Signals and Communication Theory

Communication Theory and Signals and Systems

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Introduction

Fundamental Problem of Communication[a]

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message at another point.

Analog Communication

In an analog communication, the message is a continuous-time signal. For example, a voltage signal that corresponds to a voice signal.

Digital Communication

In a digital communication the message is a symbol. Usually, binary digits. For example, the sequence of binary symbols, which are obtained from a voice signal by sampling, quantizing, and coding.

[a] C. E. Shannon, "A mathematical theory of communication," in *The Bell System Technical Journal*, vol. 27, no. 3, pp. 379-423, July 1948, doi: 10.1002/j.1538-7305.1948.tb01338.x.

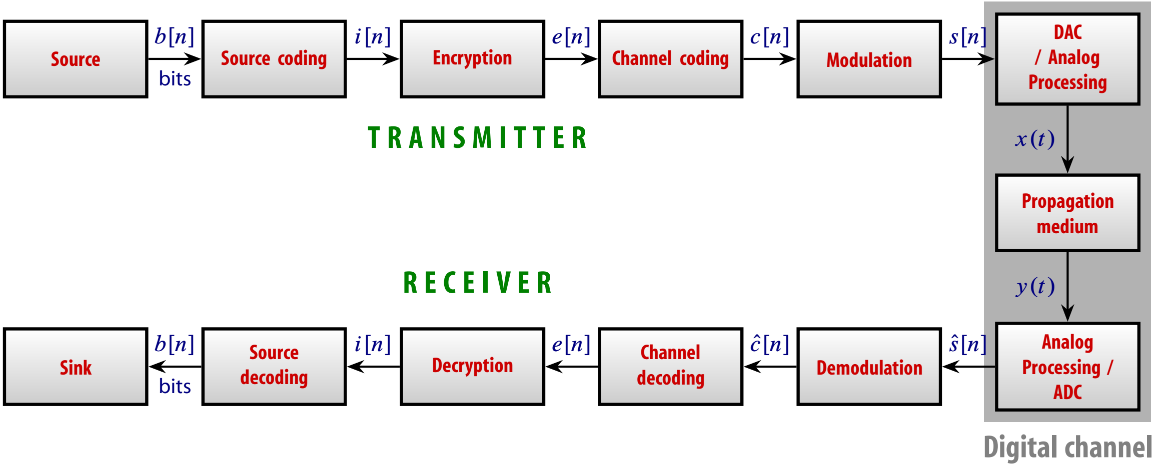
Analog and Digital Communications

Digital Communication

Digital communication offers

- higher quality
- increased security
- better robustness to noise
- reduction of power of usage
- easy integration (voice, text, video)
- easy to reconfigure (Software Defined Radio, SDR)

Typical Digital Communication System



Block Descriptions

Source Coding/Decoding **The main objective of the source coding is to compress the information while source decoding decompresses it**. In our model, $s[n]$ are the inputs bits, while $i[n]$ are de coded bits. Similarly, the input and output at the source decoding are, respectively, $i[n]$ and $s[n]$.

Encryption/Decryption **The purpose of encryption is to protect the information from unauthorized users**. To do that, the input $i[n]$ is transformed to $e[n]$, where $e[n] = p(i[n])$ and $p(\cdot)$ is a lossless transformation. Similarly, the decryption applies the inverse transformation $p^{-1}(\cdot)$ to $e[n]$ to obtain the decrypted bits $i[n]$.

Block Descriptions, cont.

Channel Coding/Decoding **In order to provide resilience to channel distortions, channel coding introduces controlled redundancy to $e[n]$ to obtain** $c[n]$. In this way, for each k bits of information there are r bits of redundancy. The code rate is defined by $k/(k+r)$. **The redundant bits may enable to detect error bits and even correct them**. In setting, because of the channel distortions, the input to the channel decoding is $\hat{c}[n]$ and the channel decoding corrects the error bits to obtain $e[n]$.

Modulation/Demodulation The first stage of the modulation maps the input bits $c[n]$ to symbols $s[n]$. At the second stage, the symbols $s[n]$ are converted to a corresponding signal $x(t)$. For the demodulation process, the signal $y(t)$ is first converted to symbols $\hat{s}[n]$ and finally they are mapped to bits $\hat{c}[n]$.

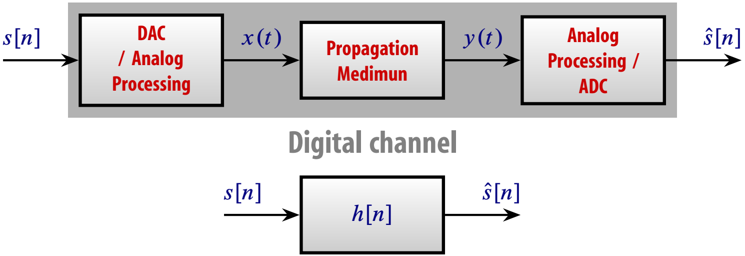
Block Descriptions, cont.

Propagation medium The signal $x(t)$ travels through a propagation medium (propagation channel). Some examples could be

- a radio wave, $x(t)$, through a wireless environment (**propagation channel includes propagation effects such as reflection, transmission, diffraction, and scattering**),
- a current, $x(t)$, through a telephone wire,
- an optical signal, $x(t)$, through a fiber.

Digital Channel

From the digital signal processing point of view, the digital channel encompasses the DAC/Analog Processing, Propagation medium, and ADC/Analog Processing blocks. The corresponding impulse response of the digital channel is denoted by $h[n]$.



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Signals

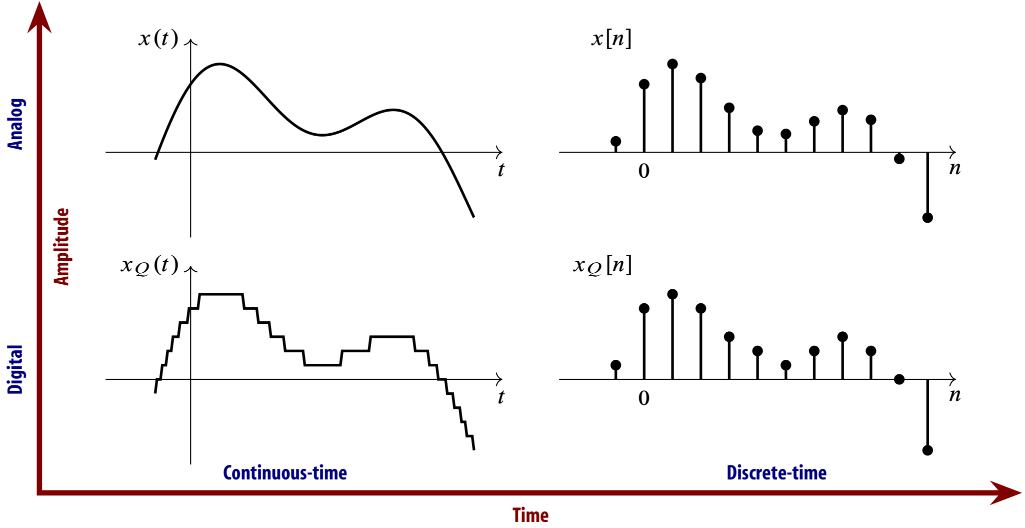
Signal

A signal is a measurable quantity (amplitude) that depends on independent measurable quantities (time and/or space).

A **continuous-time signal** possesses an amplitude value for all real values of time. On the other hand, a **discrete-time signal** possesses an amplitude for some values of time, which are obtained by sampling.

Similarly, an **analog signal** contains a continuous set of amplitude values while a **digital signal** possesses a discrete set of amplitude values. Usually, those discrete amplitude values are obtained by quantizing an analog signal.

Types of Signals



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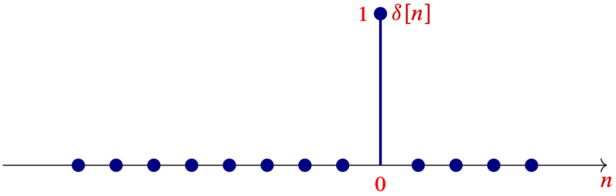
Unit-pulse sequence (impulse function)

The unit-pulse, denoted $\delta[n]$, is defined according to

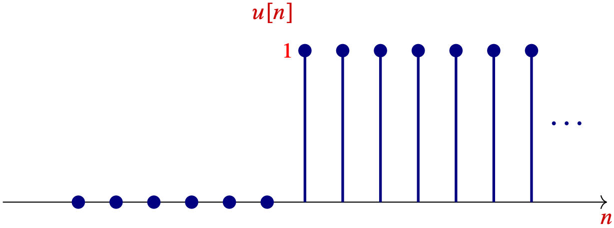
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Unit-Step sequence

The unit-step sequence is defined as

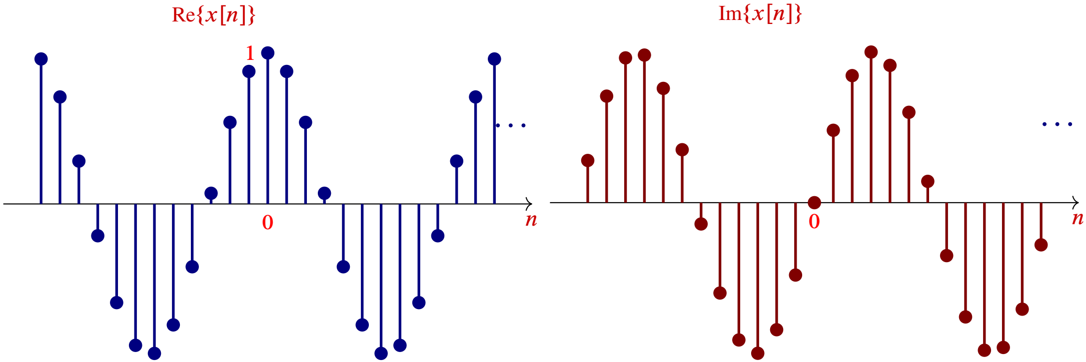
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



Complex exponential

The complex exponential sequence is expressed as

$$x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\,\sin(\omega_0 n), \qquad 0 \leq \omega_0 < 2\pi$$

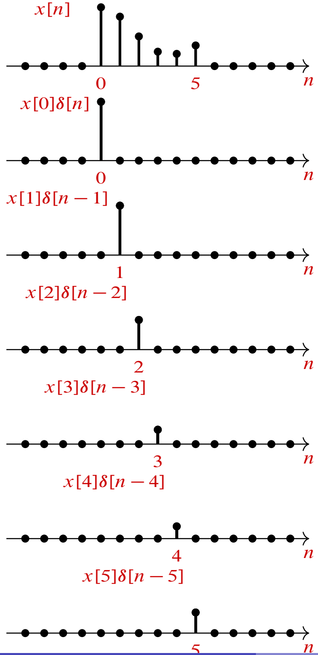


Signal $x[n]$ as superposition of unit-pulse sequences

The signal $x[n]$ can be viewed as the superposition in time of a set of scaled unit-sample sequences, that is,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Visual proof



$$\begin{aligned} x[n] &= x[0]\delta[n] + x[1]\delta[n-1] \\ &\quad + x[2]\delta[n-2] + x[3]\delta[n-3] \\ &\quad + x[4]\delta[n-4] + x[5]\delta[n-5] \\ &= \sum_{k=0}^{5} x[k]\delta[n-k] \end{aligned}$$

Homework 1

- Problem 1.47 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill

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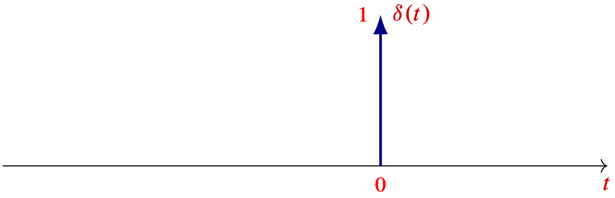
Delta Dirac function

The delta function, $\delta(t)$, is defined as

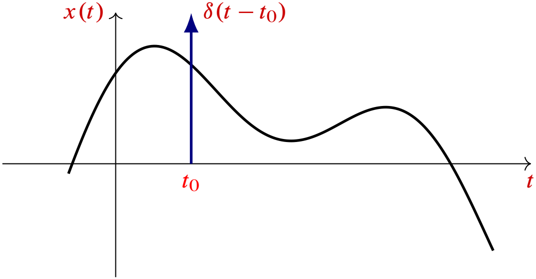
$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

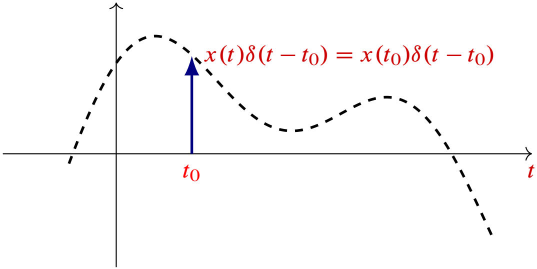
which satisfies

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$



Integral property



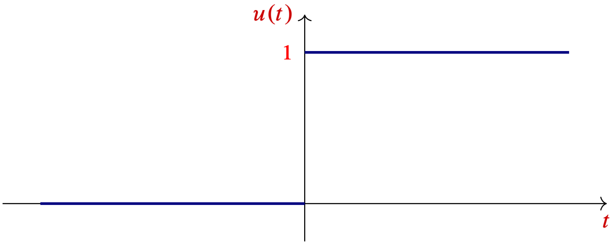


$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt &= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)\,dt \\
&= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt \\
&= x(t_0)
\end{aligned}
$$

Unit-step function

The Unit-step function is expressed as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Unit-step and delta functions are related as

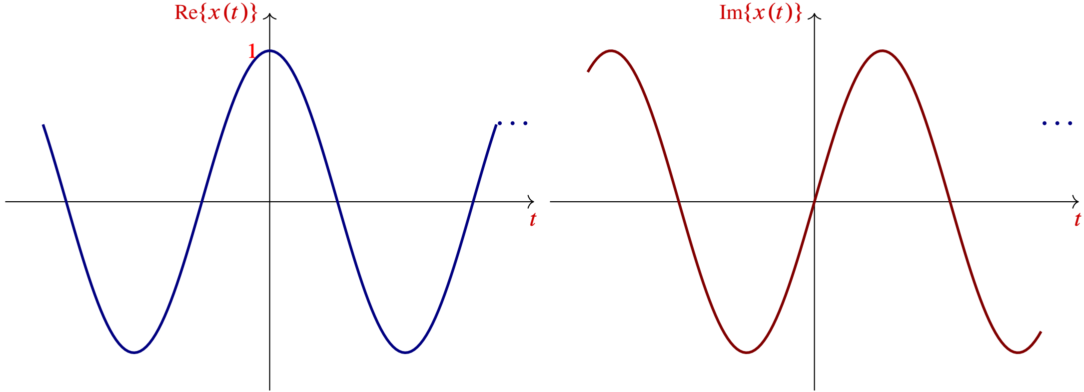
$$\frac{du(t)}{dt} = \delta(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Complex exponential

The complex exponential funtion is expressed as

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\,\sin(\omega_0 t), \qquad 0 \leq \omega_0 < \infty$$



Signal $x(t)$ as superposition of delta functions

Similar to the discrete-time case, the signal $x(t)$ can be expressed as the superposition delta functions, this means,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau$$

Homework 2

- Problem 1.55 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill

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Signal Processing

The main goal of signal processing is to modified (Process) a signal in order to obtain useful information.

Systems

Systems process signals

The input signal is processed by the system and the processed signal is obtained at the output.

Linear Systems

$$\alpha x_2[n] + \beta x_2[n] \longrightarrow \boxed{\textbf{System}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

A linear system satisfies the following Properties:

$$x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n] \qquad \textbf{Additivity Property}$$

$$\alpha x[n] \rightarrow \alpha y[n] \qquad \textbf{Homogeneity Property}$$

The signal $y_1[n]$ is the response at the input signal $x_1[n]$. Similarly, the signal $y_2[n]$ is the output for $x_2[n]$ at the input. Those Properties can be summarized as

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha y_1[n] + \beta y_2[n]$$

Example 1

Consider a system where the input-output relationship is expressed as $y[n] = ax[n] + b$*, where* a *and* b *are constants.*

Example 1: Answer

The first step is to consider the input $x_1[n]$, which produces the output $y_1[n]$ as

$$y_1[n] = ax_1[n] + b$$

In a similar form, for $x_2[n]$, we have

$$y_2[n] = ax_2[n] + b$$

As a second step, we assume that the input is given by $\alpha x_1[n] + \beta x_2[n]$. Consequently, the output satisfies

$$\begin{aligned} y[n] &= a(\alpha x_1[n] + \beta x_2[n]) + b \\ &= \alpha a x_1[n] + \beta a x_2[n] + b \\ &\neq \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Since $\alpha y_1[n] + \beta y_2[n]$ is not the response to $\alpha x_1[n] + \beta x_2[n]$, **the system is not linear.**

Example 2

In this example, we consider the following system $y[n] = x[n]\cos(\omega_0 n)$.

Example 2: Answer

First step:

$$y_1[n] = x_1[n]\cos(\omega_0 n)$$
$$y_2[n] = x_2[n]\cos(\omega_0 n)$$

Second step:

$$\begin{aligned} y[n] &= (\alpha x_1[n] + \beta x_2[n])\cos(\omega_0 n) \\ &= \alpha x_1[n]\cos(\omega_0 n) + \beta x_2[n]\cos(\omega_0 n) \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Since $\alpha y_1[n] + \beta y_2[n]$ is the output for the input $\alpha x_1[n] + \beta x_2[n]$, **the system is linear.**

Example 3: **Accumulator**

Consider the **Accumulator** *system that has the input-output relationship*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Example 3: Answer

First step:

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$$

$$y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$$

Second step:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} (\alpha x_1[k] + \beta x_2[k]) \\ &= \alpha \sum_{k=-\infty}^{n} x_1[k] + \beta \sum_{k=-\infty}^{n} x_2[k] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

The system is linear.

Example 4: **Difference Equation**

Consider the system, which is defined by the following **Difference Equation***:*

$$y[n] = \sum_{k=-M_1}^{M_2} a_k x[n-k]$$

Example 4: Answer

First step:

$$y_1[n] = \sum_{k=-M_1}^{M_2} a_k x_1[n-k]$$

$$y_2[n] = \sum_{k=-M_1}^{M_2} a_k x_2[n-k]$$

Second step:

$$\begin{aligned} y[n] &= \sum_{k=-M_1}^{M_2} a_k (\alpha x_1[n-k] + \beta x_2[n-k]) \\ &= \alpha \sum_{k=-M_1}^{M_2} a_k x_1[n-k] + \beta \sum_{k=-M_1}^{M_2} a_k x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

The system is linear.

Example 5

An special case of Example 4 is the system delay, that is,

$$y[n] = x[n - k_0],$$

which is a linear system.

Homework 3(a)

Assume that a system is represented by

$$y[n] = x^2[n],$$

Is the system linear?

Time-Invariant Systems

A time-invariant system satisfies the following Property:

$$x[n-n_0] \rightarrow y[n-n_0].$$

That means. If the input signal $x[n]$ is delayed by n_0, then the output signal $y[n]$ is delayed by n_0.

Example 6

Consider the system in Example 1, that is,

$$y[n] = ax[n] + b$$

Example 6: Answer

The first step is to consider the delayed input $x[n - n_0]$, which produces the output $y_0[n]$ as

$$y_0[n] = ax[n - n_0] + b$$

As a second step, we assume that the output is delayed by n_0. To be precise

$$\begin{aligned} y[n - n_0] &= ax[n - n_0] + b \\ &= y_0[n] \end{aligned}$$

Since the response to a delayed input $x[n - n_0]$ is $y[n - n_0]$, **the system is time-invariant**. As a consequence, the system is called nonlinear and time-invariant system.

Example 7

In this example, we consider the linear system $y[n] = x[n]\cos(\omega_0 n)$.

Example 7: Answer

First step:

$$y_0[n] = x[n - n_0] \cos(\omega_0 n)$$

Second step:

$$\begin{aligned} y[n - n_0] &= x[n - n_0] \cos\left(\omega_0 (n - n_0)\right) \\ &\neq y_0[n] \end{aligned}$$

Since $y[n - n_0]$ is not the output for the input $x[n - n_0]$, **the system is time-variant** (Linear and Time-Variant system).

Example 8: **Accumulator**

Consider the **Accumulator** *system.*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Example 8: Answer

First step:

$$y_0[n] = \sum_{k=-\infty}^{n} x[k - n_0]$$

Second step:

$$\begin{aligned} y[n - n_0] &= \sum_{k=-\infty}^{n-n_0} x[k] \qquad \text{using } m = k + n_0 \\ &= \sum_{m=-\infty}^{n} x[m - n_0] \\ &= y_0[n] \end{aligned}$$

The system is time-invariant (Linear and Time-Invariant system).

Example 9: **Difference Equation**

Now consider the system in Example 4.

$$y[n] = \sum_{k=-M_1}^{M_2} a_k x[n-k]$$

Example 9: Answer

First step:

$$y_0[n] = \sum_{k=-M_1}^{M_2} a_k x[n-k-n_0]$$

Second step:

$$\begin{aligned} y[n-n_0] &= \sum_{k=-M_1}^{M_2} a_k x[n-n_0-k] \\ &= y_0[n] \end{aligned}$$

The system is time-invariant (Linear and Time-Invariant system)

Homework 3(b)

Using the system in Homework 3(a)

$$y[n] = x^2[n],$$

answer the question Is the system time-invariant?

Linear Time-Invariant Systems

In our course, we consider that the systems are **Linear and Time-Invariant Systems** (LTI systems).

Impulse response

The impulse response $h[n]$ is the response of the system when the input is the unit-pulse $\delta[n]$.

Example 10

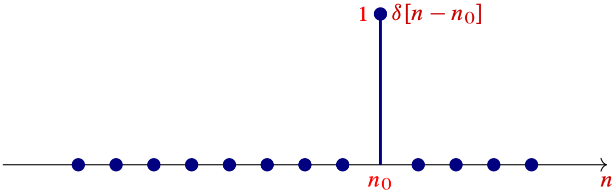
Obtain the impulse response $h[n]$ *for the following LTI system:*

$$y[n] = x[n - n_0].$$

Example 10: Answer

Using $x[n] = \delta[n]$ we have

$$h[n] = \delta[n - n_0].$$



Example 11

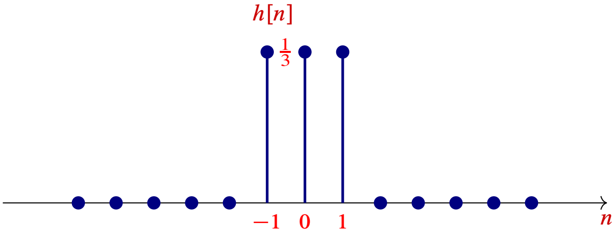
Obtain the impulse response $h[n]$ *for the following LTI system:*

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big).$$

Example 11: Answer

Using $x[n] = \delta[n]$ gives

$$h[n] = \frac{1}{3}\Big(\delta[n-1] + \delta[n] + \delta[n+1]\Big)$$



Example 12: Accumulator

For this example consider the Accumulator, i.e.,

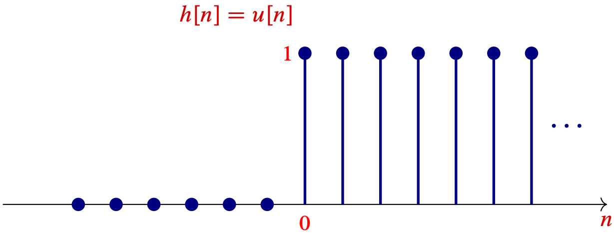
$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

Obtain the impulse response $h[n]$.

Example 12: Answer

Substituting $x[n] = \delta[n]$ into $y[n]$, we obtain

$$h[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$



Example 13: First order difference equation

Now consider that the system is given by

$$y[n] = ay[n-1] + x[n],$$

where $y[n] = 0$ *for* $n < 0$.

Example 13: Answer

Using $x[n] = \delta[n]$, we have

$$h[n] = ah[n-1] + \delta[n]$$

In order to obtain a closed form equation for $h[n]$, we substitute $n = 0, 1, \ldots,$

$$\begin{aligned}
h[0] &= ah[-1] + \delta[0] \\
&= 0 + 1 \\
h[1] &= ah[0] + \delta[1] \\
&= a + 0 \\
h[2] &= ah[1] + \delta[2] \\
&= a^2 + 0 \\
h[3] &= ah[2] + \delta[3] \\
&= a^3 + 0
\end{aligned}$$

Hence

$$h[n] = a^n u[n]$$

IIR and FIR systems

When a system produces an **impulse response** that has an **infinite duration**, it is called an **infinite-impulse response** (**IIR**) system.

On the other hand, if the system has an **impulse response** having a **finite duration**, then it is called an **finite-impulse response** (**FIR**) system.

Homework 4

Obtain the impulse response for the following system:

$$y[n] = 2r\cos(\omega_0)y[n-1] - r^2 y[n-2] + x[n] - r\cos(\omega_0)x[n-1],$$

where $y[n] = 0$ *for* $n < 0$. *Is the considered system FIR? Justify your answer.*

Causality criterion

A system is said to be **causal** when the response of the system depends on only **the present and past values** of the input.

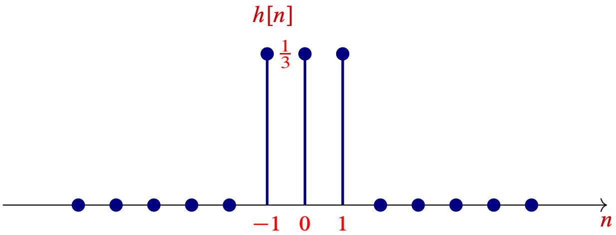
Since $h[n]$ is the response of a system to a unit-sample sequence, whose nonzero element occurs at $n = 0$, an LTI system is causal if and only if

$$h[n] = 0, \qquad n < 0.$$

Example 14

The impulse response of the system described in Example 11 is

$$h[n] = \frac{1}{3}\Big(\delta[n-1] + \delta[n] + \delta[n+1]\Big)$$

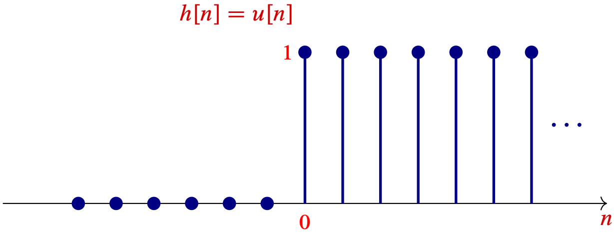


Is this system causal?

Example 15

Is the Accumulator causal?

$$h[n] = u[n]$$



Stable system

A system is stable if every bounded input sequence $x[n]$ produces a bounded output sequence $y[n]$ (**BIBO stable**)[1]

The input $x[n]$ is bounded if there exist a fixed positive finite value B_x, such that,

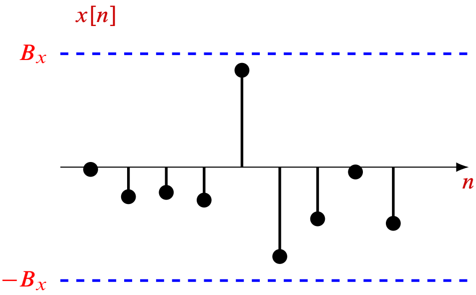
$$|x[n]| \leq B_x < \infty$$

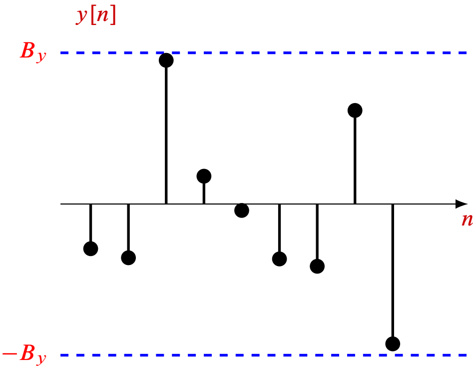
Similarly, the output $y[n]$ is bounded if there exist a fixed positive finite value B_y, such that,

$$|y[n]| \leq B_y < \infty$$

[1] Bounded Input Bounded Output, BIBO

Bounded Input and Bounded Output





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Summary

Linear Systems $\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$.

Time-Invariant System $x(t - t_0) \rightarrow y(t - t_0)$.

Impulse Response $\delta(t) \rightarrow h(t)$.

BIBO stable A bounded input $x(t)$ produces a bounded output $y(t)$.

Homework 5

Obtain the impulse response of the circuit shown below. Assume that the capacitor satisfies $v_o(0^-) = 0$.