INFERENCE FOR A STOCHASTIC GENERALIZED LOGISTIC DIFFERENTIAL EQUATION AND ITS APPLICATION

FERNANDO BALTAZAR-LARIOS, FRANCISCO DELGADO-VENCES, SAUL DIAZ-INFANTE, AND EDUARDO LINCE GÓMEZ

Abstract. The aim of this paper is to estimate three parameters in a stochastic generalized logistic differential equation (SGLDE) with a random initial value to a set of real data. We assume that the intrinsic growth rate and a shape parameter are constants but unknown, then we use Maximum Likelihood estimation (MLE) for estimate these parameters. In addition, we prove that this estimator is strongly consistent and asymptotically normal. We tested the method by fixing values for the two (three?) parameters and we simulate the solutions by using the closed solution of the SLDE; then we use our method to estimate the parameter and study a type of empirical convergence to the true value.

Finally, we apply the method to real data coming from .... Aqui falta escribir

Keywords: Maximum Likelihood; stochastic generalized logistic differential equation; Brownian bridges; biological growth; simulations.

1. Introduction

Aqui falta escribir cosas,

In this article, we study the problem of estimating two (three?) parameters for a stochastic generalized logistic differential equation. This is a highly difficult problem since the estimation of one parameter depends on the other one, therefore it is required to implement an efficient method to estimate both. We also model actual data of the biological growth of individual members of a given species. The model is a stochastic differential equation (SDE) in the Itô sense with a suitable initial random condition. We use the Girsanov theorem to obtain the Radon-Nykodim derivative of the measure generated by the solution, this allows us to obtain the Maximum Likelihood estimator for the parameters.

The SGLDE has been studied by several authors (see for instance [15]) and that takes into account the individual variability of the individuals in the population.

An SDE is essentially an ordinary differential equation in which one or more of the terms is a stochastic process, resulting in a solution that is also a stochastic process. The most driven noise is the Wiener process or some related process, and that can be additive or multiplicative type. SDEs have been applied in a wide range of disciplines such as biology, medicine, population dynamics, and engineering (see [11]).

In this work, we estimate the parameter, by using the Girsanov theorem, and then we prove that the estimator is strongly consistent and asymptotically normal.

Estimation for SDEs (e.g., the Ornstein-Uhlenbeck process, Geometric Brownian motion, etc.) have been developed for the last few decades, such as the EM method, Ozaki method, MLE method, SO routine; see for instance the books [15], [9] and the references therein.

The use of a random initial condition allow us to include in the model a randomness of the birth size.

This paper is organized as follows.

2. A stochastic generalized logistic differential equation

It is well-know in classical literature (see for instance [3]) that the classical generalized logistic model formulated via the initial value problem

$$\left.\begin{aligned} X'(t) &= \alpha X(t)\left[1-\left(\frac{X(t)}{K}\right)^{m}\right], && t>t_0 \\ X(t_0) &= x_0, && x_0\in(0,1). \end{aligned}\right\} \tag{2.1}$$

has a solution given by

$$X(t) = \frac{K}{\left(1+Qe^{-\alpha(t-t_0)}\right)^{1/m}}, \tag{2.2}$$

where

$$Q := -1 + \left(\frac{K}{x_0}\right)^{m},$$

and $\alpha, m > 0$, $K > 0$ are the parameters into the model.actualizar porque no ponemos la aplicacion a biologia Since we are interested in applying this model to biological growth, we could assume either that the parameter K is fixed using biological experience or by taking K as the maximum value of the actual observations. In this manner, we have to estimate two parameters: $\alpha, m > 0$.

When one uses the generalized logistic model for biological growth, x_0 and $X(t)$ could be interpreted as the proportion of individuals at the time instant t_0 and $t > t_0$, respectively, and in this situation, we obtain the particular case $K = 1$, which is known as Richard's equation. We will use this particular case in an application example. Moreover, α is the intrinsic growth rate and usually, it is assumed that is a constant but unknown, and m is a shape parameter that controls how fast the limiting number K is approached. This model has been applied successfully to several fields of knowledge, for instance, the growth of tumors, reaction models in chemistry, Fermi distribution in physics, etc.

In this paper we will focus in the stochastic logistic model:

$$\left.\begin{aligned} &dX(t) = \alpha X(t)\left[1 - \left(\frac{X(t)}{K}\right)^m\right]dt + \sigma X(t)dB(t), \qquad t > t_0, \\ &X(t_0) = x_0. \end{aligned}\right\} \tag{2.3}$$

where $x_0(\omega)$ is a bounded absolutely continuous random variable $x_0(\omega) : \Omega \to [a_1, a_2] \subset (0, 1)$. We further assume that $B(t)$ is a standard Brownian motion and $x_0(\omega)$, are defined on a common probability space $(\mathbb{P}, \Omega, \mathcal{F})$.

The equation (2.3) is a stochastic differential equation, in our case it is driven by a multiplicative noise, which is usually called affine noise.

Several contributions have reported formulations that are closely related to the model presented in Equation (2.3). However, to the best of our knowledge, only [19] provides details on parameter calibration using real data, but with a model based on a random differential equation. Suryawan [17] evaluates some qualitative aspects of the solution, such as long-time behavior and noise-induced transition, using the framework of Ito's stochastic calculus. Cortes et al. [21] obtain the probability density of the underlying solution processes using the Karhunen–Loève expansion and other transformations. Most of the literature focuses on the model (2.3) with $m = 1$. Other authors such as [20, 22] document the qualitative behavior of the solution process using a different form of diffusion term. Nevertheless, [23] presents a theory and exhaustive numerical analysis for other linear generalizations in the drift and nonlinear terms in the diffusion term.

This equation has been studied by some authors, here we only mention [17] buscar mas referencias, las pone Saul

By using Itô's formula we obtain the closed solution to (2.3) (see [17] for further reading)

$$\begin{aligned} X(t) = x_0 \exp &\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right] \\ &\times \left[1 + \left(\frac{x_0}{K}\right)^m \alpha m \int_0^t \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)s + \sigma B(s)\right]ds\right]^{-1/m}. \end{aligned} \tag{2.4}$$

We observe that the solution is always positive for all $t \geq 0$.

It is not difficult to see that

$$\lim_{t\to\infty} \mathbb{E}[X(t)] = K,$$

Since we are interested in the model for biological growth, then we can restrict ourselves to the case $K = 1$.

3. Estimation for the parameters

In this section, we will assume that the parameter vector $\theta = \{\alpha, m, \sigma\}$ is unknown and we denote by θ_0 the true value of parameter vector.We consider the estimation of θ based on observations of the process solution of (2.3) sampled in the time interval $[0, T]$ for continuous and discrete cases.

3.1. Continuous observation. We suppose that we have continuous observation of paths of the diffusion given by (2.3) in the time interval $[0, T]$. Based on this observations, we provide estimators for θ. We estimate σ via quadratic variation and, give the estimator of σ we find the MLEs for others parameter. We also prove the asymptotic consistency of the MLEs.

We will assume that the initial condition x_0 is a random variable with some density and that this density is the same for all possible values of θ.

For $T > 0$, we suppose that we have continuous observation in the interval [0, T] if we have a dataset $\{X(t_0) = x_{t_0}, X(t_1) = x_{t_1}, \ldots, X(t_n) = x_{t_n}\}$ for $0 = t_0 < t_1 \ldots, t_n = T$ where n large enough such that $\Delta_n = t_i - t_{i-1} = T/n$ (for $i = 1, \ldots, n$) goes to zero.

Quadratic Variation for σ. Since, for all $T \geq 0$ the quadratic variation of the diffusion $\boldsymbol{X} = \{X_t\}_{t\in[0,T]}$ solution of equation (2.3) is given by

$$< \boldsymbol{X}, \boldsymbol{X} >_T := \lim_{n\to\infty} \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))^2 = \int_0^T \sigma^2 X^2(t) dt,$$

we can estimate σ by using the quadratic variation. The corresponding estimator is given by

$$\hat{\sigma} = \sqrt{\frac{< \boldsymbol{X}, \boldsymbol{X} >_T}{\int_0^T X^2(t) dt}} = \sqrt{\frac{2\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))^2}{\Delta_n \sum_{i=1}^{n}(X(t_i)^2 + X(t_{i-1})^2)}}. \tag{3.1}$$

$\hat{\sigma}$ from the expression (3.1) is an unbiased estimator (see [18]).

Maximum Likelihood Estimators for α and m. Denote by $\mathbb{P}_{\alpha,m}$ the probability measure on the space of continuous functions $C(0, T)$ generated by $\boldsymbol{X}$. It is known that $\mathbb{P}_{\alpha,m}$ and $\mathbb{P}_{\alpha_0,m_0}$ are equivalent (see [9] or [12]).

