Signals and Communication Theory

Continuous-time Fourier Transform

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Definition

Fourier transform is a useful tool for the analysis of wireless communication systems. It is defined as

Analysis (Direct transform): $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt,$$

Synthesis (Inverse transform): $$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df.$$

Note that f and t are continuous variables.

The Fourier transform $X(f)$ is referred to as the **spectrum** of $x(t)$ and gives the information of the frequencies that are contained in the signal $x(t)$.

Magnitude and Phase Spectra

In general, the Fourier transform is a complex-valued function and can be expressed as

$$X(f) = |X(f)|e^{j\angle X(f)}$$

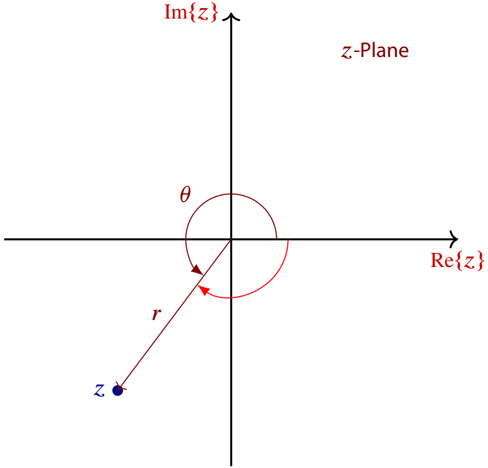
where $|X(f)|$ and $\angle X(f)$ are called the **magnitude** and **phase spectra** of $x(t)$.

They can be computed as

$$|X(f)| = \sqrt{X_R^2(f) + X_I^2(f)},$$
$$\angle X(f) = \tan^{-1}\left(\frac{X_I(f)}{X_R(f)}\right),$$

where $X_R(f)$ and $X_I(f)$ are the real and imaginary parts of $X(f)$, respectively.

Review of *modulus* and *argument* of a complex variable



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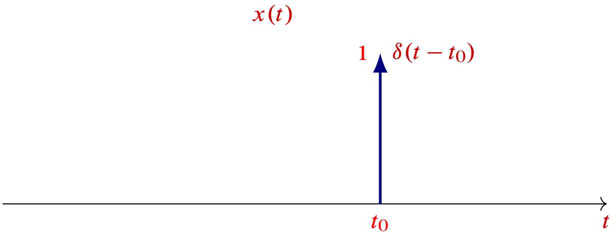
Example 1

Find the Fourier transform of $x(t) = \delta(t - t_0)$.



Example 1: Answer

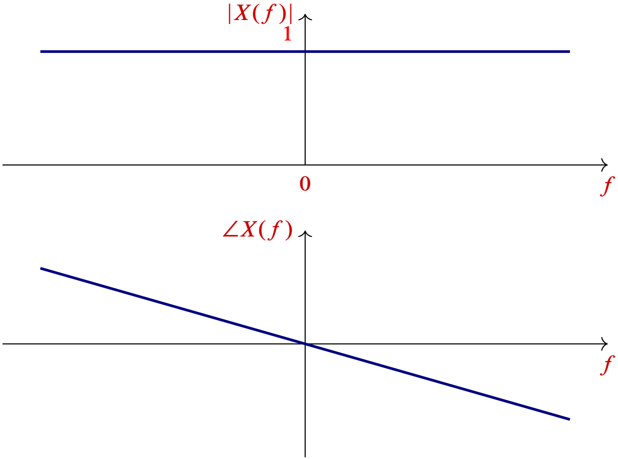
Answer: From the definition of $X(f)$, it follows that

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt, \\ &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi f t}\, dt, \\ &= e^{-j2\pi f t_0} \end{aligned}$$

Example 1: Magnitude and Phase Spectra

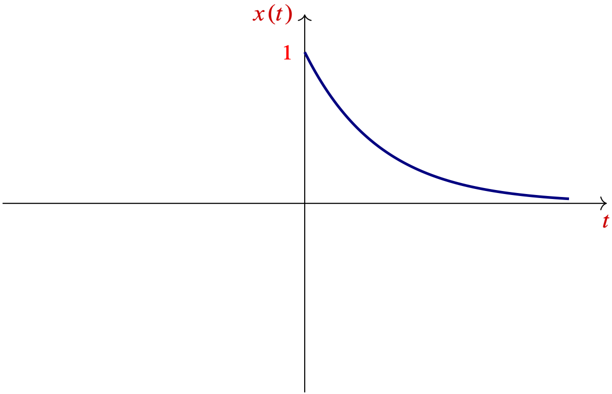
Usually, we are interesting in the magnitude $|X(f)|$ and the phase $\angle X(f)$ spectra of $x(t)$, i.e.,

$$|X(f)| = 1, \qquad \angle X(f) = -2\pi f t_0.$$



Example 2

Find the Fourier transform of $x(t) = e^{-at}u(t)$, *where* $|a| > 0$.



Example 2: Answer

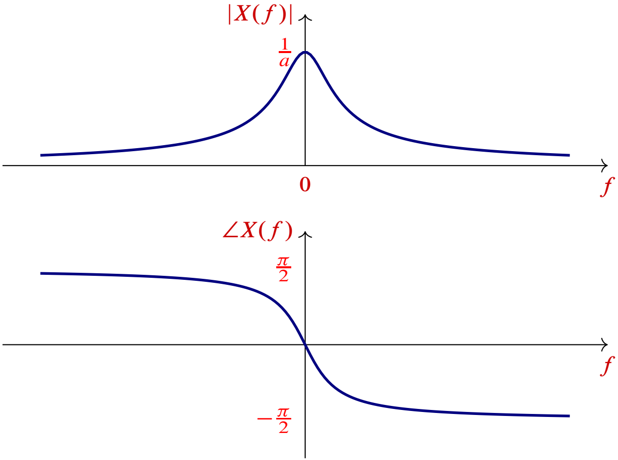
Answer: Using the definition of $X(f)$, it follows

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt, \\ &= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j2\pi f t}\, dt, \\ &= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt, \\ &= -\frac{e^{-(a+j2\pi f)t}}{a+j2\pi f}\bigg|_{0}^{\infty}, \\ &= \frac{1}{a+j2\pi f}. \end{aligned}$$

Example 2: Magnitude and Phase Spectra

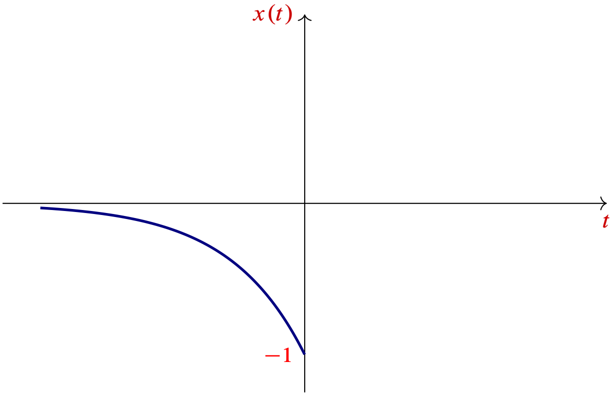
We are interesting in the magnitude $|X(f)|$ and the phase $\angle X(f)$ spectra of $x(t)$, i.e.,

$$|X(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}, \qquad \angle X(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right).$$



Example 3

Find the Fourier transform of $x(t) = -e^{at}u(-t)$, *where* $|a| > 0$.



Example 3. Answer

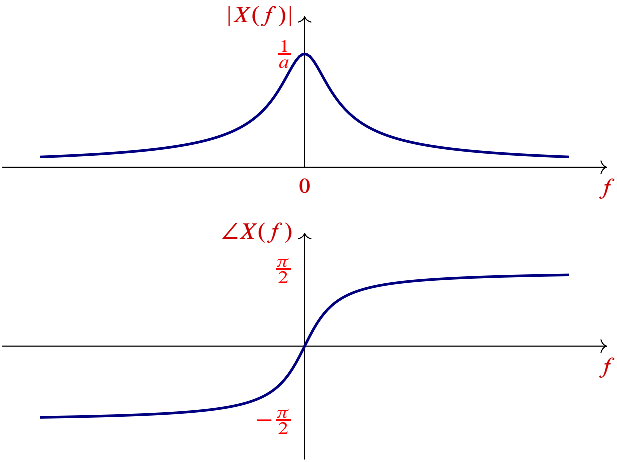
Answer: From the definition of $X(f)$, we have

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}\,dt, \\
&= \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-j2\pi f t}\,dt, \\
&= -\int_{-\infty}^{0} e^{-(-a+j2\pi f)t}\,dt, \\
&= \left.\frac{e^{-(-a+j2\pi f)t}}{-a+j2\pi f}\right|_{-\infty}^{0}, \\
&= \frac{1}{-a+j2\pi f}.
\end{aligned}$$

Example 3. Magnitude and Phase Spectra

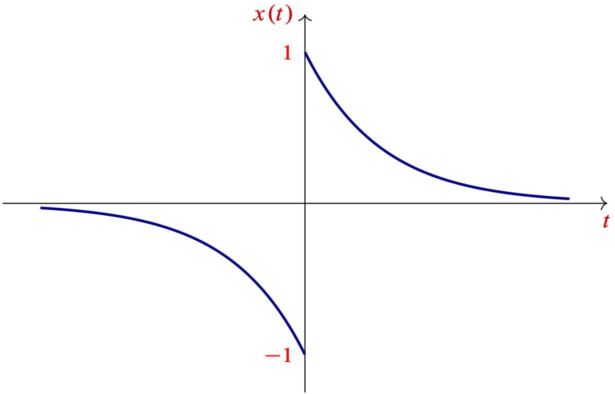
The magnitude $|X(f)|$ and phase $\angle X(f)$ spectra of $x(t)$ are given by

$$|X(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}, \qquad \angle X(f) = \tan^{-1}\left(\frac{2\pi f}{a}\right).$$



Example 4

Find the Fourier transform of $x(t) = e^{-at}u(t) - e^{at}u(-t)$, *where* $|a| > 0$.



Example 4. Answer

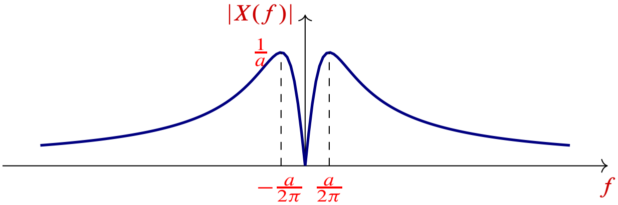
Answer: From Examples 2 and 3, we have

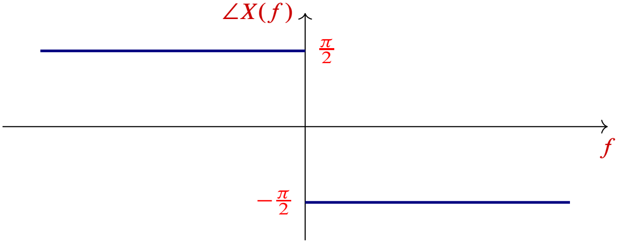
$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}\,dt, \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j2\pi f t}\,dt - \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j2\pi f t}\,dt, \\ &= \frac{1}{a + j2\pi f} + \frac{1}{-a + j2\pi f} \\ &= -\frac{j4\pi f}{a^2 + 4\pi^2 f^2} \end{aligned}$$

Example 4. Magnitude and Phase Spectra

The magnitude $|X(f)|$ and phase $\angle X(f)$ spectra of $x(t)$ are given by

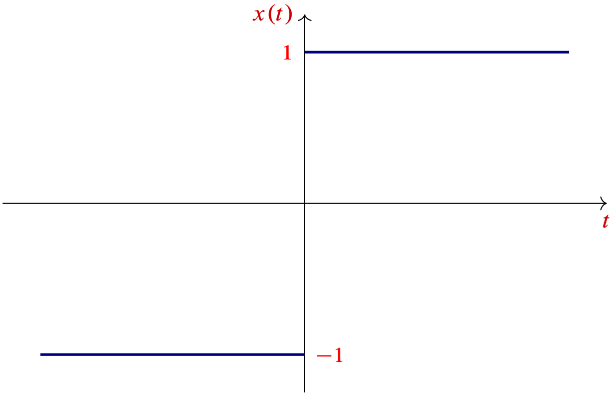
$$|X(f)| = \frac{4\pi|f|}{a^2 + 4\pi^2 f^2}, \qquad \angle X(f) = \begin{cases} -\frac{\pi}{2}, & \text{if } f > 0 \\ \frac{\pi}{2}, & \text{if } f < 0 \end{cases}.$$





Example 5

Find the Fourier transform of the sign function, that is, $x(t) = \text{sign}(t)$



Example 5. Answer

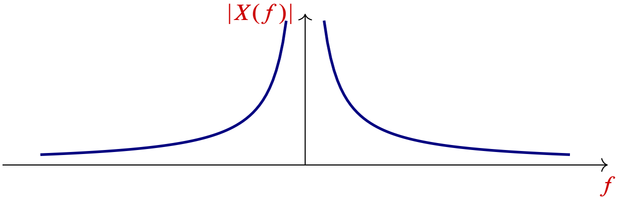
Answer: Using Example 4 and $a \to 0$, we have

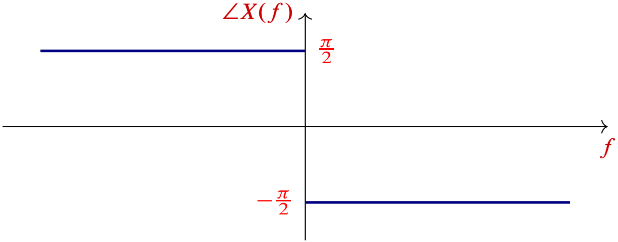
$$
\begin{aligned}
X(f) &= \lim_{a\to 0} -\frac{j4\pi f}{a^2 + 4\pi^2 f^2}, \\
&= \frac{2}{j2\pi f}.
\end{aligned}
$$

Example 5. Magnitude and Phase Spectra

The magnitude $|X(f)|$ and phase $\angle X(f)$ spectra of $x(t)$ are given by

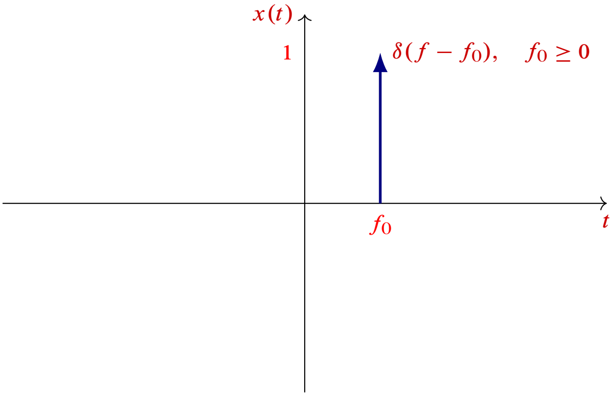
$$|X(f)| = \frac{1}{\pi|f|}, \qquad \angle X(f) = \begin{cases} -\frac{\pi}{2}, & \text{if } f > 0 \\ \frac{\pi}{2}, & \text{if } f < 0 \end{cases}.$$





Example 6

Find the Inverse Fourier Transform of $X(f) = \delta(f - f_0)$.



Example 6. Answer

Answer: Applying the definition gives

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df, \\ &= \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t}\, df, \\ &= e^{j2\pi f_0 t}. \end{aligned}$$

This implies that

$$x(t) = e^{j2\pi f_0 t} \leftrightarrow X(f) = \delta(f - f_0)$$

For the special case where $f_0 = 0$, we have

$$x(t) = 1 \leftrightarrow X(f) = \delta(f)$$

Example 7

Find the Fourier Transform of the complex Fourier Series (FS) $x(t) = \sum_{n=-\infty}^{\infty} c[n] e^{j2\pi n f_0 t}$.

Answer: Using Example 6 gives

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c[n] e^{j2\pi n f_0 t} e^{-j2\pi f t}\, dt \\
&= \sum_{n=-\infty}^{\infty} c[n] \int_{-\infty}^{\infty} e^{j2\pi n f_0 t} e^{-j2\pi f t}\, dt \\
&= \sum_{n=-\infty}^{\infty} c[n] \delta(f - n f_0)
\end{aligned}
$$

Example 7, cont.

Summarizing, we have

$$\sum_{n=-\infty}^{\infty} c[n]e^{j2\pi n f_0 t} \leftrightarrow \sum_{n=-\infty}^{\infty} c[n]\delta(f - nf_0).$$

Example 8

Find the Fourier Transform of the cosine function $x(t) = \cos(2\pi f_0 t)$.

Answer: At fist, we use the Euler identity, i.e.,

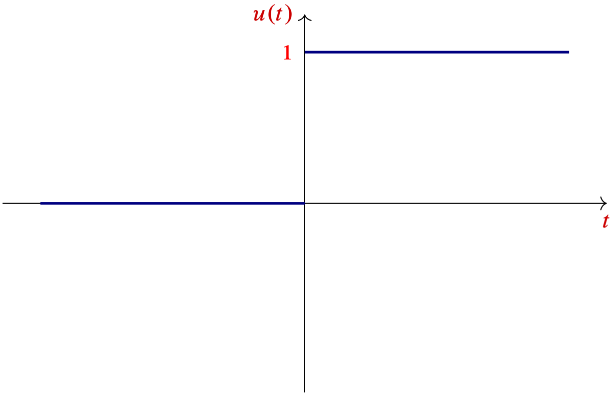
$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}.$$

Finally, using Example 6, we arrive at

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \\ &= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} e^{-j2\pi f t}\, dt \\ &= \frac{1}{2}\big(\delta(f - f_0) + \delta(f + f_0)\big). \end{aligned}$$

Example 9

Find the Fourier transform of the Unit-Step function $u(t)$.



Example 9. Answer

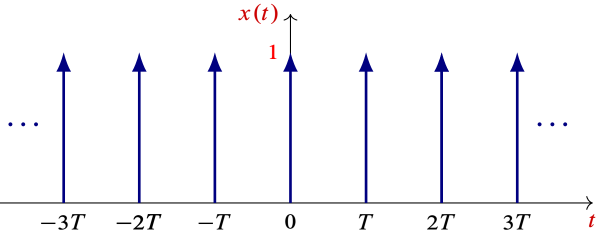
Answer: At first, note that $u(t)$ is the superposition of the constant function and the sign function. To be specific, the Unit Step function $u(t)$ is expressed as $u(t) = \frac{1}{2} + \frac{1}{2}\,\text{sign}(t)$.

Therefore, from Examples 5 and 6, we conclude that

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} u(t) e^{-j2\pi f t}\, dt, \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2} + \frac{1}{2}\,\text{sign}(t) \right) e^{-j2\pi f t}\, dt, \\
&= \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.
\end{aligned}
$$

Example 10

Find the Fourier transform of the following periodic signal: $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.



Example 10: Answer

Answer: Using $f_0 = \frac{1}{T}$, the coefficient $c[n]$ is obtained as follows

$$
\begin{aligned}
c[n] &= f_0 \int_{-1/2f_0}^{1/2f_0} \delta(t) e^{-j2\pi n f_0 t}\, dt, \\
&= f_0.
\end{aligned}
$$

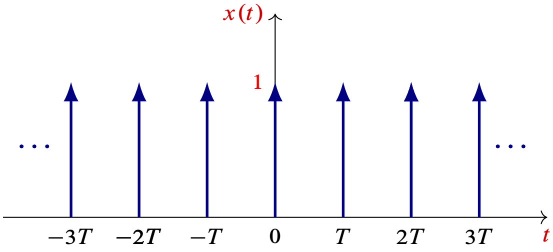
Consequently, the complex Fourier series results in

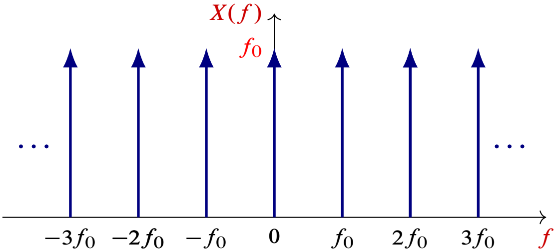
$$
\sum_{n=-\infty}^{\infty} \delta(t - nT) = f_0 \sum_{n=-\infty}^{\infty} e^{j2\pi n f_0 t}.
$$

Example 10: Fourier Transform

Using results from Example 7, the Fourier transform of $x(t)$ is expressed as

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0).$$





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Linearity

The Fourier transform satisfies the linearity property

$$\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(f) + \beta X_2(f)$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} \Big(\alpha x_1(t) + \beta x_2(t)\Big) e^{-j2\pi f t}\, dt &= \int_{-\infty}^{\infty} \alpha x_1(t) e^{-j2\pi f t}\, dt + \int_{-\infty}^{\infty} \beta x_2(t) e^{-j2\pi f t}\, dt \\
&= \alpha \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t}\, dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t}\, dt \\
&= \alpha X_1(f) + \beta X_2(f)
\end{aligned}$$

Time shifting

The time shifting property is expressed by

$$x(t-t_0) \leftrightarrow e^{-j2\pi f t_0} X(f)$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t-t_0) e^{-j2\pi f t}\, dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f (\tau + t_0)}\, d\tau \qquad \text{by using } \tau = t - t_0 \\
&= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau}\, d\tau \\
&= e^{-j2\pi f t_0} X(f)
\end{aligned}$$

Frequency shifting (Modulation Property)

The Modulation property is expressed by

$$e^{j2\pi f_0 t} x(t) \leftrightarrow X(f - f_0)$$

Proof

$$\begin{aligned}\int_{-\infty}^{\infty} e^{j2\pi f_0 t} x(t) e^{-j2\pi f t}\, dt &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_0) t}\, dt \\ &= X(f - f_0)\end{aligned}$$

Example 11

In this example, we consider an **Amplitude Modulation** signal, that is,

$$y(t) = x(t)\cos(2\pi f_0 t),$$

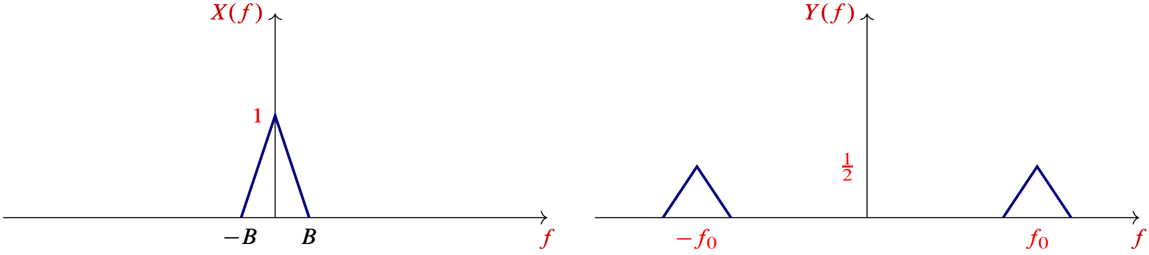
which can be rewritten as

$$y(t) = \frac{x(t)}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right),$$

Using the **modulation property**, the resulting Fourier transform is

$$Y(f) = \frac{1}{2}\Big(X(f - f_0) + X(f + f_0)\Big).$$

Illustrating Example 11



Time scaling

This property satisfies

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Proof $(a > 0)$

$$\begin{aligned}\int_{-\infty}^{\infty} x(at)e^{-j2\pi f t}\, dt &= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f \tau/a}\, d\tau \qquad \text{by using } \tau = at \\ &= \frac{1}{a} X\left(\frac{f}{a}\right)\end{aligned}$$

Time reversal

This property satisfies

$$x(-t) \leftrightarrow X(-f)$$

Proof Using $a = -1$ in the **Time scaling** property gives the desired result.

Complex conjugate

This property satisfies

$$x^*(t) \leftrightarrow X^*(-f)$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} x^*(t)e^{-j2\pi f t}\, dt &= \left(\int_{-\infty}^{\infty} x(t)e^{j2\pi f t}\right)^* dt \\
&= \left(\int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}\right)^* dt \\
&= X^*(-f)
\end{aligned}$$

Duality

The duality property is expressed as

$$X(t) \leftrightarrow x(-f)$$

Proof

$$\begin{aligned}\int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}\,dt &= \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}\,dt\\ &= \int_{-\infty}^{\infty} X(t)e^{j2\pi(-f)t}\,dt\\ &= x(-f)\end{aligned}$$

Example 12

From Example 5, we know that the Fourier transform of $\mathrm{sign}(t)$ is $\frac{1}{j\pi f}$.

Now, consider that the desired function in time is $x(t) = \frac{1}{j\pi t}$. Using the **Duality** property, the corresponding Fourier transform is $X(f) = \mathrm{sign}(-f) = -\mathrm{sign}(f)$.

Differentiation in time domain

This property is expressed as

$$\frac{dx(t)}{dt} \leftrightarrow j2\pi f X(f)$$

Proof

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$$

By definition of Inverse transform

$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f X(f) e^{j2\pi f t} \, df$$

Differentiation in frequency domain

For this property, we have

$$-j2\pi t x(t) \leftrightarrow \frac{dX(f)}{df}$$

Proof

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, df$$

By definition of Fourier transform

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} -j2\pi t x(t) e^{-j2\pi f t}\, df$$

Convolution

The Fourier transform of the convolution is related as

$$x(t) * h(t) \leftrightarrow X(f)H(f)$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t) * h(t) e^{-j2\pi f t}\, dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi f t}\, dt\, d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) H(f) e^{-j2\pi f \tau}\, d\tau \qquad \text{time shifting prop} \\
&= H(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau}\, d\tau \\
&= H(f) X(f)
\end{aligned}$$

Multiplication

The Fourier transform of the multiplication is related as

$$x(t)h(t) \leftrightarrow X(f) * H(f)$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)h(t)e^{-j2\pi f t}\,dt &= \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} H(\lambda)e^{j2\pi\lambda t}\,d\lambda e^{-j2\pi f t}\,dt \\
&= \int_{-\infty}^{\infty} H(\lambda)\int_{-\infty}^{\infty} x(t)e^{j2\pi\lambda t}e^{-j2\pi f t}\,dt\,d\lambda \\
&= \int_{-\infty}^{\infty} H(\lambda)X(f-\lambda)\,d\lambda \\
&= H(f) * X(f)
\end{aligned}$$

Integration in time domain

This property says

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f}$$

Proof First step

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\, d\tau = x(t) * u(t)$$

Second step

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{t} x(\tau)\, d\tau e^{j2\pi f t}\, dt &= X(f)U(f)\\ &= X(f)\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right)\\ &= \frac{X(f)}{2}\delta(f) + \frac{X(f)}{j2\pi f}\end{aligned}$$

Parseval's relation

Parseval's relation is defined as

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)\,df$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(f)e^{j2\pi ft}\,df\,x_2^*(t)\,dt \\
&= \int_{-\infty}^{\infty} X_1(f)\int_{-\infty}^{\infty} x_2^*(t)e^{j2\pi ft}\,dt\,df \\
&= \int_{-\infty}^{\infty} X_1(f)\int_{-\infty}^{\infty} x_2^*(t)e^{-j2\pi(-f)t}\,dt\,df \\
&= \int_{-\infty}^{\infty} X_1(f)X_2^*(f)\,df
\end{aligned}$$

Magnitude spectrum for real-valued signals

Using the fact that $x(t) = x^*(t)$ and the complex conjugate property, we have

$$X(f) = X^*(-f).$$

Thus

$$|X(f)| = |X(-f)|.$$

This means, the magnitude spectrum is an **even function**.

Phase spectrum for real-valued signals

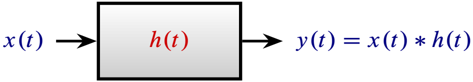
Using $X(f) = X^*(-f)$ gives

$$\angle X(f) = -\angle X(-f).$$

This means, the phase spectrum is an **odd function**.

Contents

Definition



The **frequency response** is the Fourier transform of the impulse response, $H(f)$.

Alternatively, from the convolution, $y(t) = x(t) * h(t)$, the Fourier transform of $y(t)$ is given by

$$Y(f) = X(f)H(f),$$

which can be rewritten as

$$H(f) = \frac{Y(f)}{X(f)}.$$

Magnitude and Phase Responses

The complex-valued function $H(f)$ can be expressed as

$$H(f) = |H(f)|e^{j\angle H(f)}$$

where $|H(f)|$ and $\angle H(f)$ are called the **magnitude** and **phase responses** of the system.

Example 13

$$x(t) = \cos(2\pi f_0 t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Let $x(t)$ be the cosine function with frequency f_0. Using the frequency response of the system, find the output $y(t)$. Assume that the impulse response $h(t)$ is a real-valued signal.

Answer to Example 13

In the frequency domain, we have

$$Y(f) = H(f)X(f)$$

Since $x(t) = \cos(2\pi f_0 t)$, the last equation reduces to

$$\begin{aligned} Y(f) &= \frac{H(f)}{2}\Big(\delta(f - f_0) + \delta(f + f_0)\Big) \\ &= \frac{|H(f_0)|e^{j\angle H(f_0)}}{2}\delta(f - f_0) + \frac{|H(f_0)|e^{-j\angle H(f_0)}}{2}\delta(f + f_0) \end{aligned}$$

Applying inverse Fourier transform, we arrive at

$$\begin{aligned} y(t) &= \frac{|H(f_0)|}{2}\left(e^{j\angle H(f_0)}e^{j2\pi f_0 t} + e^{-j\angle H(f_0)}e^{-j2\pi f_0 t}\right) \\ &= |H(f_0)|\cos\Big(2\pi f_0 t + \angle H(f_0)\Big) \end{aligned}$$

Example 14

Find the frequency response for the system described by

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Answer to Example 14

Performing the Fourier transform gives

$$j2\pi f Y(f) + aY(f) = X(f).$$

Solving for $H(f)$, we have

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{a + j2\pi f}.$$

See Example 2 for magnitude and phase responses.

Homework

- Problem 5.68 from [3]
- Problem 5.69 from [3]
- Problem 5.71 from [3]
- Problem 5.72 from [3]
- Problem 5.75 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill