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# Homework - Seminar 1

**5★** Let $a, b \in \mathbb{R}$ with $a > 0$. If $S$ is nonempty and bounded above, ~~S~~ prove that:

$$\sup_{x \in S}(ax+b) = a \cdot \sup(S) + b.$$

let $f(x) : S \to \mathbb{R}$, $f(x) = ax+b$.

$S$ - bounded above $\Rightarrow \exists \sup(S) \Rightarrow$

$\Rightarrow \sup(S) \geq x, \forall x \in S \mid \cdot a (>), a \in \mathbb{R}.$

$\Leftrightarrow a \cdot \sup(S) \geq a \cdot x, \forall x \in S \mid +b \Leftrightarrow, b \in \mathbb{R}.$

$\Leftrightarrow a \cdot \sup(S) + b \geq a \cdot x + b, \forall x \in S \Rightarrow$

$\Rightarrow a \cdot \sup(S) + b \geq f(x), \forall x \in S \Rightarrow$

$$\Rightarrow \begin{cases} a \cdot \sup(S) + b \in ub(f(x)) \cancel{\Rightarrow a} \\ \underline{a \cdot \sup(S) + b \geq \sup(a \cdot x + b)} \ (1) \end{cases}$$

$$\exists \sup(f(x)) \Rightarrow \begin{cases} ax+b \leq \sup(ax+b) \\ \sup(ax+b) \leq u, \forall u \in ub(f(x)) \end{cases} \Bigg| \Rightarrow$$

~~but~~ $a \cdot \sup(S) + b \in ub(f(x))$

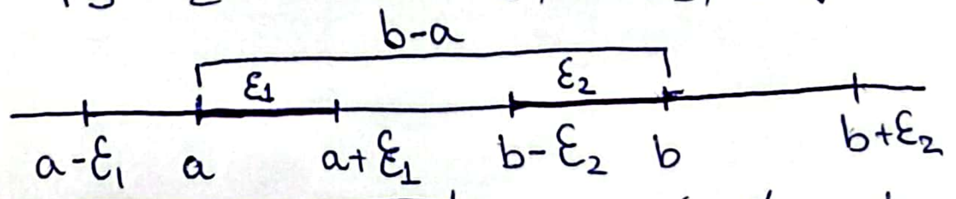
$\Rightarrow \underline{\sup(a \cdot x + b) \leq a \cdot \sup(S) + b}.$ (2)

from ~~dim~~ (1) and ~~(2)~~ (2) $\Rightarrow \sup\limits_{x \in S}(ax+b) = a \cdot \sup(S) + b.$

**8** Let $a, b \in \mathbb{R}$. ~~with $a > 0$~~ Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

$a, b \in \mathbb{R}$, $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b) \Rightarrow$

$$\Rightarrow \begin{vmatrix} \exists \varepsilon_1 > 0 : (a-\varepsilon_1, a+\varepsilon_1) \subseteq U \\ \exists \varepsilon_2 > 0 : (b-\varepsilon_2, b+\varepsilon_2) \subseteq V. \end{vmatrix}$$

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if $\boxed{|b-a| < \varepsilon_1 + \varepsilon_2}$ (1) $\Rightarrow U \cap V = \emptyset$, where $U = (a-\varepsilon_1, a+\varepsilon_1)$ and $V = (b-\varepsilon_2, b+\varepsilon_2) \in$

and there exist an infinity of possibility that satisfy ① since $a, b \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 > 0$. $\Rightarrow$

$\Rightarrow \exists\, U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t $U \cap V = \emptyset$

$\boxed{10 \star}$ Let $A = (0,1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\operatorname{int} A = \emptyset$ and $\operatorname{cl} A = [0,1]$.

$A = (0,1) \cap \mathbb{Q}$

$lb(A) = (-\infty, 0]$

$ub(A) = [1, +\infty)$.

$a > 0,\ \forall a \in (0,1) \cap \mathbb{Q}$
$u \leq 0,\ \forall u \in lb(A)$ $\Big|$ $\Rightarrow \inf A = 0$.

$a \not< 1,\ \forall a \in A$
$u \geq 1,\ \forall u \in ub(A)$ $\Big|$ $\Rightarrow \sup A = 0$.

$V \in \mathcal{V}(x)$ iff. $\exists\, \varepsilon > 0$ s.t $(x-\varepsilon, x+\varepsilon) \subseteq V$.

let $x \in (0,1) \setminus \mathbb{Q}$ $\Rightarrow$ $\underbrace{(x-\varepsilon, x+\varepsilon)}_{\text{irrational numbers, a infinity of them between } (0,1)} \not\subseteq (0,1) \cap \mathbb{Q}$ $\Big|$ $\Rightarrow$

$\Rightarrow \operatorname{int}(A) = \emptyset$.

let $x \in (0,1) \cap \mathbb{Q}$ $\Rightarrow$ $\underbrace{(x-\varepsilon, x+\varepsilon)}_{\text{rational numbers, a infinity of them between } (0,1)} \subseteq (0,1) \cap \mathbb{Q}$. $\Big|$ $\Rightarrow$

$\Rightarrow \operatorname{cl}(A) = [0,1]$.