Signals and Communication Theory

Sampling Process

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Introduction

The transmitted signal in wireless communications systems are ideally bandlimited.
The signal bandwidth is the range of frequencies where $X(f) \neq 0$.

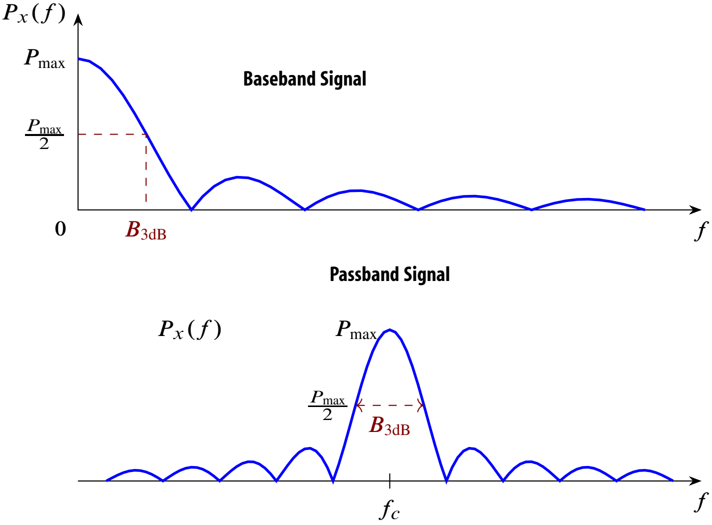
Unfortunately, real signal are not bandlimited.

There exist some alternative definitions based on energy and power spectral density.

Here we assume that the **Power Spectral Density, PSD,** is available.

Half-Power Bandwidth

The Half-Power Bandwidth is defined as the range of frequencies where the power spectral density is at least 50% of its maximum value. $B_{3\text{dB}} = f_{3\text{dB}}^{(h)} - f_{3\text{dB}}^{(\ell)}$.



XdB Bandwidth

The XdB Bandwidth is defined as the range of frequencies where the power spectral density is larger than $P_{\max} - X$ dB, that is,

$$B_{X\,\mathrm{dB}} = f_{X\,\mathrm{dB}}^{(h)} - f_{X\,\mathrm{dB}}^{(\ell)},$$

where $f_{X\,\mathrm{dB}}^{(h)}$ and $f_{X\,\mathrm{dB}}^{(\ell)}$ satisfy $P_x\big(f_{X\,\mathrm{dB}}^{(\ell)}\big) = P_x\big(f_{X\,\mathrm{dB}}^{(h)}\big) = P_{\max} - X,\ \mathrm{dB}$.

Noise Equivalent Bandwidth

The noise equivalent bandwidth is defines as

$$B = \frac{1}{P_x(f_c)} \int_{-\infty}^{\infty} P_x(f)\, df.$$

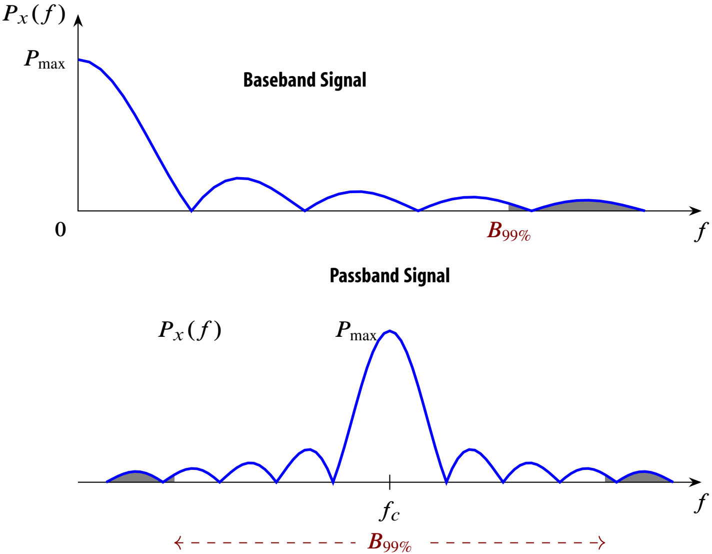
Fractional Power Containment Bandwidth

The most useful bandwidth is the Fractional Power Containment Bandwidth, which is defined by

$$\int_0^{B/2} P_x(f)\,df = \alpha \int_{-\infty}^{\infty} P_x(f)\,df,$$

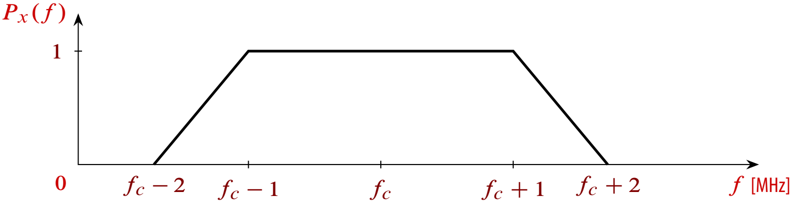
where α is the **fraction of containment**.

Illustrating Fractional Power Containment Bandwidth, $\alpha = 0.99$



Example 1

Consider the following PSD:



From the figure, we have $f_{3\text{dB}}^{(\ell)} = f_c - 1.5\,\text{MHz}$ and $f_{3\text{dB}}^{(h)} = f_c + 1.5\,\text{MHz}$, thus

$$B_{3\text{dB}} = f_{3\text{dB}}^{(h)} - f_{3\text{dB}}^{(\ell)} = 3\,\text{MHz}.$$

For the **Noise equivalent bandwidth**, the area under the curve $P_x(f)$ is 3. Consequently, the following equation holds.

$$B_{\text{NoiseEq}} = \frac{1}{P_x(f_c)} \int_{-\infty}^{\infty} P_x(f)\, df = 3\,\text{MHz}.$$

Example 1, cont.

Finally, for the **Fractional Containment Bandwidth** with $\alpha = 0.9$, we should find B such that the following relation holds.

$$\int_{f_c - B/2}^{f_c + B/2} P_x(f)\,df = \alpha \int_{-\infty}^{\infty} P_x(f)\,df = 2.7$$

Solving for B, we arrive at

$$B = 2(2 - \sqrt{0.3}) \approx 2.9\,\mathrm{MHz}.$$

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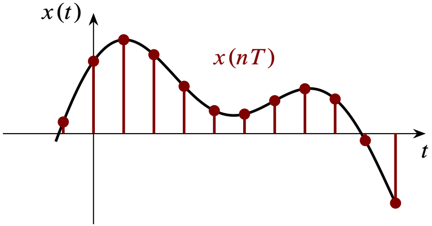
Introduction

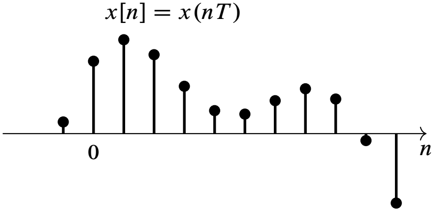
The fundamental connection between digital communication and digital signal processing is through the sampling theorem.

Communication system can be considered banlimited.

Because of the bandlimited property, it is possible to sample periodically the signal of interest.

Nyquist Sampling Theorem





Let $x(t)$ be a bandlimited signal, which means $X(f) = 0$ for $f \geq f_N$. Then $x(t)$ is uniquely determined by its samples $\{x[n] = x(nT)\}_{n=-\infty}^{\infty}$ if the **sampling frequency** satisfies

$$f_s = \frac{1}{T} > 2f_N,$$

where f_N is the **Nyquist frequency** and $2f_N$ is generally known as the **Nyquist rate**. Furthermore, the signal can be reconstructed from its samples through the reconstruction equation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}\left(\frac{t - nT}{T}\right).$$

Example 2

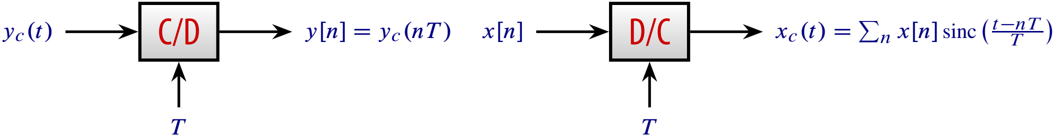
- Voice
- Sound
- Image

What is the Nyquist frequency?

What is the Nyquist rate?

Sampling Blocks

The sampling and reconstruction processes are assumed to be ideal and they are represented by the block diagrams C/D (**continuous-to-discrete**) converter and D/C (**discrete-to-continuous**) converter.



In practice, the C/D would be performed using an ADC and the D/C would be performed using a DAC. Practical ADC and DAC introduce additional distortions to the signal.

Fourier transforms of sampling signals

Let $X_c(f)$ be the Fourier transform of the original signal $x_c(t)$. Then the Fourier transform $X_s(f)$ of the sampled signal $x_c(nT)$ is given by

$$X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(f - \frac{n}{T}\right)$$

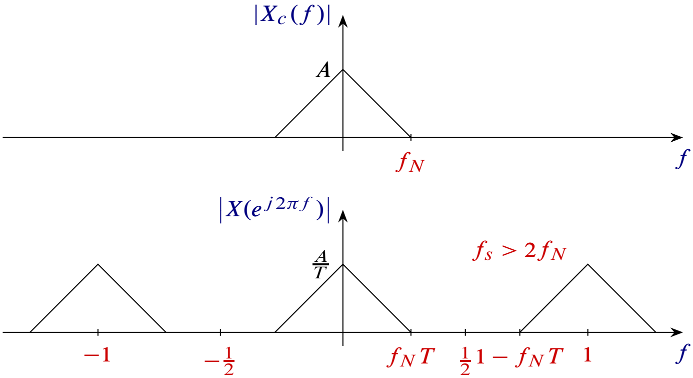
In this setting, the Discrete-time Fourier Transform $X\left(e^{j2\pi f}\right)$ of $x[n]$ is expressed as

$$\begin{aligned} X\left(e^{j2\pi f}\right) &= X_s\left(\frac{f}{T}\right) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{f-n}{T}\right). \end{aligned}$$

Signal recovery in frequency domain

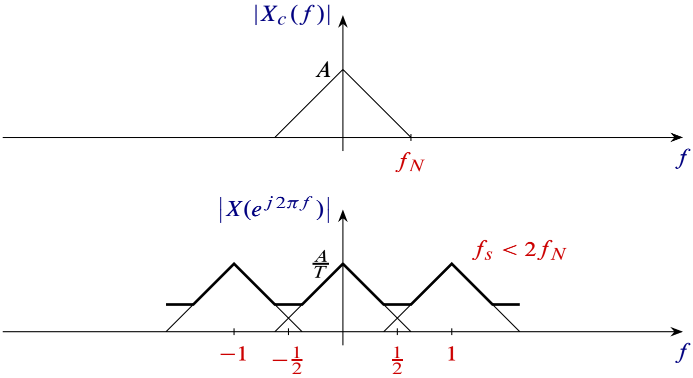
Assume $x_c(t)$ is bandlimited, that is, $X_c(f) = 0$ for $f > f_N$. If the sampling period T satisfies $T < \frac{1}{2f_N}$, then there is no overlap among replicas and

$$X\left(e^{j2\pi f}\right) = \frac{1}{T} X_c\left(\frac{f}{T}\right), \quad |f| < \frac{1}{2}.$$



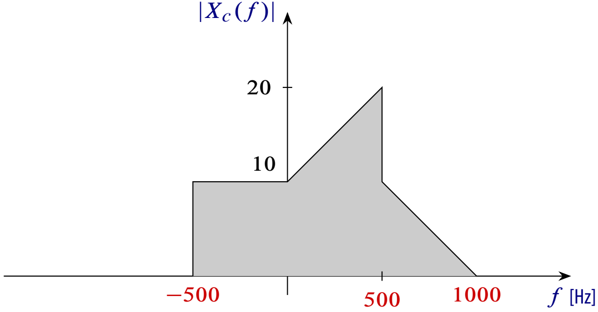
Aliasing

Now assume that the sampling period T satisfies $T > \frac{1}{2f_N}$. In this case, there is overlap among replicas, i.e., there exists **aliasing**.

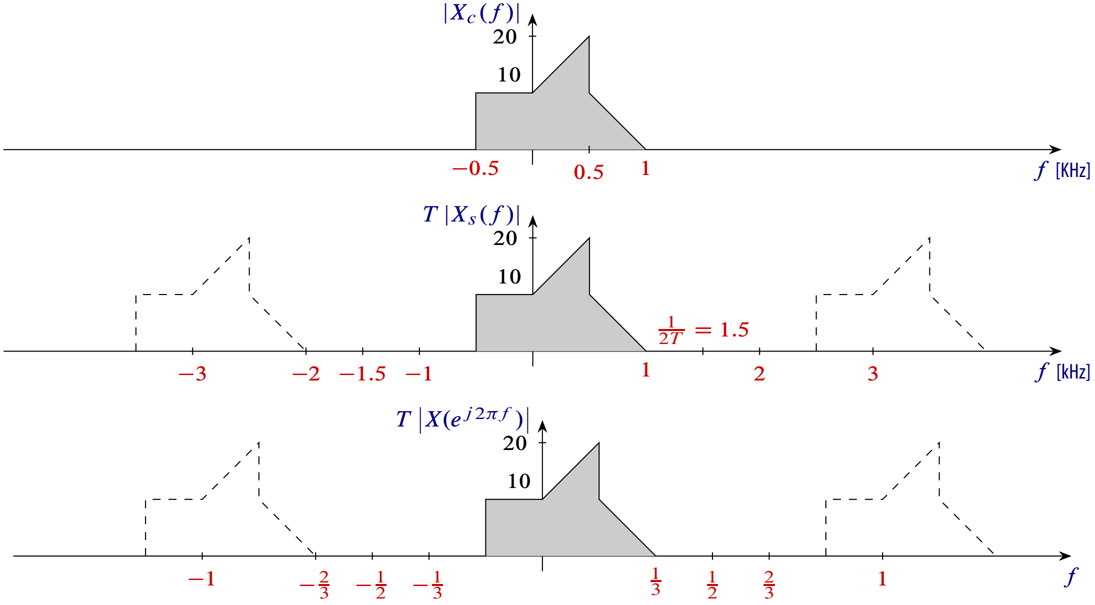


Example 3

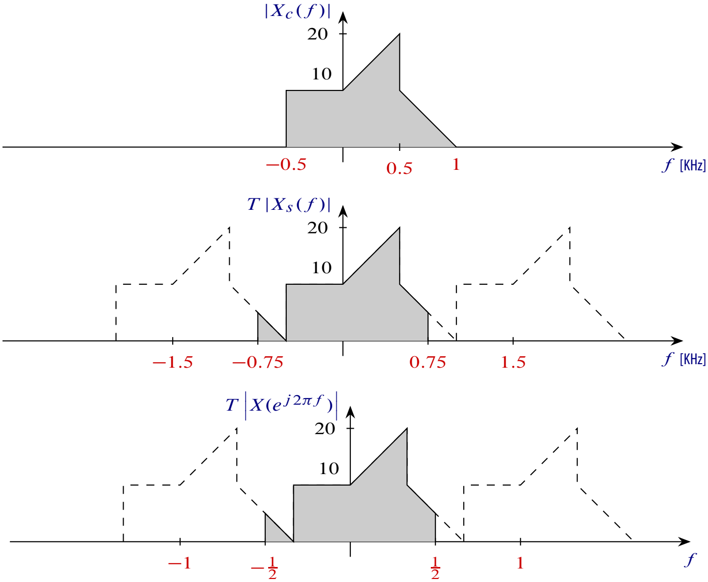
Let $x_c(t)$ be a bandlimited signal with $f_N = 1000$ Hz, where the corresponding magnitude spectrum is shown below.



Example 3. Sampling frequency $f_s = 3000$



Example 3. Sampling frequency $f_s = 1500$



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Introduction

Digital Signal Processing is the workhorse of digital communication systems.

Consider the channel system $h_c(t)$ that processes a **bandlimited signal** $x(t)$ and outputs the signal $y(t)$.

$$x(t) \longrightarrow \boxed{h_c(t)} \longrightarrow y(t)$$

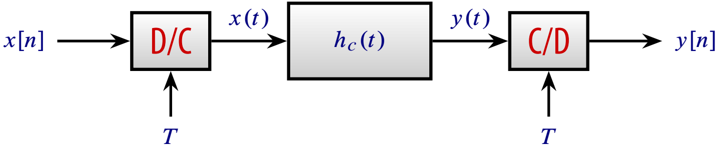
Bandlimited implications

The fact that $x(t)$ is bandlimited has three **important implications**:

1. The signal $x(t)$ can be generated in discrete time using a D/C converter operating at an appropriate sampling frequency.
2. The signal $y(t)$ is also bandlimited since $X(f)$ is bandlimited and $Y(f) = H_c(f)X(f)$. Therefore, $y(t)$ can be processed in discrete time using a C/D converter operating at an appropriate sampling frequency.
3. Only the part of the channel $H_c(f)$ that lives in the bandwidth of $X(f)$ is important to $Y(f)$.

Relationships among the signals

Using the **Implications 1 and 2**, the system can be redraw as shown below.



Now, we will obtain the equation for $y[n]$ as follows

$$
\begin{aligned}
y[n] &= y(nT) \\
y(t) &= \int_{-\infty}^{\infty} h_c(\tau)x(t-\tau)\,d\tau \\
x(t) &= \sum_{m=-\infty}^{\infty} x[m]\,\mathrm{sinc}\left(\frac{t-mT}{T}\right)
\end{aligned}
$$

Combining these equations gives

$$
y[n] = \sum_{m=-\infty}^{\infty} x[m] \int_{-\infty}^{\infty} h_c(\tau)\,\mathrm{sinc}\left(n - m - \frac{\tau}{T}\right)\,d\tau
$$

Discrete-time equivalent

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

We know that $y[n]$ can be expressed as

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m] \int_{-\infty}^{\infty} h_c(\tau)\, \mathrm{sinc}\left(n - m - \frac{\tau}{T}\right) d\tau \\ &= \sum_{m=-\infty}^{\infty} x[m] h[n-m] \end{aligned}$$

where we identify the discrete-time equivalent channel $h[n]$ as

$$h[n] = \int_{-\infty}^{\infty} h_c(\tau)\, \mathrm{sinc}\left(n - \frac{\tau}{T}\right) d\tau$$

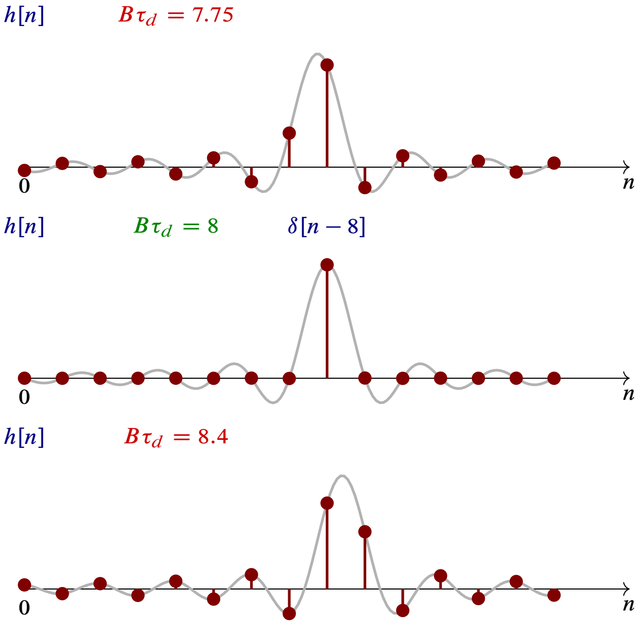
Example 4

Suppose that the channel system delays the input by an amount τ_d, that is, $h_c(t) = \delta(t - \tau_d)$. Determine the discrete-time equivalent of the channel assuming that the input signal has a bandwidth of $B/2$.

Answer: According to the Sampling Theorem, the sampling frequency f_s can be chosen as $f_s = 2(B/2) = B$ and, consequently, $T = 1/B$. Therefore, the discrete-time equivalent is expressed as

$$\begin{aligned}
h[n] &= \int_{-\infty}^{\infty} h_c(\tau)\,\mathrm{sinc}\left(n - \frac{\tau}{T}\right)\,d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau - \tau_d)\,\mathrm{sinc}\,(n - B\tau)\,d\tau \\
&= \mathrm{sinc}\,(n - B\tau_d)\,.
\end{aligned}$$

Example 4. Graphs



Homework

- A real-valued signal $x(t)$ is known to be uniquely determined by its samples when the sampling frequency is $f_s = 5000$ Hz. For what values of f is $X(f)$ guaranteed to be zero?
- The frequency which, under the sampling theorem, must be exceeded by the sampling frequency is called the **Nyquist rate**. Determine the Nyquist rate corresponding to each of the following signals:
  - $x(t) = 1 + \cos(2000\pi t) + \sin(4000\pi t)$
  - $x(t) = \frac{\sin(4000\pi t)}{\pi t}$
  - $x(t) = \left(\frac{\sin(4000\pi t)}{\pi t}\right)^2$
- Let $x(t)$ be a signal with Nyquist rate f_0. Determine the Nyquist rate for each of the following signals:
  - $x(t) + x(t-1)$
  - $\frac{dx(t)}{dt}$
  - $x^2(t)$
- Problem 5.76 from [3]