,n}}
\end{aligned}$$

then on $\overline{I_{k,n}},\ n \in \mathbb{Z},\ k \in \{1,2\}$ we have that σ satisfies:

$$\begin{aligned}
&\sigma^{-1} \circ f_2 \circ \sigma_{k,n}(\overline{I_{k,n}}) = \sigma^{-1} \circ f_2(\overline{J_{k,n+z_k}}) = \sigma_{k,n+1}^{-1}(\overline{J_{k,n+z_k+1}}) = \overline{I_{k,n+1}} = f_1(\overline{I_{k,n}}) \\
&\sigma^{-1} \circ f_2 \circ \sigma = f_1 \qquad (8)
\end{aligned}$$

So if we now define

$$\sigma(x) = \begin{cases} \varphi_2 \circ \tau \circ \varphi_1^{-1}(x) & x \in [0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{I_{k,n}} \\ \sigma_{k,n}(x) & x \in \overline{I_{k,n}} \end{cases}$$

then equations 7,8 imply that everywhere on $[0,1)$

$$\sigma^{-1} \circ f_2 \circ \sigma = f_1$$

To conclude that σ is a conjugacy we have to again prove that it is continuous (continuity of the inverse follows as we are on $[0,1)$). We have that

$$\sigma = \varphi_2 \circ \tau \circ \varphi_1^{-1}$$

is a composition of homeomorphisms on $[0,1) \setminus \bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{I_{k,n}}$ and

$$\sigma = \sigma_{k,n} : \overline{I_{k,n}} \to \overline{J_{k,n+z_k}}$$

is a homeomorphism on $\overline{I_{k,n}} = [a_{k,n}, b_{k,n}]$, where $\overline{J_{k,n+z_k}} = [c_{k,n+z_k}, d_{k,n+z_k}]$. So to prove continuity we just need to check continuity at the endpoints of the intervals. For this we need to show that $\forall n \in \mathbb{Z}$ and $k \in \{1,2\}$

$$\begin{aligned}
\lim_{x\uparrow a_{k,n}} \sigma(x) &= \lim_{x\uparrow a_{k,n}} \varphi_2 \circ \tau \circ \varphi_1^{-1}(x) = c_{k,n+z_k} \\
\lim_{x\downarrow b_{k,n}} \sigma(x) &= \lim_{x\downarrow b_{k,n}} \varphi_2 \circ \tau \circ \varphi_1^{-1}(x) = d_{k,n+z_k}
\end{aligned}$$

For the proof of the first equation we quote lemma 3.3 and continuity of τ to note that

$$\begin{aligned}
&\lim_{x\uparrow a_{k,n}} \varphi_1^{-1}(x) = f_1^n(x_k) \\
&\lim_{y\uparrow f_1^n(x_k)} \tau(y) = \tau(f_1^n(x_k)) = f_2^{n+z_k}(x'_k) \\
&\lim_{z\uparrow f_2^{n+z_k}(x'_k)} \varphi_2(z) = c_{k,n+z_k}
\end{aligned}$$

This shows that $\lim_{x\uparrow a_{k,n}} \sigma(x) = c_{k,n+z_k}$ where the limit is taken over $[0,1)\backslash\bigcup_{n\in\mathbb{Z},k\in\{1,2\}} \overline{I_{k,n}}$. It is easy to show that this can be extended to a limit over $[0,1)$. The proof of the second equation follows similarly. Therefore we have proved that f_1, f_2 are topologically conjugate and this finishes the proof of the proposition.

It is clear that in the proof of the proposition there was nothing special about $\chi(f_1) = \chi(f_2) = 2$ and the proof can be extended for any $N \in \mathbb{N}$ where $\chi(f_1) = \chi(f_2) = N$.

Proposition 3.9. *If $\chi(f_1) = \chi(f_2) = N \in \mathbb{N} \cup \{\infty\}$ are produced from transitive $\widetilde{f}_1, \widetilde{f}_2$ via surgery on*

$$Orb_{\widetilde{f}_1}(x_1), Orb_{\widetilde{f}_1}(x_2), \ldots, Orb_{\widetilde{f}_1}(x_N)$$

and

$$Orb_{\widetilde{f}_2}(x'_1), Orb_{\widetilde{f}_2}(x'_2), \ldots, Orb_{\widetilde{f}_2}(x'_N)$$

respectively, then $f_1 \sim f_2$ if and only if $\exists \tau \in Hom(S^1)$ such that $\widetilde{f}_1 = \tau^{-1} \circ \widetilde{f}_2 \circ \tau$ and there exists a bijection $i : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ such that

$$\forall j \in \{1, 2, \ldots, N\} \qquad \tau(x_j) \in Orb_{\widetilde{f}_2}(x'_{i(j)})$$

Proof. The proof of this proposition is essentially the same as of the previous proposition, only more notationaly awkward. □

The reason why this proposition is useful in deciding whether non-transitive f_1, f_2 are conjugate, is that we know exactly how the topological conjugacies between $\widetilde{f}_1, \widetilde{f}_2$ look. Recall lemma 2.4 that $\widetilde{f}_1 = \tau^{-1} \circ \widetilde{f}_2 \circ \tau$ if and only if there is $\theta \in [0, 1)$ such that $\tau = \pi_2^{-1} \circ R_\theta \circ \pi_1$. Therefore, the condition

$$\forall j \in \{1, 2, \ldots, N\} \qquad \tau(x_j) \in Orb_{\widetilde{f}_2}(x'_{i(j)})$$

Can be restated only in in terms of θ and the proposition simplifies to:

Proposition 3.10. *If $\chi(f_1) = \chi(f_2) = N \in \mathbb{N} \cup \{\infty\}$ are produced from transitive $\widetilde{f}_1, \widetilde{f}_2$ via surgery on*

$$Orb_{\widetilde{f}_1}(x_1), Orb_{\widetilde{f}_1}(x_2), \ldots, Orb_{\widetilde{f}_1}(x_N)$$

and

$$Orb_{\widetilde{f}_2}(x'_1), Orb_{\widetilde{f}_2}(x'_2), \ldots, Orb_{\widetilde{f}_2}(x'_N)$$

respectively, and π_1, π_2 are the homeomorphisms which satisfy

$$f_1 = \pi_1^{-1} \circ R_\alpha \circ \pi_1 \qquad f_2 = \pi_2^{-1} \circ R_\alpha \circ \pi_2$$

then $f_1 \sim f_2$ if and only if $\exists \theta \in [0, 1)$ and a bijection $i : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ with

$$\forall j \in \{1, 2, \ldots, N\} \qquad \pi_2^{-1} \circ R_\theta \circ \pi_1(x_j) \in Orb_{\widetilde{f}_2}(x'_{i(j)})$$

Now, we show how one would actually check in practice if f_1, f_2 are conjugate using this statement and, in addition, how many operations it would involve.

Lemma 3.11. *Let $\widetilde{f}_1, \widetilde{f}_2$ be transitive homeomorphisms and $i : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ a bijection. Then there exists a topological conjugacy τ of $\widetilde{f}_1, \widetilde{f}_2$ such that*

$$\forall j \in \{1, 2, \ldots, N\} \qquad \tau(x_j) \in Orb_{\widetilde{f}_2}(x'_{i(j)})$$

If and only if there exists a topological conjugacy τ' of $\widetilde{f}_1, \widetilde{f}_2$ such that

$$\tau'(x_1) = x'_{i(1)} \qquad \textit{and} \qquad \forall j \in \{2, \ldots, N\} \qquad \tau(x_j) \in Orb_{\widetilde{f}_2}(x'_{i(j)})$$

Proof. Suppose that $\widetilde{f_1} = \tau^{-1} \circ \widetilde{f_2} \circ \tau$ and $\forall j \in \{1, 2, \ldots, N\},\ \exists k(j) \in \mathbb{Z}$ such that

$$\tau(x_j) = \widetilde{f_2}^{k(j)}(x'_{i(j)})$$

Then let

$$\tau' = \widetilde{f_2}^{-k(0)} \circ \tau$$

Then

$$\tau'^{-1} \circ \widetilde{f_2} \circ \tau' = \tau^{-1} \circ \widetilde{f_2}^{k(j)} \circ \widetilde{f_2} \circ \widetilde{f_2}^{-k(j)} \circ \tau = \tau^{-1} \circ \widetilde{f_2} \circ \tau = \widetilde{f_1}$$

So that τ' is a topological conjugacy which satisfies

$$\tau'(x_0) = x'_{i(0)} \qquad \text{and} \qquad \forall j \in \{2, \ldots, N\} \qquad \tau(x_j) \in Orb_{\widetilde{f_2}}(x'_{i(j)})$$

The other side of the implication is trivial. $\square$

Therefore, if $N \in \mathbb{N}$ is finite, one would find $\theta_1, \theta_2, \ldots, \theta_N$ which satisfy

$$\forall k \in \{1, 2, \ldots, N\} \qquad \pi_2^{-1} \circ R_{\theta_k} \circ \pi_1(x_1) = x_k$$

which can be rewritten as

$$\theta_k = \begin{cases} \pi_2(x_k) - \pi_1(x_1) & \text{if } \ \pi_2(x_k) \geq \pi_1(x_1) \\ (\pi_2(x_k) + 1) - \pi_1(x_1) & \text{if } \ \pi_2(x_k) \leq \pi_1(x_1) \end{cases}$$

and after that one would need to check, for $k \in \{1, 2, \ldots, N\}$ if there is a bijection

$$i_k : \{2, 3, \ldots, N\} \to \{1, 2, \ldots, N\} \setminus \{k\}$$

which satisfies

$$\forall j \in \{2, \ldots, N\} \qquad \pi_2^{-1} \circ R_{\theta_k} \circ \pi_1(x_j) \in Orb_{\widetilde{f_2}}(x'_{i_k(j)})$$

This amounts to $N!$ operations of checking whether some point is in some other orbit. For large N, this increases rapidly, but still provides a finite number of operations. For example, if $N = 2$ then the problem is simplified to 2 simple operations.

Proposition 3.12. *If $\chi(f_1) = \chi(f_2) = 2$ are produced from transitive $\widetilde{f_1}, \widetilde{f_2}$ via surgery on*

$$Orb_{\widetilde{f_1}}(x_1), Orb_{\widetilde{f_1}}(x_2)$$

and

$$Orb_{\widetilde{f_2}}(x'_1), Orb_{\widetilde{f_2}}(x'_2)$$

respectively, and τ_1, τ_2 are the unique topological conjugacies of $\widetilde{f_1}, \widetilde{f_2}$ which satisfy

$$\tau_1(x_1) = x'_1 \qquad \textit{and} \qquad \tau_2(x_1) = x'_2$$

then $f_1 \sim f_2$ if and only if

$$\tau_1(x_2) \in Orb_{\widetilde{f_2}}(x'_2) \qquad \textit{or}$$
$$\tau_2(x_2) \in Orb_{\widetilde{f_2}}(x'_1)$$

3.3 Classification of non-transitive homeomorphisms

Now we will use the methods that we developed, to classify any non-transitive homeomorphism. We will show that any non-transitive homeomorphism with irrational rotation number can be produced by surgery.

First, we look at how a general non-transitive homeomorphism with rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ "looks". Suppose $f : S^1 \to S^1$ is non-transitive and $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Recalling lemma 2.7 we have

$$\forall x, y \in S^1 \qquad \omega(x) = \omega(y) =: \omega(f)$$

As S^1 is compact $\omega(f)$ is non-empty and as f is not transitive $\omega(f) \neq S^1$. Moreover $\omega(f)$ is closed in S^1 and f invariant, so that $S^1 \setminus \omega(f) = U$ is open and f invariant also. An open set in $\mathbb{R}$ can be always written as a countable union of disjoint open intervals, therefore

$$U = \bigcup_{n \in \mathbb{N}} I_n$$

for some disjoint intervals $I_m : m \in \mathbb{N}$.

Lemma 3.13. $\forall n \in \mathbb{Z}, m \in \mathbb{N}, \;\; f^n(I_m) \in \{I_j : j \in \mathbb{Z}\}$

Proof. As U is f invariant $\forall n \in \mathbb{Z}, m \in \mathbb{N}$

$$f^n(I_m) \subseteq U$$

As f is continuous $f^n(I_m)$ is an interval and hence $\exists k \in \mathbb{N}$ such that

$$f^n(I_m) \subseteq I_k$$

By repeating the same argument $\exists l \in \mathbb{N}$ such that

$$f^{-n}(I_k) \subseteq I_l$$

Therefore

$$I_m \subseteq f^{-n}(I_k) \subseteq I_l$$

But the intervals are disjoint, therefore $l = m$ and

$$f^n(I_m) = I_k$$

□

Lemma 3.14. $\forall m \in \mathbb{N}, \; Orb_f(I_m) := \{f^n(I_m) : n \in \mathbb{Z}\}$ *is an infinite set*

Proof. Using the previous lemma $\forall n_1 > n_2 \in \mathbb{Z}, m \in \mathbb{N}$,

$$f^{n_1}(I_m) \cap f^{n_2}(I_m) = \emptyset \qquad \text{or} \qquad f^{n_1}(I_m) = f^{n_2}(I_m)$$

If the latter is true then we have that $f^{n_1 - n_2} : \overline{I_m} \to \overline{I_m}$ is a homeomorphism. But a standard result from analysis states that such a function always has a fixed point. Hence f has a periodic point, which is a contradiction to $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Therefore $\forall n_1 > n_2 \in \mathbb{Z}, m \in \mathbb{N}$,

$$f^{n_1}(I_m) \cap f^{n_2}(I_m) = \emptyset$$

and hence the lemma is proved. □

The two lemmas above give that we can partition U into orbits of the intervals. In other words, there exists $N \in \mathbb{N} \cup \{\infty\}$ such that

$$U = \bigcup_{k=1}^{N} Orb_f(I_k)$$

Now if we denote

$$f^n(I_k) = I_{k,n} = (a_{k,n}, b_{k,n}) \qquad l_{k,n} = b_{k,n} - a_{k,n} \qquad d = \sum_{k=1}^{M} \left(\sum_{n\in\mathbb{Z}} l_{k,n} \right) < 1$$

and define

$$\varphi^{-1} : [0,1) \setminus \bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N\}}} (a_{k,n}, b_{k,n}] \to [0,1)$$

by

$$\varphi^{-1}(y) = \left(y - \sum_{k=1}^{N} \left(\sum_{n:a_{k,n}\in[0,x)} l_{k,n} \right) \right) \frac{1}{1-d}$$

Then we define $x_{k,n} := \varphi^{-1}(a_{k,n})$ and note lemma 3.3, that φ^{-1} is an increasing homeomorphism on

$$\varphi^{-1} : [0,1) \setminus \left(\bigcup_{\substack{n\in\mathbb{Z},\\ k\in\{1,2,\dots,N\}}} \overline{I_{k,n}} \right) \to [0,1) \setminus \{x_{k,n} : k \in \{1,2,\dots,N\},\ n \in \mathbb{Z}\}$$

so we can define $\widetilde{f} : [0,1) \to [0,1)$ by

$$\widetilde{f} = \varphi^{-1} \circ f \circ \varphi$$

Moreover, note that on $X = [0,1) \setminus \{x_{k,n} : k \in \{1,2,\dots,N\},\ n \in \mathbb{Z}\}$

$$\widetilde{f} = \varphi^{-1} \circ f \circ \varphi : X \to X$$

is a composition of homeomorphisms from a dense set (in $[0,1)$) to a dense set. This shows $\widetilde{f}$ must be a homeomorphism on $[0,1)$. In addition, $\widetilde{f}$ can be shown to be a transitive homeomorphism with $\rho(\widetilde{f}) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ by an argument analogous to arguments already seen, and by construction f can be produced from $\widetilde{f}$ by surgery.

Therefore we have shown that, every non-transitive homeomorphism with irrational rotation number can be constructed through surgery and has a Denjoy number. Therefore we can apply the propositions that we already proved for this model of non-transitive homeomorphisms and decide when two non-transitive homeomorphisms are conjugate. Therefore the question when two homeomorphisms with irrational rotation number are conjugate is answered.