Signals and Communication Theory

Discrete-time Fourier Transform

Alfonso Fernandez-Vazquez

Ingeniería en Sistemas Computacionales
Escuela Superior de Cómputo, ESCOM
Instutito Politécnico Nacional, IPN

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Definition

Most of the signal processing in communication systems is performed in discrete time. In this regard, we can use the **Discrete-Time Fourier Transform (DTFT)**, which is defined as

Analysis: $$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n},$$

Synthesis: $$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X\left(e^{j2\pi f}\right)e^{j2\pi f n}\,df.$$

We note that the frequency variable f is continuous while the time variable n is discrete. In addition, the Fourier transform $X\left(e^{j2\pi f}\right)$ is a periodic function with period 1.

Magnitude and Phase Spectra

The Fourier transform can be written as

$$X\left(e^{j2\pi f}\right) = \left|X\left(e^{j2\pi f}\right)\right| e^{j\angle X\left(e^{j2\pi f}\right)}$$

where $\left|X\left(e^{j2\pi f}\right)\right|$ and $\angle X\left(e^{j2\pi f}\right)$ are called the **magnitude** and **phase spectra** of $x[n]$.

They can be computed as

$$\left|X\left(e^{j2\pi f}\right)\right| = \sqrt{X_R^2\left(e^{j2\pi f}\right) + X_I^2\left(e^{j2\pi f}\right)},$$

$$\angle X\left(e^{j2\pi f}\right) = \tan^{-1}\left(\frac{X_I\left(e^{j2\pi f}\right)}{X_R\left(e^{j2\pi f}\right)}\right),$$

where $X_R\left(e^{j2\pi f}\right)$ and $X_I\left(e^{j2\pi f}\right)$ are the real and imaginary parts of $X\left(e^{j2\pi f}\right)$, respectively.

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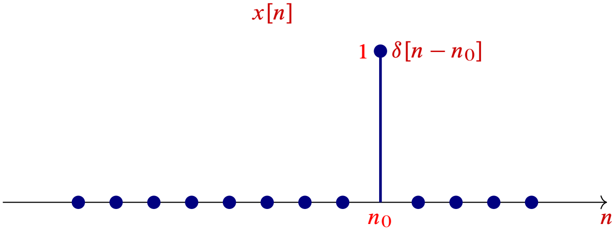
Example 1

Find the Discrete-Time Fourier transform of $x[n] = \delta[n - n_0]$.



Example 1: Answer

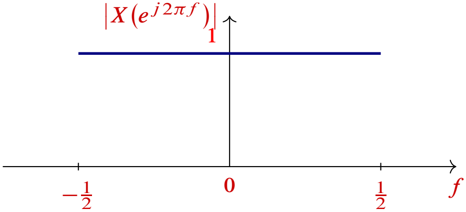
Answer: From the definition of $X\left(e^{j2\pi f}\right)$, it follows that

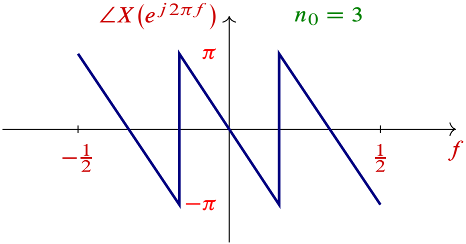
$$
\begin{aligned}
X\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi nf} \\
&= \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-j2\pi nf} \\
&= e^{-j2\pi n_0 f}.
\end{aligned}
$$

Example 1: Magnitude and Phase Spectra

We are interesting in the magnitude $\left|X\left(e^{j2\pi f}\right)\right|$ and the phase $\angle X\left(e^{j2\pi f}\right)$ spectra of $x[n]$, i.e.,

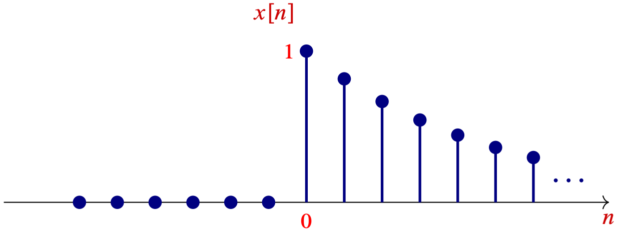
$$\left|X\left(e^{j2\pi f}\right)\right| = 1, \qquad \angle X\left(e^{j2\pi f}\right) = -2\pi n_0 f, \quad \text{for } |f| < \frac{1}{2}.$$





Example 2

Find the Discrete-Time Fourier transform of $x[n] = a^n u[n]$*, where* $|a| < 1$.



Example 2: Answer

Answer: Using the definition of $X\left(e^{j2\pi f}\right)$, it follows

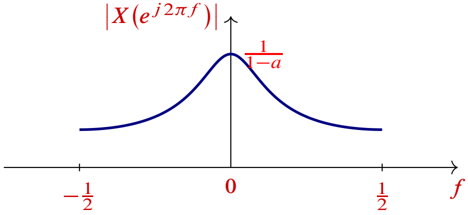
$$
\begin{aligned}
X\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi nf} \\
&= \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j2\pi nf} \\
&= \sum_{n=0}^{\infty} \left(ae^{-j2\pi f}\right)^n \\
&= \frac{1}{1-ae^{-j2\pi f}}.
\end{aligned}
$$

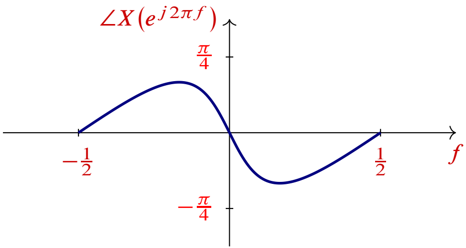
Example 2: Magnitude and Phase Spectra

The magnitude $|X(e^{j2\pi f})|$ and the phase $\angle X(e^{j2\pi f})$ spectra of $x[n]$ are given by

$$|X(e^{j2\pi f})| = \frac{1}{\sqrt{(1 - a\cos(2\pi f))^2 + a^2 \sin^2(2\pi f)}},$$

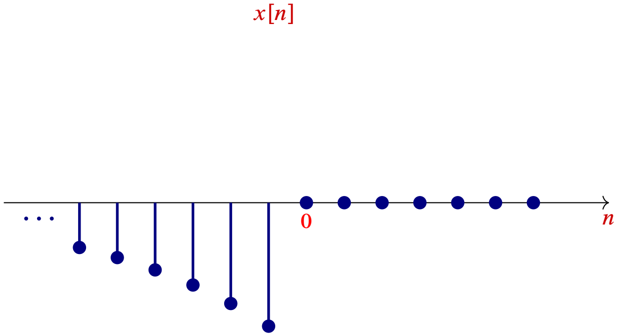
$$\angle X(e^{j2\pi f}) = -\tan^{-1}\left(\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}\right).$$





Example 3

Find the Discrete-Time Fourier transform of $x[n] = -a^{-n}u[-1-n]$, *where* $|a| < 1$.



Example 3: Answer

Answer: From the definition of $X\left(e^{j2\pi f}\right)$, we have

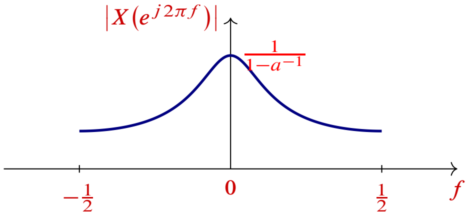
$$
\begin{aligned}
X\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi n f} \\
&= \sum_{n=-\infty}^{\infty} -a^{-n}u[-1-n]e^{-j2\pi n f} \\
&= -\sum_{n=-\infty}^{-1} \left(a^{-1}e^{-j2\pi f}\right)^n - 1 + 1 \\
&= -\sum_{n=-\infty}^{0} \left(a^{-1}e^{-j2\pi f}\right)^n + 1 \\
&= -\frac{1}{1-ae^{j2\pi f}} + 1 \\
&= \frac{1}{1-a^{-1}e^{-j2\pi f}}
\end{aligned}
$$

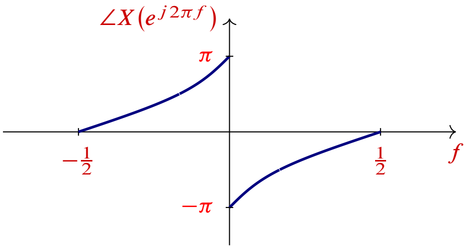
Example 3: Magnitude and Phase Spectra

The magnitude $|X(e^{j2\pi f})|$ and the phase $\angle X(e^{j2\pi f})$ spectra of $x[n]$ can be written as

$$|X(e^{j2\pi f})| = \frac{1}{\sqrt{\left(1 - a^{-1}\cos(2\pi f)\right)^2 + a^{-2}\sin^2(2\pi f)}},$$

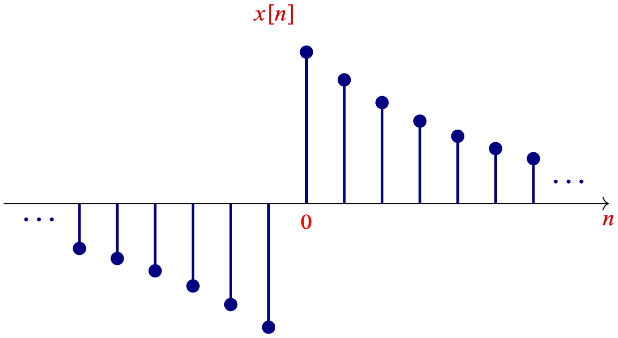
$$\angle X(e^{j2\pi f}) = -\tan^{-1}\left(\frac{a^{-1}\sin(2\pi f)}{1 - a^{-1}\cos(2\pi f)}\right) + \begin{cases} \pi, & \text{if } -\frac{1}{2\pi}\cos^{-1} a < f < 0 \\ -\pi, & \text{if } 0 < f < \frac{1}{2\pi}\cos^{-1} a \\ 0, & \text{otherwise.} \end{cases}.$$





Example 4

Find the Discrete-Time Fourier transform of $x[n] = a^n u[n] - a^{-n} u[-1-n]$*, where* $|a| < 1$.



Example 4. Answer

Answer: From Examples 2 and 3, we have

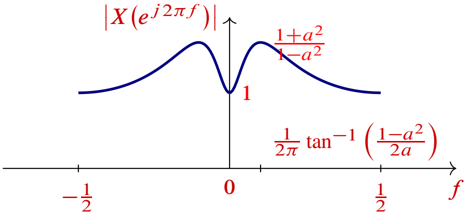
$$X\left(e^{j2\pi f}\right) = \frac{1}{1 - ae^{-j2\pi f}} + \frac{1}{1 - a^{-1}e^{-j2\pi f}}$$
$$= \frac{1 + a^2 - 2ae^{j2\pi f}}{1 - 2a\cos(2\pi f) + a^2}$$

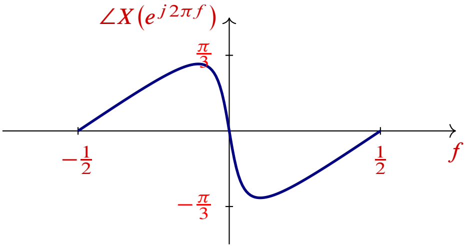
Example 4. Magnitude and Phase Spectra

The magnitude $|X(e^{j2\pi f})|$ and phase $\angle X(e^{j2\pi f})$ spectra of $x[n]$ are given by

$$|X(e^{j2\pi f})| = \frac{\sqrt{\left(1+a^2-2a\cos(2\pi f)\right)^2+4a^2\sin^2(2\pi f)}}{1-2a\cos(2\pi f)+a^2}$$

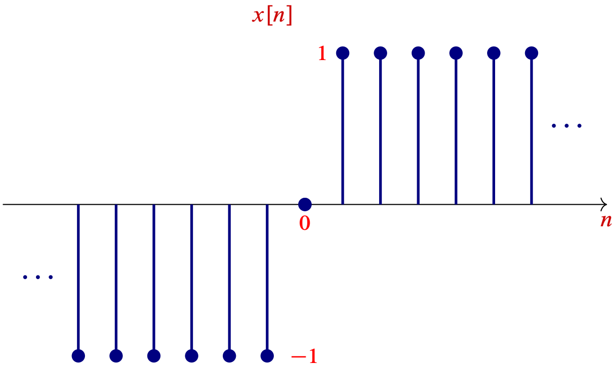
$$\angle X(e^{j2\pi f}) = \tan^{-1}\left(\frac{2a\sin(2\pi f)}{1+a^2-2a\cos(2\pi f)}\right).$$





