Simple approach to the relation between laser frequency noise and laser line shape

Gianni Di Domenico,* Stéphane Schilt, and Pierre Thomann

Laboratoire Temps-Fréquence, Université de Neuchâtel, Avenue de Bellevaux 51, CH-2009 Neuchâtel, Switzerland

*Corresponding author: gianni.didomenico@unine.ch

Received 13 May 2010; accepted 8 July 2010;
posted 26 July 2010 (Doc. ID 128398); published 30 August 2010

Frequency fluctuations of lasers cause a broadening of their line shapes. Although the relation between the frequency noise spectrum and the laser line shape has been studied extensively, no simple expression exists to evaluate the laser linewidth for frequency noise spectra that does not follow a power law. We present a simple approach to this relation with an approximate formula for evaluation of the laser linewidth that can be applied to arbitrary noise spectral densities. © 2010 Optical Society of America

OCIS codes: 140.3425, 140.3430, 140.3460, 120.0120.

1. Introduction

Lasers with a high spectral purity currently find important applications in frequency metrology, high-resolution spectroscopy, coherent optical communications, and atomic physics, to name a few uses. Advances in investigation and narrowing of laser linewidth have experienced a remarkable evolution, yielding techniques that give us unprecedented control over the optical phase/frequency [1–9]. The spectral properties of such lasers can be conveniently described either in terms of their optical line shape and associated linewidth or in terms of the power spectral density of their frequency noise. Both approaches are complementary, but the knowledge of the frequency noise spectral density provides much more information on the laser noise. A measurement of the laser linewidth (obtained by heterodyning with a reference laser source or by self-homodyne/heterodyne interferometry using a long optical delay line) is often sufficient in many applications (e.g., in high-resolution spectroscopy or coherent optical communications). Some experiments, though, require more complete knowledge of the Fourier distribution of the laser frequency fluctuations. Knowledge of the frequency noise spectral density enables one to retrieve the laser line shape and, thus, the linewidth (while the reverse process, i.e., determining the noise spectral density from the line shape, is not possible), but this operation is most often not straightforward.

The relation between frequency noise spectral density and laser linewidth has already been addressed in many papers dealing with general theoretical considerations or with more or less particular cases. In one of the earliest papers on this topic, Elliott and co-workers [10] derived theoretical formulas linking the frequency noise spectral density to the laser line shape. They also discussed the different line shapes obtained in the case of a rectangular noise spectrum of finite bandwidth in the two extreme conditions where the ratio of the frequency deviation to the noise bandwidth is either large (leading to a Gaussian line shape) or small (resulting in a Lorentzian line shape). Their work was supported by experimental results showing the transformation of the laser spectrum from Lorentzian to Gaussian for decreasing noise bandwidth. The ideal case of a pure white frequency noise spectrum has been extensively reported for a long time (see, for example, [11]), as it can be fully solved analytically leading to the well-known Lorentzian line shape described by the Schawlow–Townes–Henry linewidth [12,13]. However, the real noise spectrum of a laser is much more complicated and leads to a nonanalytical line shape that can be determined only numerically. Lasers are

0003-6935/10/254801-07$15.00/0

generally affected by flicker noise at low frequency, and this type of noise has been widely studied in the literature [14–17]. The major feature of this type of noise is to produce spectral broadening of the laser linewidth compared to the Schawlow–Townes–Henry limit, but an exact expression of the line shape cannot be obtained, and different approximations have been proposed to describe this situation. For example, Tourrenc [15] numerically showed the divergence of the linewidth with increasing observation time in the presence of $1/f$-type noise, while Mercer [16] gave an analytical approximation for this diverging Gaussian linewidth. Stéphan *et al.* [14] gave a different approximation of the $1/f$-induced Gaussian contribution to the line shape, with a linewidth that does not contain any dependency on the observation time, and Godone *et al.* [18,19] gave the rf spectra corresponding to phase noise spectral densities of arbitrary slopes. Finally, some publications also stated that the combined contribution of white noise Lorentzian line shape and $1/f$-noise Gaussian line shape resulted in a Voigt profile for the optical line shape [14,16,20].

In this paper, we present a simple geometric approach to determine the linewidth of a laser from its frequency noise spectral density. Our approach makes use of a simple approximate formula to determine the linewidth corresponding to an arbitrary noise spectrum. Starting with the ideal case of a low-pass filtered white noise of varying cutoff frequency, we show how differently the low- and high-frequency noise components affect the line shape and how the linewidth changes with respect to the noise cutoff frequency. Then, we demonstrate in which limit conditions the Lorentzian and Gaussian line shapes generally discussed in former publications are retrieved. We introduce our simple approximation of the linewidth by showing how the noise spectrum can be geometrically separated in two areas with a fully different influence on the line shape. Only one of these spectral areas contributes to the linewidth, the remaining part of the spectrum influencing only the wings of the line shape. The main benefit of our work is to make a simple link between the frequency noise spectrum of a laser and its linewidth, without any assumption on the noise spectral distribution. By showing how some spectral components of the noise determine the linewidth while others affect only the wings of the line shape, we provide a simple geometric criterion to determine those spectral components that contribute to the linewidth. As a result, a simple formula is reported to calculate the linewidth of a laser for an arbitrary frequency noise spectrum, i.e., this expression is applicable to any type of frequency noise and is thus not restricted to the ideal cases of white noise and flicker noise usually considered.

Before introducing our approach, we give a brief reminder of the important theoretical steps enabling the linking of the frequency noise spectrum of a laser and its line shape. A detailed theoretical description can be found in [10,15,16]. Given the frequency noise spectral density $S_{\delta\nu}(f)$ (we consider single-sided spectral densities throughout this article) of the laser light field $E(t) = E_0 \exp[i(2\pi\nu_0 t + \phi(t))]$ (complex representation), one can calculate the autocorrelation function $\Gamma_E(\tau) = \overline{E^*(t)E(t+\tau)}$ as follows:

$$\Gamma_E(\tau) = E_0^2 e^{i2\pi\nu_0\tau} e^{-2\int_0^\infty S_{\delta\nu}(f)\frac{\sin^2(\pi f\tau)}{f^2}\mathrm{d}f}, \tag{1}$$

where $\delta\nu = \nu - \nu_0$ is the laser frequency deviation around its average value ν_0. According to the Wiener–Khintchine theorem, the laser line shape is given by the Fourier transform of the autocorrelation function

$$S_E(\nu) = 2\int_{-\infty}^{\infty} e^{-i2\pi\nu\tau}\Gamma_E(\tau)\mathrm{d}\tau. \tag{2}$$

Unfortunately, this general formula most often cannot be analytically integrated, except for the trivial case of white frequency noise $S_{\delta\nu}(f) = h_0$ (with h_0 given in $\mathrm{Hz}^2/\mathrm{Hz}$) that leads to the well-known Lorentzian line shape with a full width at half-maximum $\mathrm{FWHM} = \pi h_0$ [10,15,16].

In the following, we will start by studying the case of a low-pass filtered white frequency noise. This will lead us to establish a simple approximate formula of the linewidth of a real laser from its frequency noise spectrum. Finally, we will apply this formula to different situations that are of practical interest to experimentalists and in which frequency noise is important.

2. Laser Spectrum in the Case of a Low-Pass Filtered White Frequency Noise

As an introduction to the derivation of our approximate expression of the laser linewidth, let us first consider a frequency noise spectral density that has a constant level $h_0(\mathrm{Hz}^2/\mathrm{Hz})$ below a cutoff frequency f_c and that drops to zero above this threshold:

$$S_{\delta\nu}(f) = \begin{cases} h_0 & \text{if } f \le f_c \\ 0 & \text{if } f > f_c \end{cases}. \tag{3}$$

In this simple case, it is possible to evaluate analytically the integral in Eq. (1) and obtain the following expression for the autocorrelation function:

$$\Gamma_E(\tau) = E_0^2 e^{i2\pi\nu_0\tau} e^{2\frac{h_0}{f_c}(\sin^2(\pi f_c\tau) - \pi f_c\tau \mathrm{Si}(2\pi f_c\tau))}, \tag{4}$$

where $\mathrm{Si}(x) = \int_0^x \sin(t)/t\,\mathrm{d}t$ is the sine integral function. On the other hand, most often, it is not possible to obtain an analytical expression for the Fourier transform in Eq. (2), and, therefore, the laser line shape must be evaluated numerically. An analytical expression of the line shape is, however, calculable in the two extreme conditions in which $f_c \to \infty$ and $f_c \to 0$:

- When $f_c \to \infty$:

$$S_E(\nu) = E_0^2 \frac{h_0}{(\nu - \nu_0)^2 + (\pi h_0/2)^2}, \tag{5}$$

and the line shape is Lorentzian with a width $\mathrm{FWHM} = \pi h_0$ (this corresponds to the white noise previously mentioned).

- When $f_c \to 0$:

$$S_E(\nu) = E_0^2 \left(\frac{2}{\pi h_0 f_c}\right)^{1/2} e^{-\frac{(\nu-\nu_0)^2}{2h_0 f_c}}, \tag{6}$$

and the line shape is Gaussian with a width $\mathrm{FWHM} = (8\ln(2) h_0 f_c)^{1/2}$ that depends on the cutoff frequency f_c.

For a fixed frequency noise level h_0, it is interesting to numerically study the evolution of the laser line shape as a function of the cutoff frequency f_c between these two extreme cases. The result is shown in Fig. 1 for $h_0 = 1\,\mathrm{Hz}^2/\mathrm{Hz}$. According to Eqs. (5) and (6), one observes that when $f_c \ll h_0$, the line shape is Gaussian and the linewidth increases with f_c. However, when $f_c \gg h_0$, the line shape becomes Lorentzian and the linewidth stops to increase (it will be shown later that the noise at high Fourier frequencies contributes only to the wings of the line shape). In order to explore the transition between these two regimes, we numerically calculated the linewidth as a function of the cutoff frequency f_c, and the results are presented in Fig. 2. We found that a good approximation valid for any f_c is given by the following expression:

$$\mathrm{FWHM} = h_0 \frac{(8\ln(2) f_c/h_0)^{1/2}}{\left[1 + \left(\frac{8\ln(2)}{\pi^2}\frac{f_c}{h_0}\right)^2\right]^{1/4}}, \tag{7}$$

with a relative error smaller than 4% over the entire range of the cutoff frequency f_c, as shown in the lower

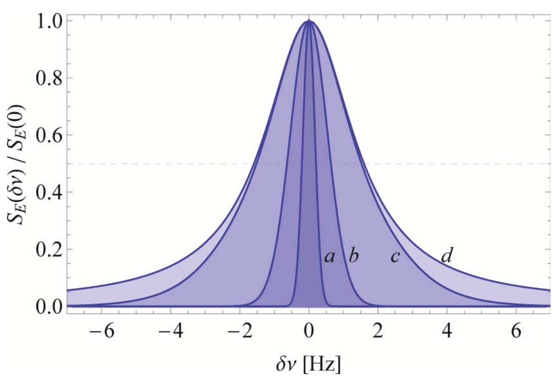

Fig. 1. (Color online) Numerical calculation of the laser line shape $S_E(\delta\nu)$ for a fixed frequency noise level $h_0 = 1\,\mathrm{Hz}^2/\mathrm{Hz}$ and different values of the cutoff frequency: a, $f_c = 0.03\,\mathrm{Hz}$; b, $f_c = 0.3\,\mathrm{Hz}$; c, $f_c = 3\,\mathrm{Hz}$; and d, $f_c = 30\,\mathrm{Hz}$. The line shapes are normalized to help the comparison of their full width at half-maximum (FWHM). The line shape evolves from a Gaussian when $f_c \ll h_0$ and to a Lorentzian when $f_c \gg h_0$.

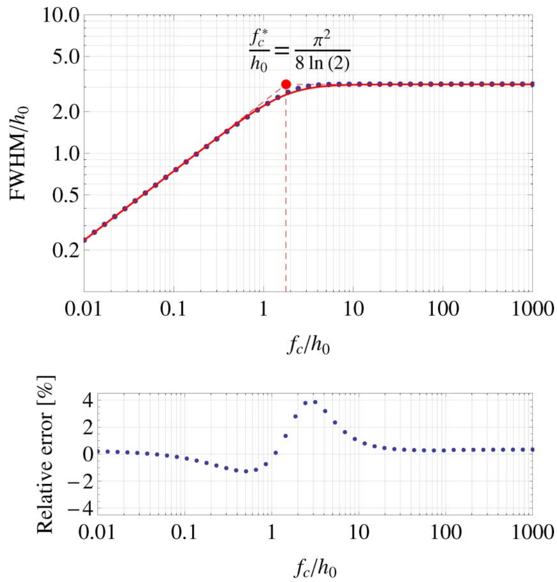

Fig. 2. (Color online) Upper graph: Numerical computation showing the evolution of the linewidth (FWHM) with the cutoff frequency f_c in the case of low-pass filtered white noise. The dots have been calculated by numerical integration of the exact relations Eqs. (1) and (2). The continuous line is given by our approximate formula Eq. (7). Both horizontal and vertical scales have been normalized to the noise level h_0. The behavior at low and high cutoff frequencies is indicated by the asymptotic response (red dashed lines). Lower graph: Relative error between the exact and approximate values.

graph of Fig. 2. The corner frequency corresponding to the transition between the two regimes is situated at the intersection of the two asymptotes shown in the upper graph of Fig. 2 and is given by

$$f_c^* = \frac{\pi^2}{8\ln(2)} h_0 \approx 1.78 h_0. \tag{8}$$

3. Simple Formula to Estimate the Laser Linewidth

The example of the low-pass filtered white noise shows that the frequency noise spectrum can be separated into two regions that affect the line shape in a radically different way. In the first region, defined by $S_{\delta\nu}(f) > 8\ln(2)f/\pi^2$, the noise contributes to the central part of the line shape, which is Gaussian, and thus to the laser linewidth. In the second region, defined by $S_{\delta\nu}(f) < 8\ln(2)f/\pi^2$, the noise contributes mainly to the wings of the line shape but does not affect the linewidth. The striking difference between the noise effects in these two regions can be understood in terms of frequency modulation theory. In the first region, the noise level is high compared to its Fourier frequency, therefore it produces a slow frequency modulation with a high modulation index [21] $\beta > 1$. Conversely, in the second region, the noise level is small compared to its Fourier frequency, and, accordingly, the modulation index β is small, which means that the modulation is too fast to have a significant effect on the laser linewidth. In the rest of this article, the line separating these two regions will be called the β-separation line. These observations are summarized in Fig. 3, where a typical laser frequency noise spectral density is represented. A careful inspection of Eqs. (1) and (2) shows that the low frequency approximation given in Eq. (6) can be extended to arbitrary noise spectra. Indeed, noise components in the high modulation index area with a spectral density higher than their Fourier frequency ($S_{\delta\nu}(f) > f$) give rise to Gaussian autocorrelation functions, which are multiplied together and then Fourier transformed to give the laser line shape. As a result, the line shape is a Gaussian function whose variance is the sum of the contributions of all high modulation index noise components. Therefore, one can obtain a good approximation of the laser linewidth by the following simple expression:

$$\mathrm{FWHM} = (8\ln(2)A)^{1/2}, \tag{9}$$

where A is the surface of the high modulation index area, i.e., the overall surface under the portions of $S_{\delta\nu}(f)$ that exceed the β-separation line (see Fig. 3)

$$A = \int_{1/T_o}^{\infty} H(S_{\delta\nu}(f) - 8\ln(2)f/\pi^2)S_{\delta\nu}(f)\mathrm{d}f, \tag{10}$$

with $H(x)$ being the Heaviside unit step function ($H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$) and T_o being the measurement time that prevents the observation of low frequencies below $1/T_o$. This low frequency limit can be set to zero when the area in Eq. (10) does not show low frequency divergence. However, this is not the case in the presence of flicker noise, for which the measurement time plays an important role, as will be shown in the next section.

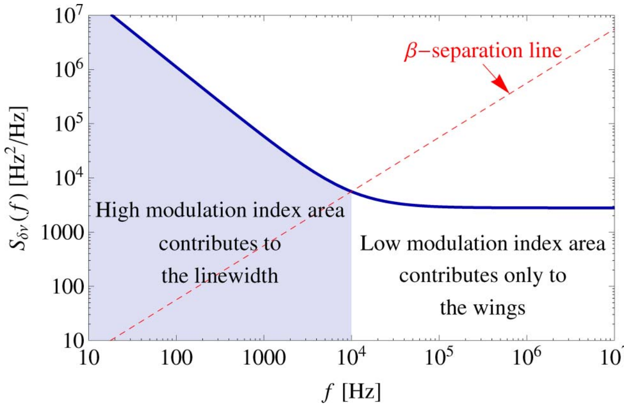

Fig. 3. (Color online) A typical laser frequency noise spectral density composed of flicker noise at low frequencies and white noise at high frequencies. The dashed line given by $S_{\delta\nu}(f) = 8\ln(2)f/\pi^2$ separates the spectrum into two regions whose contributions to the laser line shape is very different: the high modulation index area contributes to the linewidth, whereas the low modulation index area contributes only to the wings of the line shape (see the text for details).

4. Application 1: Laser Spectrum in the Case of Flicker Frequency Noise

As a first application of our approach, let us consider the case of a laser suffering from pure flicker noise, i.e., $S_{\delta\nu}(f) = af^{-\alpha}$, with $1 \leq \alpha \leq 2$. For the sake of clarity, let us write the parameter a in terms of the frequency f_m at which $S_{\delta\nu}$ intersects the β-separation line, i.e., $a = 8\ln(2)f_m^{\alpha+1}/\pi^2$. This allows a dimensionless representation of the flicker noise model

$$\frac{S_{\delta\nu}(f)}{f_m} = \frac{8\ln(2)}{\pi^2}\left(\frac{f}{f_m}\right)^{-\alpha}, \tag{11}$$

as illustrated in Fig. 4. As mentioned in the previous section, the linewidth is a function of the observation time T_o, and one can evaluate this dependence using the approximate formulas, Eqs. (9) and (10). After integrating Eq. (10), one obtains for $\alpha = 1$

$$\mathrm{FWHM} = f_m\frac{8\ln(2)}{\pi}[\ln(f_mT_o)]^{1/2}, \tag{12}$$

and for $\alpha > 1$,

$$\mathrm{FWHM} = f_m\frac{8\ln(2)}{\pi}\left[\frac{(f_mT_o)^{\alpha-1} - 1}{\alpha - 1}\right]^{1/2}. \tag{13}$$

In order to check the validity of these approximate formulas, we integrated numerically the exact relation given by Eqs. (1) and (2) to obtain the line shape for different values of the exponent α of the flicker noise ($\alpha = 1$, 1.2, 1.5, 1.7, and 2.0), from which we calculated the linewidth (FWHM). The numerical results superposed to the approximate values given by Eqs. (12) and (13) are presented in Fig. 5. They show a good agreement, the error being smaller than 10% as long as $T_of_m > 5$. Note that the discrepancy appears when the lower bound $1/T_o$ approaches f_m. This behavior was expected because the transition between high and low modulation index areas is progressive and thus can lead to deviations from the approximations given in Eqs. (9) and (10). More details on this intermediate regime will be given in the next

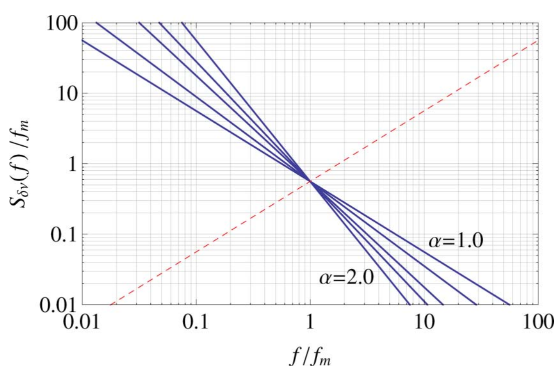

Fig. 4. (Color online) Pure flicker frequency noise model of Eq. (11) with $\alpha = 1$, 1.2, 1.5, 1.7, and 2.0. The axes are normalized with respect to the frequency f_m, at which $S_{\delta\nu}$ intersects the β-separation line.

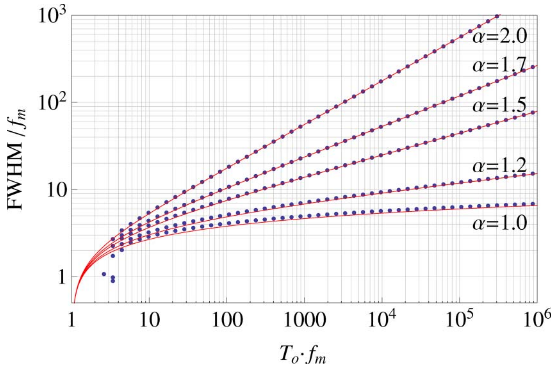

Fig. 5. (Color online) Evolution of the laser linewidth with respect to the measurement time in the case of a frequency noise spectrum composed of flicker noise as shown in Fig. 4. The dots have been obtained by numerical integration of the exact relation between the frequency noise and the line shape given by Eqs. (1) and (2). The red lines are the values given by the approximate formulas Eqs. (12) and (13).

section. As a final remark, let us discuss the relevance of this pure flicker noise model. Although this case may seem far from the frequency noise spectrum of a real laser, which has a white noise background at high Fourier frequencies, we should stress that the frequency noise below the β-separation line does not contribute to the linewidth but only to the wings of the line shape. As a consequence, our pure flicker noise model applies to any laser having flicker noise above the β-separation line, whatever the noise in the low modulation index area is.

5. Application 2: Laser Linewidth Reduction

In this section, we discuss the process of laser linewidth reduction by applying our approach to a simplifed laser frequency noise model that still keeps the main features of the problem. In this model, a free-running laser with a constant frequency noise level h_b ($\mathrm{Hz}^2/\mathrm{Hz}$) is considered, and we assume that the frequency noise is reduced to another constant level h_a with a servo loop of bandwidth f_b. The resulting frequency noise spectral density is given by $S_{\delta\nu}(f) = h_a$ if $f < f_b$ and $S_{\delta\nu}(f) = h_b$ if $f \geq f_b$, as illustrated in Fig. 6. Notice that this simplified noise model may also result from a laser showing initial flicker noise in free-running mode if the servo loop contains an integral part that reduces the flicker noise at low frequencies. With this model, it is interesting to calculate the evolution of the laser line shape and linewidth with the servo-loop bandwidth. One can evaluate Eq. (1) to obtain the autocorrelation function

$$\Gamma_E(\tau) = E_0^2 e^{i2\pi\nu_0\tau} e^{-h_b\pi^2|\tau| - \frac{h_a - h_b}{f_b}\left(\omega_b\tau\mathrm{Si}(\omega_b\tau) - 2\sin^2\left(\frac{\omega_b\tau}{2}\right)\right)}, \tag{14}$$

where $\omega_b = 2\pi f_b$, and $\mathrm{Si}(x)$ is the sine integral function. Because the Fourier transform of this autocorrelation function is difficult to solve analytically, we evaluated the laser line shape numerically and then deduced the linewidth (FWHM). The results are presented in Figs. 7 and 8. We observe that the laser linewidth tends toward πh_b when the bandwidth f_b tends toward zero. This can be understood because the noise spectrum tends toward a white-type noise of spectral density h_b, leading to a Lorentzian profile of width πh_b. On the other hand, the linewidth drops down to πh_a when the bandwidth f_b tends toward infinity, since in this case, the noise spectrum approaches a white-type noise of spectral density h_a. In Fig. 7, we reported with a dashed line the linewidth obtained with our approximate formula Eqs. (9) and (10), and the agreement with the results of the numerical integration is good, except when the value of the servo bandwidth is between h_a and h_b. In order to understand the origin of this discrepancy, we reported in Fig. 8 the laser line shape for four particular values of the bandwidth. We observe that the line shape changes considerably in this range: the servo loop

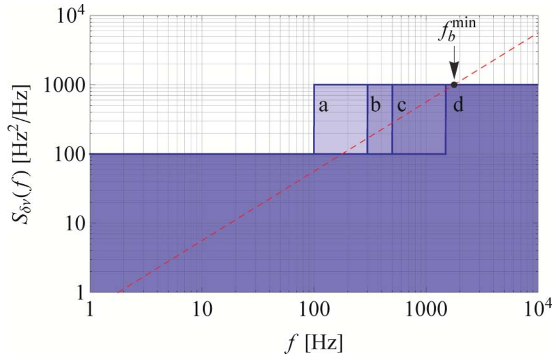

Fig. 6. (Color online) Frequency noise model used to study laser linewidth reduction using a servo loop. We assume that the free-running laser noise level $h_b = 1000\,\mathrm{Hz}^2/\mathrm{Hz}$ is reduced to $h_a = 100\,\mathrm{Hz}^2/\mathrm{Hz}$ with a servo loop having a bandwidth f_b of a, 100 Hz; b, 300 Hz; c, 500 Hz; and d, 1500 Hz. The dashed line represents the β-separation line. The minimum servo-loop bandwidth necessary to efficiently reduce the laser linewidth is $f_b^{\min} = \pi^2 h_b/(8\ln(2))$.

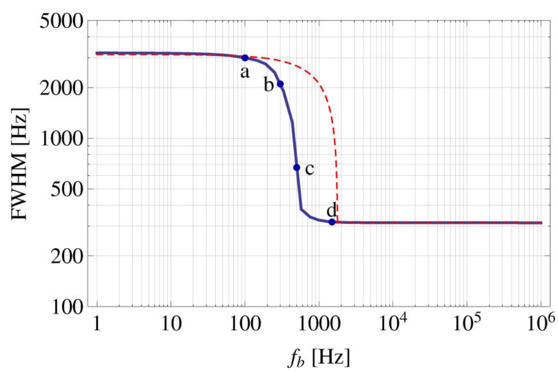

Fig. 7. (Color online) Evolution of the laser linewidth (FWHM) with the servo-loop bandwidth f_b for the frequency noise model presented in Fig. 6. Special values of the servo bandwidth, for which the line shape is represented in Fig. 8, are indicated by the following points: a, $f_b = 100$ Hz; b, $f_b = 300$ Hz; c, $f_b = 500$ Hz; and d, $f_b = 1500$ Hz. The continuous line has been obtained by numerical integration of the exact relation Eqs. (1) and (2), and the dashed line has been obtained with our approximate formula Eqs. (9) and (10).

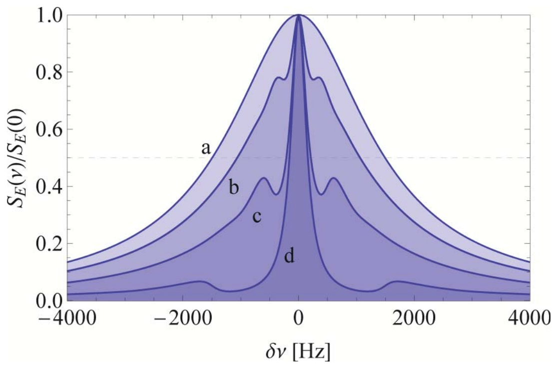

Fig. 8. (Color online) Evolution of the laser line shape with the servo-loop bandwidth for the frequency noise model presented in Fig. 6. We chose the following values of the servo bandwidth: a, $f_b = 100\,\mathrm{Hz}$; b, $f_b = 300\,\mathrm{Hz}$; c, $f_b = 500\,\mathrm{Hz}$; and d, $f_b = 1500\,\mathrm{Hz}$, which correspond to the points indicated in Fig. 7.

repels the frequency noise from the center, and, as a consequence, two sidebands appear outside of the servo bandwidth, i.e., at $\delta\nu > f_b$, while the central part strongly narrows and becomes Lorentzian. Because of this radical change of line shape, the different linewidths at half-maximum are not similar in this range, and comparison with the Gaussian linewidth approximation Eqs. (9) and (10) loses its significance, which explains the observed discrepancy. Nevertheless, our approximate formula is able to predict the minimum servo-loop bandwidth necessary to efficiently reduce the laser linewidth, which is given by $f_b^{\min} = \pi^2 h_b / (8\ln(2))$. It depends on the free-running laser noise level h_b and corresponds to the situation in which the noise level h_b is entirely below the β-separation line for frequencies outside of the servo bandwidth (see Fig. 6). As a consequence, when $f_b > f_b^{\min}$, only the low frequency part with noise level h_a is above the β-separation line and contributes to the laser linewidth, which is given by πh_a. Note that the final laser linewidth depends on the noise level h_a, and thus on the servo-loop gain at low frequency, but is independent of the servo bandwidth, provided that $f_b > f_b^{\min}$.

6. Conclusion

The study of a low-pass filtered white frequency noise has led us to the establishment of a new and simple approximation of the relation between frequency noise and laser linewidth, which is valid for arbitrary noise spectra. We have shown how the frequency noise spectrum is separated into two areas corresponding to high and low modulation index regimes (i.e., $\beta > 1$ and $\beta < 1$) by a simple line that we called the β-separation line. Then, we explained why only those spectral components for which the frequency noise is higher than the β-separation line (the high modulation index area) contribute to the linewidth. An approximate value of the linewidth is simply obtained from the geometrical surface of the high modulation index area. The application of this approach to the case of flicker noise provides an approximate formula for the linewidth, showing its dependence to the observation time. Finally, the use of this approach to the reduction of the laser linewidth emphasizes some important aspects of this problem, such as the minimal required servo-loop bandwidth and the achievable laser linewidth. Moreover, this last example showed that the limitations of this simplified approach appear only when the laser line shape is too complex to be characterized by a mere linewidth.

References

1. M. Zhu and J. L. Hall, "Stabilization of optical phase/frequency of a laser system: application to a commercial dye laser with an external stabilizer," J. Opt. Soc. Am. B **10**, 802–816 (1993).
2. B. C. Young, F. C. Cruz, W. M. Itano, and J. C. Bergquist, "Visible lasers with subhertz linewidths," Phys. Rev. Lett. **82**, 3799–3802 (1999).
3. L. Conti, M. D. Rosa, and F. Marin, "High-spectral-purity laser system for the AURIGA detector optical readout," J. Opt. Soc. Am. B **20**, 462–468 (2003).
4. M. Heurs, V. M. Quetschke, B. Willke, K. Danzmann, and I. Freitag, "Simultaneously suppressing frequency and intensity noise in a Nd:YAG nonplanar ring oscillator by means of the current-lock technique," Opt. Lett. **29**, 2148–2150 (2004).
5. J. Alnis, A. Matveev, N. Kolachevsky, T. Udem, and T. W. Hänsch, "Subhertz linewidth diode lasers by stabilization to vibrationally and thermally compensated ultralow-expansion glass Fabry–Pérot cavities," Phys. Rev. A **77**, 053809 (2008).
6. M. Yamanishi, T. Edamura, K. Fujita, N. Akikusa, and H. Kan, "Theory of the intrinsic linewidth of quantum-cascade lasers: Hidden reason for the narrow linewidth and line-broadening by thermal photons," IEEE J. Quantum Electron. **44**, 12 (2008).
7. F. Acernese, M. Alshourbagy, F. Antonucci, S. Aoudia, K. G. Arun, P. Astone, G. Ballardin, F. Barone, L. Barsotti, M. Barsuglia, T. S. Bauer, S. Bigotta, S. Birindelli, M. A. Bizouard, C. Boccara, F. Bondu, L. Bonelli, L. Bosi, S. Braccini, C. Bradaschia, A. Brillet, V. Brisson, H. J. Bulten, D. Buskulic, G. Cagnoli, E. Calloni, and E. Campagna, "Laser with an in-loop relative frequency stability of $1.0 \times 10 - 21$ on a 100 ms time scale for gravitational-wave detection," Phys. Rev. A **79**, 053824 (2009).
8. I. Galli, S. Bartalini, P. Cancio, G. Giusfredi, D. Mazzotti, and P. D. Natale, "Ultra-stable, widely tunable and absolutely linked mid-IR coherent source," Opt. Express **17**, 9582–9587 (2009).
9. S. Bartalini, S. Borri, P. Cancio, A. Castrillo, I. Galli, G. Giusfredi, D. Mazzotti, L. Gianfrani, and P. De Natale, "Observing the intrinsic linewidth of a quantum-cascade laser: beyond the Schawlow–Townes limit," Phys. Rev. Lett. **104**, 083904 (2010).
10. D. S. Elliott, R. Roy, and S. J. Smith, "Extracavity laser band shape and bandwidth modification," Phys. Rev. A **26**, 12–18 (1982).
11. P. B. Gallion and G. Debarge, "Quantum phase noise and field correlation in single frequency semiconductor laser systems," IEEE J. Quantum Electron. **20**, 343–350 (1984).
12. A. L. Schawlow and C. H. Townes, "Infrared and optical masers," Phys. Rev. **112**, 1940–1949 (1958).
13. C. Henry, "Theory of the linewidth of semiconductor lasers," IEEE J. Quantum Electron. **18**, 259–264 (1982).
14. G. M. Stéphan, T. T. Tam, S. Blin, P. Besnard, and M. Têtu, "Laser line shape and spectral density of frequency noise," Phys. Rev. A **71**, 043809 (2005).

15. J.-P. Tourrenc, "Caractérisation et modélisation du bruit d'amplitude optique, du bruit de fréquence et de la largeur de raie de VCSELs monomode," Ph.D. dissertation (Université de Montpellier II, 2005).
16. L. B. Mercer, "$1/f$ frequency noise effects on self-heterodyne linewidth measurements," J. Lightwave Technol. **9**, 485–493 (1991).
17. K. Kikuchi, "Effect of llf-type fm noise on semiconductor-laser linewidth residual in high-power limit," IEEE J. Quantum Electron. **25**, 684–688 (1989).
18. A. Godone and F. Levi, "About the radiofrequency spectrum of a phase noise-modulated carrier," in *Proceedings of the 12th European Frequency and Time Forum* (1998), pp. 392–396.
19. A. Godone, S. Micalizio, and F. Levi, "Rf spectrum of a carrier with a random phase modulation of arbitrary slope," Metrologia **45**, 313–324 (2008).
20. S. Viciani, M. Gabrysch, F. Marin, F. M. di Sopra, M. Moser, and K. H. Gulden, "Line shape of a vertical cavity surface emitting laser," Opt. Commun. **206**, 89–97 (2002).
21. The modulation index β is defined as the ratio of the frequency deviation Δf over the modulation frequency f_m, i.e., $\beta = \Delta f / f_m$.