General Equivalent Electric Circuits for Acoustic Horns*

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The continued fraction solution of the Riccati equation leads to equivalent circuits for acoustic horns of arbitrary shape. The elements of the circuits depend only on the geometric shape. Advantages and disadvantages of circuits equivalent to the input impedance and circuits equivalent to the transmission through horns are discussed. Numerical calculations are shown to be very easy, even for dissipative horns. Finally, the example of an application to the design of musical wind instruments is given.

0 INTRODUCTION

Continued fraction expansions (CFE) have found many useful applications in circuit theory (see, for example, Baker and Graves-Morris [1]). There are several uses of such expansions for the general problem of layered inhomogeneous media or the one-dimensional Schrödinger equation, but our interest lies in expansions, the coefficients of which are simple functions of frequency. Then inhomogeneous media can be described with simple elements such as inductances, capacitances, or resistances, which depend only on geometric quantities.

As an introduction to the application to acoustic horns, we will discuss the case of the cylindrical tube. The well-known ordinary form of the CFE (see Khovanskii [2]) of the tangent function is

$$\tan x = \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{-\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cfrac{1}{-\cfrac{7}{x}}}}} \tag{1a}$$

which can be written

$$\tan x = \frac{1}{1/x} + \frac{1}{-3/x} + \frac{1}{5/x} + \frac{1}{-7/x} + \cdots . \tag{1b}$$

We deduce that the input acoustic impedance of a tube terminated by a zero impedance ($Z = \mathrm{j}\rho c S^{-1} \tan kl$, where ρ and c are the density and the velocity of sound in air, respectively, S and l are the area and the length of the tube, $k = \omega/c$, ω is the angular frequency, and $\mathrm{j}^2 = -1$) can be described by the circuit of Fig. 1 with

$$L_{2n} = (4n + 1)^{-1} \frac{\rho l}{S} \qquad C_{2n+1} = (4n + 3)^{-1} \frac{lS}{\rho c^2} \tag{2}$$

while for the input impedance of a closed tube ($Z = -\mathrm{j}\rho c S^{-1} \cot kl$)

$$L_0 = \infty \qquad L_{2n} = (4n - 1)^{-1} \frac{\rho l}{S} ,$$
$$C_{2n+1} = (4n + 1)^{-1} \frac{lS}{\rho c^2} . \tag{3}$$

The major interest of this type of circuit lies in its validity at high frequencies, and even above resonance frequencies (see, for example Schroeder [3]). As an example, two terms in Eq. (1) give $\sqrt{3}$ for the first resonance ($\pi/2$), but five terms give $\pi/2$ with an error of 0.7×10^{-5}. The general advantage of the CFE is the possibility of obtaining approximations of functions with zeros and poles. Moreover, in the case of the tangent function, the convergence is very fast.

Nevertheless it appears in Eqs. (2) and (3) that the elements of the circuit depend on the termination. The assembling of several different tubes is thus rendered rather complicated. Another use of the CFE is its application to classical T- or π-shaped circuits, both of

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which describe the transmission through the tube independent of the termination. Then we obtain the circuit of Fig. 2: the series elements L^1, L^2, C^1, C^2 are given by Eqs. (2) and (3) (with $l' = l/2$), and the shunt elements are as follows:

$$C_1^S = \frac{lS}{\rho c^2} \qquad L_2^S = \frac{-\rho l}{6S} \qquad C_3^S = \frac{-7C_1}{10} . \tag{4}$$

These elements are obtained from the CFE of the $\sin^{-1} x$ function, which has only poles and no zeros. There are some disadvantages to this circuit: some elements are negative inductances or capacitances, and the calculation of the values of the elements is more difficult.

Both types of circuits C_1 and C_2 can be generalized for acoustic horns having an arbitrary shape. In Sec. 1, using the results of a previous paper [14], we show that the elements of circuit C_1 can be obtained from certain recursion relations, and circuit C_1 can be extended to the case of a dissipative termination. Moreover, we discuss the problem of viscothermal effects, including the problem of capillary tubes. In Sec. 2 we show the utility of generalized circuits such as C_2 for transmission through horns, and compare the advantages of each type of circuit in a conclusion (Sec. 3). In the Appendix we show an interesting application to the design of musical wind instruments.

1 EQUIVALENT CIRCUITS FOR INPUT IMPEDANCE OF HORNS

1.1 Horn with Reactive Termination

In [4], we applied a general solution of the Riccati equation to the input impedance of horns, defined by an area function $S(x)$.

The two differential equations defining a horn as a transmission line are the following classical equations:

$$p' = -\frac{j\omega\rho}{S} U \qquad U' = -\frac{j\omega}{\rho c^2} Sp$$

where p and U are pressure and volume velocity, respectively. One deduces the following Riccati equation for the impedance $Z = p/U$:

$$Z' = -\frac{j\omega\rho}{S} - \frac{j\omega S}{\rho c^2} Z^2 . \tag{5}$$

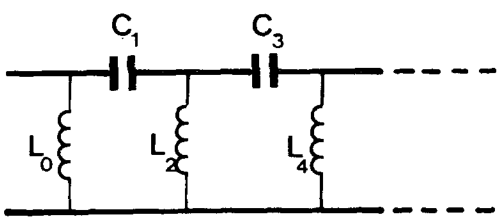

Fig. 1. Equivalent circuit for input impedance of horns with nondissipative termination.

If the numerical calculation (using, for example, the trapezoid rule for integration) is very easy, the analytical form is rather complicated, and useful only for the first elements (that is, at low frequencies). In the two simple cases $Z(x_0) = 0$ and $Z(x_0) = \infty$ the first elements are

$$L_0 = +\rho \int_x^{x_0} \frac{dx}{S(x)} \qquad C_1 = \frac{1}{\rho c^2} \frac{\int_x^{x_0} S(x) L_0^2(x)\, dx}{L_0^2(x)} \tag{6a}$$

if $Z(x_0) = 0$ and

$$L_0 = \infty \qquad C_1 = \frac{1}{\rho c^2} \int_x^{x_0} S(x)\, dx$$

$$L_2 = \rho \frac{\int_x^{x_0} S^{-1}(x)\, C_1^2(x)\, dx}{C_1^2(x)} \tag{6b}$$

if $Z(x_0) = \infty$.

The coefficient C_1 of a closed horn (proportional to its volume) is well known (see, for example, Olson [5]), but the other coefficients are more difficult to interpret. Concerning the coefficient L_0 of an "open" horn, we note the duality between horns with section $S(x)$ and horns with section $S^{-1}(x)$ (see Pyle [6]).

Note that if even $S(x)$ or $S'(x)$ is discontinuous, we can assume the continuity of both pressure and volume velocity, the algorithms thus remaining valid. (This assumption is a generalization of the fundamental assumption for the derivation of the horn equation, as explained in [7]; we discuss its validity in Sec. 1.3.) This remark allows us to explain the origin of Eqs. (6) in the example of two cylinders of length and area l_1, S_1, and l_2, S_2, respectively, the second cylinder being closed. From Eq. 6(b) we write

$$L_2 = \frac{\rho}{3} \frac{S_1 l_1^3 + 3S_2 l_2 l_1^2 + 3S_2^2 S_1^{-1} l_2^2 l_1 + S_2 l_2^3}{(S_1 l_1 + S_2 l_2)^2} . \tag{7}$$

This result can be obtained directly by using the classical impedance formulas

$$\frac{Z_1 S_1}{\rho c} = \frac{j \tan kl_1 + Z_2 S_1/\rho c}{1 + (Z_2 S_1/\rho c)\, j \tan kl_1}$$

$$Z_2 = -j \frac{\rho c}{S_2} \cot kl_2$$

and calculating an expansion with respect to kl_1 and kl_2.

1.2 Transmission through Horns with Finite Length

In order to model a horn of finite length, one needs to know the radiation impedance into an infinite space in the form of an equivalent circuit. But as an example, the circuit equivalent to Rayleigh's radiation impedance of a piston into an infinite baffle needs to involve resistive elements. One can show that a generalization of the circuit of Fig. 1 is obtained by inserting resistances as higher order terms (that is, shunt resistances with inductances, series resistances with capacitances; see Fig. 3). Then we are looking for two sequences of functions, $d_n(x)$ and $e_n(x)$, related to the values of the elements as follows:

$$
\begin{aligned}
L_{2m} &= -\frac{\rho}{d_{2m}} & R_{2m} &= -\frac{\rho c}{e_{2m}} \\
C_{2m+1} &= \frac{1}{\rho c^2 d_{2m+1}} & R_{2m+1} &= \rho c e_{2m+1} \; .
\end{aligned}
\tag{8}
$$

The algorithm is the following. At each step we know the functions $\beta_{n-1}(x)$, $\beta_n(x)$, $\gamma_n(x)$, $G'_n(x)$ and can calculate by integration $G_n(x)$ and $H_n(x)$, defined by $H'_n = 2\gamma_n G_n$. We deduce $d_n(x)$ and $e_n(x)$:

$$
d_n = -\frac{1}{\beta_n}\frac{G'_n}{G_n} \qquad e_n = -\frac{G'_n}{G_n}\frac{H_n}{\beta_n} \; . \tag{9}
$$

[To calculate the integrals we use Eqs. (9) in x_0 as the boundary conditions.]

Finally, we calculate the quantities of the succeeding step using the following recursion relations:

$$
\begin{aligned}
\beta_{n+1} &= \beta_{n-1} + \beta_n e_n^2 + 2\gamma_n e_n \\
\gamma_{n+1} &= -\gamma_n + \beta_n e_n \\
G_{n+1} &= \delta\beta_{n+1}\beta_n \frac{G_n^2}{G'_n}
\end{aligned}
\tag{10}
$$

where δ is an arbitrary constant.

The first step is defined by $\beta_0 = G'_0 = -S^{-1}(x)$, $\gamma_0 = 0$, $\beta_{-1} = S(x)$. The analytical form of the result is very complicated, but numerical solutions are easy to obtain. As an example, if

$$
\begin{aligned}
L_0(x_0) &= \frac{8}{3\pi^2}\frac{\rho}{r} \qquad \frac{1}{R_0(x_0)} = \frac{S}{\rho c}\frac{1}{2}\left(\frac{3\pi}{8}\right)^2 \\
C_1 &= -\frac{S}{\rho c^2} r\left(\frac{3\pi}{8}\right)^2\left(\frac{32}{45\pi} + \frac{13\pi}{48}\right)
\end{aligned}
$$

(Rayleigh's impedance), we obtain

$$
\begin{aligned}
L_0(x) &= L_0(x_0) + \rho\int_x^{x_0}\frac{\mathrm{d}x}{S(x)} \\
R_0(x) &= \frac{R_0(x_0)L_0(x)}{L_0(x_0)} \\
C_1(x) &= \frac{1}{L_0^2(x)}\left\{\frac{1}{\rho c^2}\int_x^{x_0} S(x)L_0^2(x)\,\mathrm{d}x + \frac{L_0^2(x_0)}{R_0^2(x_0)}\left[L_0(x) - L_0(x_0)\right] + C_1(x_0)L_0^2(x_0)\right\} \; .
\end{aligned}
\tag{11}
$$

C_1 can be either positive or negative.

Concerning horns for loudspeakers, the present solution leads to the real part of the input impedance, which is proportional to the classical transmission coefficient. Note that if it cannot be used for horns of infinite length, then we cannot find the stopband and passband of an (infinite) exponential horn. Nevertheless, the first term is reactive, and this fact is commensurate with the existence of a stopband at low frequencies.

Finally, we note that the circuit of Fig. 2 is not a unique circuit for describing tubes with finite length. Another form of circuit is given by Fig. 4, but it has some disadvantages: it does not allow one to describe horns with a reactive termination, and to our way of thinking, it is not possible to find recursion relations like those of Eqs. (10) in order to calculate the elements. Moreover, the circuit of Fig. 3 is "self-dual" when impedance is changed to admittance and vice versa, while the circuit of Fig. 4 is not.

1.3 Discontinuities in Horns

The effect on a transmission line of an abrupt change in the cross-sectional area of a tube is the insertion of a series inductance (see, for example, [8]). This type of insertion (such as a shunt capacitance) modifies the form of the circuit, and if we wish to continue the calculation using the same method, we need to calculate

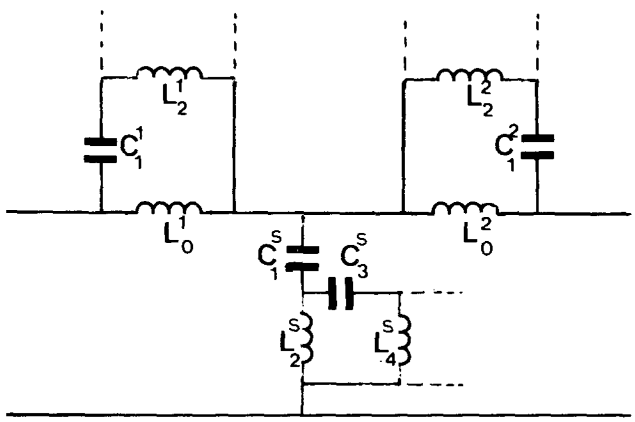

Fig. 2. Equivalent circuit for transmission through horns.

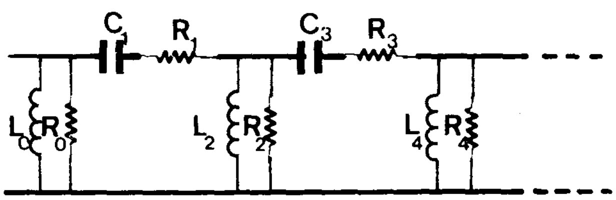

Fig. 3. Equivalent circuit for input impedance of horns with dissipative termination, or capillary tubes. From left to right, elements are arranged in order of increasing frequency.

the impedance after the discontinuity in the form of a CFE. As an example, the circuit of Fig. 5(a) can be modified as shown in Fig. 5(b) with

$$L_0' = L + L_0 \qquad C_1' = \frac{C_1 L_0^2}{(L + L_0)^2} .$$

This calculation is difficult for higher order terms, but it is possible to calculate indirectly the successive elements using numerically approximate procedures of increasing frequency. For the case of a Helmholtz resonator of any shape we obtained, in principle, the classical (L,C) circuit, with much better precision than could be calculated by means of simple formulas. Nevertheless, knowledge of the discontinuity series inductance is not sufficient to enable us to achieve excellent analytical results. Moreover, it would be necessary to describe the variation of this inductance with frequency with a circuit such as that of Fig. 1.

Note that in the case of a short open side hole on the tube, the shunt inductance does not modify the circuit of Fig. 1 or Fig. 3.

1.4 Viscothermal Effects in Horns

The simplicity of the theory of tubes without dissipation (the circuit of Fig. 3) disappears when viscothermal effects near the walls are taken into account. In [4], we used an approximate solution by perturbing each element to the first order of these effects, with terms inversely proportional to $\omega^{1/2}$.

Otherwise in the case of capillary tubes, one can show that the form of the equivalent circuit is the same form shown in Fig. 3. We did not find a general solution, even for cylindrical tubes, but it is possible to calculate it from a series expansion. As examples of this procedure, we obtained for an open cylindrical capillary with zero terminal impedance the following result (see in particular [9], [10], or [11]):

$$L_0 = \infty \qquad R_0 = \infty \qquad C_1 = \infty \qquad R_1 = 8R$$

$$L_2 = -\frac{4L}{3}(1 - 16\gamma u^2) \tag{12}$$

$$R_2 = \frac{8R(1 - 16\gamma u^2)^2}{1/32 - 12(\gamma - 1)\, l_v/l_t\, u^2 - 1536/5\, \gamma^2 u^4}$$

where

$$R = \frac{\rho c}{S} u \qquad L = \frac{\rho l}{S} \qquad u = \frac{l l_v}{r^2} \qquad l_v = \frac{\mu}{\rho c}$$

$$l_t = \frac{\lambda}{\rho c C_p} \qquad \gamma = \frac{C_p}{C_v}$$

and μ is the first coefficient of viscosity, γ the coefficient of thermal conductivity, and C_p and C_v are the specific heats at constant pressure and constant volume, respectively.

In the case of a closed cylindrical capillary we obtained

$$\begin{aligned} &L_0 = \infty \qquad R_0 = \infty \qquad C_1 = \gamma C \\ &R_1 = R\left(\frac{8}{3} + \frac{1}{u^2}\frac{\gamma - 1}{\gamma^2}\frac{1}{8}\frac{l_v}{l_t}\right) \\ &L_2 = L\left[\frac{4}{9} - \frac{64}{45}\gamma u^2 + \frac{\gamma - 1}{16\gamma^2} \times \left(\frac{1}{4}\frac{\gamma - 1}{\gamma} + \frac{1}{3}\right)\left(\frac{l_v}{l_t}\right)^2 \frac{1}{u^2}\right] \end{aligned} \tag{13}$$

where $C = lS/\rho c^2$.

It is interesting to note that the general analytical solution for cylindrical capillary tubes leads to circuits such as Fig. 3, the elements being polynomials with respect to $u = l l_v/r^2$. At high frequencies, or for large tubes, the circuit remains valid, but a large number of elements need to be taken into account. In principle, therefore, this type of circuit is possible for any large tube with viscothermal losses [when the parameter $q = (\omega/c l_v)^{1/2} r$, the ratio of radius to boundary layer thickness is large], and could avoid approximate elements involing $\omega^{1/2}$. Nevertheless, the circuits obtained in Sec. 1.2 with small perturbations to the first order in q^{-1} are, of course, more useful. The difficulty is the result of the opposition of expansions with respect to ω and expansions with respect to $\omega^{-1/2}$ (for small viscothermal effects).

2 EQUIVALENT CIRCUIT FOR TRANSMISSION THROUGH HORNS

As explained by Miller [12], it is possible to deduce the value of a wave function (that is, pressure or volume velocity) when the impedance is known in three different situations. Then the calculation of the impedance using the CFE for three different boundary conditions is sufficient to know the pressure p or volume velocity U, and the transmission, impedance, or admittance matrices. As an example, if

$$p_1 = Z_{11}U_1 - Z_{21}U_2$$

$$p_2 = Z_{21}U_1 + Z_{22}U_2$$

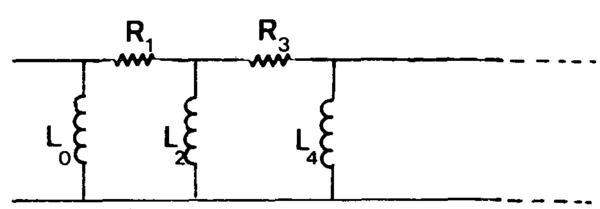

Fig. 4. Equivalent circuit for input impedance of horns with dissipative termination.

for two sections x_1 and x_2 of the horn, Z_{11} is obtained for the case $Z_2 = \infty$, Z_{22} for the case $Z_1 = \infty$, and Z_{21} is found by using the following relation:

$$Z_{21}^2 = Z_{22}(Z'_{11} - Z_{11}) \tag{14}$$

where Z'_{11} is obtained for the case $Z_2 = 0$.

Then we obtain a T-shaped circuit (Fig. 6) or a π-shaped circuit. We know the total value of the elements at each frequency, but we do not know the basic elements of Z_{21}. Of course, the calculation is still possible using an approximate numerical procedure, but it would be more interesting to find recursion relations. To our way of thinking, this is difficult for the transfer impedance Z_{21}, because it is a solution of a (second-order) wave equation, not a Riccati equation. As a matter of fact, if we write $y = 1/Z_{21} = U_2/p_1$ (with $U_1 = 0$), the unknown is $U(x_2)$ for $p(x_1)$ fixed, or $y(x_2)$ for $p(x_1)$ fixed. Then y is a solution of the well-known volume velocity horn equation,

$$y'' - \frac{S'}{S} y' + k^2 y = 0 \tag{15}$$

with $y(x_1) = 0$ and $y'(x_1) = -j\omega S(x_1)/\rho c^2$.

3 CONCLUSION

For nondissipative horns, the advantages and disadvantages of both types of circuits, that is, circuit C_3 for input impedance (Fig. 3) and circuit C_2 for transmission (Fig. 2), are now discussed.

Circuit C_3 can be calculated by means of simple recursion relations, permitting an easy determination of the first resonances and antiresonances using very few elements. On the contrary, circuit C_2 requires an approximate calculation of the elements of Z_{21} and more elements for resonance calculations.

Circuit C_2 does not depend on the terminal impedance and is well adapted for calculations on horns divided into frustums of simple shape, and including discontinuities or branched horns. Circuit C_1 needs, in general, to have a recalculation of the entire system.

Both circuits allow the calculation of transmission coefficients of horns, that is, the real part of the input impedance, but if losses were taken into account, this calculation would be possible only for circuit C_2.

Both circuits can utilize negative inductances or capacitances.

Concerning inverse problems, with applications to layered inhomogeneous media, vocal tracts, horns, and so on, Bruckstein et al. [13] or Ursin et al. [14] showed that the CFE expansions can be very useful. In the Appendix we demonstrate a particular application of our techniques to the resonances of musical wind instruments. Finally, the classical synthesis methods used for enclosures (see in particular Small [15]) utilize transfer functions with poles and no zeros. It would be possible to obtain general circuits with poles and zeros, even for loudspeaker vibration (see, for example, Bruneau and Bruneau [16]), and to study synthesis over a wide frequency range.

Of course, our method is interesting only in the limits of validity of the horn equation. (This equation is often called Webster equation, but was discovered by Lagrange.) As is well known (see, for example, [17], [18]), the range of validity is the low-frequency range, and the results for the input impedance are rather good, even when radiation properties are not exactly known [19]. As a matter of fact, by comparing the exact solution for a conical horn (considering spherical waves) and the solution of the plane-wave horn equation, we obtained the following approximate condition of validity [7]:

$$\frac{1}{2} k \int_{x_1}^{x_2} \left(\frac{dr}{dx}\right)^2 dx << 1$$

where $r(x)$ is the radius of the horn, between abscissas x_1 and x_2. This condition was in accordance with experimental results for brass musical instruments [20].

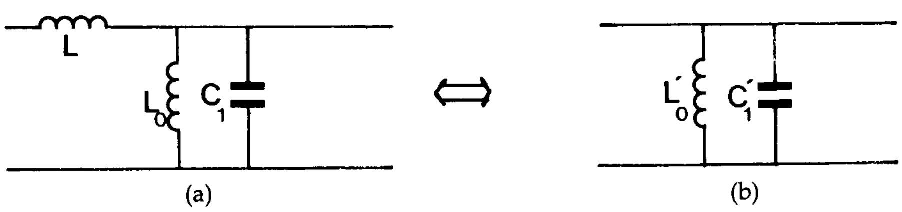

Fig. 5. Insertion of inductance L at input of horns.

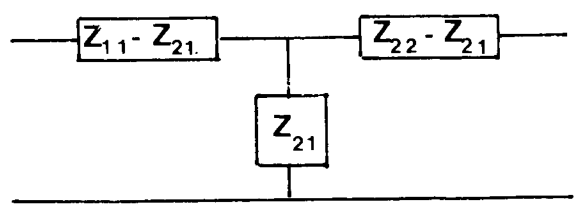

Fig. 6. Equivalent circuit for transmission through horns.

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APPENDIX
APPLICATION OF THE CONTINUED FRACTION EXPANSION TO THE ACHIEVEMENT OF HARMONICITY IN THE RESONANCES OF A SAXOPHONE

We are searching for the appropriate shape of the mouthpiece of a conical reed instrument, such as a saxophone, in order to obtain harmonically related resonances of the instrument. We ignore viscothermal dissipation and dispersion and radiation damping. The condition of resonance therefore is an infinite impedance at the input of the mouthpiece. Because the instrument is a truncated cone, the resonances are not exactly harmonic, and the design of the mouthpiece is important in order to achieve harmonicity of the resonances (see in particular Benade et al. [20]–[22]). We assume for simplicity that there is no temperature gradient and we ignore the effects of inertia and the elasticity of the reed. The condition of resonance can be obtained as follows:

$$Y = -\tilde{Y} \tag{16}$$

where Y is the admittance at the input of a truncated cone ($x = x_0$), imposed by the radiation impedance at the output ($x = x_0 + l$), and $\tilde{Y}$ is the admittance at the same point, imposed by the resonance condition ($Z = \infty$} at the position of the reed ($x = x_2$), that is, at the input of the mouthpiece (Fig. 7). For convenience, $\tilde{Y}$ is calculated with the opposite sign convention of velocity ($\tilde{x} = -x$).

At low frequencies, a good approximation for Y is the following (see, for example, Nederveen [23]):

$$Y = \frac{S}{\rho c}\left(-\mathrm{j}\cot kl - \frac{\mathrm{j}}{kx_0}\right) \tag{17}$$

where S is the area of the cone at $x = x_0$ and l is the length of the truncated cone (including radiation length correction).

In order to obtain a harmonic sequence of resonances, we write Eq. (16) as follows:

$$\tan k(l + \Delta l) = 0 \qquad \text{or} \qquad \cot kl = -\cot k\Delta l \,. \tag{18}$$

We deduce the desired value of $\tilde{Y}$, the input admittance of the *mouthpiece* in x_0, when closed at $x = x_2$:

$$\tilde{Y} = \frac{S}{j\omega\rho x_0} - \frac{S}{\rho c}\, j \cot k\Delta l \,. \tag{19}$$

Because $\tilde{Y}$ is the input impedance of a closed volume, the inductance L_0 of circuit C_1 needs to be infinite; then $\Delta l = x_0$. This result is well known [23]. Then, by identification and by use of Eq. (2), we obtain the elements of the circuit equivalent ot $\tilde{Y}$:

$$L_{2n} = (4n + 1)^{-1} \frac{\rho x_0}{S}$$

$$C_{2n+1} = (4n + 3)^{-1} \frac{x_0 S}{\rho c^2} \,.$$

As an example, we obtain

$$C_1 = \frac{1}{3} \frac{x_0 S}{\rho c^2} \qquad L_2 = \frac{1}{3} \frac{\rho x_0}{S} \,.$$

The value of C_1 corresponds to the volume of the small cone from the apex ($x = 0$) to $x = x_0$. This result (the volume of the mouthpiece should be nearly equal to the volume of the small cone) was given in [20]. The further requirement concerning the shape of the mouthpiece is obtained by using Eq. 6(b). As an example, if we choose a succession of two cylinders, four parameters (l_1, l_2, S_1, S_2) are unknown, and the value of C_1 imposes the volume (one parameter), the value of L_2, another parameter [see Eq. (7)], and we can further satisfy the values of C_3 and L_4. Then we can satisfy Eq. (18), even at high frequencies, when kx_0 is not small. The important result is the following. The identification of Eq. (19) could have been made using the coefficients of a series expansion but, using a CFE, we know that the shape obtained for the mouthpiece produces harmonic resonances even for small wavelengths compared to the length x_0 ($kx_0 = \pi/2$). As an example, if $l = 1$ m and $x_0 = 0.1$ m, the frequency error of the seventh resonance is 0.2% when three elements are identified (L_0, C_1, L_2). (At this frequency, $kx_0 = 2$.)

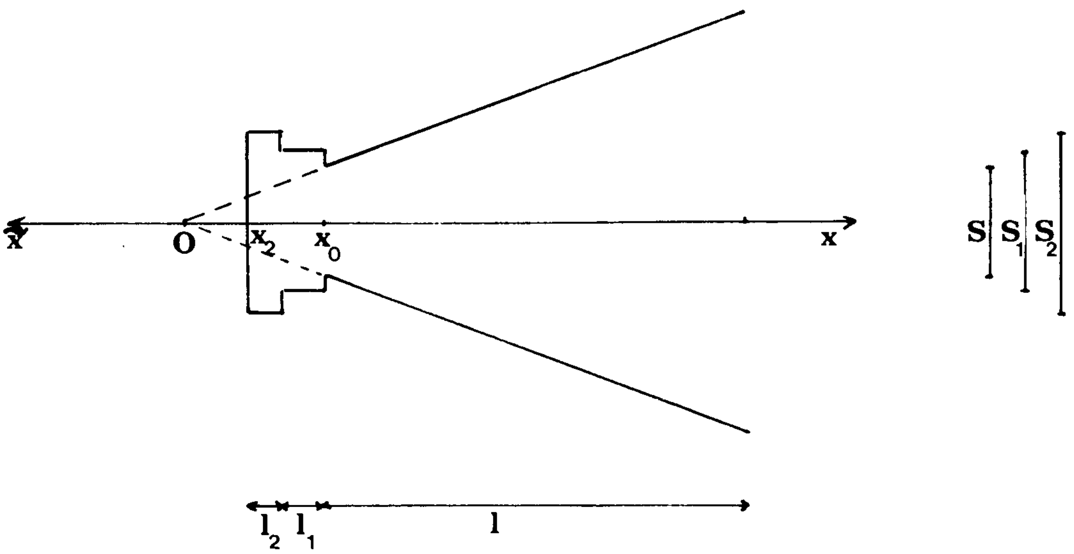
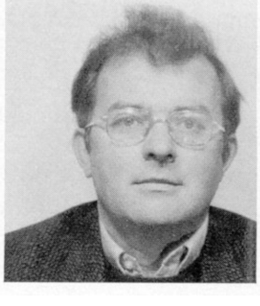


Fig. 7. Conical reed instrument with mouthpiece.

THE AUTHOR



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He gives postgraduate lectures on electroacoustics (analogies, transducers) and musical acoustics at the University of Le Mans. His publications appear in various journals on physics and acoustics. He has contributed papers to the 75th and 82nd Conventions of the Audio Engineering Society on vented-box enclosures and the measurement of acoustic impedance. He is involved in the Société Française d'Acoustique in the musical acoustics and electroacoustics groups.