Then, by the Girsanov theorem, the likelihood ratio is

$$L_T(\theta|\boldsymbol{X}) = \frac{d\mathbb{P}_{\alpha,m}}{d\mathbb{P}_{\alpha_0,m_0}}(X) = \exp\left\{ \int_0^T \left(\frac{\alpha(1 - X^m(t))}{\sigma^2 X(t)} - \frac{\alpha_0(1 - X^m(t))}{\sigma_0^2 X(t)} \right) dX(t) \right.$$
$$\left. - \frac{1}{2} \int_0^T \left(\left(\frac{\alpha^2(1 - X^m(t))^2}{\sigma^2} - \frac{\alpha_0^2(1 - X^m(t))^2}{\sigma_0^2} \right) \right) dt \right\}. \tag{3.2}$$

Deriving the log-likelihood ($l_T(\theta|\boldsymbol{X}) := \log(L_T(\theta|\boldsymbol{X}))$ with respect to α we obtain the MLE for α

$$\hat{\alpha}_{ML} = \frac{1}{\int_0^T \left(1 - \left(X(t)\right)^m\right)^2 dt} \int_0^T \frac{\left(1 - \left(X(t)\right)^m\right)}{X(t)} dX(t). \tag{3.3}$$

Now, we calculate the derivative of $l_T(\theta|\boldsymbol{X})$ with respect to m and using the MLE α_{ML} from (3.3), and we obtain

$$\frac{\partial l_T(\theta|\boldsymbol{X})}{\partial m} = \left(\int_0^T \left(1 - \left(X(t)\right)^m\right)^2 dt \right) \left(\int_0^T X(t)^{m-1}(-\log(X)) dX(t) \right)$$
$$- \left(\int_0^T \frac{\left(1 - \left(X(t)\right)^m\right)}{X(t)} dX(t) \right) \left(\int_0^T X(t)^m \left(1 - X(t)^m\right)\left(- \log(X)\right) dt \right) \tag{3.4}$$

We define the function $g(m) = \frac{\partial l_T(\theta|\boldsymbol{X})}{\partial m}$ and to find the MLE for m we find the positive root of the equation

$$g(m) = 0. \tag{3.5}$$

Saul: puedes escribir algo para justificar el uso de newton para encontrar la raiz...decir porque no hay analitica

3.1.1. *Consitencia.* Pancho: escribir el resultado de la consistencia para m y α con las pruenas de estos en el apendice

3.2. Discrete case. In this section, we consider the scenario when we have a discrete observation of the continuous observation $\boldsymbol{X}$, i.e., we have only records at times $0 = t_0 < \ldots, t_k$ when $\Delta_i = t_i - t_{i-1}$, for $i = 1, \ldots, k$ and k is small, we denote the data by $\boldsymbol{X}^{obs} = \{X(t_0), \ldots, X(t_k)\}$. Since that exact likelihood inference for discretely observed diffusion process (2.3) is not available we need to simulate paths between two observations (diffusion bridges) and complete the observations to consider that we have

Parameter	Real value	Estimator	Quantile 95%
α	1	1.00368894	(0.9256045054, 1.09949600235)
σ	0.05	0.053252	(0.04959749,0.0512)
m	2	2.0061403	(1.486703369925, 2.7423377654249)

Table 1. Average and quantiles (95%) of the parameter estimate obtained from 1 000 simulated datasets.

continuous observation and use the estimators obtained in Section 3.1. To this end, we use the method that applies to ergodic diffusion processes in [2].

We can think of the data set $\boldsymbol{X}^{obs}$ as an incomplete observation of a full data set given by $\boldsymbol{X}$ Therefore likelihood-based estimation can be done by means of the EM-algorithm (see [14]). Eduardo: debe escribir la metodologia del EM a detalle

Algorithm 1 EM

(1) Simulate a trayectories X_t
(2) Given N observations from the trayectorie X_t from 2.3

4. Simulation: calibrar

4.1. continuo. Eduardo poner un ejemplo de estimacion y de consistencia

Reportar una figura de error con los tres estimadores, es decir, log(distancia)

Las gráficas de arriba son el promedio de los estimadores con N trayectorias acumuladas(Codigo2)

Las gráficas de abajo son el promedio de los estimadores en cada iteración del algoritmo EM con condición inicial Beta(1,20) en cada trayectoria (100 trayectorias totales, 25 iteraciones del EM)(Codigo3)

4.2. discreto. poner varios ejemplos de estimacion...arrancando de estimadores arbitrarios y jugando con diferentes parametros

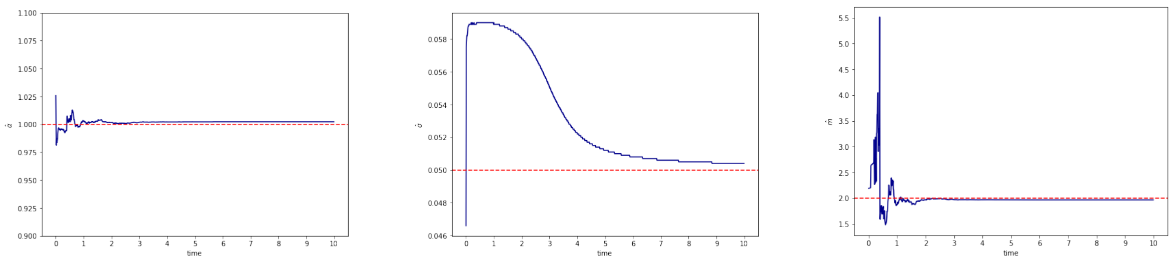
5. Validation of the proposed method

5.1. continuous case. The above graphs show the evolution of estimators with a truncated trajectory at time T (Code 1)

6. Application to real data

6.1. Data description. We are interested in modeling the biological growth of ......

Figure 1 Consistency property for σ, α, and m when $T \to \infty$



(a) Comparison between $\hat{\alpha}_{ML}$ and α when $T \to \infty$ **(b)** Comparison between $\hat{\sigma}_{ML}$ and σ when $T \to \infty$ **(c)** Comparison between $\hat{m}_{Newt}$ and m when $T \to \infty$

6.2. Results.

7. Conclusions

Appendix A. Proof of consistency and asymptotical normality

We now study the properties of the estimator $\hat{\alpha}_{ML}$.

Theorem A.1. *The estimator $\hat{\alpha}_{ML}$ is strongly consistent, i.e.*

$$\lim_{T\to\infty} \hat{\alpha}_{ML} = \alpha, \qquad \textit{with probability one} \tag{A.1}$$

and asymtotically normal, i.e.

$$\lim_{T\to\infty} C_T\Big(\hat{\alpha}_{ML} - \alpha\Big) = \mathcal{N}(0, \sigma^2), \qquad \textit{in distribution} \tag{A.2}$$

where

$$C_T := \sqrt{Var\left(\int_0^T \big(1 - X(t)^m\big) dB(t)\right)}$$

Proof. Observe that using the definition of the SDE (2.3) into (3.3) we have

$$\begin{aligned}
\hat{\alpha}_{ML} = & \frac{1}{\int_0^T (1-e^{m\sigma Y_t})^2 dt}\sigma \int_0^T (1-e^{m\sigma Y_t})\left[\left(\frac{\alpha}{\sigma}\left(1-\exp\left[m\sigma Y(t)\right]\right)-\frac{\sigma}{2}\right)dt + dB(t)\right] \\
& + \frac{1}{\int_0^T (1-e^{m\sigma Y_t})^2 dt}\frac{\sigma^2}{2}\int_0^T (1-e^{m\sigma Y_t})dt \\
= & \frac{\alpha \int_0^T (1-e^{m\sigma Y_t})^2 dt}{\int_0^T (1-e^{m\sigma Y_t})^2 dt} - \frac{\frac{\sigma^2}{2}\int_0^T (1-e^{m\sigma Y_t})dt}{\int_0^T (1-e^{m\sigma Y_t})^2 dt} + \frac{\sigma \int_0^T (1-e^{m\sigma Y_t})dB(t)}{\int_0^T (1-e^{m\sigma Y_t})^2 dt} \\
& + \frac{\frac{\sigma^2}{2}\int_0^T (1-e^{m\sigma Y_t})dt}{\int_0^T (1-e^{m\sigma Y_t})^2 dt}
\end{aligned}$$

$$= \alpha + \frac{\sigma \int_0^T (1 - e^{m\sigma Y_t}) dB(t)}{\int_0^T (1 - e^{m\sigma Y_t})^2 dt} \tag{A.3}$$

then,

$$\hat{\alpha}_{ML} - \alpha = \sigma \frac{1}{\int_0^T (1 - e^{m\sigma Y_t})^2 dt} \int_0^T (1 - e^{m\sigma Y_t}) dB(t) \tag{A.4}$$

Then, to prove the result we need to study the right side of (A.4) without the constant σ.

Focus on the right term in the equality (A.4). We can rewrite it as

$$\frac{\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)}{Var\left(\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)\right)} \times \frac{Var\left(\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)\right)}{\int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt} =: \mathbb{Q}_1(T) \times \mathbb{Q}_2(T)$$

We now study the term $\mathbb{Q}_1(T)$. By taking first and second moment we have that

$$\mathbb{E} \int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t) = 0$$

$$Var \int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t) = \mathbb{E}\left(\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)\right)^2 = \mathbb{E} \int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt,$$

therefore, $\mathbb{Q}_1(T)$ has zero mean and variance equal to 1 for all $T > 0$. From, this we deduce that

$$\lim_{T \to \infty} \mathbb{Q}_1(T) = 0, \qquad \text{in } L^2,$$

which implies that

$$\lim_{T \to \infty} \mathbb{Q}_1(T) = 0, \quad \text{in probability.}$$

For $\mathbb{Q}_2(T)$. We consider the random variable $1/\mathbb{Q}_2(T)$ and calculate the first moment,

$$\mathbb{E}\left|\frac{1}{\mathbb{Q}_2(T)}\right| = \frac{\mathbb{E} \int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt}{Var\left(\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)\right)} = 1, \quad \text{for all } T > 0,$$

From this we have the limit

$$\lim_{T \to \infty} \frac{1}{\mathbb{Q}_2(T)} = 1, \qquad \text{in } L^1,$$

and from this expression we deduce that

$$\lim_{T \to \infty} \mathbb{Q}_2(T) = 1, \qquad \text{in } L^1,$$

which implies that

$$\lim_{T\ \to\infty} \mathbb{Q}_2(T) = 1, \quad \text{in probability.}$$

Finally, by the Slutsky's theorem we conclude that

$$\lim_{T\ \to\infty} \mathbb{Q}_1(T)\mathbb{Q}_2(T) = 0, \quad \text{in probability.}$$

which proves (A.1).

To show normality, we note that

$$\mathbb{E}\left(\int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t)\right)^2 = \mathbb{E}\int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt$$

for all $T > 0$, and we deduce from this that

$$\lim_{T\to\infty} \frac{1}{\mathbb{E}\int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt} \int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t) = 1, \qquad \text{in } L^2,$$

which implies

$$\lim_{T\ \to\infty} \frac{1}{\mathbb{E}\int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt} \int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t) = 1, \qquad \text{in probability,}$$

Therefore, for the central limit theorem for martingales, we have that

$$\lim_{T\ \to\infty} \frac{1}{\sqrt{\mathbb{E}\int_0^T \left(1 - e^{m\sigma Y_t}\right)^2 dt}} \int_0^T \left(1 - e^{m\sigma Y_t}\right) dB(t) = \mathcal{N}(0,1), \qquad \text{in distribution,}$$

Moreover, since $\lim_{T\ \to\infty} \mathbb{Q}_2(T) = 1$, in probability. and by the Slutsky's theorem we conclude that

$$\lim_{T\ \to\infty} \left(\hat{\alpha}_{ML} - \alpha\right) = \mathcal{N}(0, \sigma^2), \qquad \text{in distribution.}$$

which proves (A.2). □

References

[1] J.M. Bernardo, A.F.M Smith. Bayesian Theory. John Wiley and Sons, Ltd. (2006).

[2] M. Bladt and M. Sørensen. Simple simulation of diffusion bridges with application to likelihood inference for diffusions, Bernoulli 20 645–675 (2014).

[3] Braun, M., Golubitsky, M. Differential equations and their applications (Vol. 1). New York: Springer-Verlag. (1983).

[4] R.L. Burden, D.J. Faires: Numerical Analysis (7th ed.). Brooks/Cole. (2000).

[5] S. Corlay, G. Pag s: *Functional quantization based stratified sampling methods.* Monte Carlo Methods and Applications, Volume 21, Issue 1, Pages 1–32, ISSN (Online) 1569-3961, ISSN (Print) 0929-9629. (2015).

[6] A. Gelman, J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari, D.B Rubin: Bayesian Data Analysis. Chapman and Hall/CRC Texts in Statistical Science (2013).

[7] R.G. Ghanem, P.D. Spanos: Stochastic Finite Elements: A Spectral Approach. Springer, New York, (1991).

[8] J. K. Ghosh, M. Delampady, T. Samanta: An Introduction to Bayesian Analysis Theory and Methods. Springer Texts in Statistics. (2006).

[9] Iacus, S. M. Simulation and inference for stochastic differential equations: with R examples. Springer Science & Business Media. (2009).

[10] Jiang, D., Shi, N. A note on nonautonomous logistic equation with random perturbation. Journal of Mathematical Analysis and Applications, 303(1), 164-172. (2005).

[11] Kloeden, P. E., Platen, E. Numerical solution of stochastic differential equations (Vol. 23). Springer Science & Business Media. (2013).

[12] Liptser, R. S., Shiryaev, A. N. