Signals and Communication Theory

Communication Theory and Signals and Systems

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Geometric sequence and geometric sum

A **geometric sequence** is a series in which consecutive elements differ by a constant ratio. Such sequence can be expressed as

$$x[n] = r^n$$

where r is constant.

The sum of finite-duration geometric sequence is given by

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

By setting $N = \infty$ and $|r| < 1$ gives the sum of infinite-duration geometric sequence, i.e.,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

Convolution in LTI systems

$$\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n-k]$$

We first consider a unit-step sequence delayed by k at the input of the system. Since the considered system is **Time-Invariance**, the output is the impulse response delayed by k, $h[n-k]$.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Now considering $x[n]$ as superposition of unit-pulse sequences, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$, and the **linearity** of the system, the output $y[n]$ is expressed as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k], \\ &= x[n] * h[n], \end{aligned}$$

which is the **convolution operation**.

Commutative Property

By definition we have

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k],$$

Using $m = n - k$, the following relation holds

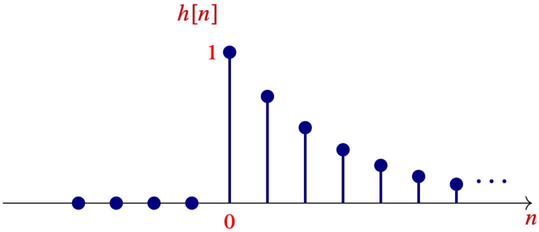
$$\begin{aligned}\sum_{k=-\infty}^{\infty} x[k]h[n-k] &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \\ &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= h[n] * x[n]\end{aligned}$$

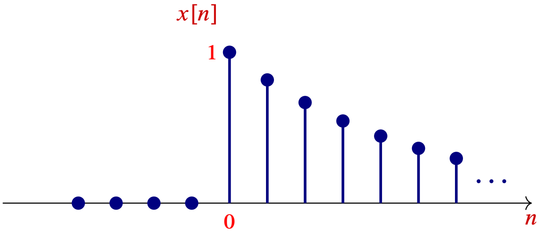
Consequently

$$x[n] * h[n] = h[n] * x[n]$$

Example 1

Assume that $h[n] = a^n u[n]$ *and* $x[n] = b^n u[n]$. *Compute the convolution.*



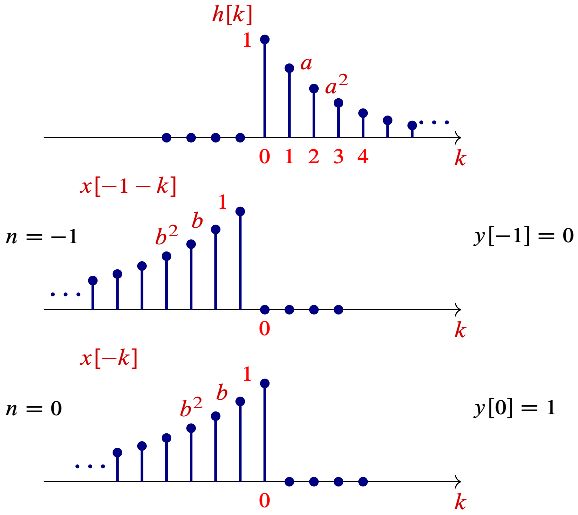


Answer to Example 1: Analytic method

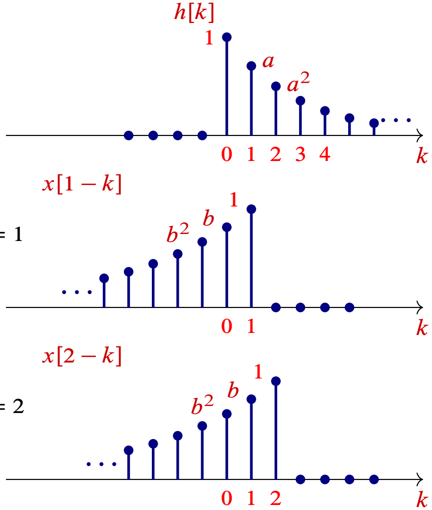
Substituting $h[n]$ and $x[n]$ into the definition of convolution, we have

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} b^k u[k] a^{n-k} u[n-k] \\
&= \sum_{k=0}^{\infty} b^k a^{n-k} u[n-k] && \text{because } u[k] = 0 \text{ for } k < 0 \\
&= \sum_{k=0}^{n} b^k a^{n-k} && \text{because } u[n-k] = 0 \text{ for } k > n \\
&= a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k && \text{applying geometric sum} \\
&= a^n \frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}} && \text{simplifying} \\
&= \frac{a^{n+1}-b^{n+1}}{a-b}
\end{aligned}
$$

Answer to Example 1: Visual approach



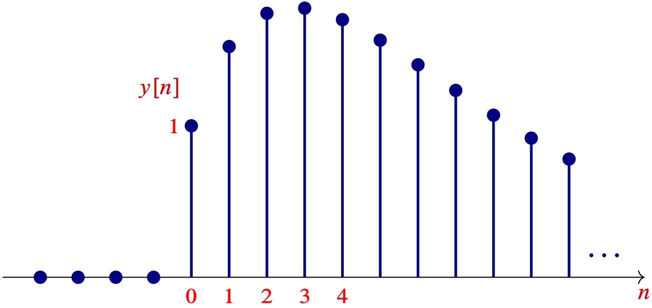
Answer to Example 1: Visual approach, cont.



$n = 1$ $\quad y[1] = a + b$

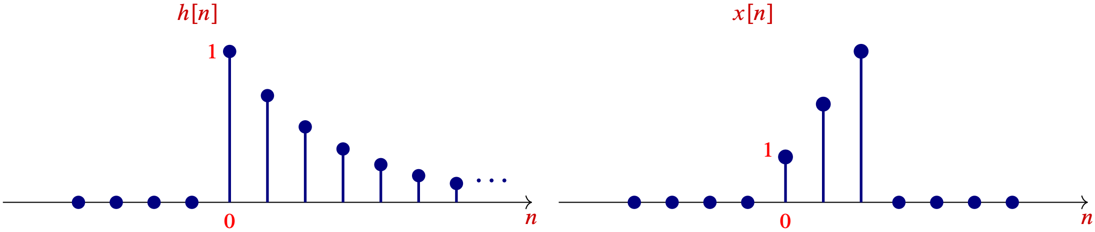
$n = 2$ $\quad y[2] = a^2 + ab + b^2$

Answer to Example 1: Output



Example 2

Assume that $h[n] = a^n u[n]$ *and* $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$. *Compute the convolution.*

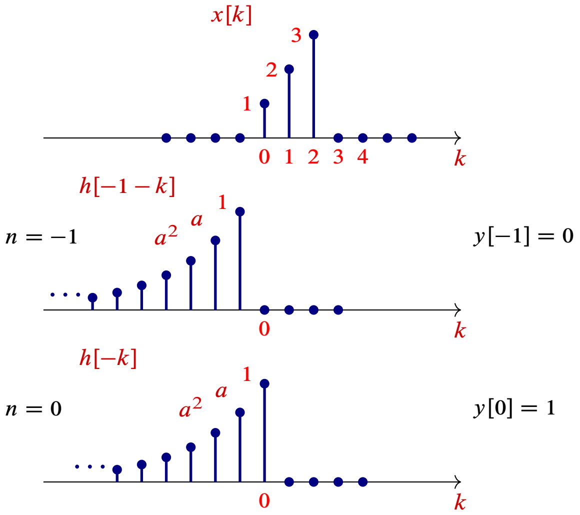


Answer to Example 2: Analytic method

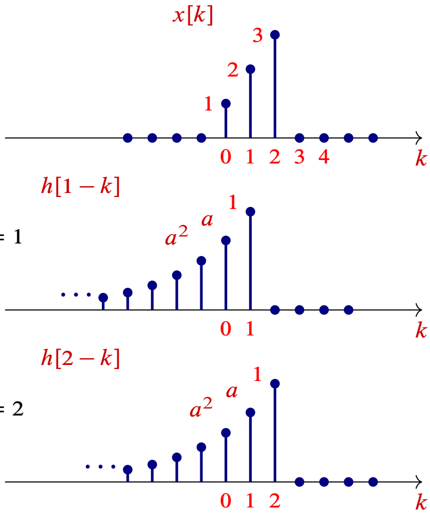
Substituting $h[n]$ and $x[n]$ gives

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} \Big(\delta[k] + 2\delta[k-1] + 3\delta[k-2]\Big) a^{n-k} u[n-k] \\
&= \sum_{k=-\infty}^{\infty} \delta[k] a^{n-k} u[n-k] + 2 \sum_{k=-\infty}^{\infty} \delta[k-1] a^{n-k} u[n-k] \\
&\quad + 3 \sum_{k=-\infty}^{\infty} \delta[k-2] a^{n-k} u[n-k] \\
&= a^n u[n] + 2a^{n-1} u[n-1] + 3a^{n-2} u[n-2]
\end{aligned}
$$

Answer to Example 2: Visual approach



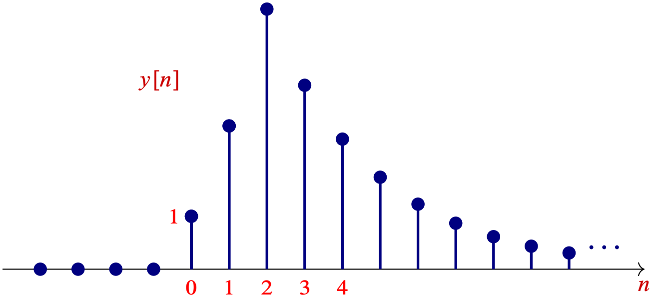
Answer to Example 2: Visual approach, cont.



$n = 1$ $\quad y[1] = a + 2$

$n = 2$ $\quad y[2] = a^2 + 2a + 3$

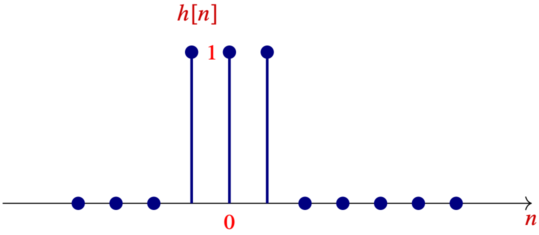
Answer to Example 2: Output



Example 3

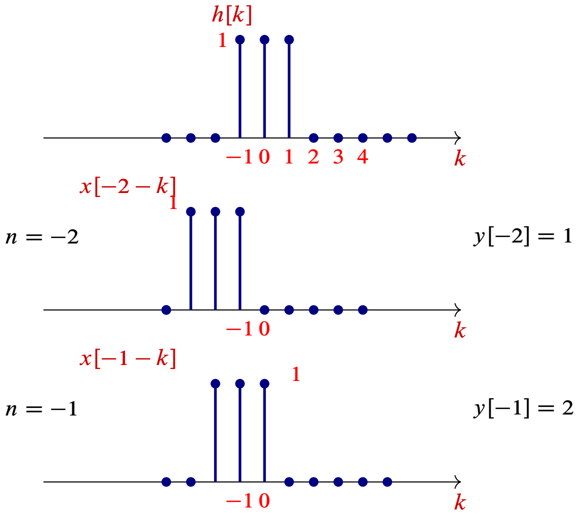
Now let's consider

$$h[n] = x[n] = \begin{cases} 1, & n = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

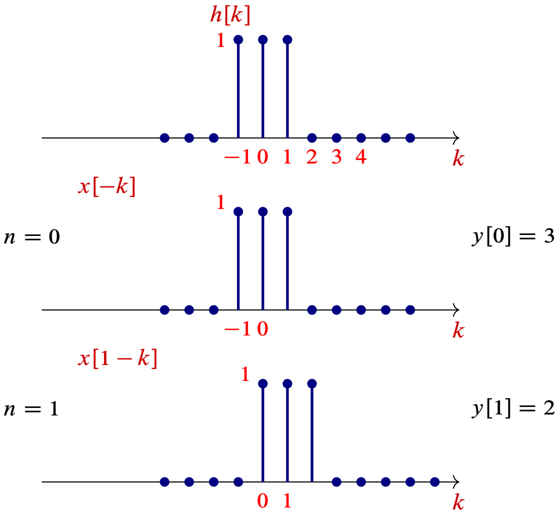


Compute the convolution.

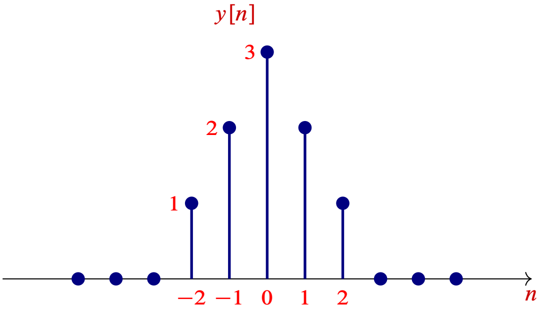
Answer to Example 3



Answer to Example 3 cont.



Answer to Example 3: Output



Finite-duration sequences

If the sequence $x(n)$ and $h(n)$ contain N_x and N_h samples, respectively, then the output sequence $y(n)$ contains $N_y = N_x + N_h - 1$ samples.

Convolution: Toeplitz Matrix

First assume that $h[n]$, for $n = 0, \ldots, N_h - 1$, and $x[n]$, for $n = 0, \ldots, N_x - 1$, are FIR sequences.

$$y[n] = \sum_{k=0}^{N_h} h[k]x[n-k]$$

Expanding for each value of n, we have

$$\begin{aligned}
y[0] &= h[0]x[0] \\
y[1] &= h[1]x[0] + h[0]x[1] \\
y[2] &= h[2]x[0] + h[1]x[1] + h[0]x[2] \\
\vdots &\qquad \vdots
\end{aligned}$$

Convolution: Toeplitz Matrix cont.

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ \vdots \\ y[N_x+N_h-2] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ h[1] & h[0] & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & \cdots & 0 \\ h[3] & h[2] & h[1] & h[0] & 0 & 0 & 0 & \cdots & 0 \\ h[4] & h[3] & h[2] & h[1] & h[0] & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ \vdots \\ x[N_x-1] \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}$$

Ejemplo 4

Obtain the matrix $\boldsymbol{H}$ *if* $h[n]$ *is defined as*

$$h[n] = \begin{cases} 1, & n = 0, \\ 0.95, & n = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and the input sequence is ranging from $n = 0$ *to* 9.

Answer to Example 4

$$
\boldsymbol{H} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.95 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.95 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.95 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.95 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.95 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.95 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95
\end{bmatrix}
$$

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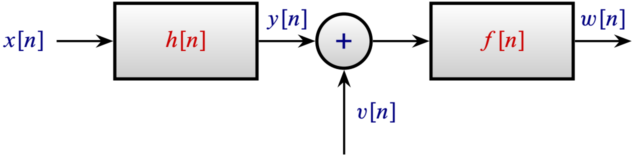
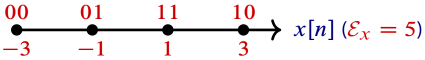
Digital Communication Application: Equalizer





$$\boldsymbol{f} = \boldsymbol{H}^{\mathrm{T}} \left(\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} + \frac{1}{\mathrm{SNR}} \boldsymbol{I} \right)^{-1} \mathbf{1}_{n_0}$$

$$\mathrm{SNR} = \frac{\mathcal{E}_x}{\mathcal{N}_0}$$

Computer Simulation

For this simulation the following parameters are involved.

- We consider $\mathrm{SNR} = 100$.
- Considering $\mathrm{SNR} = 100$ and $\mathcal{E}_x = 5$ gives $\mathcal{N}_0 = 0.05$.
- The noise signal $v[n]$ is a white gaussian random process of zero mean and variance $\mathcal{N}_0$.
- The discrete channel is $h[n] = \delta[n] + 0.95\delta[n-1]$ (see Example 4).
- The length of the considered system $f[n]$ is $N_f = 10$.

Results

bits	$x[n]$	$y[n]$	$v[n]$	$y[n]+v[n]$	$\widehat{x_1}[n]$	$\widehat{b_1}$	$w[n]$	$\widehat{x_{eq}}[n]$	$\widehat{b}$
11	1	−1.85	0.08	−1.77	−1	01	1.58	1	11
10	3	3.95	0.16	4.11	3	10	2.46	3	10
01	−1	1.85	−0.29	1.56	1	11	−0.80	−1	01
10	3	2.05	−0.23	1.82	1	11	2.33	3	10
01	−1	1.85	0.18	2.03	3	10	−0.31	−1	01
10	3	2.05	−0.03	2.02	3	10	1.88	1	11
01	−1	1.85	0.12	1.97	1	11	0.45	1	11
01	−1	−1.95	−0.21	−2.16	−3	00	−2.20	−3	00
00	−3	−3.95	−0.37	−4.32	−3	00	−2.22	−3	00
00	−3	−5.85	0.17	−5.68	−3	00	−3.51	−3	00
01	−1	−3.85	0.27	−3.58	−3	00	−0.05	−1	01

bits	$\cdots 1110011001100101000001 \cdots$
without equalizer	$\cdots 0110111110101100000000 \cdots$
with equalizer	$\cdots 1110011001111100000001 \cdots$

Contents

Stability for LTI systems

An LTI system is BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Proof. Lets first consider a bounded input, i.e., $|x[n]| < B_x$. Now using the convolution operation, the following relation holds

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

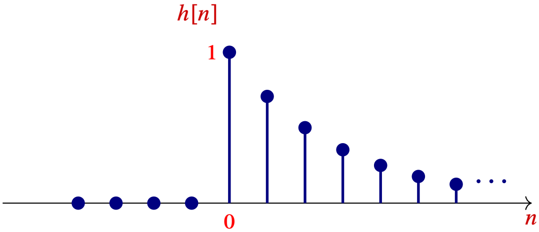
$$\leq B_x \sum_{k=-\infty}^{\infty} |h[k]| < B_y$$

Consequently

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Example 5

Assume that $h[n] = a^n u[n]$*. Is the system stable?*



Answer to Example 5

Using the geometric sum, we have

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |a^n u[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|}, \quad |a| < 1$$

The system is stable if $|a| < 1$.

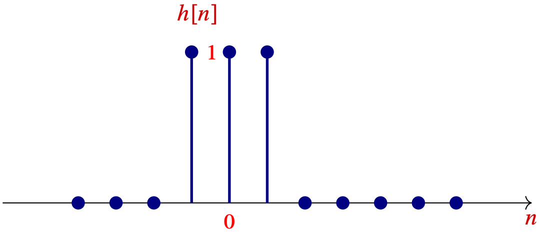
Example 6

Is the Accumulator stable ($h[n] = u[n]$)?

Example 7

Now let's consider

$$h[n] = \begin{cases} 1, & n = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$



Is the system stable?

Homework 1

- Problem 2.47 from [3]
- Problem 2.56 from [3]
- Problem 2.57 from [3]
- Problem 2.58 from [3]
- Problem 2.63 from [3]
- Problem 2.64 from [3]

[3] Hwei Hsu, *Schaum's Outline of Signals and Systems*, Second Edition, 2010, McGraw Hill