Homework-seminar 07

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914.

1c) ★ $\lim_{n\to\infty} \frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\dots+n\sqrt[n]{e^n}}{n^2}$

$$\frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\dots+n\sqrt[n]{e^n}}{n^2} = \sum_{k=1}^{n} \frac{k\cdot\sqrt[n]{e^k}}{n^2} = \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{k}{n}\cdot e^{\frac{k}{n}} =$$

$$= \sum_{k=1}^{n} \frac{1}{n}\cdot f\left(\frac{k}{n}\right), \text{ where } f\left(\frac{k}{n}\right) = \frac{k}{n}\cdot e^{\frac{k}{n}} \Rightarrow f(x) = x\cdot e^x.$$

$$\hookrightarrow \int_0^1 f(x)\,dx = \int_0^1 x\cdot e^x\,dx = \int_0^1 x\,(e^x)'\,dx =$$

$$= e^x\cdot x\Big|_0^1 - \int_0^1 e^x\,dx = e\cdot 1 - 0 - e^x\Big|_0^1 = e - e + 1 = \cancel{2e}\ 1$$

3. ★ $\int_0^\infty e^{-x}\cdot\sin x\,dx = -\int_0^\infty (e^{-x})'\cdot\sin x\,dx =$

$$= \underbrace{-e^{-x}\cdot\sin x\Big|_0^\infty}_{L_1} + \int_0^\infty e^{-x}\cdot\cos x\,dx =$$

$$= -L_1 - \underbrace{e^{-x}\cdot\cos x\Big|_0^\infty}_{L_2} + \int_0^\infty e^{-x}\cdot(-\sin x). \Rightarrow$$

$$\Rightarrow \int_0^\infty e^{-x}\cdot\sin x\,dx = \frac{1}{2}\left(-L_1-L_2\right). = \frac{1}{2}\cdot 0 = 0.$$

$$L_1 = \lim_{x\to\infty}\frac{\sin x}{e^x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\cos x}{e^x}$$

$$\left.\begin{array}{l} L_2 = \lim_{x\to\infty}\frac{\cos x}{e^x} \\ \cos x \in [-1,1] \\ e^x \to \infty \end{array}\right\} \Rightarrow L_1 = L_2 = 0$$

d) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n}\cdot \sin\frac{2\pi}{2n} \cdots \sin\frac{(n+1)\pi}{2n}}$.

$$\sqrt[n]{\sin\frac{\pi}{2n}\cdot \sin\frac{2\pi}{2n} \cdots \sin\frac{(n+1)\pi}{2n}} = e^{\frac{1}{n}\ln\left(\sin\frac{\pi}{2n}\cdots \sin\frac{(n+1)\pi}{2n}\right)} \Rightarrow$$

$$\Rightarrow \frac{1}{n}\ln\left(\sin\frac{\pi}{2n}\cdot \sin\frac{2\pi}{2n} \cdots \sin\frac{(n+1)\pi}{2n}\right) = \frac{1}{n}\sum_{k=1}^{n-1} \ln\sin\left(\frac{\pi}{2}\cdot\frac{k}{n}\right) \Rightarrow$$

$$\begin{cases} f\left(\frac{k}{n}\right) = \ln\sin\left(\frac{\pi}{2}\frac{k}{n}\right) \Rightarrow \\ \Rightarrow f(x) = \ln\sin\left(\frac{\pi}{2}x\right) \end{cases}$$

$$\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 \ln\left(\sin\frac{\pi}{2}x\right)dx =$$

let $u = \frac{\pi}{2}x \Rightarrow du = \frac{\pi}{2}dx \Rightarrow dx = \frac{2}{\pi}du$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln(\sin u)\,du \quad ①$$

$$\left.\begin{aligned} &\int_0^{\frac{\pi}{2}} \ln(\sin u)\,du \overset{t=\cancel{x}\frac{\pi}{2}-x}{=\!=} \int_0^{\frac{\pi}{2}} \ln\cos t\,dt \\ &\qquad\text{and} \\ &\int_{\frac{\pi}{2}}^{\pi} \ln(\sin u)\,du \overset{t=x-\frac{\pi}{2}}{=\!=} \int_0^{\frac{\pi}{2}} \ln\cos t\,dt \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \ln(\sin u)\,du = \int_0^{\frac{\pi}{2}} \ln(\sin u)\,du + \int_0^{\frac{\pi}{2}} \ln(\overset{\cos u}{\cancel{\sin t}})\,du \neq$$

$$= \int_0^{\frac{\pi}{2}} \ln\sin 2u\,du - \frac{\pi}{2}\ln 2 = \frac{1}{2}\int_0^{\pi} \ln\sin u\,du - \frac{\pi}{2}\ln 2 =$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln\sin u\,du + \int_{\frac{\pi}{2}}^{\pi} \ln\sin u\,du\right) - \frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln\sin u\,du - \frac{\pi}{2}\ln 2$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin u)\,du = -\frac{\pi}{2}\ln 2 \quad ②$$

- from ① and ② $\Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \cdots \sin\frac{(n-1)\pi}{2n}} = \frac{2}{\pi} \cdot \left(-\frac{\pi}{2}\right) \ln 2 = -\ln 2.$$


```python
import numpy as np
import matplotlib.pyplot as plt

1 usage
def gaussian_integral_approximation(a, num_points=1000):
    # define the Gaussian function
    f = lambda x: np.exp(-x**2) # calculates e^(x^2)

    # generates num_points evenly spaced values between -a and a
    # (inclusive) to create the x-coordinates for the trapezoidal rule.
    x_points = np.linspace(-a, a, num_points)

    # evaluates the Gaussian function at each x-coordinate
    # to get the corresponding y-coordinates.
    y_points = f(x_points)

    # the trapezoidal rule to estimate the integral of the
    # function over the specified interval.
    integral_approximation = np.trapz(y_points, x_points)
    return integral_approximation
```

```python
# sets the value of a for the interval [-a, a]
a_values = np.arange(0.1, 5.1, 0.1)

# computes the numerical approximations for different values of a
results = [gaussian_integral_approximation(a) for a in a_values]

# Plot the results
```

```python
# plot the result
plt.plot( *args: a_values, results, label='Numerical Approximation')
plt.axhline(y=np.sqrt(np.pi), color='r', linestyle='--', label='Square root of pi')
plt.xlabel('a')
plt.ylabel('Integral Approximation')
plt.title('Convergence of Numerical Approximation to sqrt(pi)')
plt.legend()
plt.show()
```

Result: