Signals and Communication Theory

Discrete Fourier Transform

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Introduction

In wireless communication, we use the **Discrete Fourier Transform (DFT)** for equalization strategies and OFDM. The DFT equations are given by

$$\textbf{Analysis:} \qquad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk},$$

$$\textbf{Synthesis:} \qquad x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}.$$

In this case, note that the frequency variable f is discrete, i.e., $f = \frac{k}{N}$. Similarly, the time variable n is discrete. Additionally, the signal $x[n]$ and its corresponding DFT $X[k]$ are periodic signals with period N.

Modulo operation

Since the signals $x[n]$ and $X[k]$ are periodic with period N, shifts are given in terms of a modulo by N operation denoted as $(\cdot)_N$. This ensures that the argument falls in $[0, N-1]$ as shown below

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$(n)_4$	3	0	1	2	3	0	1	2	3	0	1

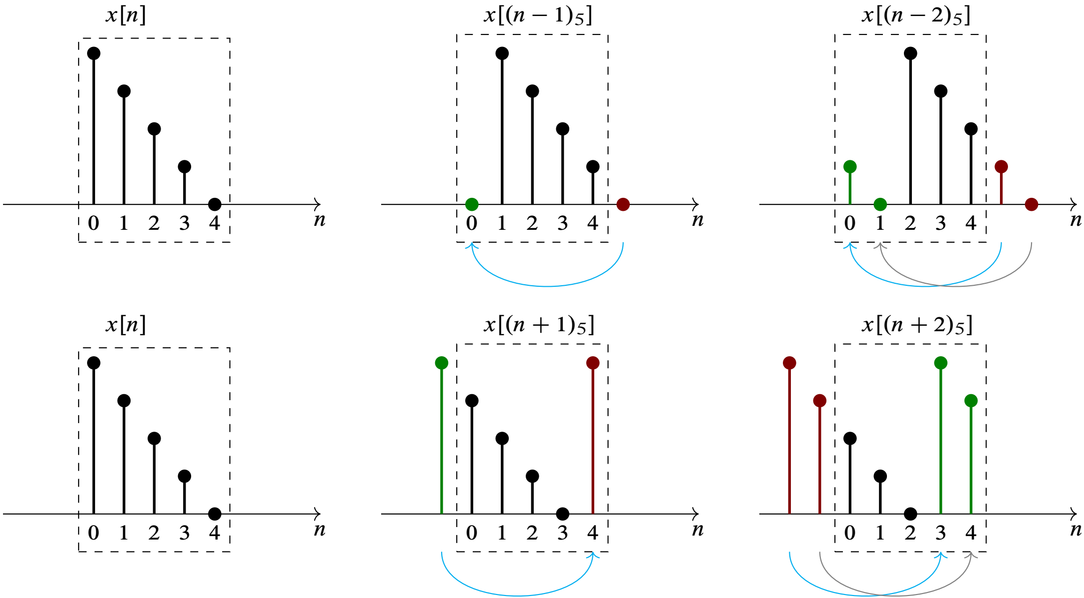
The modulo operation $(n)_N$ can be computed as

$$(n)_N = r,$$

where r is a positive integer $r < N$ such that $n = qN + r$ and q is an integer.

Circular Shift

Using modulo operation, the circular shift is denoted as $x[(n-k)_N]$.



Circular Convolution

The convolution between two discrete-time periodic signal, $x_1[n]$ and $x_2[n]$, is called Circular convolution and is defined as

$$y[n] = x_1[n] \circledast x_2[n] = \sum_{\ell=0}^{N-1} x_1[\ell] x_2[(n-\ell)_N].$$

The resulting signal $y[n]$ is periodic with period N.

Example 1

Calculate the circular convolution between the following signals:

$$x_1[n] = \begin{cases} 1, & \text{if } n = 0 \\ -1, & \text{if } n = 1 \\ 2, & \text{if } n = 2. \end{cases} \qquad x_2[n] = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2. \end{cases}$$

Example 1. Answer

Answer: Substituting $x_1[n]$ and $x_2[n]$ with $N = 3$ into the definition of circular convolution, we obtain

$$y[n] = \sum_{\ell=0}^{N-1} x_1[\ell] x_2[(n-\ell)_N]$$

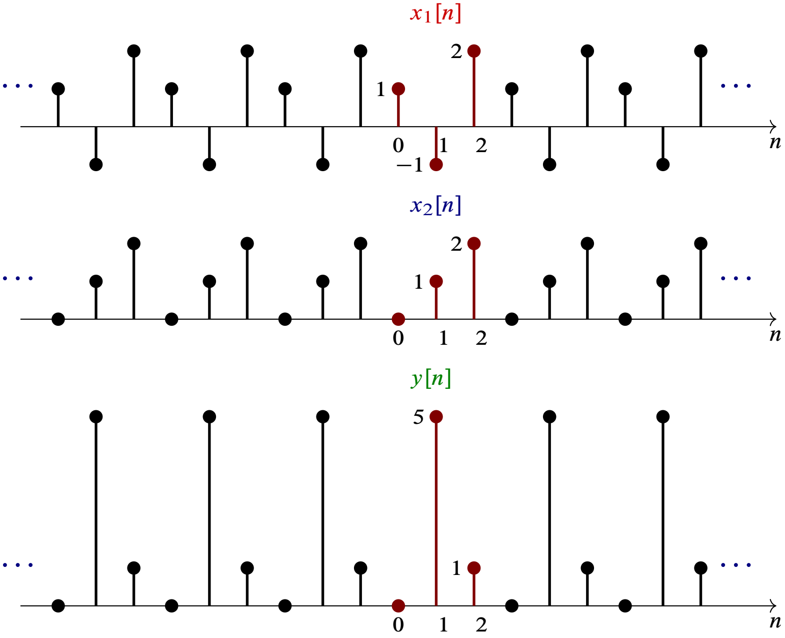
$$y[0] = \sum_{\ell=0}^{2} x_1[\ell] x_2[(-\ell)_N] = x_1[0]x_2[\mathbf{0}] + x_1[1]x_2[\mathbf{2}] + x_1[2]x_2[\mathbf{1}] = 0$$

$$y[1] = \sum_{\ell=0}^{2} x_1[\ell] x_2[(1-\ell)_N] = x_1[0]x_2[\mathbf{1}] + x_1[1]x_2[\mathbf{0}] + x_1[2]x_2[\mathbf{2}] = 5$$

$$y[2] = \sum_{\ell=0}^{2} x_1[\ell] x_2[(2-\ell)_N] = x_1[0]x_2[\mathbf{2}] + x_1[1]x_2[\mathbf{1}] + x_1[2]x_2[\mathbf{0}] = 1$$

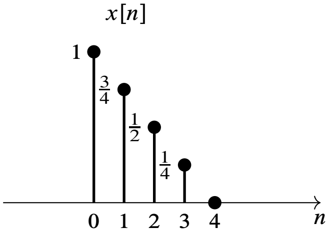
$$y[n] = \begin{cases} 0, & \text{if } n = 0 \\ 5, & \text{if } n = 1 \\ 1, & \text{if } n = 2. \end{cases}$$

Example 1. Graphs



Example 2

Find the DFT of the following signal:



Example 2. Answer

Substituting $x[n]$ into the analysis equation, we have

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{4} x[n] e^{-j\frac{2\pi}{5}kn}$$

$$= 1 + \frac{3}{4} e^{-j\frac{2\pi}{5}k} + \frac{1}{2} e^{-j\frac{4\pi}{5}k} + \frac{1}{4} e^{-j\frac{6\pi}{5}k}$$

k	$X[k]$	$\lvert X[k] \rvert$	$\angle X[k]$
0	2.5	2.5	0
1	$0.625 - 0.860239j$	1.063314	-0.3π
2	$0.625 - 0.203075j$	0.657164	-0.1π
3	$0.625 + 0.203075j$	0.657164	0.1π
4	$0.625 + 0.860239j$	1.063314	0.3π

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Linearity

The DFT satisfies the linearity property

$$\alpha x_1[n] + \beta x_2[n] \leftrightarrow \alpha X_1[k] + \beta X_2[k]$$

Proof

$$\begin{aligned}\sum_{n=0}^{N-1} \Big(\alpha x_1[n] + \beta x_2[n]\Big) e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1} \alpha x_1[n] e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} \beta x_2[n] e^{-j\frac{2\pi}{N}kn} \\ &= \alpha \sum_{n=0}^{N-1} \beta x_1[n] e^{-j\frac{2\pi}{N}kn} + \beta \sum_{n=0}^{N-1} \beta x_2[n] e^{-j\frac{2\pi}{N}kn} \\ &= \alpha X_1[k] + \beta X_2[k]\end{aligned}$$

Periodicity

The Discrete Fourier Transform and inverse Discrete Fourier Transform are periodic sequences with period N.

$$X[k] = X[k+N]$$
$$x[n] = x[n+N]$$

Proof

$$\begin{aligned} x[n+N] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}k(n+N)} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}kn}e^{-j2\pi k} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}kn}, \qquad \text{Since } e^{-j2\pi k} = 1 \\ &= x[n] \end{aligned}$$

Time shifting

The time shifting property is expressed by

$$x[(n-n_0)_N] \leftrightarrow e^{-j\frac{2\pi}{N}kn_0}X[k]$$

Proof. Considering $n - n_0 = qN + m$, we obtain $(n-n_0)_N = m$. Thus

$$\begin{aligned}
\sum_{n=0}^{N-1} x[(n-n_0)_N]e^{-j\frac{2\pi}{N}kn} &= e^{-j\frac{2\pi}{N}kn_0}\sum_{n=0}^{N-1} x[(n-n_0)_N]e^{-j\frac{2\pi}{N}k(n-n_0)} \\
&= e^{-j\frac{2\pi}{N}kn_0}\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km} \\
&= e^{-j\frac{2\pi}{N}kn_0}X[k]
\end{aligned}$$

Frequency shifting

The Modulation property is expressed by

$$e^{j\frac{2\pi}{N}k_0 n}x[n] \leftrightarrow X[(k-k_0)_N]$$

Proof. By considering $k - k_0 = qN + m$, we have $(k - k_0)_N = m$. Thus

$$\begin{aligned}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n}x[n]e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}(k-k_0)n} \\ &= X[(k-k_0)_N]\end{aligned}$$

Time reversal

This property satisfies

$$x[(-n)_N] \leftrightarrow X[(-k)_N]$$

Proof. Using $-n = qN + m$, we have $(-n)_N = m$. Therefore

$$\begin{aligned}
\sum_{n=0}^{N-1} x[(-n)_N] e^{-j\frac{2\pi}{N}kn} &= \sum_{m} x[m] e^{-j\frac{2\pi}{N}k(-qN-m)} \\
&= \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}(-k)m} \\
&= \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}(-k)_N m} \\
&= X[(-k)_N]
\end{aligned}$$

Complex conjugate

This property satisfies

$$x^*[n] \leftrightarrow X^*[(-k)_N]$$

Proof

$$\begin{aligned}
\sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi}{N}kn} &= \left(\sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}kn}\right)^* \\
&= \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(-k)n}\right)^* \\
&= \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(-k)_N n}\right)^* \\
&= X^*[(-k)_N]
\end{aligned}$$

Duality

The duality property is expressed as

$$X[n] \leftrightarrow Nx[(-k)_N]$$

Proof

$$\begin{aligned}
\sum_{n=0}^{N-1} X[n]e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1} X[n]e^{j\frac{2\pi}{N}(-k)n} \\
&= \sum_{n=0}^{N-1} X[n]e^{j\frac{2\pi}{N}(-k)_N n} \\
&= Nx[(-k)_N]
\end{aligned}$$

Circular Convolution

The Discrete Fourier transform of the circular convolution is related as

$$x[n] \circledast h[n] \leftrightarrow X[k]H[k]$$

Proof

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n] \circledast h[n] e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[m]h[(n-m)_N]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} h[(n-m)_N]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km} H[k] \qquad \text{time shifting prop.} \\
&= X[k]H[k]
\end{aligned}$$

Multiplication

The Discrete Fourier transform of the multiplication is related as

$$x[n]h[n] \leftrightarrow \frac{1}{N} X[k] \circledast H[k]$$

Proof

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n]h[n]e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1} x[n]\frac{1}{N}\sum_{m=0}^{N-1} H[m]e^{j\frac{2\pi}{N}mn}e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} H[m]\sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}mn}e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} H[m]X[(n-m)_N] \\
&= \frac{1}{N}H[k] \circledast X[k]
\end{aligned}$$

Parseval's relation

Parseval's relation is defined as

$$\sum_{n=0}^{N-1} x_1[n]x_2^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2^*[k]$$

Proof

$$\begin{aligned}
\sum_{n=0}^{N-1} x_1[n]x_2^*[n] &= \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]e^{j\frac{2\pi}{N}kn} x_2^*[n] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} x_2^*[n] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2^*[k]
\end{aligned}$$

Magnitude spectrum for real-valued signals

Using the fact that $x[n] = x^*[n]$ and the complex conjugate property, we have

$$X[k] = X^*[(-k)_N].$$

Thus

$$|X[k]| = |X[(-k)_N]|.$$

This means, the magnitude spectrum is an **even function**.

Phase spectrum for real-valued signals

Using $X[k] = X^*[(-k)_N]$ gives

$$\angle X[k] = -\angle X[(-k)_N].$$

This means, the phase spectrum is an **odd function**.

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Multiplications and Additions in DFT Computation

Consider the following example with $N = 8$:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{7} x[n] e^{-j\frac{\pi}{4}kn} \\
&= x[0]\cdot 1 + x[1]e^{-j\frac{\pi}{4}k} + x[2]e^{-j\frac{\pi}{2}k} + x[3]e^{-j\frac{3\pi}{4}k} \\
&\quad + x[4]e^{-j\pi k} + x[5]e^{-j\frac{5\pi}{4}k} + x[6]e^{-j3\pi k} + x[7]e^{-j\frac{7\pi}{4}k}.
\end{aligned}
$$

In this particular example, each value of $X[k]$ requires 8 complex multiplications and 7 complex additions. Thus, the total computation of the DFT requires 64 complex multiplications and 56 complex additions.

In general, the computation of the DFT performs N^2 complex multiplications and $N(N-1)$ complex additions.

FFT Algorithm

We assume that N can be expressed as $N = 2^{\ell}$. Now in order to obtain the N-point DFT, we split the signal into even and odd-indexing signals as follows

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N/2-1} x[2n]e^{-j\frac{2\pi}{N}k2n} + \sum_{n=0}^{N/2-1} x[2n+1]e^{-j\frac{2\pi}{N}k(2n+1)} \\
&= \underbrace{\sum_{n=0}^{N/2-1} x[2n]e^{-j\frac{2\pi}{N/2}kn}}_{N/2\text{-point DFT}} + e^{-j\frac{2\pi}{N}k} \underbrace{\sum_{n=0}^{N/2-1} x[2n+1]e^{-j\frac{2\pi}{N/2}kn}}_{N/2\text{-point DFT}}
\end{aligned}
$$

In this way, neglecting the multiplications by $e^{-j\frac{2\pi}{N}k}$, the evaluation of the DFT requires $\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 = \frac{N^2}{2}$ complex multiplications. That is, a reduction by 2.

FFT Algorithm, cont.

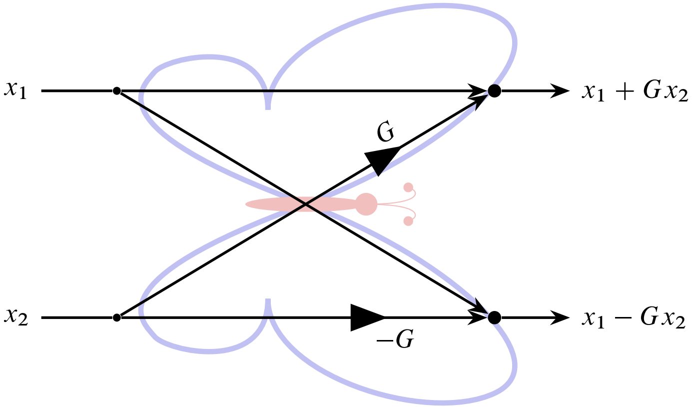
The splitting operation can continue until only pairs of the original signal occur. In that case the 2-point DFT is given by

$$X[k] = \sum_{n=0}^{1} x[n]e^{-j\pi kn},$$
$$X[0] = x[0] + x[1],$$
$$X[1] = x[0] - x[1],$$

which does not require multiplications.

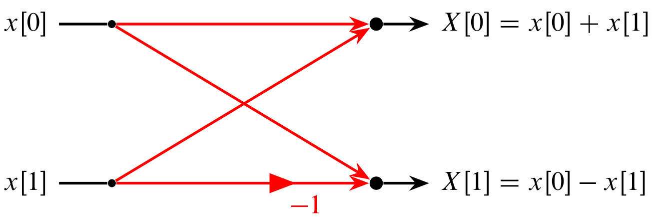
Evaluating the DFT with the FFT algorithm requires $N \log_2 N$ complex multiplications.

Butterfly Pattern



2-FFT Butterfly Structure

$$X[k] = \sum_{n=0}^{1} x[n]e^{-j\pi kn},$$
$$X[0] = x[0] + x[1],$$
$$X[1] = x[0] - x[1].$$

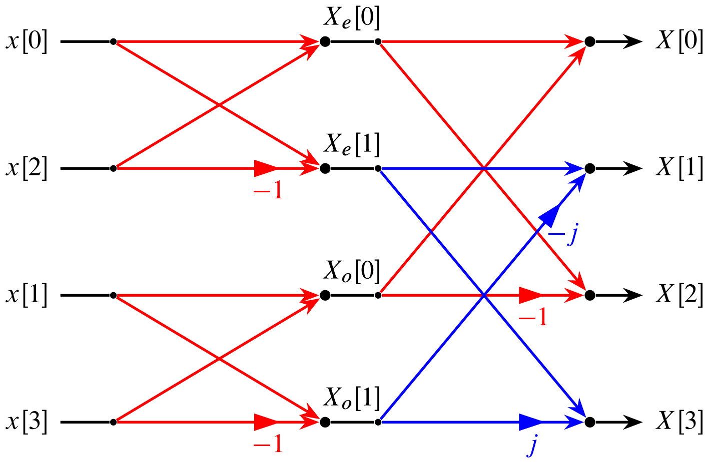


4-FFT Algorithm

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{3} x[n] e^{-j\frac{\pi}{2}kn}, \\
&= \underbrace{\sum_{n=0}^{1} x[2n] e^{-j\pi kn}}_{\text{2-DFT}} + e^{-j\frac{\pi}{2}k} \underbrace{\sum_{n=0}^{1} x[2n+1] e^{-j\pi kn}}_{\text{2-DFT}} \\
&= X_e[k] + e^{-j\frac{\pi}{2}k} X_o[k]
\end{aligned}
$$

$$
\begin{aligned}
X[0] &= X_e[0] + X_o[0] & \qquad X_e[0] &= x[0] + x[2] \\
X[1] &= X_e[1] - jX_o[1] & \qquad X_e[1] &= x[0] - x[2] \\
X[2] &= X_e[0] - X_o[0] & \qquad X_o[0] &= x[1] + x[3] \\
X[3] &= X_e[1] + jX_o[1] & \qquad X_o[1] &= x[1] - x[3]
\end{aligned}
$$

4-FFT Butterfly Structure



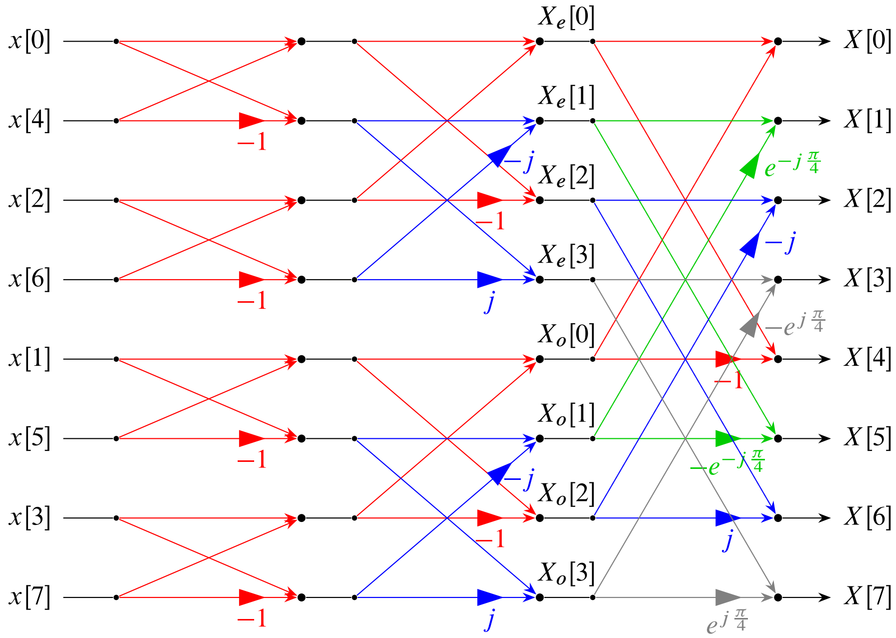
8-FFT Algorithm

$$X[k] = \sum_{n=0}^{7} x[n]e^{-j\frac{\pi}{4}kn} = \underbrace{\sum_{n=0}^{3} x[2n]e^{-j\frac{\pi}{2}kn}}_{\text{4-DFT}} + e^{-j\frac{\pi}{4}k} \underbrace{\sum_{n=0}^{3} x[2n+1]e^{-j\frac{\pi}{2}kn}}_{\text{4-DFT}}$$

$$= X_e[k] + e^{-j\frac{\pi}{4}k} X_o[k]$$

$$\begin{aligned}
X[0] &= X_e[0] + X_o[0] \\
X[1] &= X_e[1] + e^{-j\frac{\pi}{4}} X_o[1] \\
X[2] &= X_e[2] - jX_o[2] \\
X[3] &= X_e[3] - e^{j\frac{\pi}{4}} X_o[3] \\
X[4] &= X_e[0] - X_o[0] \\
X[5] &= X_e[1] - e^{-j\frac{\pi}{4}} X_o[1] \\
X[6] &= X_e[2] + jX_o[2] \\
X[7] &= X_e[3] + e^{j\frac{\pi}{4}} X_o[3]
\end{aligned}$$

8-FFT Butterfly Structure



Comparison of Complex Multiplications

N	DFT(N^2)	FFT($N \log_2 N$)
32	1024	160
64	4096	384
128	16384	896
256	65536	2024
512	262144	4608
1024	1048576	10240

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Summary of Fourier Analysis/Synthesis

	Continuous-Time	Discrete-Time
Continuous-Frequency	**Fourier Transform (FT)** $$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$ $$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$	**Discrete-Time Fourier Transform (DTFT)** $$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$$ $$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X\left(e^{j2\pi f}\right)e^{j2\pi fn}\,df$$
Discrete-Frequency	**Fourier Series (FS)** $$c[n] = f_0\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi nf_0t}\,dt$$ $$x(t) = \sum_{n=-\infty}^{\infty} c[n]e^{j2\pi nf_0t}$$	**Discrete Fourier Transform (DFT)** $$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}$$ $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk}$$

Homework

- Problems 6.79–6.82 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill