Extra homework 2 :

1. Let $\alpha \in (0,1)$ and $x_1, x_2 \in \mathbb{R}$. Study the convergence of the sequence (x_n) given by

$$x_{n+2} = \alpha x_{n+1} + (1-\alpha)x_n, \ \forall n \in \mathbb{N}.$$

if (x_n) has a limit L as n approaches $+\infty$, then this limit must satisfy:

$$L = \alpha L + (1-\alpha)L$$

Solve:

$$L = \alpha L + (1-\alpha)L \Leftrightarrow$$

$$\Leftrightarrow L = \cancel{\alpha L} + L - \cancel{\alpha L} \Leftrightarrow$$

$\Leftrightarrow L = L$ (True) $\Rightarrow$ doesn't provide the value of the limit, but it surely exists $\Rightarrow$ the sequence converges, and the exact value depends on x_1 and x_2 values.

2. Give an example of a sequence having the set of limit points equal to $[0,1]$? Justify.

- let (x_n) be a sequence defined as follows:

$$x_n = \sin^2(n).$$

$-1 \leq \sin(n) \leq 1$, $\forall n \in \mathbb{N} \Rightarrow \sin^2(n) \leq 1$, $\forall n \in \mathbb{N} \Rightarrow$

$\Rightarrow \sin^2(n) \in [0,1],\ \forall n \in \mathbb{N} \Rightarrow$ all terms are within the closed interval $[0,1]$.

Limit points in $[0,1]$

$(x_n) \to \infty$, $\sin^2(n)$ gets arbitrarily close to 0 and 1 $\Rightarrow$ 0, 1 - limit points

- for any value K in $(0,1)$ it is also a limit point of the sequence because $0 \leq \sin^2(n) \leq 1$, $\forall n \in \mathbb{N}$. As n becomes larger, the values of $\sin^2(n)$ get arbitrarily close to K, making it a limit point to (x_n).