Example 5

Find the Discrete-Fourier transform of the sign function, that is, $x[n] = \text{sign}[n]$



Example 5. Answer

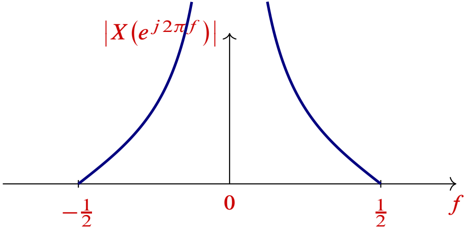
Answer: Using Examples 1 and 4 with $a \to 1$, we have $\mathrm{sign}[n] = \lim_{a\to 1} a^n u[n] - a^{-n} u[-1-n] - \delta[n]$. Therefore, it follows that

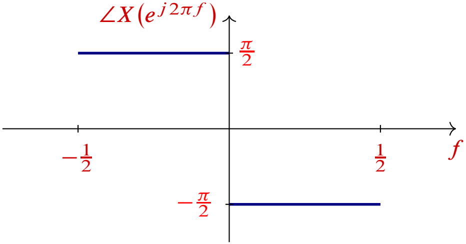
$$X\left(e^{j2\pi f}\right) = \lim_{a\to 1} \frac{1 + a^2 - 2ae^{j2\pi f}}{1 - 2a\cos(2\pi f) + a^2} - 1,$$
$$= \frac{-j\sin(2\pi f)}{1 - \cos(2\pi f)}.$$

Example 5. Magnitude and Phase Spectra

The magnitude $|X(e^{j2\pi f})|$ and phase $\angle X(e^{j2\pi f})$ spectra of $x[n]$ are given by

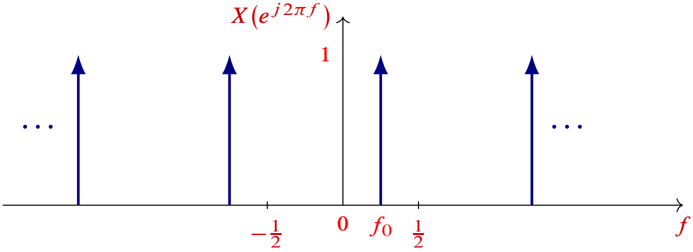
$$|X(e^{j2\pi f})| = \frac{|\sin(2\pi f)|}{1-\cos(2\pi f)}, \qquad \angle X(e^{j2\pi f}) = \begin{cases} -\frac{\pi}{2}, & \text{if } f > 0 \\ \frac{\pi}{2}, & \text{if } f < 0 \end{cases}.$$





Example 6

Find the Discrete-Time Inverse Fourier Transform of $X\left(e^{j2\pi f}\right) = \sum_{\ell=-\infty}^{\infty} \delta(f - f_0 - \ell)$ *for* $|f_0| < \frac{1}{2}$.



Example 6. Answer

Answer: Applying the definition gives

$$\begin{aligned} x[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(e^{j2\pi f}) e^{j2\pi f n} \, df, \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(f - f_0) e^{j2\pi f n} \, df, \\ &= e^{j2\pi f_0 n}. \end{aligned}$$

This implies that

$$x[n] = e^{j2\pi f_0 n} \leftrightarrow X(e^{j2\pi f}) = \sum_{\ell=-\infty}^{\infty} \delta(f - f_0 - \ell)$$

For the special case where $f_0 = 0$, we have

$$x[n] = 1 \leftrightarrow X(e^{j2\pi f}) = \sum_{\ell=-\infty}^{\infty} \delta(f - \ell)$$

Example 7

Find the Discrete-Time Fourier Transform (DTFT) of the cosine function $x[n] = \cos(2\pi f_0 n)$.

Answer: At first, we use the Euler identity, i.e.,

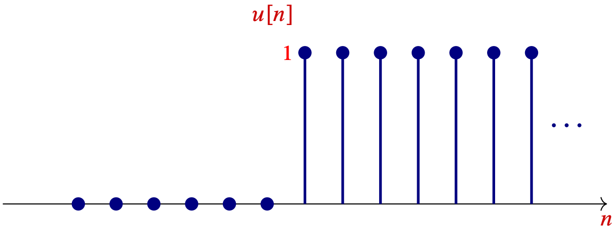
$$\cos(2\pi f_0 n) = \frac{e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}}{2}.$$

Finally, using Example 6, we arrive at

$$\begin{aligned} X\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} \\ &= \sum_{n=-\infty}^{\infty} \frac{e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}}{2} e^{-j2\pi f n} \\ &= \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \Big(\delta(f - f_0 - \ell) + \delta(f + f_0 - \ell)\Big). \end{aligned}$$

Example 8

Find the DTFT of the Unit Step function $u[n]$.



Example 8. Answer

Answer: At first, note that the unit step function $u[n]$ can be expressed as $u[n] = \frac{1}{2}\,\mathrm{sign}[n] + \frac{1}{2}\delta[n] + \frac{1}{2}$.

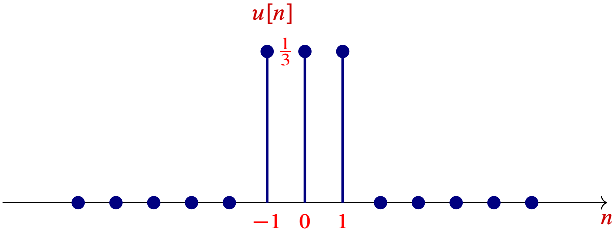
Therefore, from Examples 5 and 6, we conclude that

$$U\left(e^{j2\pi f}\right) = \frac{1}{1 - e^{-j2\pi f}} + \frac{1}{2}\sum_{\ell=-\infty}^{\infty} \delta(f - \ell)$$

Example 9

Find the DTFT of $x[n]$ defined by.

$$x[n] = \begin{cases} \frac{1}{3}, & n = -1, 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$



Answer to Example 9

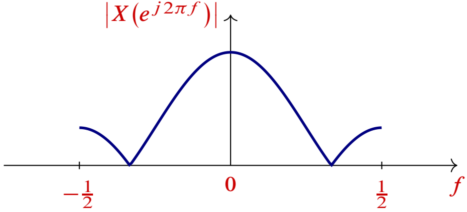
The DTFT is given by

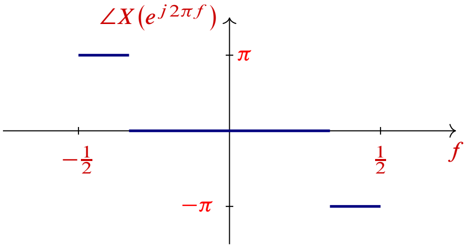
$$
\begin{aligned}
X\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-2\pi f n} \\
&= \sum_{n=-1}^{1} \frac{1}{3} e^{-2\pi f n} \\
&= \frac{1}{3}\left(e^{j2\pi f} + 1 + e^{-j2\pi f}\right) \\
&= \frac{1}{3}\left(1 + 2\cos(2\pi f)\right)
\end{aligned}
$$

Example 9. Magnitude and Phase Spectra

The magnitude $|X(e^{j2\pi f})|$ and phase $\angle X(e^{j2\pi f})$ spectra of $x[n]$ are given by

$$|X(e^{j2\pi f})| = \frac{1}{3}\big|1 + 2\cos(2\pi f)\big|, \qquad \angle X(e^{j2\pi f}) = \begin{cases} \pm\pi, & 1 + 2\cos(2\pi f) < 0 \\ 0, & 1 + 2\cos(2\pi f) > 0 \end{cases}.$$





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Linearity

The Discrete-time Fourier transform satisfies the linearity property

$$\alpha x_1[n] + \beta x_2[n] \leftrightarrow \alpha X_1\left(e^{j2\pi f}\right) + \beta X_2\left(e^{j2\pi f}\right)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\Big(\alpha x_1[n] + \beta x_2[n]\Big)e^{-j2\pi f n} &= \sum_{n=-\infty}^{\infty} \alpha x_1[n]e^{-j2\pi f n} + \int_{-\infty}^{\infty} \beta x_2[n]e^{-j2\pi f n} \\ &= \alpha \sum_{n=-\infty}^{\infty} x_1[n]e^{-j2\pi f n}\, dt + \beta \sum_{n=-\infty}^{\infty} x_2[n]e^{-j2\pi f n} \\ &= \alpha X_1\left(e^{j2\pi f}\right) + \beta X_2\left(e^{j2\pi f}\right)\end{aligned}$$

