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Homework - seminar 12

1. Use Lagrange multipliers to find the extrema of the following functions subject to constraints

c) $f(x,y) = x^2 - y^2$ subject to $g(x,y) = x^2 + y^2 - 1 = 0$

$L(x,y,\lambda) = f(x,y) + \lambda(g(x,y) - c) = x^2 - y^2 + \lambda(x^2 + y^2 - 1)$.

$\frac{\partial L}{\partial x} = 2x + 2\lambda x = 2x(\lambda + 1) = 0 \Rightarrow \begin{cases} x = 0 \\ \text{or} \\ \lambda = -1 \end{cases}$

$\frac{\partial L}{\partial y} = -2y + 2\lambda y = 2y(\lambda - 1) = 0 \Rightarrow \begin{cases} y = 0 \\ \text{or} \\ \lambda = 1 \end{cases}$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$.

I. $x = 0, y = 0$ - not possible because $x^2 + y^2 - 1 = 0 \Leftrightarrow -1 = 0$ (F)

II. $x = 0, \lambda = 1 \Rightarrow y^2 - 1 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow$

$\Rightarrow$ the critical points are $(0, -1, 1)$ and $(0, 1, 1)$.

III. $y = 0, \lambda = -1 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow$

$\Rightarrow$ the critical points are $(1, 0, -1)$ and $(-1, 0, -1)$.

$\left.\begin{array}{l} f(0,-1) = f(0,1) = -1 \\ f(-1,0) = f(1,0) = 1 \end{array}\right| \Rightarrow$

$\Rightarrow \begin{cases} \text{the MAX is } (1,0,-1), (-1,0,-1) \\ \text{the MIN is } (0,-1,1), (0,1,1) \end{cases}$

f) $f(x,y,z) = x^3 + y^3 + z^3$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = 1$.

$L(x,y,z,\lambda) = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2 - 1)$

$$\frac{dL}{dx} = 3x^2 + 2x\lambda = x(3x+2\lambda) = 0 \Rightarrow \begin{cases} x = 0 \\ x = \frac{-2\lambda}{3} \end{cases}$$

$$\frac{dL}{dy} = 3y^2 + 2y\lambda = y(3y+2\lambda) = 0 \Rightarrow \begin{cases} y = 0 \\ y = \frac{-2\lambda}{3} \end{cases}$$

$$\frac{dL}{dx} = 3z^2 + 2z\lambda = z(3z+2\lambda) = 0 \Rightarrow \begin{cases} z = 0 \\ z = -\frac{2\lambda}{3} \end{cases}$$

$$\frac{dL}{d\lambda} = x^2 + y^2 + z^2 - 1 = 0.$$

I. $x=0, y=0, z=0$ - not possible because $-1 = 0$ (F).

II. $x = y = z = -\frac{2\lambda}{3}$

$$\left(-\frac{2\lambda}{3}\right)^2 + \left(-\frac{2\lambda}{3}\right)^2 + \left(-\frac{2\lambda}{3}\right)^2 - 1 = 0 \Rightarrow$$

$$\Rightarrow 3 \cdot \frac{4\lambda^2}{9} = 1 \Rightarrow \frac{4\lambda^2}{3} = 1 \Rightarrow \lambda^2 = \frac{3}{4} \Rightarrow \begin{cases} \lambda_1 = -\frac{\sqrt{3}}{2} \\ \lambda_2 = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = y_1 = z_1 = -\frac{\cancel{2}}{3} \cdot \left(-\frac{\sqrt{3}}{\cancel{2}}\right) = \frac{\sqrt{3}}{3} \\ x_2 = y_2 = z_2 = -\frac{\cancel{2}}{3} \cdot \frac{\sqrt{3}}{\cancel{2}} = -\frac{\sqrt{3}}{3} \end{cases} \Rightarrow \text{the critical points are: } \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{2}\right)$$

and $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2}\right)$.

III. 2 variables are equal $\left(x = y = -\frac{2\lambda}{3}\right)$ and one is 0 $(z=0)$.

$$\left(-\frac{2\lambda}{3}\right)^2 + \left(-\frac{2\lambda}{3}\right)^2 + 0 - 1 = 0 \Rightarrow$$

$$\Rightarrow \frac{8\lambda^2}{9} = 1 \Rightarrow \lambda^2 = \frac{9}{8} \Rightarrow \begin{cases} \lambda_1 = \frac{3\sqrt{2}}{4} \\ \lambda_2 = -\frac{3\sqrt{2}}{4} \end{cases} \Rightarrow \begin{cases} x_1 = y_1 = -\frac{\sqrt{2}}{2} \\ x_2 = y_2 = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow$$

$\Rightarrow$ the critical point are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0, \frac{3\sqrt{2}}{4}\right)$ and $\left(+\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}, 0, -\frac{3\sqrt{2}}{4}\right)$.

- analogue for x and y as the variable $= 0$.

$\underline{\text{IV}}$. 1 variable equal to $-\frac{2\lambda}{3}$ $\left(x = -\frac{2\lambda}{3}\right)$ and 2 equal to 0 $(z = y = 0)$

$\left(-\frac{2\lambda}{3}\right)^2 + 0 + 0 - 1 = 0 \Rightarrow$

$\Rightarrow \frac{4\lambda^2}{9} = 1 \Rightarrow \begin{cases} \lambda_1 = \frac{3}{2} \Rightarrow x_1 = -1 \\ \lambda_2 = -\frac{3}{2} \Rightarrow x_2 = 1. \end{cases}$ $\Rightarrow$ the critical points

are : $\left(1, 0, 0, -\frac{3}{2}\right)$ and $\left(-1, 0, 0, \frac{3}{2}\right)$.

- analogue for y an z as the variable $= -\frac{2\lambda}{3}$