Homework: S03

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2. Find the sum for each of the following series:

a) $\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right)$.

$$\ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln\left(\frac{(n+1)(n-1)}{n^2}\right) = \ln\frac{n+1}{n} + \ln\frac{n-1}{n}.$$

$$\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln\frac{n+1}{n} + \ln\frac{n-1}{n} =$$

$$= \ln\frac{1}{2} + \ln\frac{3}{2} + \ln\frac{4}{3} + \ln\frac{2}{3} + \dots + \ln\frac{n}{n-1} + \ln\frac{n+1}{n} =$$

$$= \ln\left(\frac{1}{2}\cdot\cancel{\frac{3}{2}}\cdot\frac{4}{\cancel{3}}\cdot\frac{\cancel{2}}{3}\cdot\dots\cdot\frac{\cancel{n}}{\cancel{n-1}}\cdot\frac{n+1}{n}\right) = \ln\frac{1}{2}\frac{n+1}{n}$$

$$\lim_{n\to+\infty} \ln\frac{1}{2}\frac{n+1}{n} = \ln\lim_{n\to+\infty}\frac{1}{2}\frac{n+1}{n} = \ln\frac{1}{2}$$

b) $$\sum_{n\geq 1}\frac{n+1}{3^n} = \underbrace{\sum_{n\geq 1}\frac{n}{3^n}}_{S^*} + \underbrace{\sum_{n\geq 1}\frac{1}{3^n}}_{**} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}.$$

* $S = \sum_{n\geq 1}\frac{n}{3^n}$

$$\frac{1}{3}S = \sum_{n\geq 1}\frac{n}{3^{n+1}} = \sum_{n\geq 2}\frac{n+1}{3^n} = \sum_{n\geq 2}\frac{n}{3^n} + \sum_{n\geq 2}\frac{1}{3^n}$$

$$S - \frac{1}{3}S = \frac{1}{3} + \sum_{n\geq 2}\frac{1}{3^n} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \Rightarrow \frac{2}{3}S = \frac{1}{2} \Rightarrow S = \frac{3}{4}.$$

$$\sum_{n\geq 2}\frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} - 1 - \frac{1}{3} = \frac{3}{2} - 1 - \frac{1}{3} = \frac{9-6-2}{6} = \frac{1}{6}.$$

** $$\sum_{n\geq 1}\frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

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4. Study if the following series are convergent or divergent:

a) $\sum\limits_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$.

- we use the ratio test.

$$\lim_{n\to+\infty} \frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n} = \lim \frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\cdot\left(\frac{n}{n+1}\right)^p =$$

$$= x\cdot\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p\right) \longrightarrow 1 = x.$$

- if $x>1$ => the series diverges
- if $0<x<1$ => the series converges.
- if $x=1$ => further analysis is needed to determine.

b) $\sum\limits_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$.

$n < n+1$ => , $\forall n\geq 2$

=> $\ln n < \ln n+1$ =>

=> $(\ln n)^{\ln n} < (\ln n+1)^{\ln n+1}$ =>

=> $\frac{1}{(\ln n+1)^{\ln n+1}} < \frac{1}{(\ln n)^{\ln n}}$ => $(x_n) = \frac{1}{(\ln n)^{\ln n}}$ - decreasing =>

=> Cauchy Condensation Test: $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}} \underset{\text{nature}}{\overset{\text{the same}}{\sim}} \sum\limits_{n=2} 2^n\cdot\frac{1}{(\ln 2^n)^{\ln 2^n}}$

let $(b_n) = 2^n\cdot\frac{1}{(\ln 2^n)^{\ln 2^n}} = \frac{2^n}{(n\cdot\ln 2)^{n\cdot\ln 2}} = \left(\frac{2}{(n\cdot\ln 2)^{\ln 2}}\right)^n$

- we apply the root test.

$$\lim_{n\to\infty}\sqrt[n]{b_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2}{(n\cdot\ln 2)^{\ln 2}}\right)^n} = \lim_{n\to\infty}\frac{2}{(n\cdot\ln 2)^{\ln 2}} = 0$$

$l < 1 \Rightarrow \sum_{n=2}^{\infty} (b_n)$ - convergent.

but $\sum_{n=2}^{\infty} b_n$ has the same nature as $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $\Bigg| \Rightarrow$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ - convergent

c). $\sum_{n \geq 1} (\sqrt[n]{n} - 1)$

let $(a_n) = \sqrt[n]{n} - 1 = n^{\frac{1}{n}} - 1 = e^{\ln(n^{\frac{1}{n}})} - 1 = e^{\frac{\ln n}{n}} - 1.$ $\Bigg\} \Rightarrow$

let $(b_n) = \frac{\ln n}{n}.$

$\Rightarrow \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{e^{\frac{\ln n}{n}} - 1}{\frac{\ln n}{n}} = 1 \Rightarrow (a_n)$ and (b_n) have the same nature. ①

$\ln > 1, \forall n \geq 3 \Rightarrow$

$\Rightarrow \frac{\ln n}{n} > \frac{1}{n}, \forall n \geq 3 \Leftrightarrow$

$\Leftrightarrow \sum_{n \geq 3} \frac{\ln n}{n} > \underbrace{\sum_{n \geq 3} \frac{1}{n}}_{\lim \to \infty} \Rightarrow \sum_{n \geq 3} \frac{\ln n}{n} = \infty.$

$\sum_{n \geq 1} \frac{\ln n}{n} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \sum_{n \geq 3} = \infty \Rightarrow$

$\Rightarrow \sum_{n \geq 1} \frac{\ln n}{n}$ - diverges $\Bigg\} \Rightarrow \sum_{n \geq 1} \sqrt[n]{n} - 1$ - diverges.

+ ①

$\boxed{2}$ c) $\sum_{n \geq 1} \frac{n}{n^4 + n^2 + 1}.$

$\underline{n^4 + n^2 + 1} = (n^2 + 1)^2 - n^2 = (n^2 + n + 1)(n^2 - n + 1)$

$$\frac{n}{n^4+n^2+1} = \frac{n}{(n^2+n+1)(n^2-n+1)} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)$$

$$\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right) =$$

$$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\left(\frac{1}{1} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{7}} + \dots\right.$$

$$\left. - \cancel{\frac{1}{3}} - \cancel{\frac{1}{7}} + \dots\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

$\boxed{5}$ - from an equilateral triangle $\to$ replace each side with 4 segments $\Rightarrow$ the number of sides at each iteration increases by a factor of 4.

it 0: 3 sides

it 1: $3 * 4 = 12$ sides

it 2: $12 * 4^2 = 48$ sides

...... $\to$ the number of sides at it $n = 3 * 4^n$

Perimeter:

- The length of each side at iteration n is $(1/3)^n$ the length of the original size: $\Rightarrow$

$$\Rightarrow P_n = 3 * 4^n * \left(\frac{1}{3}\right)^n \ell = \cancel{4^n \cdot \frac{1}{3^{n-1}}}, \quad \ell - \text{side of original } \Delta$$

$$\sum_{n=1} \frac{4^n}{3^n}\cdot \cancel{\frac{1}{3^{n-1}}} * 3 \cdot \ell = \lim_{n\to\infty} \underbrace{\left(\frac{4}{3}\right)}_{>1}^n \cdot 3\cdot\ell = +\infty$$

Area:

- the area of a Δ - equilateral, with side ℓ:

$$A = \frac{\sqrt{3}}{4}\cdot \ell^2$$

iteration 1 : $s = \frac{\ell}{3}$, the side is $\frac{1}{3}$ shorter each time s

$$\Rightarrow A = \underbrace{\frac{\sqrt{3}}{4} s^2}_{\text{original } \Delta} + 3 \cdot \frac{\sqrt{3}}{4} \left(\frac{s}{3}\right)^2 = \frac{\sqrt{3}}{4} s^2 \left(1 + \frac{3}{9}\right).$$

iteration 2 : $s = \frac{\ell}{3^2}$, —"— $\Rightarrow$

$$\Rightarrow A = \frac{\sqrt{3}}{4} s^2 \left(1 + \frac{3}{9}\right) + 3 \cdot 4 \cdot \frac{\sqrt{3}}{4} \left(\frac{s}{9}\right)^2 = \frac{\sqrt{3}}{4} s^2 \left(1 + \frac{3}{9}\right)$$

. . . .

iteration K : $s = \frac{\ell}{3^K}$

$$A_K = \underbrace{3 \cdot 4^{K-1}}_{\text{additional triangles}} \cdot \frac{\sqrt{3}}{4} \left(\frac{s}{3^K}\right)^2 = \frac{\sqrt{3}}{4} s^2 \left(\frac{3 \cdot 4^{K-1}}{9^K}\right)$$

of area, K-1 triangles

$$A_t = \frac{\sqrt{3}}{4} s^2 \left(1 + \underbrace{\sum_{K=1}^{m} \frac{3 \cdot 4^{K-1}}{9^K}}_{\text{geometric series with } q = \frac{4}{9}} \right) =$$

$$= \frac{\sqrt{3}}{4} s^2 \left(1 + \frac{\frac{3}{9}}{1 - \frac{4}{9}}\right) = \frac{\sqrt{3}}{4} s^2 \left(\frac{8}{5}\right) = \frac{2\sqrt{3}}{5} s^2$$