Periodicity

The Discrete-time Fourier transform is periodic function of the variable f, with period 1.

$$X\left(e^{j2\pi(f+k)}\right) = X\left(e^{j2\pi f}\right).$$

Proof

$$\begin{aligned}
X\left(e^{j2\pi(f+k)}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi(f+k)n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n}e^{-j2\pi k n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n}, \qquad \text{Since } e^{-j2\pi k n} = 1 \\
&= X\left(e^{j2\pi f}\right)
\end{aligned}$$

Time shifting

The time shifting property is expressed by

$$x[n-n_0] \leftrightarrow e^{-j2\pi f n_0} X\left(e^{j2\pi f}\right)$$

Proof

$$\sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x[m] e^{-j2\pi f (m+n_0)} \qquad \text{by using } m = n - n_0$$

$$= e^{-j2\pi f n_0} \sum_{n=-\infty}^{\infty} x[m] e^{-j2\pi f m}$$

$$= e^{-j2\pi f n_0} X\left(e^{j2\pi f}\right)$$

Frequency shifting (Modulation Property)

The Modulation property is expressed by

$$e^{j2\pi f_0 n} x[n] \leftrightarrow X\left(e^{j2\pi(f-f_0)}\right)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} e^{j2\pi f_0 n} x[n] e^{-j2\pi f n} &= \sum_{n=-\infty}^{\infty} x(t) e^{-j2\pi (f-f_0) n} \\ &= X\left(e^{j2\pi(f-f_0)}\right)\end{aligned}$$

Time reversal

This property satisfies

$$x[-n] \leftrightarrow X\left(e^{-j2\pi f}\right)$$

Proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[-n]e^{-j2\pi f n} &= \sum_{m=\infty}^{-\infty} x[m]e^{-j2\pi(-f)m} \\ &= X\left(e^{-j2\pi f}\right)\end{aligned}$$

Complex conjugate

This property satisfies

$$x^*[n] \leftrightarrow X^*\left(e^{-j2\pi f}\right)$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^*[n]e^{-j2\pi fn} &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j2\pi ft}\right)^* \\
&= \left(\int_{n=-\infty}^{\infty} x[n]e^{-j2\pi(-f)n}\right)^* \\
&= X^*\left(e^{-j2\pi f}\right)
\end{aligned}$$

Differentiation in frequency domain

For this property, we have

$$-j2\pi n x[n] \leftrightarrow \frac{dX\left(e^{j2\pi f}\right)}{df}$$

Proof

$$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n}$$

By definition of Fourier transform

$$\frac{dX\left(e^{j2\pi f}\right)}{df} = \sum_{n=-\infty}^{\infty} -j2\pi n x[n]e^{-j2\pi f t}$$

Convolution

The Discrete-time Fourier transform of the convolution is related as

$$x[n] * h[m] \leftrightarrow X\left(e^{j2\pi f}\right) H\left(e^{j2\pi f}\right)$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n] * h[n] e^{-j2\pi f n} &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[n-m] e^{-j2\pi f n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m] e^{-j2\pi f n} \\
&= \sum_{m-\infty}^{\infty} x[m] H\left(e^{j2\pi f}\right) e^{-j2\pi f m} \qquad \text{time shifting prop.} \\
&= H\left(e^{j2\pi f}\right) \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f m} \\
&= H\left(e^{j2\pi f}\right) X\left(e^{j2\pi f}\right)
\end{aligned}$$

Multiplication

The Discrete-time Fourier transform of the multiplication is related as

$$x[n]h[n] \leftrightarrow X\left(e^{j2\pi f}\right) \circledast H\left(e^{j2\pi f}\right)$$

where $X\left(e^{j2\pi f}\right) \circledast H\left(e^{j2\pi f}\right)$ stands for the **circular convolution** of $X\left(e^{j2\pi f}\right)$ and $H\left(e^{j2\pi f}\right)$.

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n]h[n]e^{-j2\pi f n} &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\frac{1}{2}}^{\frac{1}{2}} H\left(e^{j2\pi\lambda n}\right)e^{j2\pi\lambda n}\, d\lambda e^{-j2\pi f n} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} H\left(e^{j2\pi\lambda n}\right) \sum_{n=-\infty}^{\infty} x[n]e^{j2\pi\lambda n} e^{-j2\pi f n}\, d\lambda \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} H\left(e^{j2\pi\lambda n}\right) X\left(e^{j2\pi(f-\lambda)n}\right) d\lambda \\
&= H\left(e^{j2\pi f}\right) \circledast X\left(e^{j2\pi f}\right)
\end{aligned}$$

Accumulation

This property says

$$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{X(1)}{2} \sum_{\ell=-\infty}^{\infty} \delta(f-\ell) + \frac{X\left(e^{j2\pi f}\right)}{1-e^{-j2\pi f}}$$

Proof First step

$$\sum_{m=-\infty}^{n} x[m]\tau = \sum_{m=-\infty}^{\infty} x[m]u[n-m] = x[n] * u[n]$$

Second step

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{n} x[m]e^{j2\pi f n}\, dt &= X\left(e^{j2\pi f}\right)U\left(e^{j2\pi f}\right) \\
&= X(f)\left(\frac{1}{1-e^{-j2\pi f}} + \frac{1}{2}\sum_{\ell=-\infty}^{\infty}\delta(f-\ell)\right) \\
&= \frac{X\left(e^{j2\pi f}\right)}{2} \sum_{\ell=-\infty}^{\infty} \delta(f-\ell) + \frac{X\left(e^{j2\pi f}\right)}{1-e^{-j2\pi f}}
\end{aligned}$$

Parseval's relation

Parseval's relation is defined as

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1\left(e^{j2\pi f}\right) X_2^*\left(e^{j2\pi f}\right) df$$

Proof

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] &= \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1\left(e^{j2\pi f}\right) e^{j2\pi f n}\, df\, x_2^*[n] \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1\left(e^{j2\pi f}\right) \sum_{n=-\infty}^{\infty} x_2^*[n] e^{j2\pi f n}\, df \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1\left(e^{j2\pi f}\right) \sum_{n=-\infty}^{\infty} x_2^*[n] e^{-j2\pi(-f)n}\, df \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1\left(e^{j2\pi f}\right) X_2^*\left(e^{j2\pi f}\right) df
\end{aligned}$$

Magnitude spectrum for real-valued signals

For real-valued signal, we have $x[n] = x^*[n]$. Now using the complex conjugate property gives

$$X\left(e^{j2\pi f}\right) = X^*\left(e^{-j2\pi f}\right).$$

Thus

$$|X\left(e^{j2\pi f}\right)| = |X\left(e^{-j2\pi f}\right)|.$$

This means, the magnitude spectrum is an **even function**.

Phase spectrum for real-valued signals

Using $X\left(e^{j2\pi f}\right) = X^*\left(e^{-j2\pi f}\right)$, we have

$$\angle X\left(e^{j2\pi f}\right) = -\angle X\left(e^{-j2\pi f}\right).$$

This means, the phase spectrum is an **odd function**.

Contents

Definition

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

The **frequency response** is the Discrete-time Fourier transform of the impulse response, $H\left(e^{j2\pi f}\right)$.

Applying DTFT to the convolution, $y[n] = x[n] * h[n]$, we have

$$Y\left(e^{j2\pi f}\right) = X\left(e^{j2\pi f}\right) H\left(e^{j2\pi f}\right).$$

In this setting, the frequency response is given by

$$H\left(e^{j2\pi f}\right) = \frac{Y\left(e^{j2\pi f}\right)}{X\left(e^{j2\pi f}\right)}.$$

Magnitude and Phase Responses

The polar form of the complex-valued function $H\left(e^{j2\pi f}\right)$ is

$$H\left(e^{j2\pi f}\right) = |H\left(e^{j2\pi f}\right)| e^{j\angle H\left(e^{j2\pi f}\right)}$$

where $|H\left(e^{j2\pi f}\right)|$ and $\angle H\left(e^{j2\pi f}\right)$ are called the **magnitude** and **phase responses** of the system.

Example 10

$$x[n] = \cos(2\pi f_0 n) \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Let $x[n]$ be the cosine function with frequency f_0. Using the frequency response of the system, find the output $y[n]$. Assume that the impulse response $h[n]$ is a real-valued signal.

Answer to Example 10

In the frequency domain, we have

$$Y\left(e^{j2\pi f}\right) = H\left(e^{j2\pi f}\right) X\left(e^{j2\pi f}\right)$$

Since $x[n] = \cos(2\pi f_0 n)$, the last equation reduces to

$$\begin{aligned} Y\left(e^{j2\pi f}\right) &= \frac{H\left(e^{j2\pi f}\right)}{2} \sum_{\ell=-\infty}^{\infty} \Big(\delta(f - f_0 - \ell) + \delta(f + f_0 - \ell)\Big) \\ &= \frac{|H\left(e^{j2\pi f_0}\right)| e^{j\angle H\left(e^{j2\pi f_0}\right)}}{2} \sum_{\ell=-\infty}^{\infty} \delta(f - f_0 - \ell) \\ &+ \frac{|H\left(e^{j2\pi f_0}\right)| e^{-j\angle H\left(e^{j2\pi f_0}\right)}}{2} \sum_{\ell=-\infty}^{\infty} \delta(f + f_0 - \ell) \end{aligned}$$

Applying inverse Fourier transform, we arrive at

$$y[n] = |H\left(e^{j2\pi f_0}\right)| \cos\left(2\pi f_0 n + \angle H\left(e^{j2\pi f_0}\right)\right)$$

Example 11

Find the frequency response for the system described by the following difference equation:

$$y[n] + ay[n-1] = x[n]$$

Answer to Example 11

Performing the Fourier transform gives

$$Y\left(e^{j2\pi f}\right) + ae^{-j2\pi f}Y\left(e^{j2\pi f}\right) = X\left(e^{j2\pi f}\right).$$

Solving for $H\left(e^{j2\pi f}\right)$, we have

$$H\left(e^{j2\pi f}\right) = \frac{Y\left(e^{j2\pi f}\right)}{X\left(e^{j2\pi f}\right)} = \frac{1}{1 + ae^{-j2\pi f}}.$$

See Example 2 for magnitude and phase responses.

Example 12: System defined by difference equation

A **causal, linear and time-invariant** *can be described by a difference equation as*

$$y[n] + \sum_{m=1}^{M} a_m y[n-m] = \sum_{k=1}^{N} b_k x[n-k],$$

where a_m, *for* $m = 1, \ldots, M$, *and* b_k, *for* $k = 0, \ldots, N$, *are constants. Find the transfer function.*

Answer to Example 12

Applying Discrete-time Fourier transform, we obtain

$$Y\left(e^{j2\pi f}\right) + \sum_{m=1}^{M} a_m e^{-j2\pi f m} Y\left(e^{j2\pi f}\right) = \sum_{k=1}^{N} b_k e^{-j2\pi f k} X\left(e^{j2\pi f}\right),$$

Solving for $H\left(e^{j2\pi f}\right)$, we have

$$H\left(e^{j2\pi f}\right) = \frac{Y\left(e^{j2\pi f}\right)}{X\left(e^{j2\pi f}\right)} = \frac{\sum_{k=1}^{N} b_k e^{-j2\pi f k}}{1 + \sum_{m=1}^{N} a_m e^{-j2\pi f m}}.$$

Homework

- Find the transfer function of the following systems

$$y[n] = 2r\cos(\omega_0)y[n-1] - r^2 y[n-2] + x[n] - r\cos(\omega_0)x[n-1].$$

- Find the Inverse Discrete-time Fourier transform of $X\left(e^{j2\pi f}\right)$, where

$$X\left(e^{j2\pi f}\right) = \begin{cases} 1 & |f| < B < \frac{1}{2} \\ 0 & B < |f| < \frac{1}{2} \end{cases}$$

- Problems 6.65–6.69 from [3]
- Problem 6.71 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill