Limits and Continuity

Exercise Set 1.1

1. **(a)** 3 **(b)** 3 **(c)** 3 **(d)** 3

2. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0

3. **(a)** -1 **(b)** 3 **(c)** does not exist **(d)** 1

4. **(a)** 2 **(b)** 0 **(c)** does not exist **(d)** 2

5. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 3

6. **(a)** 1 **(b)** 1 **(c)** 1 **(d)** 0

7. **(a)** $-\infty$ **(b)** $-\infty$ **(c)** $-\infty$ **(d)** 1

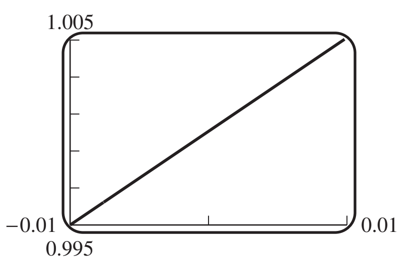
8. **(a)** $+\infty$ **(b)** $+\infty$ **(c)** $+\infty$ **(d)** can not be found from graph

9. **(a)** $+\infty$ **(b)** $+\infty$ **(c)** 2 **(d)** 2 **(e)** $-\infty$ **(f)** $x=-2$, $x=0$, $x=2$

10. **(a)** does not exist **(b)** $-\infty$ **(c)** 0 **(d)** -1 **(e)** $+\infty$ **(f)** 3 **(g)** $x=-2$, $x=2$

11. **(i)**

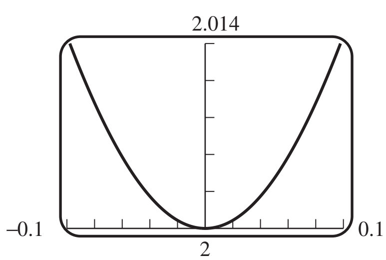
-0.01	-0.001	-0.0001	0.0001	0.001	0.01
0.9950166	0.9995002	0.9999500	1.0000500	1.0005002	1.0050167

(ii) 

The limit appears to be 1.

12. **(i)**

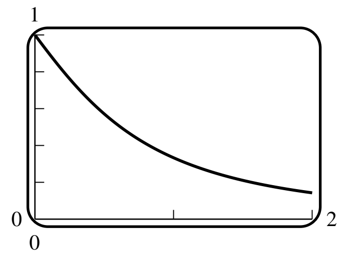
-0.1	-0.01	-0.001	0.001	0.01	0.1
2.0135792	2.0001334	2.0000013	2.0000013	2.0001334	2.0135792

(ii) 

The limit appears to be 2.

13. (a)

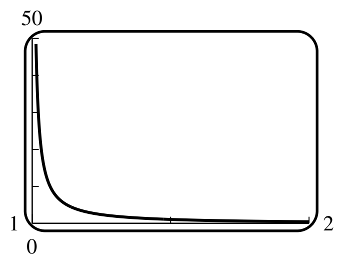
2	1.5	1.1	1.01	1.001	0	0.5	0.9	0.99	0.999
0.1429	0.2105	0.3021	0.3300	0.3330	1.0000	0.5714	0.3690	0.3367	0.3337



The limit is $1/3$.

(b)

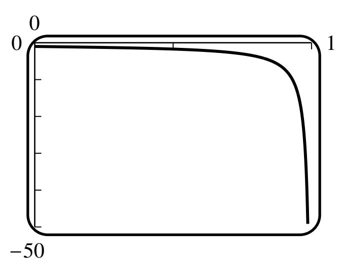
2	1.5	1.1	1.01	1.001	1.0001
0.4286	1.0526	6.344	66.33	666.3	6666.3



The limit is $+\infty$.

(c)

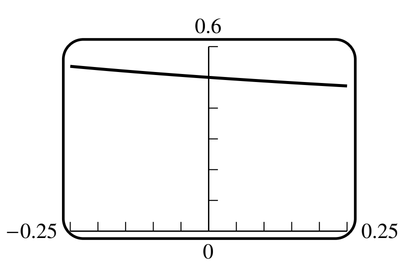
0	0.5	0.9	0.99	0.999	0.9999
-1	-1.7143	-7.0111	-67.001	-667.0	-6667.0



The limit is $-\infty$.

14. (a)

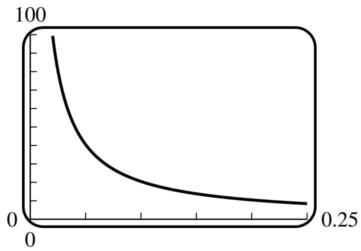
-0.25	-0.1	-0.001	-0.0001	0.0001	0.001	0.1	0.25
0.5359	0.5132	0.5001	0.5000	0.5000	0.4999	0.4881	0.4721



The limit is $1/2$.

(b)

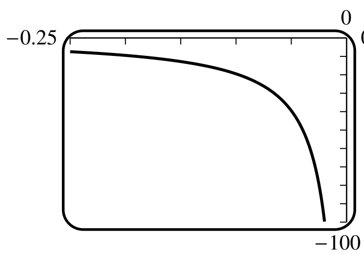
0.25	0.1	0.001	0.0001
8.4721	20.488	2000.5	20001



The limit is $+\infty$.

(c)

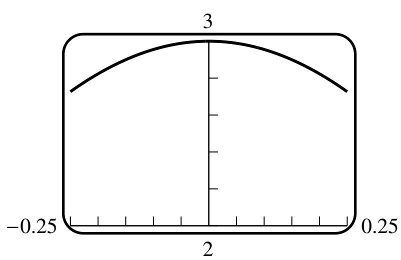
-0.25	-0.1	-0.001	-0.0001
-7.4641	-19.487	-1999.5	-20000



The limit is $-\infty$.

15. (a)

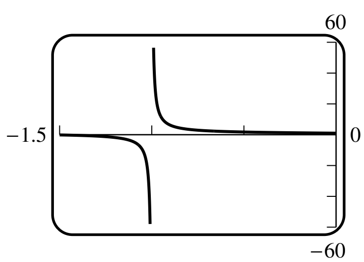
-0.25	-0.1	-0.001	-0.0001	0.0001	0.001	0.1	0.25
2.7266	2.9552	3.0000	3.0000	3.0000	3.0000	2.9552	2.7266



The limit is 3.

(b)

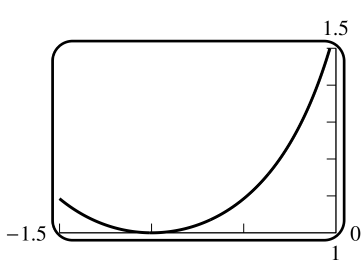
0	-0.5	-0.9	-0.99	-0.999	-1.5	-1.1	-1.01	-1.001
1	1.7552	6.2161	54.87	541.1	-0.1415	-4.536	-53.19	-539.5



The limit does not exist.

16. (a)

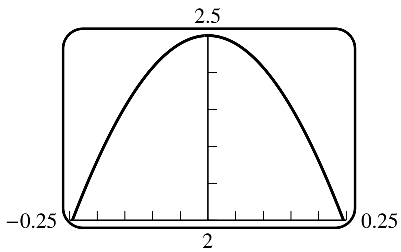
0	-0.5	-0.9	-0.99	-0.999	-1.5	-1.1	-1.01	-1.001
1.5574	1.0926	1.0033	1.0000	1.0000	1.0926	1.0033	1.0000	1.0000



The limit is 1.

(b)

−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
1.9794	2.4132	2.5000	2.5000	2.5000	2.5000	2.4132	1.9794



The limit is 5/2.

17. False; define $f(x) = x$ for $x \neq a$ and $f(a) = a + 1$. Then $\lim_{x\to a} f(x) = a \neq f(a) = a + 1$.

18. True; by 1.1.3.

19. False; define $f(x) = 0$ for $x < 0$ and $f(x) = x + 1$ for $x \geq 0$. Then the left and right limits exist but are unequal.

20. False; define $f(x) = 1/x$ for $x > 0$ and $f(0) = 2$.

27. $m_{\text{sec}} = \dfrac{x^2 - 1}{x + 1} = x - 1$ which gets close to -2 as x gets close to -1, thus $y - 1 = -2(x + 1)$ or $y = -2x - 1$.

28. $m_{\text{sec}} = \dfrac{x^2}{x} = x$ which gets close to 0 as x gets close to 0, thus $y = 0$.

29. $m_{\text{sec}} = \dfrac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1$ which gets close to 4 as x gets close to 1, thus $y - 1 = 4(x - 1)$ or $y = 4x - 3$.

30. $m_{\text{sec}} = \dfrac{x^4 - 1}{x + 1} = x^3 - x^2 + x - 1$ which gets close to -4 as x gets close to -1, thus $y - 1 = -4(x + 1)$ or $y = -4x - 3$.

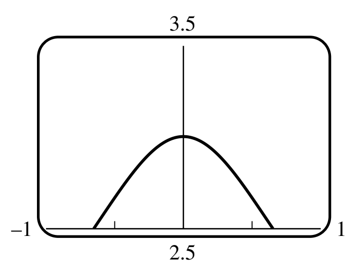
31. **(a)** The length of the rod while at rest.

(b) The limit is zero. The length of the rod approaches zero as its speed approaches c.

32. **(a)** The mass of the object while at rest.

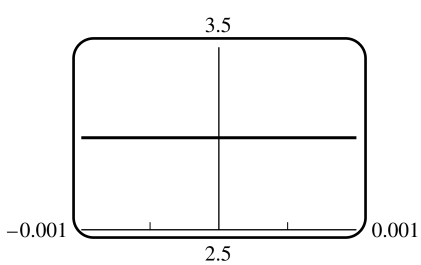
(b) The limiting mass as the velocity approaches the speed of light; the mass is unbounded.

33. **(a)**

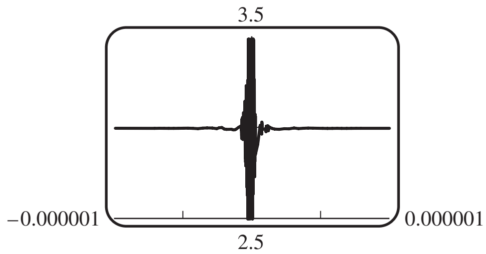


The limit appears to be 3.

(b)



The limit appears to be 3.

(c) 

The limit does not exist.

Exercise Set 1.2

1. **(a)** By Theorem 1.2.2, this limit is $2 + 2 \cdot (-4) = -6$.

(b) By Theorem 1.2.2, this limit is $0 - 3 \cdot (-4) + 1 = 13$.

(c) By Theorem 1.2.2, this limit is $2 \cdot (-4) = -8$.

(d) By Theorem 1.2.2, this limit is $(-4)^2 = 16$.

(e) By Theorem 1.2.2, this limit is $\sqrt[3]{6+2} = 2$.

(f) By Theorem 1.2.2, this limit is $\dfrac{2}{(-4)} = -\dfrac{1}{2}$.

2. **(a)** By Theorem 1.2.2, this limit is $0 + 0 = 0$.

(b) The limit doesn't exist because $\lim f$ doesn't exist and $\lim g$ does.

(c) By Theorem 1.2.2, this limit is $-2 + 2 = 0$.

(d) By Theorem 1.2.2, this limit is $1 + 2 = 3$.

(e) By Theorem 1.2.2, this limit is $0/(1+0) = 0$.

(f) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

(g) The limit doesn't exist because $\sqrt{f(x)}$ is not defined for $0 < x < 2$.

(h) By Theorem 1.2.2, this limit is $\sqrt{1} = 1$.

3. By Theorem 1.2.3, this limit is $2 \cdot 1 \cdot 3 = 6$.

4. By Theorem 1.2.3, this limit is $3^3 - 3 \cdot 3^2 + 9 \cdot 3 = 27$.

5. By Theorem 1.2.4, this limit is $(3^2 - 2 \cdot 3)/(3+1) = 3/4$.

6. By Theorem 1.2.4, this limit is $(6 \cdot 0 - 9)/(0^3 - 12 \cdot 0 + 3) = -3$.

7. After simplification, $\dfrac{x^4-1}{x-1} = x^3 + x^2 + x + 1$, and the limit is $1^3 + 1^2 + 1 + 1 = 4$.

8. After simplification, $\dfrac{t^3+8}{t+2} = t^2 - 2t + 4$, and the limit is $(-2)^2 - 2 \cdot (-2) + 4 = 12$.

9. After simplification, $\dfrac{x^2+6x+5}{x^2-3x-4} = \dfrac{x+5}{x-4}$, and the limit is $(-1+5)/(-1-4) = -4/5$.

10. After simplification, $\dfrac{x^2-4x+4}{x^2+x-6}=\dfrac{x-2}{x+3}$, and the limit is $(2-2)/(2+3)=0$.

11. After simplification, $\dfrac{2x^2+x-1}{x+1}=2x-1$, and the limit is $2\cdot(-1)-1=-3$.

12. After simplification, $\dfrac{3x^2-x-2}{2x^2+x-3}=\dfrac{3x+2}{2x+3}$, and the limit is $(3\cdot 1+2)/(2\cdot 1+3)=1$.

13. After simplification, $\dfrac{t^3+3t^2-12t+4}{t^3-4t}=\dfrac{t^2+5t-2}{t^2+2t}$, and the limit is $(2^2+5\cdot 2-2)/(2^2+2\cdot 2)=3/2$.

14. After simplification, $\dfrac{t^3+t^2-5t+3}{t^3-3t+2}=\dfrac{t+3}{t+2}$, and the limit is $(1+3)/(1+2)=4/3$.

15. The limit is $+\infty$.

16. The limit is $-\infty$.

17. The limit does not exist.

18. The limit is $+\infty$.

19. The limit is $-\infty$.

20. The limit does not exist.

21. The limit is $+\infty$.

22. The limit is $-\infty$.

23. The limit does not exist.

24. The limit is $-\infty$.

25. The limit is $+\infty$.

26. The limit does not exist.

27. The limit is $+\infty$.

28. The limit is $+\infty$.

29. After simplification, $\dfrac{x-9}{\sqrt{x}-3}=\sqrt{x}+3$, and the limit is $\sqrt{9}+3=6$.

30. After simplification, $\dfrac{4-y}{2-\sqrt{y}}=2+\sqrt{y}$, and the limit is $2+\sqrt{4}=4$.

31. **(a)** 2 **(b)** 2 **(c)** 2

32. **(a)** does not exist **(b)** 1 **(c)** 4

33. True, by Theorem 1.2.2.

34. False; e.g. $\lim\limits_{x\to 0}\dfrac{x^2}{x}=0$.

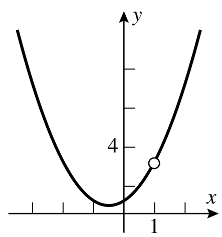
35. False; e.g. $f(x) = 2x$, $g(x) = x$, so $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} f(x)/g(x) = 2$.

36. True, by Theorem 1.2.4.

37. After simplification, $\dfrac{\sqrt{x+4}-2}{x} = \dfrac{1}{\sqrt{x+4}+2}$, and the limit is $1/4$.

38. After simplification, $\dfrac{\sqrt{x^2+4}-2}{x} = \dfrac{x}{\sqrt{x^2+4}+2}$, and the limit is 0.

39. **(a)** After simplification, $\dfrac{x^3-1}{x-1} = x^2+x+1$, and the limit is 3.

(b) 

40. **(a)** After simplification, $\dfrac{x^2-9}{x+3} = x-3$, and the limit is -6, so we need that $k = -6$.

(b) On its domain (all real numbers), $f(x) = x - 3$.

41. **(a)** Theorem 1.2.2 doesn't apply; moreover one cannot subtract infinities.

(b) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{x-1}{x^2}\right) = -\infty.$

42. **(a)** Theorem 1.2.2 assumes that L_1 and L_2 are real numbers, not infinities. It is in general not true that "$\infty\cdot 0 = 0$".

(b) $\dfrac{1}{x} - \dfrac{2}{x^2+2x} = \dfrac{x^2}{x(x^2+2x)} = \dfrac{1}{x+2}$ for $x \neq 0$, so that $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{x^2+2x}\right) = \frac{1}{2}$.

43. For $x \neq 1$, $\dfrac{1}{x-1} - \dfrac{a}{x^2-1} = \dfrac{x+1-a}{x^2-1}$ and for this to have a limit it is necessary that $\lim_{x\to 1}(x+1-a) = 0$, i.e. $a = 2$. For this value, $\dfrac{1}{x-1} - \dfrac{2}{x^2-1} = \dfrac{x+1-2}{x^2-1} = \dfrac{x-1}{x^2-1} = \dfrac{1}{x+1}$ and $\displaystyle\lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}$.

44. **(a)** For small x, $1/x^2$ is much bigger than $\pm 1/x$.

(b) $\dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{x+1}{x^2}$. Since the numerator has limit 1 and x^2 tends to zero from the right, the limit is $+\infty$.

45. The left and/or right limits could be plus or minus infinity; or the limit could exist, or equal any preassigned real number. For example, let $q(x) = x - x_0$ and let $p(x) = a(x - x_0)^n$ where n takes on the values $0, 1, 2$.

46. If on the contrary $\lim_{x\to a} g(x)$ did exist then by Theorem 1.2.2 so would $\lim_{x\to a}[f(x) + g(x)]$, and that would be a contradiction.

47. Clearly, $g(x) = [f(x) + g(x)] - f(x)$. By Theorem 1.2.2, $\lim_{x\to a}[f(x) + g(x)] - \lim_{x\to a} f(x) = \lim_{x\to a}[f(x) + g(x) - f(x)] = \lim_{x\to a} g(x)$.

48. By Theorem 1.2.2, $\lim\limits_{x\to a} f(x) = \left(\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}\right)\lim\limits_{x\to a} g(x) = \left(\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}\right)\cdot 0 = 0$, since $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ exists.

Exercise Set 1.3

1. **(a)** $-\infty$ **(b)** $+\infty$

2. **(a)** 2 **(b)** 0

3. **(a)** 0 **(b)** -1

4. **(a)** does not exist **(b)** 0

5. **(a)** $3+3\cdot(-5)=-12$ **(b)** $0-4\cdot(-5)+1=21$ **(c)** $3\cdot(-5)=-15$ **(d)** $(-5)^2=25$

(e) $\sqrt[3]{5+3}=2$ **(f)** $3/(-5)=-3/5$ **(g)** 0

(h) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

6. **(a)** $2\cdot 7-(-6)=20$ **(b)** $6\cdot 7+7\cdot(-6)=0$ **(c)** $+\infty$ **(d)** $-\infty$ **(e)** $\sqrt[3]{-42}$

(f) $-6/7$ **(g)** 7 **(h)** $-7/12$

7. **(a)**

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	1.471128	1.560797	1.569796	1.570696	1.570786	1.570795

The limit appears to be $\approx 1.57079\ldots$.

(b) The limit is $\pi/2$.

8.

x	10	100	1000	10000	100000	1000000
$f(x)$	1.258925	1.047129	1.006932	1.000921	1.000115	1.000014

The limit appears to be 1.

9. The limit is $-\infty$, by the highest degree term.

10. The limit is $+\infty$, by the highest degree term.

11. The limit is $+\infty$.

12. The limit is $+\infty$.

13. The limit is $3/2$, by the highest degree terms.

14. The limit is $5/2$, by the highest degree terms.

15. The limit is 0, by the highest degree terms.

16. The limit is 0, by the highest degree terms.

17. The limit is 0, by the highest degree terms.

18. The limit is $5/3$, by the highest degree terms.

19. The limit is $-\infty$, by the highest degree terms.

20. The limit is $+\infty$, by the highest degree terms.

21. The limit is $-1/7$, by the highest degree terms.

22. The limit is $4/7$, by the highest degree terms.

23. The limit is $\sqrt[3]{-5/8} = -\sqrt[3]{5}/2$, by the highest degree terms.

24. The limit is $\sqrt[3]{3/2}$, by the highest degree terms.

25. $\dfrac{\sqrt{5x^2-2}}{x+3} = \dfrac{\sqrt{5-\frac{2}{x^2}}}{-1-\frac{3}{x}}$ when $x<0$. The limit is $-\sqrt{5}$.

26. $\dfrac{\sqrt{5x^2-2}}{x+3} = \dfrac{\sqrt{5-\frac{2}{x^2}}}{1+\frac{3}{x}}$ when $x>0$. The limit is $\sqrt{5}$.

27. $\dfrac{2-y}{\sqrt{7+6y^2}} = \dfrac{-\frac{2}{y}+1}{\sqrt{\frac{7}{y^2}+6}}$ when $y<0$. The limit is $1/\sqrt{6}$.

28. $\dfrac{2-y}{\sqrt{7+6y^2}} = \dfrac{\frac{2}{y}-1}{\sqrt{\frac{7}{y^2}+6}}$ when $y>0$. The limit is $-1/\sqrt{6}$.

29. $\dfrac{\sqrt{3x^4+x}}{x^2-8} = \dfrac{\sqrt{3+\frac{1}{x^3}}}{1-\frac{8}{x^2}}$ when $x<0$. The limit is $\sqrt{3}$.

30. $\dfrac{\sqrt{3x^4+x}}{x^2-8} = \dfrac{\sqrt{3+\frac{1}{x^3}}}{1-\frac{8}{x^2}}$ when $x>0$. The limit is $\sqrt{3}$.

31. $\displaystyle\lim_{x\to+\infty}(\sqrt{x^2+3}-x)\frac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x} = \lim_{x\to+\infty}\frac{3}{\sqrt{x^2+3}+x} = 0$, by the highest degree terms.

32. $\displaystyle\lim_{x\to+\infty}(\sqrt{x^2-3x}-x)\frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x} = \lim_{x\to+\infty}\frac{-3x}{\sqrt{x^2-3x}+x} = -3/2$, by the highest degree terms.

33. $\displaystyle\lim_{x\to-\infty}\frac{1-e^x}{1+e^x} = \frac{1-0}{1+0} = 1.$

34. Divide the numerator and denominator by e^x: $\displaystyle\lim_{x\to+\infty}\frac{1-e^x}{1+e^x} = \lim_{x\to+\infty}\frac{e^{-x}-1}{e^{-x}+1} = \frac{0-1}{0+1} = -1.$

35. Divide the numerator and denominator by e^x: $\displaystyle\lim_{x\to+\infty}\frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1+0}{1-0} = 1.$

36. Divide the numerator and denominator by e^{-x}: $\displaystyle\lim_{x\to-\infty}\frac{e^{2x}+1}{e^{2x}-1} = \frac{0+1}{0-1} = -1.$

37. The limit is $-\infty$.

38. The limit is $+\infty$.

39. $\dfrac{x+1}{x} = 1+\dfrac{1}{x}$, so $\displaystyle\lim_{x\to+\infty}\frac{(x+1)^x}{x^x} = e$ from Figure 1.3.4.

40. $\left(1+\dfrac{1}{x}\right)^{-x} = \dfrac{1}{(1+\frac{1}{x})^x}$, so the limit is e^{-1}.

41. False: $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{2x} = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right]^2 = e^2.$

42. False; $y=0$ is a horizontal asymptote for the curve $y=e^x$ yet $\displaystyle\lim_{x\to+\infty} e^x$ does not exist.

43. True: for example $f(x) = \sin x/x$ crosses the x-axis infinitely many times at $x = n\pi, n = 1, 2, \ldots$.

44. False: if the asymptote is $y = 0$, then $\lim_{x\to\pm\infty} p(x)/q(x) = 0$, and clearly the degree of $p(x)$ is strictly less than the degree of $q(x)$. If the asymptote is $y = L \neq 0$, then $\lim_{x\to\pm\infty} p(x)/q(x) = L$ and the degrees must be equal.

45. It appears that $\lim_{t\to+\infty} n(t) = +\infty$, and $\lim_{t\to+\infty} e(t) = c$.

46. **(a)** It is the initial temperature of the potato (400° F).

(b) It is the ambient temperature, i.e. the temperature of the room.

47. **(a)** $+\infty$ **(b)** -5

48. **(a)** 0 **(b)** -6

49. $\lim_{x\to-\infty} p(x) = +\infty$. When n is even, $\lim_{x\to+\infty} p(x) = +\infty$; when n is odd, $\lim_{x\to+\infty} p(x) = -\infty$.

50. **(a)** $p(x) = q(x) = x$. **(b)** $p(x) = x$, $q(x) = x^2$. **(c)** $p(x) = x^2$, $q(x) = x$. **(d)** $p(x) = x + 3$, $q(x) = x$.

51. **(a)** No. **(b)** Yes, $\tan x$ and $\sec x$ at $x = n\pi + \pi/2$ and $\cot x$ and $\csc x$ at $x = n\pi, n = 0, \pm1, \pm2, \ldots$.

52. If $m > n$ the limit is zero. If $m = n$ the limit is c_m/d_m. If $n > m$ the limit is $+\infty$ if $c_n d_m > 0$ and $-\infty$ if $c_n d_m < 0$.

53. **(a)** Every value taken by e^{x^2} is also taken by e^t: choose $t = x^2$. As x and t increase without bound, so does $e^t = e^{x^2}$. Thus $\lim_{x\to+\infty} e^{x^2} = \lim_{t\to+\infty} e^t = +\infty$.

(b) If $f(t) \to +\infty$ (resp. $f(t) \to -\infty$) then $f(t)$ can be made arbitrarily large (resp. small) by taking t large enough. But by considering the values $g(x)$ where $g(x) > t$, we see that $f(g(x))$ has the limit $+\infty$ too (resp. limit $-\infty$). If $f(t)$ has the limit L as $t \to +\infty$ the values $f(t)$ can be made arbitrarily close to L by taking t large enough. But if x is large enough then $g(x) > t$ and hence $f(g(x))$ is also arbitrarily close to L.

(c) For $\lim_{x\to-\infty}$ the same argument holds with the substitutiion "x decreases without bound" instead of "x increases without bound". For $\lim_{x\to c^-}$ substitute "x close enough to $c, x < c$", etc.

54. **(a)** Every value taken by e^{-x^2} is also taken by e^t: choose $t = -x^2$. As x increases without bound and t decreases without bound, the quantity $e^t = e^{-x^2}$ tends to 0. Thus $\lim_{x\to+\infty} e^{-x^2} = \lim_{t\to-\infty} e^t = 0$.

(b) If $f(t) \to +\infty$ (resp. $f(t) \to -\infty$) then $f(t)$ can be made arbitrarily large (resp. small) by taking t small enough. But by considering the values $g(x)$ where $g(x) < t$, we see that $f(g(x))$ has the limit $+\infty$ too (resp. limit $-\infty$). If $f(t)$ has the limit L as $t \to -\infty$ the values $f(t)$ can be made arbitrarily close to L by taking t small enough. But if x is large enough then $g(x) < t$ and hence $f(g(x))$ is also arbitrarily close to L.

(c) For $\lim_{x\to-\infty}$ the same argument holds with the substitutiion "x decreases without bound" instead of "x increases without bound". For $\lim_{x\to c^-}$ substitute "x close enough to $c, x < c$", etc.

55. $t = 1/x$, $\lim_{t\to+\infty} f(t) = +\infty$.

56. $t = 1/x$, $\lim_{t\to-\infty} f(t) = 0$.

57. $t = \csc x$, $\lim_{t\to+\infty} f(t) = +\infty$.

58. $t = \csc x$, $\lim_{t\to-\infty} f(t) = 0$.

59. Let $t = \ln x$. Then t also tends to $+\infty$, and $\dfrac{\ln 2x}{\ln 3x} = \dfrac{t + \ln 2}{t + \ln 3}$, so the limit is 1.

60. With $t = x - 1, [\ln(x^2 - 1) - \ln(x + 1)] = \ln(x + 1) + \ln(x - 1) - \ln(x + 1) = \ln t$, so the limit is $+\infty$.

61. Set $t = -x$, then get $\lim\limits_{t\to-\infty}\left(1+\dfrac{1}{t}\right)^t = e$ by Figure 1.3.4.

62. With $t = x/2$, $\lim\limits_{x\to+\infty}\left(1+\dfrac{2}{x}\right)^x = \left(\lim\limits_{t\to+\infty}[1+1/t]^t\right)^2 = e^2$

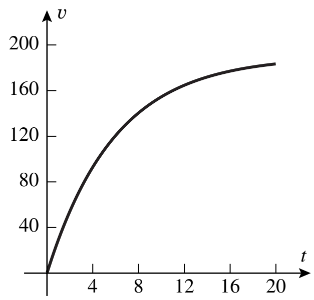
63. From the hint, $\lim\limits_{x\to+\infty} b^x = \lim\limits_{x\to+\infty} e^{(\ln b)x} = \begin{cases} 0 & \text{if } b<1, \\ 1 & \text{if } b=1, \\ +\infty & \text{if } b>1. \end{cases}$

64. It suffices by Theorem 1.1.3 to show that the left and right limits at zero are equal to e.

(a) $\lim\limits_{x\to+\infty}(1+x)^{1/x} = \lim\limits_{t\to0^+}(1+1/t)^t = e$.

(b) $\lim\limits_{x\to-\infty}(1+x)^{1/x} = \lim\limits_{t\to0^-}(1+1/t)^t = e$.

65. (a)

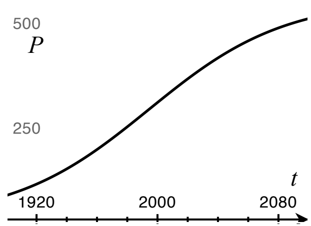


(b) $\lim\limits_{t\to\infty} v = 190\left(1-\lim\limits_{t\to\infty}e^{-0.168t}\right) = 190$, so the asymptote is $v = c = 190$ ft/sec.

(c) Due to air resistance (and other factors) this is the maximum speed that a sky diver can attain.

66. (a) $p(1990) = 525/(1+1.1) = 250$ (million).

(b)



(c) $\lim\limits_{t\to\infty} p(t) = \dfrac{525}{1+1.1\lim_{t\to\infty}e^{-0.02225(t-1990)}} = 525$ (million).

(d) The population becomes stable at this number.

67. (a)

n	2	3	4	5	6	7
$1+10^{-n}$	1.01	1.001	1.0001	1.00001	1.000001	1.0000001
$1+10^n$	101	1001	10001	100001	1000001	10000001
$(1+10^{-n})^{1+10^n}$	2.7319	2.7196	2.7184	2.7183	2.71828	2.718282

The limit appears to be e.

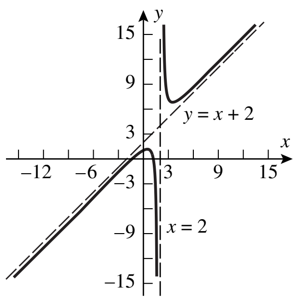
(b) This is evident from the lower left term in the chart in part (a).

(c) The exponents are being multiplied by a, so the result is e^a.

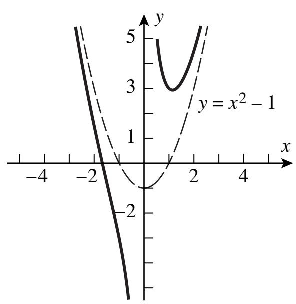
68. **(a)** $f(-x) = \left(1 - \dfrac{1}{x}\right)^{-x} = \left(\dfrac{x-1}{x}\right)^{-x} = \left(\dfrac{x}{x-1}\right)^{x}$, $f(x-1) = \left(\dfrac{x}{x-1}\right)^{x-1} = \left(\dfrac{x-1}{x}\right) f(-x)$.

(b) $\displaystyle\lim_{x\to-\infty}\left(1 + \frac{1}{x}\right)^{x} = \lim_{x\to+\infty} f(-x) = \left[\lim_{x\to+\infty}\frac{x}{x-1}\right]\lim_{x\to+\infty} f(x-1) = \lim_{x\to+\infty} f(x-1) = e.$

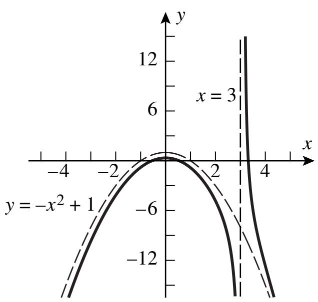
69. After a long division, $f(x) = x + 2 + \dfrac{2}{x-2}$, so $\displaystyle\lim_{x\to\pm\infty}(f(x) - (x+2)) = 0$ and $f(x)$ is asymptotic to $y = x + 2$. The only vertical asymptote is at $x = 2$.



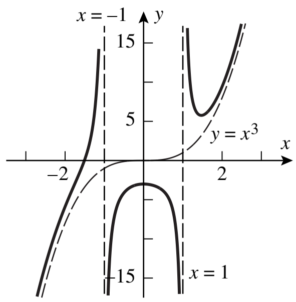
70. After a simplification, $f(x) = x^2 - 1 + \dfrac{3}{x}$, so $\displaystyle\lim_{x\to\pm\infty}(f(x) - (x^2 - 1)) = 0$ and $f(x)$ is asymptotic to $y = x^2 - 1$. The only vertical asymptote is at $x = 0$.



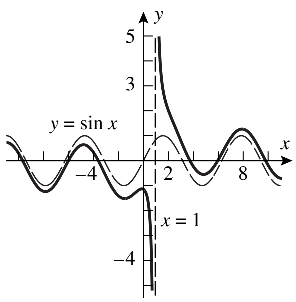
71. After a long division, $f(x) = -x^2 + 1 + \dfrac{2}{x-3}$, so $\displaystyle\lim_{x\to\pm\infty}(f(x) - (-x^2 + 1)) = 0$ and $f(x)$ is asymptotic to $y = -x^2 + 1$. The only vertical asymptote is at $x = 3$.



72. After a long division, $f(x) = x^3 + \dfrac{3}{2(x-1)} - \dfrac{3}{2(x+1)}$, so $\displaystyle\lim_{x\to\pm\infty}(f(x) - x^3) = 0$ and $f(x)$ is asymptotic to $y = x^3$. The vertical asymptotes are at $x = \pm 1$.



73. $\lim_{x\to\pm\infty}(f(x)-\sin x)=0$ so $f(x)$ is asymptotic to $y=\sin x$. The only vertical asymptote is at $x=1$.



Exercise Set 1.4

1. **(a)** $|f(x)-f(0)|=|x+2-2|=|x|<0.1$ if and only if $|x|<0.1$.

(b) $|f(x)-f(3)|=|(4x-5)-7|=4|x-3|<0.1$ if and only if $|x-3|<(0.1)/4=0.025$.

(c) $|f(x)-f(4)|=|x^2-16|<\epsilon$ if $|x-4|<\delta$. We get $f(x)=16+\epsilon=16.001$ at $x=4.000124998$, which corresponds to $\delta=0.000124998$; and $f(x)=16-\epsilon=15.999$ at $x=3.999874998$, for which $\delta=0.000125002$. Use the smaller δ: thus $|f(x)-16|<\epsilon$ provided $|x-4|<0.000125$ (to six decimals).

2. **(a)** $|f(x)-f(0)|=|2x+3-3|=2|x|<0.1$ if and only if $|x|<0.05$.

(b) $|f(x)-f(0)|=|2x+3-3|=2|x|<0.01$ if and only if $|x|<0.005$.

(c) $|f(x)-f(0)|=|2x+3-3|=2|x|<0.0012$ if and only if $|x|<0.0006$.

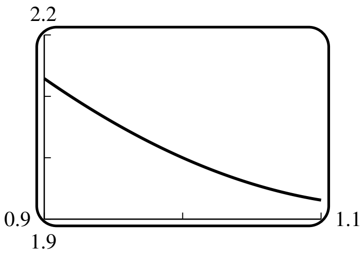
3. **(a)** $x_0=(1.95)^2=3.8025, x_1=(2.05)^2=4.2025$.

(b) $\delta=\min\left(|4-3.8025|,|4-4.2025|\right)=0.1975$.

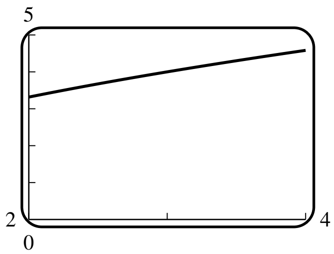
4. **(a)** $x_0=1/(1.1)=0.909090\ldots, x_1=1/(0.9)=1.111111\ldots$

(b) $\delta=\min(|1-0.909090|,|1-1.111111|)=0.0909090\ldots$

5. $|(x^3-4x+5)-2|<0.05$ is equivalent to $-0.05<(x^3-4x+5)-2<0.05$, which means $1.95<x^3-4x+5<2.05$. Now $x^3-4x+5=1.95$ at $x=1.0616$, and $x^3-4x+5=2.05$ at $x=0.9558$. So $\delta=\min\left(1.0616-1,1-0.9558\right)=0.0442$.



6. $\sqrt{5x+1} = 3.5$ at $x = 2.25$, $\sqrt{5x+1} = 4.5$ at $x = 3.85$, so $\delta = \min(3 - 2.25, 3.85 - 3) = 0.75$.



7. With the TRACE feature of a calculator we discover that (to five decimal places) $(0.87000, 1.80274)$ and $(1.13000, 2.19301)$ belong to the graph. Set $x_0 = 0.87$ and $x_1 = 1.13$. Since the graph of $f(x)$ rises from left to right, we see that if $x_0 < x < x_1$ then $1.80274 < f(x) < 2.19301$, and therefore $1.8 < f(x) < 2.2$. So we can take $\delta = 0.13$.

8. From a calculator plot we conjecture that $\lim_{x\to 0} f(x) = 2$. Using the TRACE feature we see that the points $(\pm 0.2, 1.94709)$ belong to the graph. Thus if $-0.2 < x < 0.2$, then $1.95 < f(x) \le 2$ and hence $|f(x) - L| < 0.05 < 0.1 = \epsilon$.

9. $|2x - 8| = 2|x - 4| < 0.1$ when $|x - 4| < 0.1/2 = 0.05 = \delta$.

10. $|(5x - 2) - 13| = 5|x - 3| < 0.01$ when $|x - 3| < 0.01/5 = 0.002 = \delta$.

11. If $x \neq 3$, then $\left|\dfrac{x^2-9}{x-3} - 6\right| = \left|\dfrac{x^2 - 9 - 6x + 18}{x-3}\right| = \left|\dfrac{x^2-6x+9}{x-3}\right| = |x-3| < 0.05$ when $|x - 3| < 0.05 = \delta$.

12. If $x \neq -1/2$, then $\left|\dfrac{4x^2-1}{2x+1} - (-2)\right| = \left|\dfrac{4x^2 - 1 + 4x + 2}{2x+1}\right| = \left|\dfrac{4x^2+4x+1}{2x+1}\right| = |2x+1| = 2|x-(-1/2)| < 0.05$ when $|x - (-1/2)| < 0.025 = \delta$.

13. Assume $\delta \le 1$. Then $-1 < x - 2 < 1$ means $1 < x < 3$ and then $|x^3 - 8| = |(x-2)(x^2+2x+4)| < 19|x-2|$, so we can choose $\delta = 0.001/19$.

14. Assume $\delta \le 1$. Then $-1 < x - 4 < 1$ means $3 < x < 5$ and then $|\sqrt{x} - 2| = \left|\dfrac{x-4}{\sqrt{x}+2}\right| < \dfrac{|x-4|}{\sqrt{3}+2}$, so we can choose $\delta = 0.001 \cdot (\sqrt{3} + 2)$.

15. Assume $\delta \le 1$. Then $-1 < x - 5 < 1$ means $4 < x < 6$ and then $\left|\dfrac{1}{x} - \dfrac{1}{5}\right| = \left|\dfrac{x-5}{5x}\right| < \dfrac{|x-5|}{20}$, so we can choose $\delta = 0.05 \cdot 20 = 1$.

16. $||x| - 0| = |x| < 0.05$ when $|x - 0| < 0.05 = \delta$.

17. Let $\epsilon > 0$ be given. Then $|f(x) - 3| = |3 - 3| = 0 < \epsilon$ regardless of x, and hence any $\delta > 0$ will work.

18. Let $\epsilon > 0$ be given. Then $|(x + 2) - 6| = |x - 4| < \epsilon$ provided $\delta = \epsilon$ (although any smaller δ would work).

19. $|3x - 15| = 3|x - 5| < \epsilon$ if $|x - 5| < \epsilon/3$, $\delta = \epsilon/3$.

20. $|7x + 5 + 2| = 7|x + 1| < \epsilon$ if $|x + 1| < \epsilon/7$, $\delta = \epsilon/7$.

21. $\left|\dfrac{2x^2 + x}{x} - 1\right| = |2x| < \epsilon$ if $|x| < \epsilon/2$, $\delta = \epsilon/2$.

22. $\left|\dfrac{x^2 - 9}{x + 3} - (-6)\right| = |x + 3| < \epsilon$ if $|x + 3| < \epsilon$, $\delta = \epsilon$.

23. $|f(x) - 3| = |x + 2 - 3| = |x - 1| < \epsilon$ if $0 < |x - 1| < \epsilon$, $\delta = \epsilon$.

24. $|9 - 2x - 5| = 2|x - 2| < \epsilon$ if $0 < |x - 2| < \epsilon/2$, $\delta = \epsilon/2$.

25. If $\epsilon > 0$ is given, then take $\delta = \epsilon$; if $|x - 0| = |x| < \delta$, then $|x - 0| = |x| < \epsilon$.

26. If $x < 2$ then $|f(x)-5| = |9-2x-5| = 2|x-2| < \epsilon$ if $|x-2| < \epsilon/2, \delta_1 = \epsilon/2$. If $x > 2$ then $|f(x)-5| = |3x-1-5| = 3|x - 2| < \epsilon$ if $|x - 2| < \epsilon/3, \delta_2 = \epsilon/3$ Now let $\delta = \min(\delta_1, \delta_2)$ then for any x with $|x - 2| < \delta, |f(x) - 5| < \epsilon$.

27. For the first part, let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \epsilon$. For the left limit replace $a < x < a + \delta$ with $a - \delta < x < a$.

28. **(a)** Given $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $||f(x) - L| - 0| < \epsilon$, or $|f(x) - L| < \epsilon$.

(b) From part (a) it follows that $|f(x) - L| < \epsilon$ is the defining condition for each of the two limits, so the two limit statements are equivalent.

29. **(a)** $|(3x^2 + 2x - 20 - 300| = |3x^2 + 2x - 320| = |(3x + 32)(x - 10)| = |3x + 32| \cdot |x - 10|$.

(b) If $|x - 10| < 1$ then $|3x + 32| < 65$, since clearly $x < 11$.

(c) $\delta = \min(1, \epsilon/65)$; $\quad |3x + 32| \cdot |x - 10| < 65 \cdot |x - 10| < 65 \cdot \epsilon/65 = \epsilon$.

30. **(a)** $\left|\dfrac{28}{3x + 1} - 4\right| = \left|\dfrac{28 - 12x - 4}{3x + 1}\right| = \left|\dfrac{-12x + 24}{3x + 1}\right| = \left|\dfrac{12}{3x + 1}\right| \cdot |x - 2|$.

(b) If $|x - 2| < 4$ then $-2 < x < 6$, so x can be very close to $-1/3$, hence $\left|\dfrac{12}{3x + 1}\right|$ is not bounded.

(c) If $|x - 2| < 1$ then $1 < x < 3$ and $3x + 1 > 4$, so $\left|\dfrac{12}{3x + 1}\right| < \dfrac{12}{4} = 3$.

(d) $\delta = \min(1, \epsilon/3)$; $\quad \left|\dfrac{12}{3x + 1}\right| \cdot |x - 2| < 3 \cdot |x - 2| < 3 \cdot \epsilon/3 = \epsilon$.

31. If $\delta < 1$ then $|2x^2 - 2| = 2|x - 1||x + 1| < 6|x - 1| < \epsilon$ if $|x - 1| < \epsilon/6$, so $\delta = \min(1, \epsilon/6)$.

32. If $\delta < 1$ then $|x^2 + x - 12| = |x + 4| \cdot |x - 3| < 5|x - 3| < \epsilon$ if $|x - 3| < \epsilon/5$, so $\delta = \min(1, \epsilon/5)$.

33. If $\delta < 1/2$ and $|x - (-2)| < \delta$ then $-5/2 < x < -3/2$, $x + 1 < -1/2$, $|x + 1| > 1/2$; then $\left|\dfrac{1}{x + 1} - (-1)\right| = \dfrac{|x + 2|}{|x + 1|} < 2|x + 2| < \epsilon$ if $|x + 2| < \epsilon/2$, so $\delta = \min(1/2, \epsilon/2)$.

34. If $\delta < 1/4$ and $|x-(1/2)| < \delta$ then $\left|\dfrac{2x + 3}{x} - 8\right| = \dfrac{|6x - 3|}{|x|} < \dfrac{6|x - (1/2)|}{1/4} = 24|x-(1/2)| < \epsilon$ if $|x-(1/2)| < \epsilon/24$, so $\delta = \min(1/4, \epsilon/24)$.

35. $|\sqrt{x}-2| = \left|(\sqrt{x}-2)\dfrac{\sqrt{x}+2}{\sqrt{x}+2}\right| = \left|\dfrac{x-4}{\sqrt{x}+2}\right| < \dfrac{1}{2}|x-4| < \epsilon$ if $|x-4| < 2\epsilon$, so $\delta = \min(2\epsilon, 4)$.

36. If $\delta < 1$ and $|x-2| < \delta$ then $|x| < 3$ and $x^2+2x+4 < 9+6+4 = 19$, so $|x^3-8| = |x-2|\cdot|x^2+2x+4| < 19\delta < \epsilon$ if $\delta = \min(\epsilon/19, 1)$.

37. Let $\epsilon > 0$ be given and take $\delta = \epsilon$. If $|x| < \delta$, then $|f(x)-0| = 0 < \epsilon$ if x is rational, and $|f(x)-0| = |x| < \delta = \epsilon$ if x is irrational.

38. If the limit did exist, then for $\epsilon = 1/2$ there would exist $\delta > 0$ such that if $|x| < \delta$ then $|f(x)-L| < 1/2$. Some of the x-values are rational, for which $|L| < 1/2$; some are irrational, for which $|1-L| < 1/2$. But $1 = |1| = L + (1-L) < 1/2 + 1/2$, or $1 < 1$, a contradiction. Hence the limit cannot exist.

39. **(a)** We have to solve the equation $1/N^2 = 0.1$ here, so $N = \sqrt{10}$.

(b) This will happen when $N/(N+1) = 0.99$, so $N = 99$.

(c) Because the function $1/x^3$ approaches 0 from below when $x \to -\infty$, we have to solve the equation $1/N^3 = -0.001$, and $N = -10$.

(d) The function $x/(x+1)$ approaches 1 from above when $x \to -\infty$, so we have to solve the equation $N/(N+1) = 1.01$. We obtain $N = -101$.

40. **(a)** $N = \sqrt[3]{10}$ **(b)** $N = \sqrt[3]{100}$ **(c)** $N = \sqrt[3]{1000} = 10$

41. **(a)** $\dfrac{x_1^2}{1+x_1^2} = 1-\epsilon$, $x_1 = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$; $\dfrac{x_2^2}{1+x_2^2} = 1-\epsilon$, $x_2 = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$

(b) $N = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$ **(c)** $N = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$

42. **(a)** $x_1 = -1/\epsilon^3$; $x_2 = 1/\epsilon^3$ **(b)** $N = 1/\epsilon^3$ **(c)** $N = -1/\epsilon^3$

43. $\dfrac{1}{x^2} < 0.01$ if $|x| > 10$, $N = 10$.

44. $\dfrac{1}{x+2} < 0.005$ if $|x+2| > 200$, $x > 198$, $N = 198$.

45. $\left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $x > 999$, $N = 999$.

46. $\left|\dfrac{4x-1}{2x+5} - 2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $2x > 105$, $N = 52.5$.

47. $\left|\dfrac{1}{x+2} - 0\right| < 0.005$ if $|x+2| > 200$, $-x-2 > 200$, $x < -202$, $N = -202$.

48. $\left|\dfrac{1}{x^2}\right| < 0.01$ if $|x| > 10$, $-x > 10$, $x < -10$, $N = -10$.

49. $\left|\dfrac{4x-1}{2x+5} - 2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $-2x-5 > 110$, $2x < -115$, $x < -57.5$, $N = -57.5$.

50. $\left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $-x-1 > 1000$, $x < -1001$, $N = -1001$.

51. $\left|\dfrac{1}{x^2}\right| < \epsilon$ if $|x| > \dfrac{1}{\sqrt{\epsilon}}$, so $N = \dfrac{1}{\sqrt{\epsilon}}$.

52. $\left|\dfrac{1}{x+2}\right| < \epsilon$ if $|x+2| > \dfrac{1}{\epsilon}$, i.e. when $x+2 > \dfrac{1}{\epsilon}$, or $x > \dfrac{1}{\epsilon} - 2$, so $N = \dfrac{1}{\epsilon} - 2$.

53. $\left|\dfrac{4x-1}{2x+5} - 2\right| = \left|\dfrac{11}{2x+5}\right| < \epsilon$ if $|2x+5| > \dfrac{11}{\epsilon}$, i.e. when $-2x-5 > \dfrac{11}{\epsilon}$, which means $2x < -\dfrac{11}{\epsilon} - 5$, or $x < -\dfrac{11}{2\epsilon} - \dfrac{5}{2}$, so $N = -\dfrac{5}{2} - \dfrac{11}{2\epsilon}$.

54. $\left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < \epsilon$ if $|x+1| > \dfrac{1}{\epsilon}$, i.e. when $-x-1 > \dfrac{1}{\epsilon}$, or $x < -1 - \dfrac{1}{\epsilon}$, so $N = -1 - \dfrac{1}{\epsilon}$.

55. $\left|\dfrac{2\sqrt{x}}{\sqrt{x}-1} - 2\right| = \left|\dfrac{2}{\sqrt{x}-1}\right| < \epsilon$ if $\sqrt{x} - 1 > \dfrac{2}{\epsilon}$, i.e. when $\sqrt{x} > 1 + \dfrac{2}{\epsilon}$, or $x > \left(1 + \dfrac{2}{\epsilon}\right)^2$, so $N = \left(1 + \dfrac{2}{\epsilon}\right)^2$.

56. $2^x < \epsilon$ if $x < \log_2 \epsilon$, so $N = \log_2 \epsilon$.

57. **(a)** $\dfrac{1}{x^2} > 100$ if $|x| < \dfrac{1}{10}$ **(b)** $\dfrac{1}{|x-1|} > 1000$ if $|x-1| < \dfrac{1}{1000}$

(c) $\dfrac{-1}{(x-3)^2} < -1000$ if $|x-3| < \dfrac{1}{10\sqrt{10}}$ **(d)** $-\dfrac{1}{x^4} < -10000$ if $x^4 < \dfrac{1}{10000}$, $|x| < \dfrac{1}{10}$

58. **(a)** $\dfrac{1}{(x-1)^2} > 10$ if and only if $|x-1| < \dfrac{1}{\sqrt{10}}$

(b) $\dfrac{1}{(x-1)^2} > 1000$ if and only if $|x-1| < \dfrac{1}{10\sqrt{10}}$

(c) $\dfrac{1}{(x-1)^2} > 100000$ if and only if $|x-1| < \dfrac{1}{100\sqrt{10}}$

59. If $M > 0$ then $\dfrac{1}{(x-3)^2} > M$ when $0 < (x-3)^2 < \dfrac{1}{M}$, or $0 < |x-3| < \dfrac{1}{\sqrt{M}}$, so $\delta = \dfrac{1}{\sqrt{M}}$.

60. If $M < 0$ then $\dfrac{-1}{(x-3)^2} < M$ when $0 < (x-3)^2 < -\dfrac{1}{M}$, or $0 < |x-3| < \dfrac{1}{\sqrt{-M}}$, so $\delta = \dfrac{1}{\sqrt{-M}}$.

61. If $M > 0$ then $\dfrac{1}{|x|} > M$ when $0 < |x| < \dfrac{1}{M}$, so $\delta = \dfrac{1}{M}$.

62. If $M > 0$ then $\dfrac{1}{|x-1|} > M$ when $0 < |x-1| < \dfrac{1}{M}$, so $\delta = \dfrac{1}{M}$.

63. If $M < 0$ then $-\dfrac{1}{x^4} < M$ when $0 < x^4 < -\dfrac{1}{M}$, or $|x| < \dfrac{1}{(-M)^{1/4}}$, so $\delta = \dfrac{1}{(-M)^{1/4}}$.

64. If $M > 0$ then $\dfrac{1}{x^4} > M$ when $0 < x^4 < \dfrac{1}{M}$, or $x < \dfrac{1}{M^{1/4}}$, so $\delta = \dfrac{1}{M^{1/4}}$.

65. If $x > 2$ then $|x+1-3| = |x-2| = x-2 < \epsilon$ if $2 < x < 2+\epsilon$, so $\delta = \epsilon$.

66. If $x < 1$ then $|3x+2-5| = |3x-3| = 3|x-1| = 3(1-x) < \epsilon$ if $1-x < \epsilon/3$, or $1-\epsilon/3 < x < 1$, so $\delta = \epsilon/3$.

67. If $x > 4$ then $\sqrt{x-4} < \epsilon$ if $x - 4 < \epsilon^2$, or $4 < x < 4 + \epsilon^2$, so $\delta = \epsilon^2$.

68. If $x < 0$ then $\sqrt{-x} < \epsilon$ if $-x < \epsilon^2$, or $-\epsilon^2 < x < 0$, so $\delta = \epsilon^2$.

69. If $x > 2$ then $|f(x) - 2| = |x - 2| = x - 2 < \epsilon$ if $2 < x < 2 + \epsilon$, so $\delta = \epsilon$.

70. If $x < 2$ then $|f(x) - 6| = |3x - 6| = 3|x - 2| = 3(2 - x) < \epsilon$ if $2 - x < \epsilon/3$, or $2 - \epsilon/3 < x < 2$, so $\delta = \epsilon/3$.

71. (a) Definition: For every $M < 0$ there corresponds a $\delta > 0$ such that if $1 < x < 1 + \delta$ then $f(x) < M$. In our case we want $\dfrac{1}{1-x} < M$, i.e. $1 - x > \dfrac{1}{M}$, or $x < 1 - \dfrac{1}{M}$, so we can choose $\delta = -\dfrac{1}{M}$.

(b) Definition: For every $M > 0$ there corresponds a $\delta > 0$ such that if $1 - \delta < x < 1$ then $f(x) > M$. In our case we want $\dfrac{1}{1-x} > M$, i.e. $1 - x < \dfrac{1}{M}$, or $x > 1 - \dfrac{1}{M}$, so we can choose $\delta = \dfrac{1}{M}$.

72. (a) Definition: For every $M > 0$ there corresponds a $\delta > 0$ such that if $0 < x < \delta$ then $f(x) > M$. In our case we want $\dfrac{1}{x} > M$, i.e. $x < \dfrac{1}{M}$, so take $\delta = \dfrac{1}{M}$.

(b) Definition: For every $M < 0$ there corresponds a $\delta > 0$ such that if $-\delta < x < 0$ then $f(x) < M$. In our case we want $\dfrac{1}{x} < M$, i.e $x > \dfrac{1}{M}$, so take $\delta = -\dfrac{1}{M}$.

73. (a) Given any $M > 0$, there corresponds an $N > 0$ such that if $x > N$ then $f(x) > M$, i.e. $x + 1 > M$, or $x > M - 1$, so $N = M - 1$.

(b) Given any $M < 0$, there corresponds an $N < 0$ such that if $x < N$ then $f(x) < M$, i.e. $x + 1 < M$, or $x < M - 1$, so $N = M - 1$.

74. (a) Given any $M > 0$, there corresponds an $N > 0$ such that if $x > N$ then $f(x) > M$, i.e. $x^2 - 3 > M$, or $x > \sqrt{M+3}$, so $N = \sqrt{M+3}$.

(b) Given any $M < 0$, there corresponds an $N < 0$ such that if $x < N$ then $f(x) < M$, i.e. $x^3 + 5 < M$, or $x < (M - 5)^{1/3}$, so $N = (M - 5)^{1/3}$.

75. (a) $\dfrac{3.0}{7.5} = 0.4$ (amperes) **(b)** $[0.3947, 0.4054]$ **(c)** $\left[\dfrac{3}{7.5+\delta}, \dfrac{3}{7.5-\delta}\right]$ **(d)** 0.0187

(e) It approaches infinity.

Exercise Set 1.5

1. (a) No: $\lim_{x\to 2} f(x)$ does not exist. **(b)** No: $\lim_{x\to 2} f(x)$ does not exist. **(c)** No: $\lim_{x\to 2^-} f(x) \neq f(2)$.

(d) Yes. **(e)** Yes. **(f)** Yes.

2. (a) No: $\lim_{x\to 2} f(x) \neq f(2)$. **(b)** No: $\lim_{x\to 2} f(x) \neq f(2)$. **(c)** No: $\lim_{x\to 2^-} f(x) \neq f(2)$.

(d) Yes. **(e)** No: $\lim_{x\to 2^+} f(x) \neq f(2)$. **(f)** Yes.

3. (a) No: $f(1)$ and $f(3)$ are not defined. **(b)** Yes. **(c)** No: $f(1)$ is not defined.

(d) Yes. **(e)** No: $f(3)$ is not defined. **(f)** Yes.

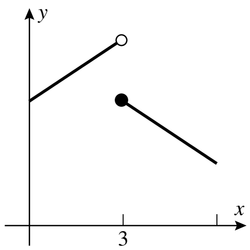
4. **(a)** No: $f(3)$ is not defined. **(b)** Yes. **(c)** Yes.

(d) Yes. **(e)** No: $f(3)$ is not defined. **(f)** Yes.

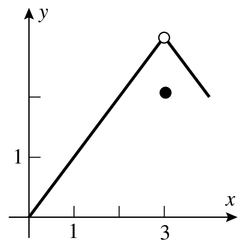
5. **(a)** No. **(b)** No. **(c)** No. **(d)** Yes. **(e)** Yes. **(f)** No. **(g)** Yes.

6. **(a)** No. **(b)** No. **(c)** No. **(d)** No. **(e)** Yes. **(f)** Yes. **(g)** Yes.

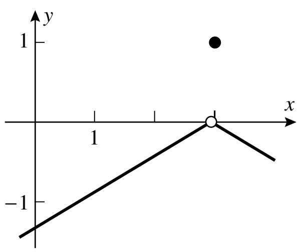
7. **(a)**



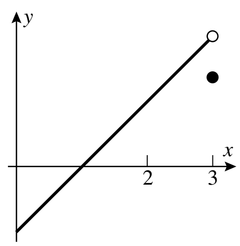
(b)



(c)

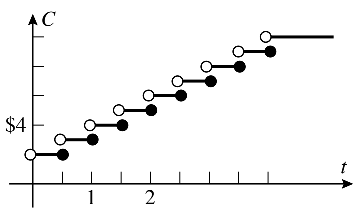


(d)



8. The discontinuities probably correspond to the times when the patient takes the medication. We see a jump in the concentration values here, which are followed by continuously decreasing concentration values as the medication is being absorbed.

9. **(a)**



(b) One second could cost you one dollar.

10. **(a)** Not continuous, since the values are integers.

(b) Continuous.

(c) Not continuous, again, the values are integers (if we measure them in cents).

(d) Continuous.

11. None, this is a continuous function on the real numbers.

12. None, this is a continuous function on the real numbers.

13. None, this is a continuous function on the real numbers.

14. The function is not continuous at $x = 2$ and $x = -2$.

15. The function is not continuous at $x = -1/2$ and $x = 0$.

16. None, this is a continuous function on the real numbers.

17. The function is not continuous at $x = 0$, $x = 1$ and $x = -1$.

18. The function is not continuous at $x = 0$ and $x = -4$.

19. None, this is a continuous function on the real numbers.

20. The function is not continuous at $x = 0$ and $x = -1$.

21. None, this is a continuous function on the real numbers. $f(x) = 2x + 3$ is continuous on $x < 4$ and $f(x) = 7 + \dfrac{16}{x}$ is continuous on $4 < x$; $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4) = 11$ so f is continuous at $x = 4$.

22. The function is not continuous at $x = 1$, as $\lim_{x\to 1} f(x)$ does not exist.

23. True; by Theorem 1.5.5.

24. False; e.g. $f(x) = 1$ if $x \neq 3$, $f(3) = -1$.

25. False; e.g. $f(x) = g(x) = 2$ if $x \neq 3$, $f(3) = 1$, $g(3) = 3$.

26. False; e.g. $f(x) = g(x) = 2$ if $x \neq 3$, $f(3) = 1$, $g(3) = 4$.

27. True; use Theorem 1.5.3 with $g(x) = \sqrt{f(x)}$.

28. Generally, this statement is false because $\sqrt{f(x)}$ might not even be defined. If we suppose that $f(c)$ is nonnegative, and $f(x)$ is also nonnegative on some interval $(c - \alpha, c + \alpha)$, then the statement is true. If $f(c) = 0$ then given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|x - c| < \delta, 0 \leq f(x) < \epsilon^2$. Then $|\sqrt{f(x)}| < \epsilon$ and $\sqrt{f}$ is continuous at $x = c$. If $f(c) \neq 0$ then given $\epsilon > 0$ there corresponds $\delta > 0$ such that whenever $|x-c| < \delta, |f(x)-f(c)| < \epsilon\sqrt{f(c)}$. Then $|\sqrt{f(x)} - \sqrt{f(c)}| = \dfrac{|f(x) - f(c)|}{|\sqrt{f(x)} + \sqrt{f(c)}|} \leq \dfrac{|f(x) - f(c)|}{\sqrt{f(c)}} < \epsilon$.

29. **(a)** f is continuous for $x < 1$, and for $x > 1$; $\lim_{x\to 1^-} f(x) = 5$, $\lim_{x\to 1^+} f(x) = k$, so if $k = 5$ then f is continuous for all x.

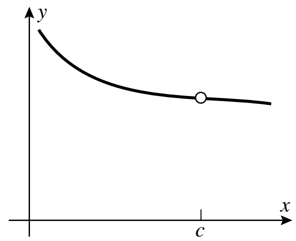
(b) f is continuous for $x < 2$, and for $x > 2$; $\lim_{x\to 2^-} f(x) = 4k$, $\lim_{x\to 2^+} f(x) = 4 + k$, so if $4k = 4 + k$, $k = 4/3$ then f is continuous for all x.

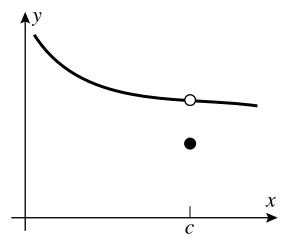
30. **(a)** f is continuous for $x < 3$, and for $x > 3$; $\lim_{x\to 3^-} f(x) = k/9$, $\lim_{x\to 3^+} f(x) = 0$, so if $k = 0$ then f is continuous for all x.

(b) f is continuous for $x < 0$, and for $x > 0$; $\lim_{x\to 0^-} f(x)$ doesn't exist unless $k = 0$, and if so then $\lim_{x\to 0^-} f(x) = 0$; $\lim_{x\to 0^+} f(x) = 9$, so there is no k value which makes the function continuous everywhere.

31. f is continuous for $x < -1$, $-1 < x < 2$ and $x > 2$; $\lim_{x\to -1^-} f(x) = 4$, $\lim_{x\to -1^+} f(x) = k$, so $k = 4$ is required. Next, $\lim_{x\to 2^-} f(x) = 3m + k = 3m + 4$, $\lim_{x\to 2^+} f(x) = 9$, so $3m + 4 = 9, m = 5/3$ and f is continuous everywhere if $k = 4$ and $m = 5/3$.

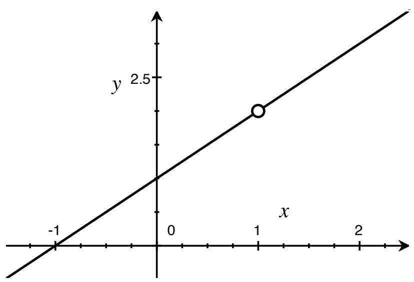
32. **(a)** No, f is not defined at $x = 2$. **(b)** No, f is not defined for $x \leq 2$. **(c)** Yes. **(d)** No, see (b).

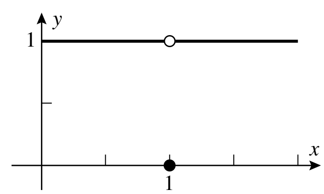
33. (a) 

(b) 

34. (a) $f(c) = \lim_{x \to c} f(x)$

(b) $\lim_{x \to 1} f(x) = 2$, $\lim_{x \to 1} g(x) = 1$.





(c) Define $f(1) = 2$ and redefine $g(1) = 1$.

35. (a) $x = 0$, $\lim_{x \to 0^-} f(x) = -1 \neq +1 = \lim_{x \to 0^+} f(x)$ so the discontinuity is not removable.

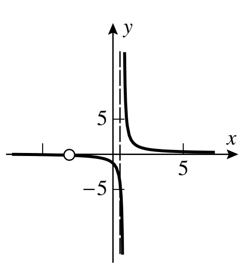
(b) $x = -3$; define $f(-3) = -3 = \lim_{x \to -3} f(x)$, then the discontinuity is removable.

(c) f is undefined at $x = \pm 2$; at $x = 2$, $\lim_{x \to 2} f(x) = 1$, so define $f(2) = 1$ and f becomes continuous there; at $x = -2$, $\lim_{x \to -2} f(x)$ does not exist, so the discontinuity is not removable.

36. (a) f is not defined at $x = 2$; $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x+2}{x^2 + 2x + 4} = \dfrac{1}{3}$, so define $f(2) = \dfrac{1}{3}$ and f becomes continuous there.

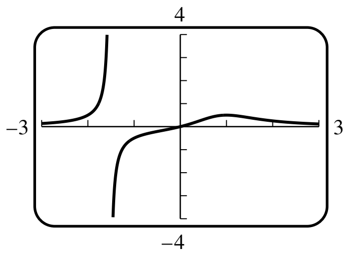
(b) $\lim_{x \to 2^-} f(x) = 1 \neq 4 = \lim_{x \to 2^+} f(x)$, so f has a nonremovable discontinuity at $x = 2$.

(c) $\lim_{x \to 1} f(x) = 8 \neq f(1)$, so f has a removable discontinuity at $x = 1$.

37. (a) 

Discontinuity at $x = 1/2$, not removable; at $x = -3$, removable.

(b) $2x^2 + 5x - 3 = (2x - 1)(x + 3)$

38. **(a)**  There appears to be one discontinuity near $x = -1.52$.

(b) One discontinuity at $x \approx -1.52$.

39. Write $f(x) = x^{3/5} = (x^3)^{1/5}$ as the composition (Theorem 1.5.6) of the two continuous functions $g(x) = x^3$ and $h(x) = x^{1/5}$; it is thus continuous.

40. $x^4 + 7x^2 + 1 \geq 1 > 0$, thus $f(x)$ is the composition of the polynomial $x^4 + 7x^2 + 1$, the square root $\sqrt{x}$, and the function $1/x$ and is therefore continuous by Theorem 1.5.6.

41. Since f and g are continuous at $x = c$ we know that $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. In the following we use Theorem 1.2.2.

(a) $f(c) + g(c) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \lim_{x\to c}(f(x) + g(x))$ so $f + g$ is continuous at $x = c$.

(b) Same as (a) except the $+$ sign becomes a $-$ sign.

(c) $f(c)g(c) = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = \lim_{x\to c} f(x)g(x)$ so fg is continuous at $x = c$.

42. A rational function is the quotient $f(x)/g(x)$ of two polynomials $f(x)$ and $g(x)$. By Theorem 1.5.2 f and g are continuous everywhere; by Theorem 1.5.3 f/g is continuous except when $g(x) = 0$.

43. **(a)** Let $h = x - c, x = h + c$. Then by Theorem 1.5.5, $\lim_{h\to 0} f(h + c) = f(\lim_{h\to 0}(h + c)) = f(c)$.

(b) With $g(h) = f(c+h)$, $\lim_{h\to 0} g(h) = \lim_{h\to 0} f(c+h) = f(c) = g(0)$, so $g(h)$ is continuous at $h = 0$. That is, $f(c+h)$ is continuous at $h = 0$, so f is continuous at $x = c$.

44. The function $h(x) = f(x) - g(x)$ is continuous on the interval $[a, b]$, and satisfies $h(a) > 0$, $h(b) < 0$. The Intermediate Value Theorem or Theorem 1.5.8 tells us that there is at least one solution of the equation on this interval $h(x) = 0$, i.e. $f(x) = g(x)$.

45. Of course such a function must be discontinuous. Let $f(x) = 1$ on $0 \leq x < 1$, and $f(x) = -1$ on $1 \leq x \leq 2$.

46. **(a)** (i) No. (ii) Yes. **(b)** (i) No. (ii) No. **(c)** (i) No. (ii) No.

47. If $f(x) = x^3 + x^2 - 2x - 1$, then $f(-1) = 1$, $f(1) = -1$. The Intermediate Value Theorem gives us the result.

48. Since $\lim_{x\to -\infty} p(x) = -\infty$ and $\lim_{x\to +\infty} p(x) = +\infty$ (or vice versa, if the leading coefficient of p is negative), it follows that for $M = -1$ there corresponds $N_1 < 0$, and for $M = 1$ there is $N_2 > 0$, such that $p(x) < -1$ for $x < N_1$ and $p(x) > 1$ for $x > N_2$. We choose $x_1 < N_1$ and $x_2 > N_2$ and use Theorem 1.5.8 on the interval $[x_1, x_2]$ to show the existence of a solution of $p(x) = 0$.

49. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.3) < 0$ and $f(-1.2) > 0$, the midpoint $x = -1.25$ of $[-1.3, -1.2]$ is the required approximation of the root. For the positive root use the interval $[0, 1]$; since $f(0.7) < 0$ and $f(0.8) > 0$, the midpoint $x = 0.75$ of $[0.7, 0.8]$ is the required approximation.

50. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.7) < 0$ and $f(-1.6) > 0$, use the interval $[-1.7, -1.6]$. Since $f(-1.61) < 0$ and $f(-1.60) > 0$ the midpoint $x = -1.605$ of $[-1.61, -1.60]$ is the required approximation of the root. For the positive root use the interval $[1,2]$; since $f(1.3) > 0$ and $f(1.4) < 0$, use the interval $[1.3, 1.4]$. Since $f(1.37) > 0$ and $f(1.38) < 0$, the midpoint $x = 1.375$ of $[1.37, 1.38]$ is the required approximation.

51. For the positive root, use intervals on the x-axis as follows: $[2,3]$; since $f(2.2) < 0$ and $f(2.3) > 0$, use the interval $[2.2, 2.3]$. Since $f(2.23) < 0$ and $f(2.24) > 0$ the midpoint $x = 2.235$ of $[2.23, 2.24]$ is the required approximation of the root.

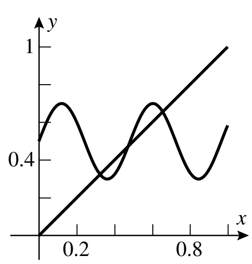
52. Assume the locations along the track are numbered with increasing $x \geq 0$. Let $T_S(x)$ denote the time during the sprint when the runner is located at point $x, 0 \leq x \leq 100$. Let $T_J(x)$ denote the time when the runner is at the point x on the return jog, measured so that $T_J(100) = 0$. Then $T_S(0) = 0, T_S(100) > 0, T_J(100) = 0, T_J(0) > 0$, so that Exercise 44 applies and there exists an x_0 such that $T_S(x_0) = T_J(x_0)$.

53. Consider the function $f(\theta) = T(\theta + \pi) - T(\theta)$. Note that T has period 2π, $T(\theta + 2\pi) = T(\theta)$, so that $f(\theta + \pi) = T(\theta + 2\pi) - T(\theta + \pi) = -(T(\theta + \pi) - T(\theta)) = -f(\theta)$. Now if $f(\theta) \equiv 0$, then the statement follows. Otherwise, there exists θ such that $f(\theta) \neq 0$ and then $f(\theta + \pi)$ has an opposite sign, and thus there is a t_0 between θ and $\theta + \pi$ such that $f(t_0) = 0$ and the statement follows.

54. Let the ellipse be contained between the horizontal lines $y = a$ and $y = b$, where $a < b$. The expression $|f(z_1) - f(z_2)|$ expresses the area of the ellipse that lies between the vertical lines $x = z_1$ and $x = z_2$, and thus $|f(z_1) - f(z_2)| \leq (b-a)|z_1 - z_2|$. Thus for a given $\epsilon > 0$ there corresponds $\delta = \epsilon/(b-a)$, such that if $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| \leq (b-a)|z_1 - z_2| < (b-a)\delta = \epsilon$ which proves that f is a continuous function.

55. Since R and L are arbitrary, we can introduce coordinates so that L is the x-axis. Let $f(z)$ be as in Exercise 54. Then for large z, $f(z) =$ area of ellipse, and for small z, $f(z) = 0$. By the Intermediate Value Theorem there is a z_1 such that $f(z_1) =$ half of the area of the ellipse.

56. (a)



(b) Let $g(x) = x - f(x)$. Then $g(x)$ is continuous, $g(1) \geq 0$ and $g(0) \leq 0$; by the Intermediate Value Theorem there is a solution c in $[0,1]$ of $g(c) = 0$, which means $f(c) = c$.

Exercise Set 1.6

1. This is a composition of continuous functions, so it is continuous everywhere.

2. Discontinuity at $x = \pi$.

3. Discontinuities at $x = n\pi, n = 0, \pm 1, \pm 2, \ldots$

4. Discontinuities at $x = \dfrac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

5. Discontinuities at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

6. Continuous everywhere.

7. Discontinuities at $x = \dfrac{\pi}{6} + 2n\pi$, and $x = \dfrac{5\pi}{6} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

8. Discontinuities at $x = \dfrac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

9. $\sin^{-1} u$ is continuous for $-1 \le u \le 1$, so $-1 \le 2x \le 1$, or $-1/2 \le x \le 1/2$.

10. $\cos^{-1} u$ is defined and continuous for $-1 \le u \le 1$ which means $-1 \le \ln x \le 1$, or $1/e \le x \le e$.

11. $(0,3) \cup (3,\infty)$.

12. $(-\infty,0) \cup (0,+\infty)$.

13. $(-\infty,-1] \cup [1,\infty)$.

14. $(-3,0) \cup (0,\infty)$.

15. **(a)** $f(x) = \sin x$, $g(x) = x^3 + 7x + 1$. **(b)** $f(x) = |x|$, $g(x) = \sin x$. **(c)** $f(x) = x^3$, $g(x) = \cos(x+1)$.

16. **(a)** $f(x) = |x|$, $g(x) = 3 + \sin 2x$. **(b)** $f(x) = \sin x$, $g(x) = \sin x$. **(c)** $f(x) = x^5 - 2x^3 + 1$, $g(x) = \cos x$.

17. $\displaystyle \lim_{x\to+\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x\to+\infty} \frac{1}{x}\right) = \cos 0 = 1.$

18. $\displaystyle \lim_{x\to+\infty} \sin\left(\frac{\pi x}{2-3x}\right) = \sin\left(\lim_{x\to+\infty} \frac{\pi x}{2-3x}\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$

19. $\displaystyle \lim_{x\to+\infty} \sin^{-1}\left(\frac{x}{1-2x}\right) = \sin^{-1}\left(\lim_{x\to+\infty} \frac{x}{1-2x}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$

20. $\displaystyle \lim_{x\to+\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left(\lim_{x\to+\infty} \frac{x+1}{x}\right) = \ln(1) = 0.$

21. $\displaystyle \lim_{x\to 0} e^{\sin x} = e^{\left(\lim\limits_{x\to 0} \sin x\right)} = e^0 = 1.$

22. $\displaystyle \lim_{x\to+\infty} \cos(2\tan^{-1} x) = \cos(\lim_{x\to+\infty} 2\tan^{-1} x) = \cos(2(\pi/2)) = -1.$

23. $\displaystyle \lim_{\theta\to 0} \frac{\sin 3\theta}{\theta} = 3 \lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta} = 3.$

24. $\displaystyle \lim_{h\to 0} \frac{\sin h}{2h} = \frac{1}{2} \lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{2}.$

25. $\displaystyle \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta^2} = \left(\lim_{\theta\to 0^+} \frac{1}{\theta}\right) \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = +\infty.$

26. $\displaystyle \lim_{\theta\to 0^+} \frac{\sin^2\theta}{\theta} = \left(\lim_{\theta\to 0} \sin\theta\right) \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0.$

27. $\displaystyle \frac{\tan 7x}{\sin 3x} = \frac{7}{3\cos 7x} \cdot \frac{\sin 7x}{7x} \cdot \frac{3x}{\sin 3x}$, so $\displaystyle \lim_{x\to 0} \frac{\tan 7x}{\sin 3x} = \frac{7}{3\cdot 1} \cdot 1 \cdot 1 = \frac{7}{3}.$

28. $\displaystyle \frac{\sin 6x}{\sin 8x} = \frac{6}{8} \cdot \frac{\sin 6x}{6x} \cdot \frac{8x}{\sin 8x}$, so $\displaystyle \lim_{x\to 0} \frac{\sin 6x}{\sin 8x} = \frac{6}{8} \cdot 1 \cdot 1 = \frac{3}{4}.$

29. $\lim_{x\to 0^+} \frac{\sin x}{5\sqrt{x}} = \frac{1}{5} \lim_{x\to 0^+} \sqrt{x} \lim_{x\to 0^+} \frac{\sin x}{x} = 0.$

30. $\lim_{x\to 0} \frac{\sin^2 x}{3x^2} = \frac{1}{3}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = \frac{1}{3}.$

31. $\lim_{x\to 0} \frac{\sin x^2}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{\sin x^2}{x^2}\right) = 0.$

32. $\frac{\sin h}{1-\cos h} = \frac{\sin h}{1-\cos h}\cdot\frac{1+\cos h}{1+\cos h} = \frac{\sin h(1+\cos h)}{1-\cos^2 h} = \frac{1+\cos h}{\sin h}$; this implies that $\lim_{h\to 0^+}$ is $+\infty$, and $\lim_{h\to 0^-}$ is $-\infty$, therefore the limit does not exist.

33. $\frac{t^2}{1-\cos^2 t} = \left(\frac{t}{\sin t}\right)^2$, so $\lim_{t\to 0} \frac{t^2}{1-\cos^2 t} = 1.$

34. $\cos(\frac{1}{2}\pi - x) = \cos(\frac{1}{2}\pi)\cos x + \sin(\frac{1}{2}\pi)\sin x = \sin x$, so $\lim_{x\to 0} \frac{x}{\cos\left(\frac{1}{2}\pi - x\right)} = 1.$

35. $\frac{\theta^2}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{\theta^2(1+\cos\theta)}{1-\cos^2\theta} = \left(\frac{\theta}{\sin\theta}\right)^2(1+\cos\theta)$, so $\lim_{\theta\to 0} \frac{\theta^2}{1-\cos\theta} = (1)^2\cdot 2 = 2.$

36. $\frac{1-\cos 3h}{\cos^2 5h - 1}\cdot\frac{1+\cos 3h}{1+\cos 3h} = \frac{\sin^2 3h}{-\sin^2 5h}\cdot\frac{1}{1+\cos 3h}$, so (using the result of problem 28)

$$\lim_{x\to 0} \frac{1-\cos 3h}{\cos^2 5h - 1} = \lim_{x\to 0} \frac{\sin^2 3h}{-\sin^2 5h}\cdot\frac{1}{1+\cos 3h} = -\left(\frac{3}{5}\right)^2\cdot\frac{1}{2} = -\frac{9}{50}$$

37. $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right) = \lim_{t\to+\infty} \sin t$, so the limit does not exist.

38. $\lim_{x\to 0} \frac{x^2 - 3\sin x}{x} = \lim_{x\to 0} x - 3\lim_{x\to 0} \frac{\sin x}{x} = -3.$

39. $\frac{2-\cos 3x - \cos 4x}{x} = \frac{1-\cos 3x}{x} + \frac{1-\cos 4x}{x}$. Note that $\frac{1-\cos 3x}{x} = \frac{1-\cos 3x}{x}\cdot\frac{1+\cos 3x}{1+\cos 3x} = \frac{\sin^2 3x}{x(1+\cos 3x)} = \frac{\sin 3x}{x}\cdot\frac{\sin 3x}{1+\cos 3x}$. Thus

$$\lim_{x\to 0} \frac{2-\cos 3x - \cos 4x}{x} = \lim_{x\to 0} \frac{\sin 3x}{x}\cdot\frac{\sin 3x}{1+\cos 3x} + \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\frac{\sin 4x}{1+\cos 4x} = 3\cdot 0 + 4\cdot 0 = 0.$$

40. $\frac{\tan 3x^2 + \sin^2 5x}{x^2} = \frac{3}{\cos 3x^2}\cdot\frac{\sin 3x^2}{3x^2} + 25\cdot\frac{\sin^2 5x}{(5x)^2}$, so

$$\lim_{x\to 0} \frac{\tan 3x^2 + \sin^2 5x}{x^2} = \lim_{x\to 0} \frac{3}{\cos 3x^2} \lim_{x\to 0} \frac{\sin 3x^2}{3x^2} + 25\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)^2 = 3 + 25 = 28.$$

41. (a)

4	4.5	4.9	5.1	5.5	6
0.093497	0.100932	0.100842	0.098845	0.091319	0.076497

The limit appears to be 0.1.

(b) Let $t = x - 5$. Then $t \to 0$ as $x \to 5$ and $\lim_{x\to 5} \frac{\sin(x-5)}{x^2 - 25} = \lim_{x\to 5} \frac{1}{x+5} \lim_{t\to 0} \frac{\sin t}{t} = \frac{1}{10}\cdot 1 = \frac{1}{10}.$

42. **(a)**

-2.1	-2.01	-2.001	-1.999	-1.99	-1.9
-1.09778	-1.00998	-1.00100	-0.99900	-0.98998	-0.89879

The limit appears to be -1.

(b) Let $t = (x+2)(x+1)$. Then $t \to 0$ as $x \to -2$, and $\lim_{x\to -2} \frac{\sin[(x+2)(x+1)]}{x+2} = \lim_{x\to -2}(x+1) \lim_{t\to 0} \frac{\sin t}{t} = -1 \cdot 1 = -1$ by the Substitution Principle (Exercise 1.3.53).

43. True: let $\epsilon > 0$ and $\delta = \epsilon$. Then if $|x-(-1)| = |x+1| < \delta$ then $|f(x)+5| < \epsilon$.

44. True; from the proof of Theorem 1.6.5 we have $\tan x \ge x \ge \sin x$ for $0 < x < \pi/2$, and the desired inequalities follow immediately.

45. False; consider $f(x) = \tan^{-1} x$.

46. True; by the Squeezing Theorem 1.6.4 $|\lim_{x\to 0} xf(x)| \le M \lim_{x\to 0} |x| = 0$ and $\left|\lim_{x\to +\infty} \frac{f(x)}{x}\right| \le M \lim_{x\to +\infty} \frac{1}{x} = 0$.

47. **(a)** The student calculated x in degrees rather than radians.

(b) $\sin x^\circ = \sin t$ where x° is measured in degrees, t is measured in radians and $t = \frac{\pi x^\circ}{180}$. Thus $\lim_{x^\circ \to 0} \frac{\sin x^\circ}{x^\circ} = \lim_{t\to 0} \frac{\sin t}{(180t/\pi)} = \frac{\pi}{180}$.

48. Denote θ by x in accordance with Figure 1.6.4. Let P have coordinates $(\cos x, \sin x)$ and Q coordinates $(1,0)$ so that $c^2(x) = (1-\cos x)^2 + \sin^2 x = 2(1-\cos x)$. Since $s = r\theta = 1 \cdot x = x$ we have $\lim_{x\to 0^+} \frac{c^2(x)}{s^2(x)} = \lim_{x\to 0^+} 2\frac{1-\cos x}{x^2} = \lim_{x\to 0^+} 2\frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^2 \frac{2}{1+\cos x} = 1$.

49. $\lim_{x\to 0^-} f(x) = k \lim_{x\to 0} \frac{\sin kx}{kx \cos kx} = k$, $\lim_{x\to 0^+} f(x) = 2k^2$, so $k = 2k^2$, and the nonzero solution is $k = \frac{1}{2}$.

50. No; $\sin x/|x|$ has unequal one-sided limits ($+1$ and -1).

51. **(a)** $\lim_{t\to 0^+} \frac{\sin t}{t} = 1$.

(b) $\lim_{t\to 0^-} \frac{1-\cos t}{t} = 0$ (Theorem 1.6.3).

(c) $\sin(\pi - t) = \sin t$, so $\lim_{x\to\pi} \frac{\pi - x}{\sin x} = \lim_{t\to 0} \frac{t}{\sin t} = 1$.

52. Let $t = \frac{\pi}{2} - \frac{\pi}{x}$. Then $\cos\left(\frac{\pi}{2} - t\right) = \sin t$, so $\lim_{x\to 2} \frac{\cos(\pi/x)}{x-2} = \lim_{t\to 0} \frac{(\pi - 2t)\sin t}{4t} = \lim_{t\to 0} \frac{\pi - 2t}{4} \lim_{t\to 0} \frac{\sin t}{t} = \frac{\pi}{4}$.

53. $t = x-1$; $\sin(\pi x) = \sin(\pi t + \pi) = -\sin \pi t$; and $\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} = -\lim_{t\to 0} \frac{\sin \pi t}{t} = -\pi$.

54. $t = x - \pi/4$; $\tan x - 1 = \frac{2\sin t}{\cos t - \sin t}$; $\lim_{x\to\pi/4} \frac{\tan x - 1}{x - \pi/4} = \lim_{t\to 0} \frac{2\sin t}{t(\cos t - \sin t)} = 2$.

55. $t = x - \pi/4$, $\cos(t + \pi/4) = (\sqrt{2}/2)(\cos t - \sin t)$, $\sin(t + \pi/4) = (\sqrt{2}/2)(\sin t + \cos t)$, so $\dfrac{\cos x - \sin x}{x - \pi/4} = -\dfrac{\sqrt{2}\sin t}{t}$; $\lim\limits_{x\to\pi/4} \dfrac{\cos x - \sin x}{x - \pi/4} = -\sqrt{2}\lim\limits_{t\to 0}\dfrac{\sin t}{t} = -\sqrt{2}$.

56. Let $g(x) = f^{-1}(x)$ and $h(x) = f(x)/x$ when $x \neq 0$ and $h(0) = L$. Then $\lim\limits_{x\to 0} h(x) = L = h(0)$, so h is continuous at $x = 0$. Apply Theorem 1.5.5 to $h \circ g$ to obtain that on the one hand $h(g(0)) = L$, and on the other $h(g(x)) = \dfrac{f(g(x))}{g(x)}$, $x \neq 0$, and $\lim\limits_{x\to 0} h(g(x)) = h(g(0))$. Since $f(g(x)) = x$ and $g = f^{-1}$ this shows that $\lim\limits_{x\to 0}\dfrac{x}{f^{-1}(x)} = L$.

57. $\lim\limits_{x\to 0}\dfrac{x}{\sin^{-1} x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

58. $\tan(\tan^{-1} x) = x$, so $\lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\tan x} = (\lim\limits_{x\to 0}\cos x)\lim\limits_{x\to 0}\dfrac{x}{\sin x} = 1$.

59. $5\lim\limits_{x\to 0}\dfrac{\sin^{-1} 5x}{5x} = 5\lim\limits_{x\to 0}\dfrac{5x}{\sin 5x} = 5$.

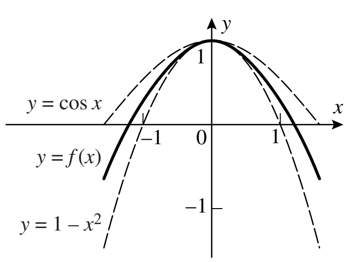
60. $\lim\limits_{x\to 1}\dfrac{1}{x+1}\lim\limits_{x\to 1}\dfrac{\sin^{-1}(x-1)}{x-1} = \dfrac{1}{2}\lim\limits_{x\to 1}\dfrac{x-1}{\sin(x-1)} = \dfrac{1}{2}$.

61. $-|x| \le x\cos\left(\dfrac{50\pi}{x}\right) \le |x|$, which gives the desired result.

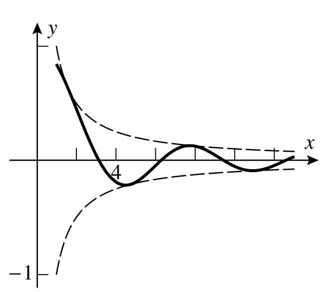
62. $-x^2 \le x^2\sin\left(\dfrac{50\pi}{\sqrt[3]{x}}\right) \le x^2$, which gives the desired result.

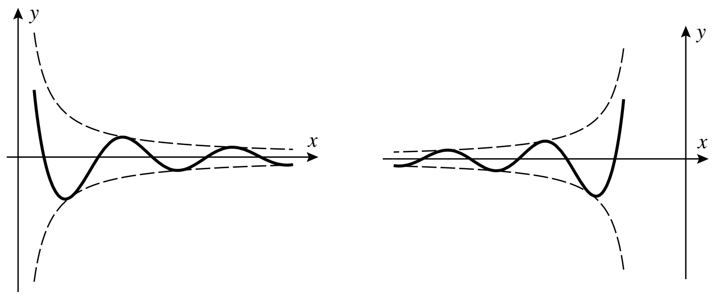
63. Since $\lim\limits_{x\to 0}\sin(1/x)$ does not exist, no conclusions can be drawn.

64. $\lim\limits_{x\to 0} f(x) = 1$ by the Squeezing Theorem.



65. $\lim\limits_{x\to +\infty} f(x) = 0$ by the Squeezing Theorem.

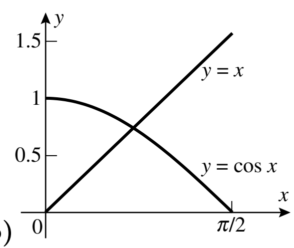




66.

67. **(a)** Let $f(x) = x - \cos x$; $f(0) = -1$, $f(\pi/2) = \pi/2$. By the IVT there must be a solution of $f(x) = 0$.

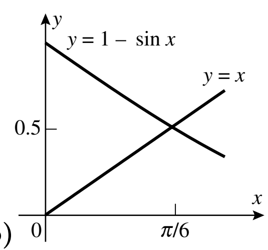
(b)



(c) 0.739

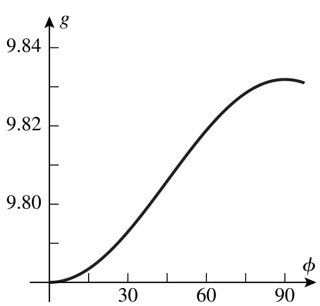
68. **(a)** $f(x) = x + \sin x - 1$; $f(0) = -1$, $f(\pi/6) = \pi/6 - 1/2 > 0$. By the IVT there must be a solution of $f(x) = 0$ in the interval.

(b)



(c) 0.511

69. **(a)** Gravity is strongest at the poles and weakest at the equator.



(b) Let $g(\phi)$ be the given function. Then $g(38) < 9.8$ and $g(39) > 9.8$, so by the Intermediate Value Theorem there is a value c between 38 and 39 for which $g(c) = 9.8$ exactly.

Chapter 1 Review Exercises

1. **(a)** 1 **(b)** Does not exist. **(c)** Does not exist. **(d)** 1 **(e)** 3 **(f)** 0 **(g)** 0

(h) 2 **(i)** 1/2

2. **(a)**

x	2.00001	2.0001	2.001	2.01	2.1	2.5
$f(x)$	0.250	0.250	0.250	0.249	0.244	0.222

For $x \neq 2$, $f(x) = \dfrac{1}{x+2}$, so the limit is $1/4$.

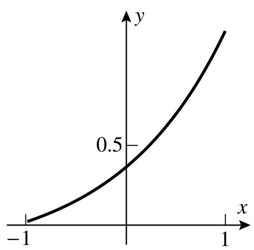
(b)

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	4.0021347	4.0000213	4.0000002	4.0000002	4.0000213	4.0021347

Use $\dfrac{\tan 4x}{x} = \dfrac{\sin 4x}{x \cos 4x} = \dfrac{4}{\cos 4x} \cdot \dfrac{\sin 4x}{4x}$; the limit is 4.

3. **(a)**

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	0.402	0.405	0.405	0.406	0.406	0.409

(b) 

4.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	5.357	5.526	5.543	5.547	5.564	5.742

5. The limit is $\dfrac{(-1)^3 - (-1)^2}{-1-1} = 1$.

6. For $x \neq 1$, $\dfrac{x^3 - x^2}{x-1} = x^2$, so $\lim\limits_{x\to 1} \dfrac{x^3 - x^2}{x-1} = 1$.

7. If $x \neq -3$ then $\dfrac{3x+9}{x^2+4x+3} = \dfrac{3}{x+1}$ with limit $-\dfrac{3}{2}$.

8. The limit is $-\infty$.

9. By the highest degree terms, the limit is $\dfrac{2^5}{3} = \dfrac{32}{3}$.

10. $\dfrac{\sqrt{x^2+4}-2}{x^2} \cdot \dfrac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} = \dfrac{x^2}{x^2(\sqrt{x^2+4}+2)} = \dfrac{1}{\sqrt{x^2+4}+2}$, so $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+4}-2}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x^2+4}+2} = \dfrac{1}{4}$.

11. **(a)** $y = 0$. **(b)** None. **(c)** $y = 2$.

12. **(a)** $\sqrt{5}$, no limit, $\sqrt{10}$, $\sqrt{10}$, no limit, $+\infty$, no limit.

(b) $-1, +1, -1, -1$, no limit, $-1, +1$

13. If $x \neq 0$, then $\dfrac{\sin 3x}{\tan 3x} = \cos 3x$, and the limit is 1.

14. If $x \neq 0$, then $\dfrac{x \sin x}{1-\cos x} \cdot \dfrac{1+\cos x}{1+\cos x} = \dfrac{x}{\sin x}(1+\cos x)$, so the limit is 2.

15. If $x \neq 0$, then $\dfrac{3x - \sin(kx)}{x} = 3 - k\dfrac{\sin(kx)}{kx}$, so the limit is $3-k$.

16. $\lim\limits_{\theta\to 0} \tan\left(\dfrac{1-\cos\theta}{\theta}\right) = \tan\left(\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}\right) = \tan\left(\lim\limits_{\theta\to 0} \dfrac{1-\cos^2\theta}{\theta(1+\cos\theta)}\right) = \tan\left(\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \cdot \dfrac{\sin\theta}{(1+\cos\theta)}\right) = 0$.

17. As $t \to \pi/2^+$, $\tan t \to -\infty$, so the limit in question is 0.

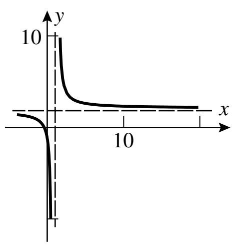
18. $\ln(2\sin\theta\cos\theta) - \ln\tan\theta = \ln 2 + 2\ln\cos\theta$, so the limit is $\ln 2$.

19. $\left(1+\dfrac{3}{x}\right)^{-x} = \left[\left(1+\dfrac{3}{x}\right)^{x/3}\right]^{(-3)}$, so the limit is e^{-3}.

20. $\left(1+\dfrac{a}{x}\right)^{bx} = \left[\left(1+\dfrac{a}{x}\right)^{x/a}\right]^{(ab)}$, so the limit is e^{ab}.

21. \$2,001.60, \$2,009.66, \$2,013.62, \$2013.75.

23. **(a)** $f(x) = 2x/(x-1)$.



(b)

24. Given any window of height 2ϵ centered at the point $x = a, y = L$ there exists a width 2δ such that the window of width 2δ and height 2ϵ contains all points of the graph of the function for x in that interval.

25. **(a)** $\lim_{x\to 2} f(x) = 5$.

(b) $\delta = (3/4)\cdot(0.048/8) = 0.0045$.

26. $\delta \approx 0.07747$ (use a graphing utility).

27. **(a)** $|4x - 7 - 1| < 0.01$ means $4|x-2| < 0.01$, or $|x-2| < 0.0025$, so $\delta = 0.0025$.

(b) $\left|\dfrac{4x^2-9}{2x-3} - 6\right| < 0.05$ means $|2x+3-6| < 0.05$, or $|x-1.5| < 0.025$, so $\delta = 0.025$.

(c) $|x^2 - 16| < 0.001$; if $\delta < 1$ then $|x+4| < 9$ if $|x-4| < 1$; then $|x^2-16| = |x-4||x+4| \le 9|x-4| < 0.001$ provided $|x-4| < 0.001/9 = 1/9000$, take $\delta = 1/9000$, then $|x^2-16| < 9|x-4| < 9(1/9000) = 1/1000 = 0.001$.

28. **(a)** Given $\epsilon > 0$ then $|4x-7-1| < \epsilon$ provided $|x-2| < \epsilon/4$, take $\delta = \epsilon/4$.

(b) Given $\epsilon > 0$ the inequality $\left|\dfrac{4x^2-9}{2x-3} - 6\right| < \epsilon$ holds if $|2x+3-6| < \epsilon$, or $|x-1.5| < \epsilon/2$, take $\delta = \epsilon/2$.

29. Let $\epsilon = f(x_0)/2 > 0$; then there corresponds a $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$, $-\epsilon < f(x) - f(x_0) < \epsilon$, $f(x) > f(x_0) - \epsilon = f(x_0)/2 > 0$, for $x_0 - \delta < x < x_0 + \delta$.

30. **(a)**

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	0.49	0.54	0.540	0.5403	0.54030	0.54030

(b) $\cos 1$

31. **(a)** f is not defined at $x = \pm 1$, continuous elsewhere.

(b) None; continuous everywhere.

(c) f is not defined at $x = 0$ and $x = -3$, continuous elsewhere.

32. **(a)** Continuous everywhere except $x = \pm 3$.

(b) Defined and continuous for $x \le -1$, $x \ge 1$.

(c) Defined and continuous for $x > 0$.

33. For $x < 2$ f is a polynomial and is continuous; for $x > 2$ f is a polynomial and is continuous. At $x = 2$, $f(2) = -13 \neq 13 = \lim_{x\to 2^+} f(x)$, so f is not continuous there.

35. $f(x) = -1$ for $a \le x < \dfrac{a+b}{2}$ and $f(x) = 1$ for $\dfrac{a+b}{2} \le x \le b$; f does not take the value 0.

36. If, on the contrary, $f(x_0) < 0$ for some x_0 in $[0, 1]$, then by the Intermediate Value Theorem we would have a solution of $f(x) = 0$ in $[0, x_0]$, contrary to the hypothesis.

37. $f(-6) = 185$, $f(0) = -1$, $f(2) = 65$; apply Theorem 1.5.8 twice, once on $[-6, 0]$ and once on $[0, 2]$.

Chapter 1 Making Connections

1. Let $P(x, x^2)$ be an arbitrary point on the curve, let $Q(-x, x^2)$ be its reflection through the y-axis, let $O(0, 0)$ be the origin. The perpendicular bisector of the line which connects P with O meets the y-axis at a point $C(0, \lambda(x))$, whose ordinate is as yet unknown. A segment of the bisector is also the altitude of the triangle ΔOPC which is isosceles, so that $CP = CO$.

Using the symmetrically opposing point Q in the second quadrant, we see that $\overline{OP} = \overline{OQ}$ too, and thus C is equidistant from the three points O, P, Q and is thus the center of the unique circle that passes through the three points.

2. Let R be the midpoint of the line segment connecting P and O, so that $R(x/2, x^2/2)$. We start with the Pythagorean Theorem $\overline{OC}^2 = \overline{OR}^2 + \overline{CR}^2$, or $\lambda^2 = (x/2)^2 + (x^2/2)^2 + (x/2)^2 + (\lambda - x^2/2)^2$. Solving for λ we obtain $\lambda x^2 = (x^2 + x^4)/2, \lambda = 1/2 + x^2/2$.

3. Replace the parabola with the general curve $y = f(x)$ which passes through $P(x, f(x))$ and $S(0, f(0))$. Let the perpendicular bisector of the line through S and P meet the y-axis at $C(0, \lambda)$, and let $R(x/2, (f(x) - \lambda)/2)$ be the midpoint of P and S. By the Pythagorean Theorem, $\overline{CS}^2 = \overline{RS}^2 + \overline{CR}^2$, or $(\lambda - f(0))^2 = x^2/4 + \left[\dfrac{f(x)+f(0)}{2} - f(0)\right]^2 + x^2/4 + \left[\dfrac{f(x)+f(0)}{2} - \lambda\right]^2$,
which yields $\lambda = \dfrac{1}{2}\left[f(0) + f(x) + \dfrac{x^2}{f(x) - f(0)}\right]$.

4. **(a)** $f(0) = 0, C(x) = \frac{1}{8} + 2x^2, x^2 + (y - \frac{1}{8})^2 = \left(\frac{1}{8}\right)^2$.

(b) $f(0) = 0, C(x) = \frac{1}{2}(\sec x + x^2), x^2 + (y - \frac{1}{2})^2 = \left(\frac{1}{2}\right)^2$.

(c) $f(0) = 0, C(x) = \frac{1}{2}\dfrac{x^2 + |x|^2}{|x|}, x^2 + y^2 = 0$ (not a circle).

(d) $f(0) = 0, C(x) = \frac{1}{2}\dfrac{x(1 + \sin^2 x)}{\sin x}, x^2 + \left(y - \dfrac{1}{2}\right)^2 = \left(\dfrac{1}{2}\right)^2$.

(e) $f(0) = 1, C(x) = \frac{1}{2}\dfrac{x^2 - \sin^2 x}{\cos x - 1}, x^2 + y^2 = 1.$

(f) $f(0) = 0, C(x) = \dfrac{1}{2g(x)} + \dfrac{x^2 g(x)}{2}, x^2 + \left(y - \dfrac{1}{2g(0)}\right)^2 = \left(\dfrac{1}{2g(0)}\right)^2.$

(g) $f(0) = 0, C(x) = \frac{1}{2}\dfrac{1 + x^6}{x^2}$, limit does not exist, osculating circle does not exist.