Signals and Communication Theory

z-Transform

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Introduction

In digital communications, a useful tool for implementing digital systems in hardware is the z-transform.

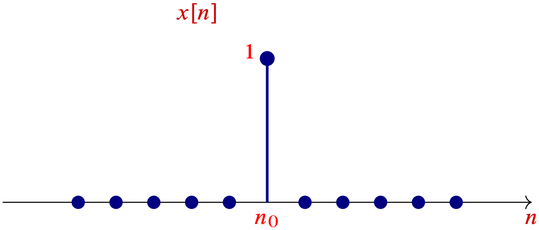
The direct and inverse z-transforms are

$$\textbf{Analysis:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

$$\textbf{Synthesis:} \qquad x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}\, dz.$$

Example 1

Find the z*-transform of* $x[n] = \delta[n - n_0]$.



Answer to Example 1

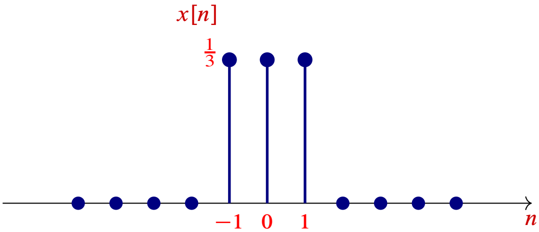
Substituting $x[n] = \delta[n - n_0]$ gives

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \delta[n - n_0] z^{-n} \\ &= z^{-n_0}. \end{aligned}$$

Example 2

Let

$$x[n] = \begin{cases} \frac{1}{3}, & \text{if } n = -1, 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$



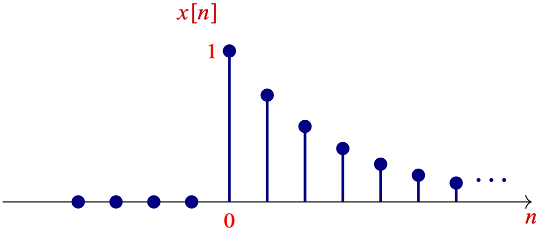
Answer to Example 2

Substituting $x[n]$ into the definition of z-transform, we obtain

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= \frac{1}{3}\left(z + 1 + z^{-1}\right).
\end{aligned}
$$

Example 3

Obtain the z-transform of the signal $x[n] = a^n u[n]$.



Answer to Example 3

Using the definition of $X(z)$, it follows that

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} a^n u[n]z^{-n} \\
&= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \\
&= \frac{1}{1-az^{-1}}, \quad \text{for } |az^{-1}| < 1
\end{aligned}
$$

Example 4

Let

$$x[n] = \begin{cases} r^n \cos(\omega_0 n), & \text{if } n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the corresponding z-transform.

Answer to Example 4

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} r^n \cos(\omega_0 n) z^{-n} \\
&= \sum_{n=0}^{\infty} r^n \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right) z^{-n} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(r e^{j\omega_0} z^{-1} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(r e^{-j\omega_0} z^{-1} \right)^n \\
&= \frac{1}{2} \frac{1}{1 - r e^{j\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - r e^{-j\omega_0} z^{-1}} \qquad \text{for } |r z^{-1}| < 1 \\
&= \frac{1}{2} \frac{2 - r\left(e^{j\omega_0} + e^{-j\omega_0}\right) z^{-1}}{\left(1 - r e^{j\omega_0} z^{-1}\right)\left(1 - r e^{-j\omega_0} z^{-1}\right)} \\
&= \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}
\end{aligned}$$

Example 5

Assume

$$x[n] = \begin{cases} r^n \sin(\omega_0 n), & \text{if } n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

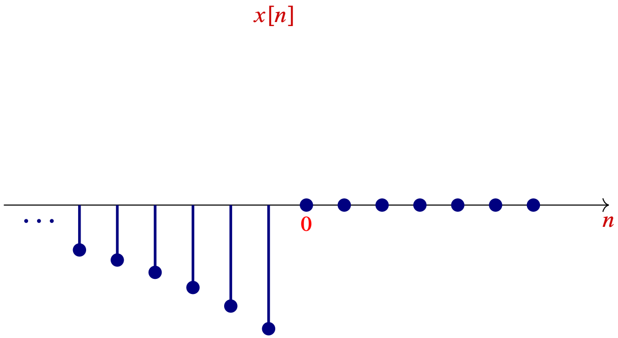
Find the corresponding z-transform.

Answer to Example 5

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} r^n \sin(\omega_0 n) z^{-n} \\
&= \sum_{n=0}^{\infty} r^n \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \right) z^{-n} \\
&= \frac{1}{2j} \sum_{n=0}^{\infty} \left(r e^{j\omega_0} z^{-1} \right)^n + \frac{1}{2j} \sum_{n=0}^{\infty} \left(r e^{-j\omega_0} z^{-1} \right)^n \\
&= \frac{1}{2j} \frac{1}{1 - r e^{j\omega_0} z^{-1}} - \frac{1}{2j} \frac{1}{1 - r e^{-j\omega_0} z^{-1}} \qquad \text{for } |r z^{-1}| < 1 \\
&= \frac{1}{2j} \frac{r \left(e^{j\omega_0} - e^{-j\omega_0} \right) z^{-1}}{\left(1 - r e^{j\omega_0} z^{-1} \right) \left(1 - r e^{-j\omega_0} z^{-1} \right)} \\
&= \frac{r \sin(\omega_0) z^{-1}}{1 - 2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}
\end{aligned}
$$

Example 6

Find the z*-transform of* $x[n] = -a^n u[-1-n]$.



Compare the obtained result with that in Example 3.

Answer to Example 6

I this case, we have

$$
\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} + 1 - 1 \\
&= -\sum_{n=-\infty}^{0} a^n z^{-n} + 1 \\
&= -\sum_{m=0}^{\infty} a^{-m} z^{m} + 1 \\
&= -\frac{1}{1-a^{-1}z} + 1 \quad \text{for } |a^{-1}z| < 1 \\
&= \frac{1}{1-az^{-1}}.
\end{aligned}
$$

Answer to Example 6, Cont.

The z-transform in Example 3 is

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{for } |az^{-1}| < 1,$$

while this example gives

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{for } |a^{-1}z| < 1.$$

Contents

Definition

The **region of convergence** (ROC) is the set of points in the z-plane for which the z-transform converges. In other words,

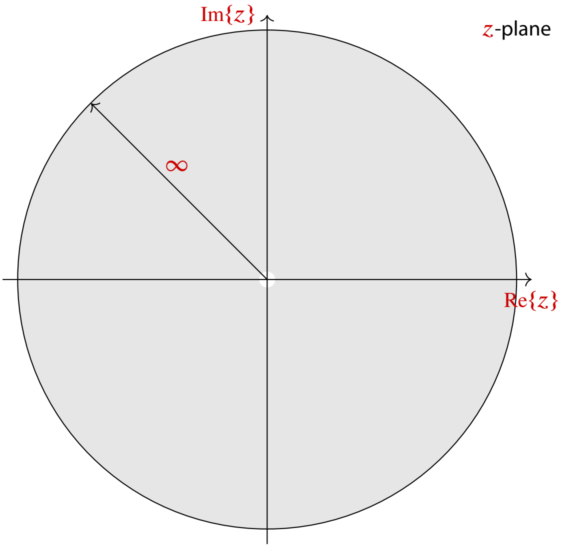
$$\mathrm{ROC} = \{z : |X(z)| < \infty\},$$

Example 7

Use Example 1 with $n_0 = 1$*. Find the corresponding ROC. That is,*

$$X(z) = z^{-1}.$$

Answer to Example 7



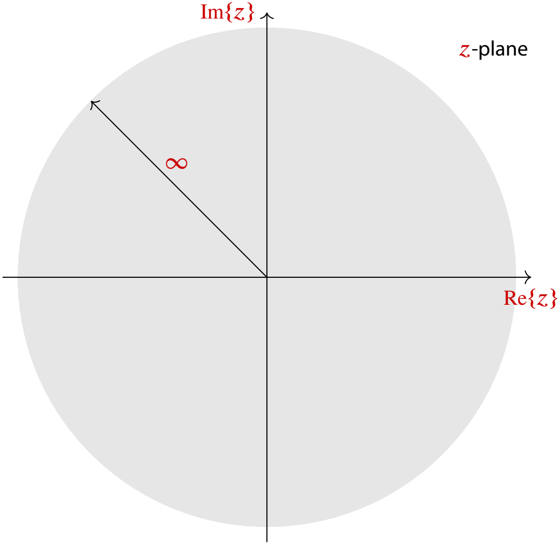
$$\text{ROC} = \{z : |z| > 0\}.$$

Example 8

Find the ROC in Example 1 for $n_0 = -1$*, i.e.,*

$$X(z) = z.$$

Answer to Example 8



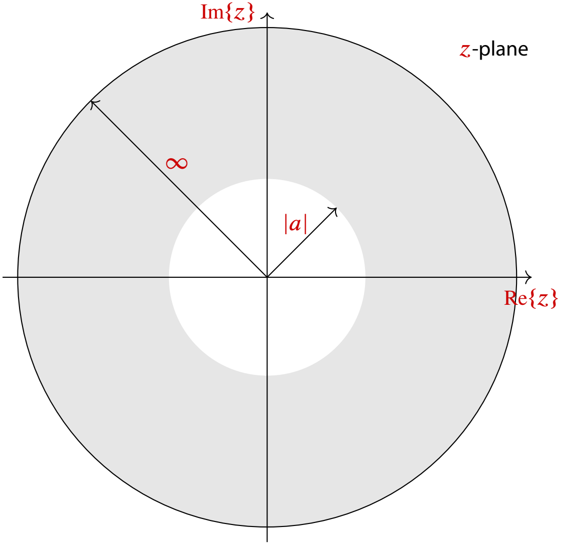
$$\text{ROC} = \{z : |z| < \infty\}.$$

Example 9

Find the ROC in Example 3.

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{for } |az^{-1}| < 1.$$

Answer to Example 9



$$\mathrm{ROC} = \{z : |a| < |z|\}.$$

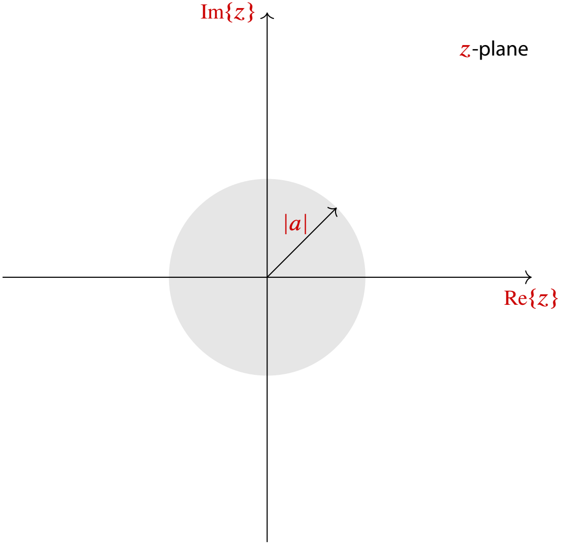
Example 10

From Example 6, we have

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{for } |a^{-1}z| < 1.$$

Obtain the ROC.

Answer to Example 10



$$\text{ROC} = \{z : |z| < |a|\}.$$

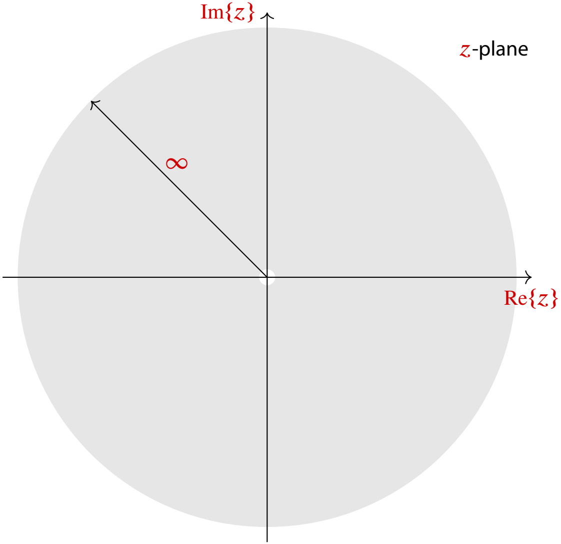
Example 11

Using Example 2, we have

$$X(z) = \frac{1}{3}\left(z + 1 + z^{-1}\right).$$

Obtain the ROC.

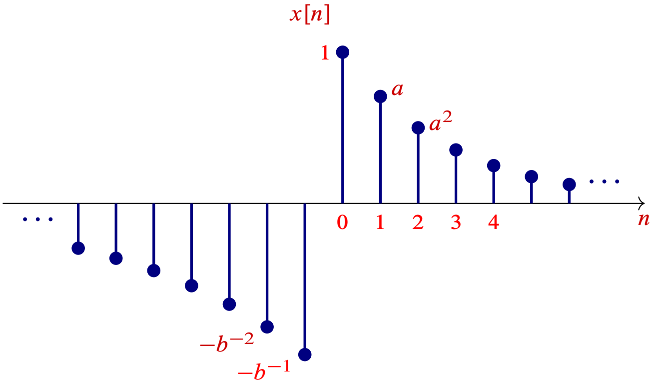
Answer to Example 11



$$\text{ROC} = \{z : 0 < |z| < \infty\}.$$

Example 12

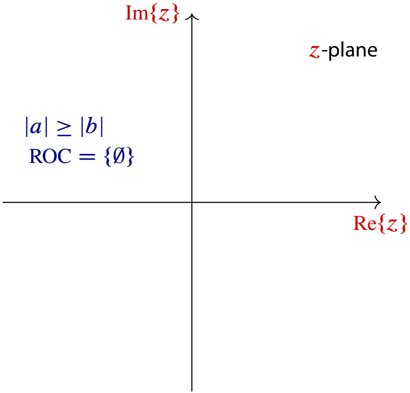
Assume that $x[n] = a^n u[n] - b^n u[-n-1]$. *Find the corresponding ROC.*

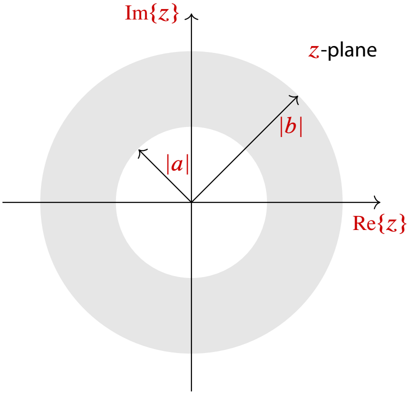


Answer to Example 12

Using Examples 3 and 6, the z-transform results in

$$X(z) = \underbrace{\frac{1}{1 - az^{-1}}}_{|a|<|z|} + \underbrace{\frac{1}{1 - bz^{-1}}}_{|z|<|b|}$$





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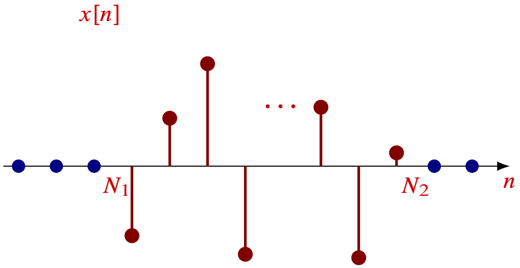
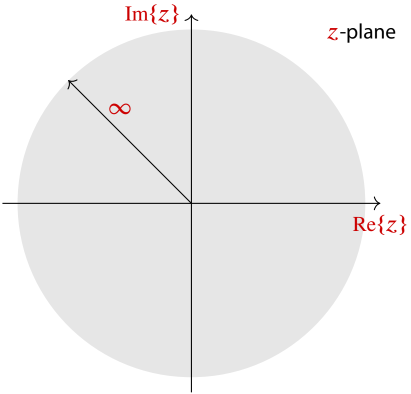
Property 1

The **ROC** does not contain any **poles**.

Property 2

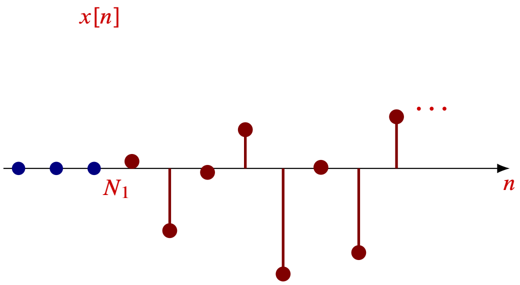
If $x[n]$ is of **finite** duration, then the **ROC** is the **entire** z-plane, except possibly $z = 0$ and/or $z = \infty$.

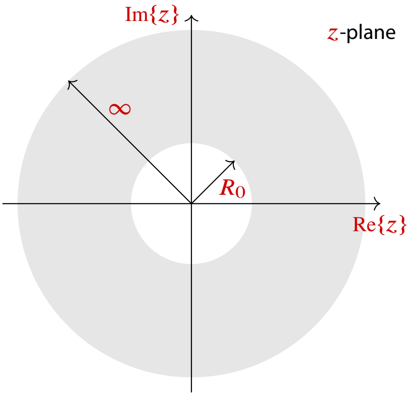




Property 3

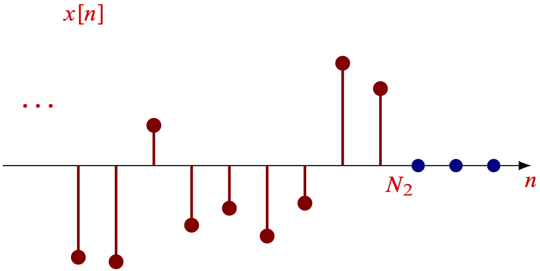
If $x[n]$ is a **right-sided** sequence, then the **ROC** is the **exterior** of the circle $|z| = R_0$ in the z-plane with the possible exception of $|z| = \infty$.

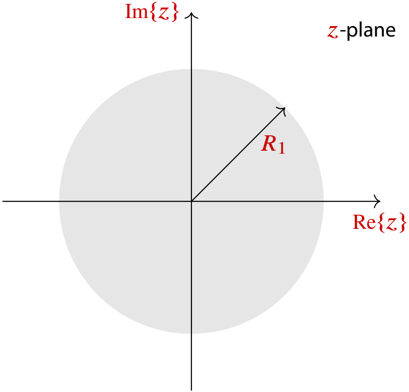




Property 4

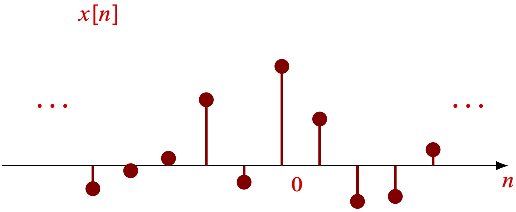
If $x[n]$ is a **left-sided** sequence, then the **ROC** is the **interior** of the circle $|z| = R_1$ in the z-plane with the possible exception of $z = 0$.

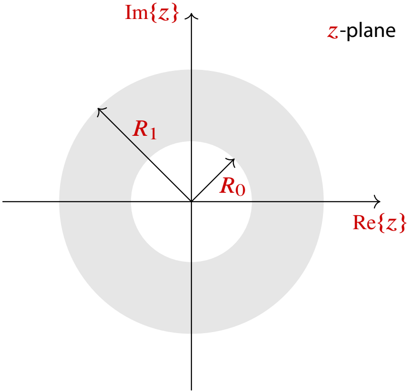




Property 5

If $x[n]$ is a **two-sided** sequence, then the **ROC** is an **annular ring** in the z-plane between the circles $|z| = R_1$ and $|z| = R_0$.





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Linearity

The z-transform satisfies the linearity property

$$\alpha x_1[n] + \beta x_2[n] \leftrightarrow \alpha X_1(z) + \beta X_2(z)$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \Big(\alpha x_1[n] + \beta x_2[n]\Big) z^{-n} &= \sum_{n=-\infty}^{\infty} \alpha x_1[n] z^{-n} + \int_{-\infty}^{\infty} \beta x_2[n] z^{-n} \\
&= \alpha \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + \beta \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} \\
&= \alpha X_1(z) + \beta X_2(z)
\end{aligned}$$

Time shifting

The time shifting property is expressed by

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z)$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} &= \sum_{n=-\infty}^{\infty} x[m]z^{-(m+n_0)} \qquad \text{by using } m=n-n_0 \\
&= z^{-n_0} \sum_{n=-\infty}^{\infty} x[m]z^{-m} \\
&= z^{-n_0}X(z)
\end{aligned}$$

Scaling in the z-Domain

The Scaling in the z-Domain property is expressed by

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} &= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} \\ &= X\left(\frac{z}{z_0}\right)\end{aligned}$$

Time reversal

This property satisfies

$$x[-n] \leftrightarrow X\left(z^{-1}\right)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[-n]z^{-n} &= \sum_{m=\infty}^{-\infty} x[m]z^{m} \\ &= X\left(z^{-1}\right)\end{aligned}$$

Complex conjugate

This property satisfies

$$x^*[n] \leftrightarrow X^*(z^*)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} &= \left(\sum_{n=-\infty}^{\infty} x[n]\left(z^{-n}\right)^*\right)^* \\ &= \left(\int_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\ &= X^*\left(z^*\right)\end{aligned}$$

Differentiation in z-Domain

For this property, we have

$$-nx[n] \leftrightarrow \frac{dX(z)}{dz}$$

Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -nx[n]z^{-n}$$

Convolution

The z-transform of the convolution is related as

$$x[n] * h[m] \leftrightarrow X(z)H(z)$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n] * h[n] z^{-n} &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[n-m] z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m] z^{-n} \\
&= \sum_{m-\infty}^{\infty} x[m] H(z) z^{-m} \qquad \text{time shifting prop.} \\
&= H(z) \sum_{m=-\infty}^{\infty} x[m] z^{-m} \\
&= H(z) X(z)
\end{aligned}$$

Multiplication

The z-transform of the multiplication is related as

$$x[n]h[n] \leftrightarrow \frac{1}{2\pi j} \oint_C H(\zeta) X\left(\frac{z}{\zeta}\right) \zeta^{-1}\, d\zeta$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n]h[n]z^{-n} &= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j} \oint_C H(\zeta)\zeta^{n-1}\, d\zeta z^{-n} \\
&= \frac{1}{2\pi j} \oint_C H(\zeta) \sum_{n=-\infty}^{\infty} x[n]\zeta^{n-1} z^{-n}\, d\zeta \\
&= \frac{1}{2\pi j} \oint_C H(\zeta) X\left(\frac{z}{\zeta}\right) \zeta^{-1}\, d\zeta
\end{aligned}$$

Accumulation

This property says

$$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{X(z)}{1-z^{-1}}$$

Proof First step

$$\sum_{m=-\infty}^{n} x[m] = \sum_{m=-\infty}^{\infty} x[m]u[n-m] = x[n] * u[n]$$

Second step

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{n} x[m]z^{-n} &= X(z)U(z) \\ &= X(z)\frac{1}{1-z^{-1}}\end{aligned}$$

Parseval's relation

Parseval's relation is defined as

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(z)X_2^*(z^{*-1})z^{-1}\,dz$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j}\oint_C X_1(z)z^{n-1}\,dz\,x_2^*[n] \\
&= \frac{1}{2\pi j}\oint_C X_1(z)\sum_{n=-\infty}^{\infty} x_2^*[n]z^n z^{-1}\,dz \\
&= \frac{1}{2\pi j}\oint_C X_1(z)\sum_{n=-\infty}^{\infty} x_2^*[n](z^{-1})^{-n} z^{-1}\,dz \\
&= \frac{1}{2\pi j}\oint_C X_1(z)X_2^*(z^{*-1})z^{-1}\,dz
\end{aligned}$$

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Definition

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

The **system function** $H(z)$ is the z-transform of the impulse response $h[n]$.

Applying z-transform to $y[n] = x[n] * h[n]$, we have

$$Y(z) = X(z)H(z).$$

In this setting, we thus have

$$H(z) = \frac{Y(z)}{X(z)}.$$

Example 13

$$x[n] = r^n \cos(\omega_0 n)u[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = r^n \sin(\omega_0 n)u[n]$$

Let $x[n] = r^n \cos(\omega_0 n)u[n]$ *and* $y[n] = r^n \sin(\omega_0 n)u[n]$ *be the input and output signals to the system. Find the system function.*

Answer to Example 13

Using Examples 4 and 5, we have

$$Y(z) = \frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}},$$
$$X(z) = \frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}.$$

This gives

$$H(z) = \frac{Y(z)}{X(z)} = \frac{r\sin(\omega_0)z^{-1}}{1 - r\cos(\omega_0)z^{-1}}$$

Example 14

Find the system function for the system described by the following difference equation:

$$y[n] + ay[n-1] = x[n]$$

Answer to Example 14

Performing the z-transform gives

$$Y(z) + az^{-1}Y(z) = X(z).$$

Solving for $H(z)$, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + az^{-1}}.$$

Example 15: System defined by difference equation

A **causal, linear and time-invariant** *can be described by a difference equation as*

$$y[n] + \sum_{m=1}^{M} a_m y[n-m] = \sum_{k=0}^{N} b_k x[n-k],$$

where a_m*, for* $m = 1, \ldots, M$*, and* b_k*, for* $k = 0, \ldots, N$*, are constants. Find the system function.*

Answer to Example 15

Applying z-transform, we obtain

$$Y(z) + \sum_{m=1}^{M} a_m z^{-m} Y(z) = \sum_{k=0}^{N} b_k z^{-k} X(z),$$

Solving for $H(z)$, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{m=1}^{M} a_m z^{-m}}.$$

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Singularities in the z-plane

Consider that the z-transform is given by

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{m=1}^{M} a_m z^{-m}} = A \frac{\prod_{k=1}^{N}(1 - c_k z^{-1})}{\prod_{m=1}^{M}(1 - d_m z^{-1})}.$$

The corresponding singularities are the following:

- N **zeros** at $z = c_k$, for $k = 1, \ldots, N$, and N poles at $z = 0$,
- M **poles** at $z = d_m$, for $m = 1, \ldots, M$, and M zeros at $z = 0$.

The values of z for which $H(z) = 0$ define the locations of the **zeros** in the z-plane. Similarly, the values of z for which $H(z)$ becomes infinity define the locations of the **poles** in the z-plane.

Example 16

Consider the z-transform in Example 2, i.e.,

$$X(z) = \frac{1}{3}\left(z + 1 + z^{-1}\right).$$

Obtain the pole/zero pattern.

Answer to Example 16

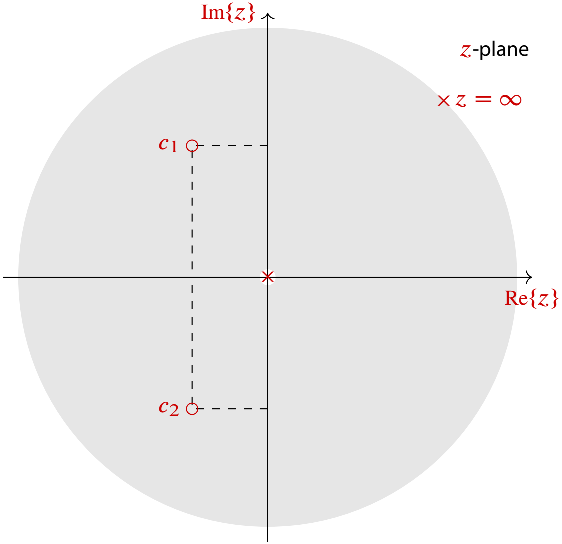
The z-transform can be rewritten as

$$X(z) = \frac{z}{3}\left(1 - c_1 z^{-1}\right)\left(1 - c_2 z^{-1}\right),$$

where

$$\begin{aligned} c_1 &= \frac{-1 + j\sqrt{3}}{2} = e^{j2\pi/3}, \\ c_2 &= \frac{-1 - j\sqrt{3}}{2} = e^{-j2\pi/3}, \\ d_1 &= 0, \\ d_2 &= \infty. \end{aligned}$$

Answer to Example 16 cont.



Example 17

Now consider

$$X(z) = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right).$$

Obtain the pole/zero pattern.

Answer to Example 17

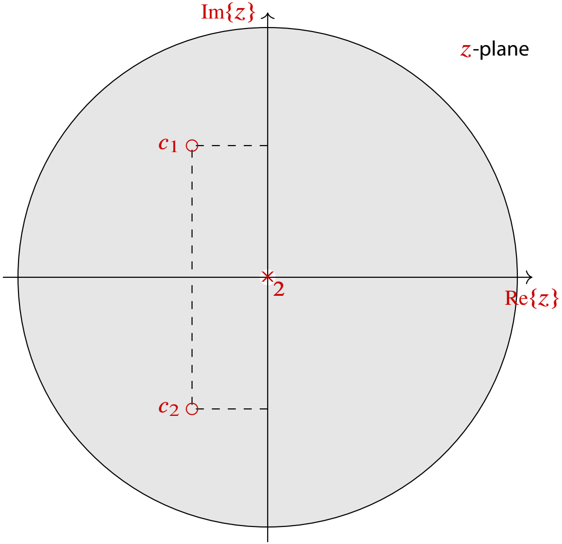
The z-transform can be rewritten as

$$X(z) = \frac{1}{3}\left(1 - c_1 z^{-1}\right)\left(1 - c_2 z^{-1}\right),$$

where

$$\begin{aligned} c_1 &= \frac{-1 + j\sqrt{3}}{2} = e^{j2\pi/3}, \\ c_2 &= \frac{-1 - j\sqrt{3}}{2} = e^{-j2\pi/3}, \\ d_1 &= d_2 = 0. \end{aligned}$$

Answer to Example 17 cont.



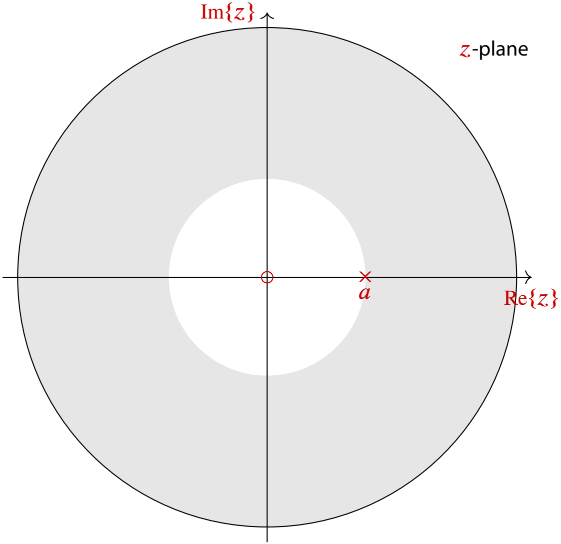
Example 18

From Example 3, we have

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{for } |az^{-1}| < 1.$$

Assuming $a > 0$*, find the pole/zero pattern.*

Answer to Example 18



$$c_1 = 0,$$
$$d_1 = a.$$

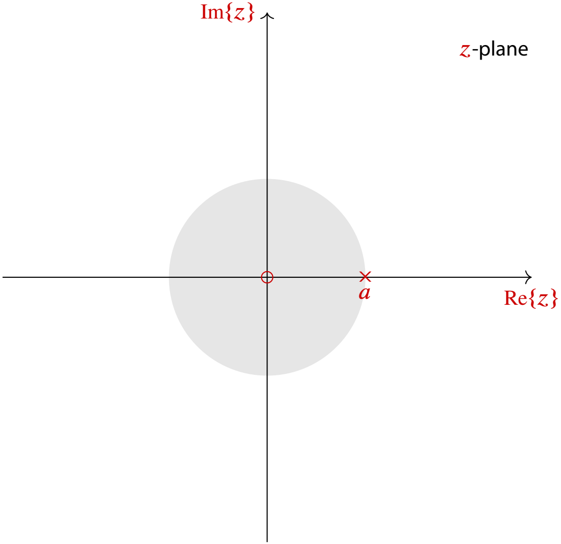
Example 19

Using Example 6 gives

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{for } |a^{-1}z| < 1.$$

Assuming $a > 0$*, obtain the pole/zero pattern.*

Answer to Example 19



$$c_1 = 0,$$
$$d_1 = a.$$

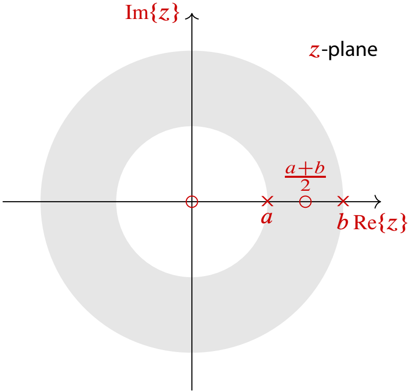
Example 20

Find the corresponding pole/zero pattern in Example 12, i.e.,

$$X(z) = \frac{2 - (a + b)z^{-1}}{\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)}.$$

Consider $a > 0$, $b > 0$, *and* $b > a$

Answer to Example 20



$$c_1 = \frac{a+b}{2}$$
$$c_2 = 0$$
$$d_1 = a$$
$$d_2 = b$$

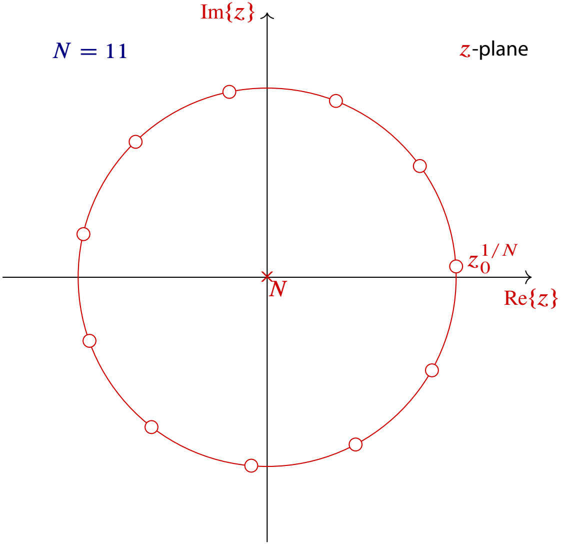
Example 21: Bracelet of zeros

Consider the following system function:

$$H(z) = 1 - z_0 z^{-N}.$$

where N *is an integer and* $N > 0$.

Answer to Example 21



Since

$$H(z) = \frac{z^N - z_0}{z^N},$$

there are N poles at $z = 0$. The zeros occur at $z = z_0^{1/N} e^{j2\pi m/N}$, for $m = 0, \ldots, N-1$.

Contents

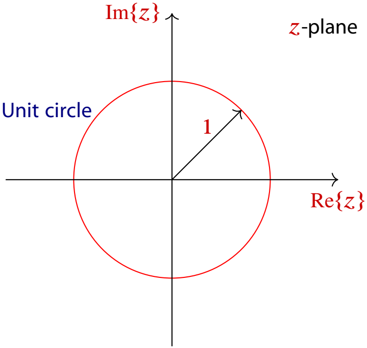
Complex form of the z-transform

Using the polar form of the complex variable z, we have

$$H\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]r^n e^{-j\omega n},$$

which becomes the DTFT when $r = 1$ and $\omega = 2\pi f$.

The locus $z = e^{j\omega}$ is called the **unit circle**.



From the z-transform to the DTFT

Hence the DTFT $H\left(e^{j2\pi f}\right)$ is equal to $H(z)$ evaluated along the **unit circle**, that is,

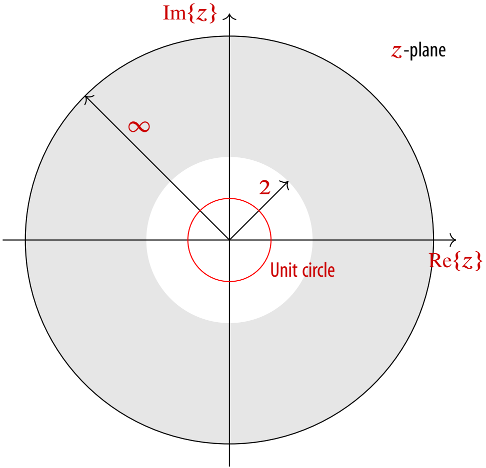
$$H\left(e^{j\omega}\right) = H(z)\big|_{z=e^{j2\pi f}}.$$

For $H\left(e^{j2\pi f}\right)$ to exist, the ROC of $H(z)$ must include the unit circle.

Example 22

Find the DTFT $X\left(e^{j2\pi f}\right)$ *if*

$$X(z) = \frac{1}{1 - 2z^{-1}}, \quad \text{ROC} = \left\{z : 2 < |z|\right\}.$$



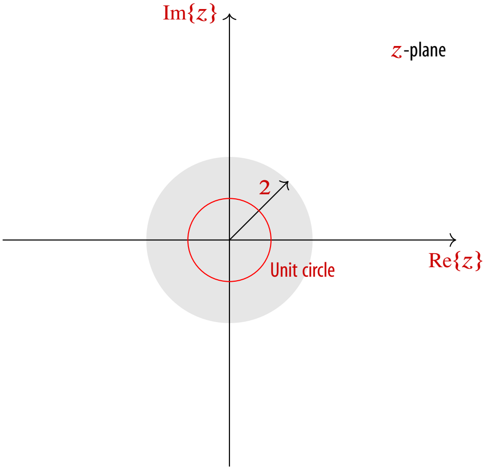
Answer to Example 22

Because the ROC of $X(z)$ does not include the unit circle, the DTFT $X\left(e^{j2\pi f}\right)$ does not exist.

Example 23

Find the DTFT $X\left(e^{j2\pi f}\right)$ *if*

$$X(z) = \frac{1}{1 - 2z^{-1}}, \quad \text{ROC} = \{z : |z| < 2\}.$$



Answer to Example 23

For this case, the DTFT $X\left(e^{j2\pi f}\right)$ is given by

$$X\left(e^{j2\pi f}\right) = \frac{1}{1 - 2e^{-j2\pi f}}.$$

Contents

Methods for computing inverse z-transform

The problem we consider is the following: Given $X(z)$ and the ROC, how do we determine $x[n]$?

There are three methods to determine the values of the discrete-time sequence:

1. **long division**,
2. **Taylor expansion and partial fraction expansion**,
3. **application of the residue theorem from complex-variable theory**.

Contents

Example 24

Consider the following z-transform and its corresponding ROC:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \mathrm{ROC} = \{z : |a| < |z|\}.$$

Obtain $x[n]$.

Answer to Example 24

Since the ROC is outside the pole, we should obtain a **right-Sided sequence**, i.e., a polynomial in z^{-n} for $n \geq 0$.

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + a^4z^{-4} + a^5z^{-5} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{-1 + az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{-az^{-1} + a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{-a^2z^{-2} + a^3z^{-3}} \\
 & \qquad\quad a^3z^{-3} \\
 & \qquad\quad \underline{-a^3z^{-3} + a^4z^{-4}} \\
 & \qquad\qquad a^4z^{-4} \\
 & \qquad\qquad \underline{-a^4z^{-4} + a^5z^{-5}} \\
 & \qquad\qquad\quad a^5z^{-5}
\end{array}
$$

Answer to Example 24. Cont.

We can rewrite $X(z)$ as

$$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + a^4z^{-4} + a^5z^{-5} + \cdots$$

Since $x[n]$ is the coefficient of z^{-n}, we find that

$$x[n] = \begin{cases} a^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 25

Now consider.

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \mathrm{ROC} = \{z : |z| < |a|\}.$$

Obtain $x[n]$.

Answer to Example 25

In this case, we obtain a **left-sided sequence**.

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 - a^{-5}z^5 - a^{-6}z^6 - \cdots \\
\hline
-az^{-1}+1 & 1 \\
 & \underline{-\;1 + a^{-1}z} \\
 & \qquad a^{-1}z \\
 & \qquad \underline{-\;a^{-1}z + a^{-2}z^2} \\
 & \qquad\qquad a^{-2}z^2 \\
 & \qquad\qquad \underline{-\,a^{-2}z^2 + a^{-3}z^3} \\
 & \qquad\qquad\qquad a^{-3}z^3 \\
 & \qquad\qquad\qquad \underline{-\,a^{-3}z^3 + a^{-4}z^4} \\
 & \qquad\qquad\qquad\qquad a^{-4}z^4 \\
 & \qquad\qquad\qquad\qquad \underline{-\,a^{-4}z^4 + a^{-5}z^5} \\
 & \qquad\qquad\qquad\qquad\qquad a^{-5}z^5
\end{array}
$$

Answer to Example 25. Cont.

Rewriting $X(z)$ gives

$$X(z) = \frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - a^{-4}z^{4} - a^{-5}z^{5} - a^{-6}z^{6} - \cdots$$

Since $x[n]$ is the coefficient of z^{-n}, we find that

$$x[n] = \begin{cases} -a^{n}, & \text{if } n \leq -1 \\ 0, & \text{otherwise.} \end{cases}$$

Example 26

Let

$$X(z) = \frac{z^K}{1 - bz^{-1}}, \quad \mathrm{ROC} = \{z : |b| < |z| < \infty\}.$$

Obtain $x[n]$.

Answer to Example 26

Right-sided sequence

$$
\begin{array}{r|l}
 & z^K + bz^{K-1} + b^2 z^{K-2} + \quad \cdots \quad + b^K + b^{K+1} z^{-1} + b^{K+2} z^{-2} + \cdots \\
\hline
1 - bz^{-1} & z^K \\
 & \underline{-z^K + bz^{K-1}} \\
 & \qquad bz^{K-1} \\
 & \qquad \underline{-bz^{K-1} + b^2 z^{K-2}} \\
 & \qquad\qquad b^2 z^{K-2} \\
 & \qquad\qquad \underline{-b^2 z^{K-2} + b^3 z^{K-3}} \\
 & \qquad\qquad\qquad \vdots \\
 & \qquad\qquad\qquad \underline{\vdots \qquad + b^K} \\
 & \qquad\qquad\qquad\qquad b^K \\
 & \qquad\qquad\qquad\qquad \underline{-b^K + b^{K+1} z^{-1}} \\
 & \qquad\qquad\qquad\qquad\qquad b^{K+1} z^{-1} \\
 & \qquad\qquad\qquad\qquad\qquad \underline{-b^{K+1} z^{-1} + b^{K+2} z^{-2}} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad b^{K+2} z^{-2}
\end{array}
$$

Answer to Example 26. Cont.

Rewriting $X(z)$ gives

$$X(z) = \frac{z^K}{1 - bz^{-1}} = z^K + bz^{K-1} + b^2 z^{K-2} + \cdots + b^K + b^{K+1} z^{-1} + b^{K+2} z^{-2} + \cdots$$

Equating the coefficient of z^{-n} with $x[n]$, it follows that

$$x[n] = \begin{cases} b^{n+K}, & \text{if } n \geq -K \\ 0, & \text{otherwise.} \end{cases}$$

Example 27

Now assume that

$$X(z) = \frac{z^K}{1 - bz^{-1}}, \quad \mathrm{ROC} = \{z : |z| < |b|\}.$$

Obtain $x[n]$.

Answer to Example 27

Left-sided sequence

$$
\begin{array}{r|llllllll}
 & -b^{-1}z^{K+1} & -b^{-2}z^{K+2} & -b^{-3}z^{K+3} & -b^{-4}z^{K+4} & -b^{-5}z^{K+5} & -b^{-6}z^{K+6} & -b^{-7}z^{K+7} & -\cdots \\ \hline
-bz^{-1}+1 & z^{K} & & & & & & & \\
 & -\; z^{K} & +b^{-1}z^{K+1} & & & & & & \\ \cline{2-3}
 & & b^{-1}z^{K+1} & & & & & & \\
 & & -b^{-1}z^{K+1} & +b^{-2}z^{K+2} & & & & & \\ \cline{3-4}
 & & & b^{-2}z^{K+2} & & & & & \\
 & & & -b^{-2}z^{K+2} & +b^{-3}z^{K+3} & & & & \\ \cline{4-5}
 & & & & b^{-3}z^{K+3} & & & & \\
 & & & & -b^{-3}z^{K+3} & +b^{-4}z^{K+4} & & & \\ \cline{5-6}
 & & & & & b^{-4}z^{K+4} & & & \\
 & & & & & -b^{-4}z^{K+4} & +b^{-5}z^{K+5} & & \\ \cline{6-7}
 & & & & & & b^{-5}z^{K+5} & & \\
 & & & & & & -b^{-5}z^{K+5} & +b^{-6}z^{K+6} & \\ \cline{7-8}
 & & & & & & & b^{-6}z^{K+6} &
\end{array}
$$

Answer to Example 27. Cont.

From the last result, we have

$$X(z) = \frac{z^K}{1 - bz^{-1}} = -b^{-1}z^{K+1} - b^{-2}z^{K+2} - b^{-3}z^{K+3} - b^{-4}z^{K+4} - b^{-5}z^{K+5} - \cdots$$

Equating the coefficient of z^{-n} with $x[n]$, it follows that

$$x[n] = \begin{cases} -b^{n+K}, & \text{if } n \leq -K - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Example 28

Obtain $x[n]$ from the following z-transform:

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}}, \quad \text{ROC} = \{z : |a| < |z|\}.$$

Answer to Example 28

Right-sided sequence

$$
\begin{array}{r|l}
 & 1-(a^2+b^2)z^{-2}+a^2(a^2+b^2)z^{-4}-a^4(a^2+b^2)z^{-6}+a^6(a^2+b^2)z^{-8}-a^8(a^2+b^2)z^{-10}+\cdots \\
\hline
1+a^2z^{-2} & 1-b^2z^{-2} \\
 & \underline{-1-a^2z^{-2}} \\
 & -(a^2+b^2)z^{-2} \\
 & \underline{(a^2+b^2)z^{-2}+a^2(a^2+b^2)z^{-4}} \\
 & a^2(a^2+b^2)z^{-4} \\
 & \underline{-a^2(a^2+b^2)z^{-4}-a^4(a^2+b^2)z^{-6}} \\
 & -a^4(a^2+b^2)z^{-6} \\
 & \underline{a^4(a^2+b^2)z^{-6}+a^6(a^2+b^2)z^{-8}} \\
 & a^6(a^2+b^2)z^{-8} \\
 & \underline{-a^6(a^2+b^2)z^{-8}-a^8(a^2+b^2)z^{-10}} \\
 & -a^8(a^2+b^2)z^{-10}
\end{array}
$$

Answer to Example 28. Cont.

From the last result, we find

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}} = 1 - (a^2 + b^2)z^{-2} + a^2(a^2 + b^2)z^{-4} - a^4(a^2 + b^2)z^{-6} + a^6(a^2 + b^2)z^{-8} - a^8(a^2 + b^2)z^{-10} + \cdots$$

Equating the coefficient of z^{-n} with $x[n]$, it follows that

$$x[n] = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^{n/2} a^{n-2}(a^2 + b^2), & \text{if } n > 0 \text{ and even,} \\ 0, & \text{otherwise.} \end{cases}$$

Example 29

Consider Example 29 with different ROC, i.e.,

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}}, \quad \mathrm{ROC} = \{z : |z| < |a|\}.$$

Answer to Example 29

Left-sided sequence

$$
\begin{array}{r|l}
 & -a^{-2}b^2 + a^{-2}(1+a^{-2}b^2)z^2 - a^{-4}(1+a^{-2}b^2)z^4 + a^{-6}(1+a^{-2}b^2)z^6 - a^{-8}(1+a^{-2}b^2)z^8 + \cdots \\
\hline
a^2z^{-2}+1 & -b^2z^{-2} + 1 \\
 & b^2z^{-2} + a^{-2}b^2 \\
\cline{2-2}
 & (1+a^{-2}b^2) \\
 & -(1+a^{-2}b^2) - a^{-2}(1+a^{-2}b^2)z^2 \\
\cline{2-2}
 & -a^{-2}(1+a^{-2}b^2)z^2 \\
 & a^{-2}(1+a^{-2}b^2)z^2 + a^{-4}(1+a^{-2}b^2)z^4 \\
\cline{2-2}
 & a^{-4}(1+a^{-2}b^2)z^4 \\
 & -a^{-4}(1+a^{-2}b^2)z^4 - a^{-6}(1+a^{-2}b^2)z^6 \\
\cline{2-2}
 & -a^{-6}(1+a^{-2}b^2)z^6
\end{array}
$$

Answer to Example 29. Cont.

Using the last result, we rewrite $X(z)$ as

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}} = -a^{-2}b^2 + a^{-2}(1 + a^{-2}b^2)z^2 - a^{-4}(1 + a^{-2}b^2)z^4 + a^{-6}(1 + a^{-2}b^2)z^6 - a^{-8}(1 + a^{-2}b^2)z^8 + \cdots$$

Thus, we have

$$x[n] = \begin{cases} -\left(\frac{b}{a}\right)^2, & \text{if } n = 0, \\ (-1)^{n/2+1} a^n \left(1 + \left(\frac{b}{a}\right)^2\right), & \text{if } n < 0 \text{ and even}, \\ 0, & \text{otherwise.} \end{cases}$$

Contents

Taylor Expansions

Consider the following Taylor expansions:

$$\frac{1}{(1-c)^{K+1}} = \sum_{n=K}^{\infty} \binom{n}{K} c^{n-K}$$

where $|c| < 1$.

The denominator of the z-transform must be factored to find the locations of the poles.

Then a partial fraction expansion can be performed to obtain the appropriate form with possibly complex coefficients.

Example 30

Let

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}}, \quad \mathrm{ROC} = \left\{z : |a| < |z|\right\}.$$

Obtain $x[n]$.

Answer to Example 30

The z-transform can be factored as

$$
\begin{aligned}
X(z) &= \frac{1}{1+a^2z^{-2}} - \frac{b^2z^{-2}}{1+a^2z^{-2}} \\
&= \sum_{n=0}^{\infty}\left(-a^2z^{-2}\right)^n - b^2z^{-2}\sum_{n=0}^{\infty}\left(-a^2z^{-2}\right)^n \\
&= \sum_{n=0}^{\infty}(-1)^n a^{2n}z^{-2n} - b^2z^{-2}\sum_{n=0}^{\infty}(-1)^n a^{2n}z^{-2n} \\
&= \sum_{n=0}^{\infty}(-1)^n a^{2n}z^{-2n} + b^2\sum_{n=1}^{\infty}(-1)^n a^{2n-2}z^{-2n}
\end{aligned}
$$

which results in

$$
\begin{aligned}
x[2n] &= (-1)^n a^{2n}u[n] + b^2(-1)^n a^{2n-2}u[n-1] \\
x[2n+1] &= 0
\end{aligned}
$$

Example 31

Now consider

$$X(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}}, \quad \mathrm{ROC} = \{z : |z| < |a|\}.$$

Obtain $x[n]$.

Answer to Example 31

We factored $X(z)$ as

$$
\begin{aligned}
X(z) &= \frac{a^{-2}z^2 - a^{-2}b^2}{1 + a^{-2}z^2} = \frac{a^{-2}z^2}{1 + a^{-2}z^2} - \frac{a^{-2}b^2}{1 + a^{-2}z^2} \\
&= a^{-2}z^2 \sum_{n=0}^{\infty} \left(- a^{-2}z^2 \right)^n - b^2 a^{-2} \sum_{n=0}^{\infty} \left(- a^{-2}z^2 \right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n a^{-2n-2} z^{2n+2} - b^2 \sum_{n=0}^{\infty} (-1)^n a^{-2n-2} z^{2n} \\
&= - \sum_{n=-\infty}^{-1} (-1)^n a^{2n} z^{-2n} - b^2 \sum_{n=-\infty}^{0} (-1)^n a^{2n-2} z^{-2n}
\end{aligned}
$$

Thus

$$
\begin{aligned}
x[2n] &= -(-1)^n a^{2n} u[-n-1] - b^2 (-1)^n a^{2n-2} u[-n] \\
x[2n+1] &= 0
\end{aligned}
$$

Example 32

Now consider

$$X(z) = \frac{b(a+b)z^{-2} - (2a+3b)z^{-1} + 3}{(1-az^{-1})(1-bz^{-1})^2}, \quad \mathrm{ROC} = \{z : |a| < |z| < |b|\}.$$

Obtain $x[n]$.

Answer to Example 32

Applying partial fraction expansion, we obtain

$$X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}} + \frac{1}{(1 - bz^{-1})^2}$$

Considering the ROC, we rewrite $X(z)$ as

$$\begin{aligned} X(z) &= \frac{1}{1 - az^{-1}} - \frac{b^{-1}z}{1 - b^{-1}z} + \frac{b^{-2}z^2}{(1 - b^{-1}z)^2} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} - b^{-1}z \sum_{n=0}^{\infty} b^{-n} z^n + b^{-2}z^2 \sum_{n=1}^{\infty} n b^{-n+1} z^{n-1} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n} - \sum_{n=-\infty}^{-2} (n+1) b^n z^{-n} \end{aligned}$$

Finally

$$x[n] = a^n u[n] - b^n u[-1-n] - (n+1) b^n u[-2-n].$$

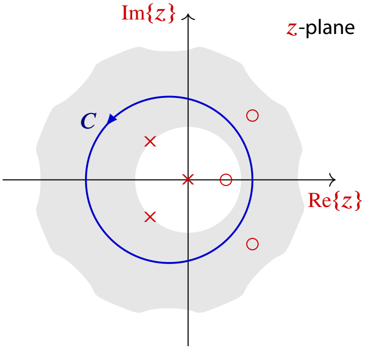
Contents

Definition

The inverse z-transform is given by

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

where C lies in the region of convergence of $X(z)$ and completely encloses the origin.



Residue Theorem

If $H(z)z^{n-1}$ has N poles (d_k for $k = 1, \ldots, N$), then

$$x[n] = \sum_{k=1}^{N} \operatorname{Res}\left\{X(z)z^{n-1} \text{ at } d_k\right\}$$

where $\operatorname{Res}\{\cdot\}$ is the **residue**.

Assuming that $X(z)z^{n-1}$ has an m-order pole at d_k, i.e.,

$$H(z)z^{n-1} = \frac{P(z)}{(z-d_k)^m},$$

the residue is defined as

$$\operatorname{Res}\left\{H(z)z^{n-1} \text{ at } d_k\right\} = \frac{1}{(m-1)!} \left.\frac{d^{m-1}P(z)}{dz^{m-1}}\right|_{z=d_k}.$$

Example 33

Consider

$$X(z) = \frac{b(a+b)z^{-2} - (2a+3b)z^{-1} + 3}{(1-az^{-1})(1-bz^{-1})^2}, \quad \text{ROC} = \{z : |a| < |z| < |b|\}.$$

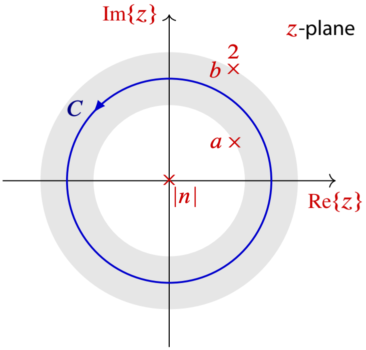
Obtain $x[n]$.

Answer to Example 33 (n negative)

The function $X(z)z^{n-1}$ is written as

$$X(z)z^{n-1} = \frac{b(a+b) - (2a+3b)z + 3z^2}{(z-a)(z-b)^2} z^n.$$

Using $n < 0$, the contour C encloses the poles d_1 and d_2, where $d_1 = 0$ and $d_2 = a$.



Answer to Example 33 (n negative). Cont.

Using $m = -n$, the first residue is computed by

$$
\begin{aligned}
R_1 &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ \frac{b(a+b) - (2a+3b)z + 3z^2}{(z-a)(z-b)^2} \right\}_{z=0} \\
&= -a^{-m} + (m-2)b^{-m},
\end{aligned}
$$

Similarly, the second residue is

$$
\begin{aligned}
R_2 &= \left. \frac{b(a+b) - (2a+3b)z + 3z^2}{z^m (z-b)^2} \right|_{z=a} \\
&= a^{-m}
\end{aligned}
$$

Thus for $n < 0$, we have

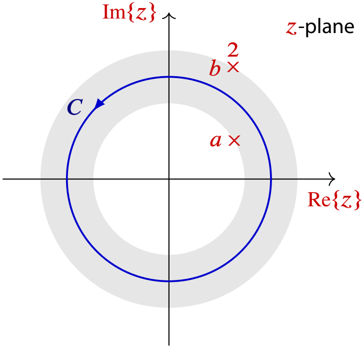
$$
\begin{aligned}
x[n] &= \left[R_1 + R_2 \right]_{m=-n} \\
&= -(n+2)b^n
\end{aligned}
$$

Answer to Example 33 ($n \geq 0$)

For $n \geq 0$,

$$X(z)z^{n-1} = \frac{b(a+b) - (2a+3b)z + 3z^2}{(z-a)(z-b)^2} z^n,$$

In this case, the contour C encloses the pole $d_1 = a$.



Answer to Example 33 ($n \geq 0$). Cont.

The residue is computed by

$$\begin{aligned} R_1 &= \left. \frac{b(a+b) - (2a+3b)z + 3z^2}{(z-b)^2} z^n \right|_{z=a} \\ &= a^n \end{aligned}$$

Consequently,

$$x[n] = a^n u[n]$$

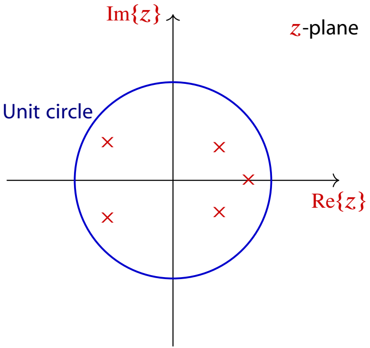
Finally,

$$x[n] = a^n u[n] - (n+2) b^n u[-1-n].$$

Contents

Definition

Each pole of a **casual stable system** must lie within the unit circle in the z-plane.



Homework

- Textbook [3] Problems 4.41, 4.43, 4.48, 4.49, 4.50, 4.52, 4.53, 4.54, 4.58

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill