Homework - Seminar 2

Vităreanu Eduard - David

1* Prove using the ε-definition that $\lim_{n \to +\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

$x_n \to l : \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon, \forall n \geq N$

$\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon, \forall n \geq N_\varepsilon \Leftrightarrow$

$\Leftrightarrow \left|\frac{2(n+1) - (2n+3)}{2(2n+3)}\right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \left|\frac{2n+2-2n-3}{4n+6}\right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \left|\frac{-1}{4n+6}\right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{4n+6} < \varepsilon.$

- we need to find N_ε st n is greater or equal to N

$\frac{1}{4n+6} < \varepsilon \Rightarrow 4n+6 > \frac{1}{\varepsilon} \Leftrightarrow 4n > \frac{1}{\varepsilon} - 6 \Leftrightarrow$

$\Leftrightarrow n > \frac{1}{4\varepsilon} - \frac{3}{2} \Rightarrow N_\varepsilon$ is $\left[\frac{1}{4\varepsilon} - \frac{3}{2}\right] \Rightarrow \forall n \geq N_\varepsilon.$

$n > \frac{1}{4\varepsilon} - \frac{3}{2}$ and thus $\frac{1}{4(n)+6} < \varepsilon.$

$\boxed{2}$. Study if the sequence (x_n) is bounded, monotone and convergent, for each of the following:

(c)* $x_n = \frac{\sin(n)}{n}$.

1. Bounded:

the sine function is bounded as : $-1 \leq \sin(n) \leq 1$, $\forall n \in \mathbb{R}$. $\Rightarrow$

$\Rightarrow -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

n - increasing.

- both $-\frac{1}{n}$ and $\frac{1}{n}$ approach 0

$\Rightarrow x_n = \frac{\sin n}{n}$ - bounded in 0.

2. monotony:

- the sine function ($\sin n$) oscilates between -1 and 1 and n increases $\Rightarrow$ doesn't have a set monotony.

3. convergency:

$-1 \leq \sin n \leq 1 \Rightarrow | \cdot n, n \in \mathbb{N}$

$\Rightarrow -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \Rightarrow$

Squeeze Th.

$\Rightarrow \lim_{n\to+\infty} -\frac{1}{n} \leq \lim_{n\to+\infty} \frac{\sin n}{n} \leq \lim_{n\to+\infty} \frac{1}{n} \Longrightarrow$

$\longrightarrow 0 \longleftarrow$

$\Rightarrow \lim_{n\to+\infty} \frac{\sin n}{n} = 0 \Rightarrow (x_n)$ - convergent.

$\boxed{3}$ Find the limit for each of the following sequence:

d) * $\sqrt[n]{1+2+\dots+n}$

$$\lim_{n\to+\infty} \sqrt[n]{1+2+\dots+n} = (1+2+\dots+n)^{\frac{1}{n}} \overset{\lim_{n\to+\infty}}{=} \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}} =$$

$$= \lim_{n\to+\infty} e^{\ln\left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}}} = \cancel{\lim_{n\to+\infty} e^{\frac{1}{n}\cdot\ln\left(\frac{n^2}{2}+\frac{n}{2}\right)}}$$

$$= \lim_{n\to+\infty} e^{\frac{1}{n}(\ln n + \ln(n+1) - \ln 2)} = \lim_{n\to+\infty} e^{\frac{\ln n}{n} + \frac{\ln(n+1)}{n} - \frac{\ln 2}{n}}$$

$$= \underbrace{e^{\lim_{n\to+\infty}\frac{\ln n}{n}}}_{0} \cdot e^{\lim_{n\to+\infty}\frac{\ln(n+1)}{n}} : \underbrace{e^{\lim_{n\to+\infty}\frac{\ln 2}{n}}}_{1}$$

$$= 1 \cdot e^{\lim_{n\to+\infty}\frac{\ln(n+1)}{n}} = 1 \cdot e^{0} = 1 \cdot 1 = 1.$$

$$\lim_{n\to+\infty} \frac{\ln(n+1)}{n} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{n\to+\infty} \frac{1}{n+1} = 0$$

$\boxed{5}$ Find the limit for each of the following seque

c). $\lim\limits_{n \to +\infty} \left(\frac{\ln(n+1)}{\ln n} \right)^n =$

$= \lim\limits_{n \to +\infty} \left[\frac{\ln(n+1)}{\ln n} \right]^n = \lim\limits_{n \to +\infty} \left(\frac{\ln\left[n\left(1+\frac{1}{n}\right)\right]}{\ln n} \right)^n =$

$= \lim\limits_{n \to +\infty} \left(\frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n} \right)^n =$

$= \lim\limits_{n \to +\infty} \left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n} \right)^n =$

$= \left[\lim\limits_{n \to +\infty} \left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n} \right)^{\frac{\ln n}{\ln\left(1+\frac{1}{n}\right)}} \right]^{\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n} \cdot n} =$

$= e^{\frac{\lim\limits_{n \to +\infty} \ln\left(1+\frac{1}{n}\right)^n}{\ln n}} = e^{\frac{\ln \lim\limits_{n \to +\infty} \left(1+\frac{1}{n}\right)^n}{\ln n}} =$ solved at $\boxed{4}$

$= e^{\frac{\ln e}{\infty}} = e^{\frac{1}{\infty}} = e^0 = 1.$

$\boxed{6}$ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent

- its limit is denoted by γ

$$x_{n+1} = 1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1)$$

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$$

$(-)$

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln n.$$

- $\frac{1}{n+1} > 0$, but aproaches $0 \to$ when $\to \infty$. ①
- $\ln$ - increasing function $\Rightarrow \ln(n+1) > \ln n, \forall n \in \mathbb{N} \Rightarrow$

$\Rightarrow -\ln(n+1) + \ln n < 0$. ②

- from ① and ② $\Rightarrow x_{n+1} - x_n < 0 \Rightarrow (x_n)$ - decreasing

Bounded:

$$\left\{\begin{array}{l}(x_n) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \\ \text{- all the terms} > 0 \\ \text{- } \ln n \text{ grows slower than the sum.}\end{array}\right| \Rightarrow lb = 0.$$

Method 1
(M2 last page).

$$\left. x_n \le 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \le \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n} - n}_{\text{harmonic series that is finit}} \right| \Rightarrow$$

$\Rightarrow (x_n)$ - bounded.

$$\left.\begin{array}{l}\text{Bounded} \\ + \\ \text{~~mon~~ decreasing}\end{array}\right| \xrightarrow{\text{M.C.Th}} \text{the sequence is convergent.}$$

$\boxed{6.}$ Method 2: lower bound (idea taken from Office hours)

$$\frac{1}{2} < \cancel{\ln(2)} - \ln 1 < \frac{1}{1}$$

$$\frac{1}{3} < \cancel{\ln(3)} - \cancel{\ln 2} < \frac{1}{2}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\frac{1}{n+1} < \ln(n+1) - \cancel{\ln n} < \frac{1}{n} \qquad +$$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} < \underbrace{\ln(n+1) - \underset{0}{\ln 1} < 1 + \frac{1}{2} + \dots + \frac{1}{n}}_{\text{we take this further}} \Rightarrow$$

$$\Rightarrow 0 < 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n+1) \quad \Big| + \frac{1}{n+1} \Leftrightarrow$$

$$\Leftrightarrow 0 < \frac{1}{n+1} < \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1)}_{x_{n+1}} \Bigg| \Rightarrow$$

but $x_{n+1} < x_n$

$\Rightarrow$ $0 < x_n$ $\Rightarrow$ bounded below.

$\boxed{8}$ b) $\lim_{n\to+\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} = ?$

- let $(a_n)_{n\geq 0} = n^n$ and $(b_n)_{n\geq 0} = 1+2^2+\dots+n^n$, $b_n \to +\infty$,

b_n - increasing $\Rightarrow$

Scanned with CamScanner

Stolz-cezaro $\Rightarrow$ IF $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l$ Then $\lim_{n\to+\infty} \frac{a_n}{b_n} = l$.

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to+\infty} \frac{(n+1)^{n+1}-n^n}{\cancel{1}+\cancel{2^2}+\cancel{3^3}+\ldots+\cancel{n^n}+(n+1)^{n+1}-\cancel{1}-\cancel{2^2}-\cancel{3^3}-\ldots-\cancel{n^n}}$$

$$= \lim_{n\to+\infty} \frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} = \lim_{n\to+\infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1}} - \lim_{n\to+\infty} \frac{n^n}{(n+1)^{n+1}} =$$

$$= 1 - \lim_{n\to+\infty} \frac{n^n}{(n+1)^n\cdot(n+1)} = 1 - 0 = 1 \Rightarrow$$

$$\cdot \lim_{n\to+\infty} \frac{n^n}{(n+1)^n}\cdot\frac{1}{n+1} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} =$$

$$= \lim_{n\to+\infty} \left(1-\frac{1}{n+1}\right)^n \cdot \frac{1}{n+1} = \left[\lim_{n\to+\infty}\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{-1}{n+1}\cdot n} \cdot \frac{1}{n+1} =$$

$$= e^{\lim_{n\to+\infty} \frac{\cancel{n}\cdot(-1)}{\cancel{n}(1+\frac{1}{n})}} \cdot \lim_{n\to+\infty} \frac{1}{n+1} = e^{-1}\cdot 0 = 0.$$

$$\Rightarrow \lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = 0 \Rightarrow \lim_{n\to+\infty} \frac{a_n}{b_n} = 0.$$

10. Study the convergence and find the limit of:

b) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $x_1 = 1$, $a > 1$.

$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = \frac{1}{2}\left(1 + \frac{a}{1}\right) = \frac{1}{2} + \frac{a}{2}$.

$x_3 = \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{a}{2} + \frac{a}{\frac{1}{2}+\frac{a}{2}}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{a}{2} + \frac{2a}{1+a}\right)$

$= \frac{1}{4} + \frac{a}{4} + \frac{a}{1+a} = \frac{1+a+a+a^2+a}{4(1+a)} = \frac{a^2+3a+1}{4(1+a)}$ X

X wrong lead

$x_{m+1} = \frac{1}{2}\left(x_m + \frac{a}{x_m}\right)$

- let $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$
- we need to find a fixed point for g, which means finding x such that $g(x) = x$.

$g(x)$ is a contraction $\Leftrightarrow |g'(x)| < 1$.

$g'(x) = \left[\frac{1}{2}\left(x + \frac{a}{x}\right)\right]' = \frac{1}{2}\left(x' + \left(\frac{a}{x}\right)'\right) = \frac{1}{2}\left(1 + \frac{a' \cdot x - a \cdot x'}{x^2}\right) =$

$= \frac{1}{2}\left(1 + \frac{0 - a}{x^2}\right) = \frac{1}{2} - \frac{a}{2x^2}$.

we know for a fact: $a > 1 \Rightarrow \frac{a}{2x^2} > \frac{1}{2x^2}$ $\Bigg\} \Rightarrow$

$\Rightarrow |g'(x)| < \frac{1}{2} < 1 \Rightarrow g(x)$ - contraction mapping

- $x \in [1, a]$ since $x \geq 1$ and $a > 0 \Rightarrow$ a closed interval such that $g(x)$ maps points in interval to other points.
- the contraction factor $< 1 \Rightarrow g(x)$ maps closer to a fixed point which is the limit of the sequence. The solution to $g(x) = x$ is the said fixed point.

$\frac{1}{2}\left(x + \frac{a}{x}\right) = x \Leftrightarrow$

$\Leftrightarrow x + \frac{a}{x} = 2x \Rightarrow$

$\Rightarrow \frac{a}{x} = x \Leftrightarrow$

$\Leftrightarrow a = x^2 \Rightarrow x = \sqrt{a}$

$a > 0$

$x \in [1, a]$

$\Rightarrow \lim_{k \to \infty} x_m = \sqrt{a}$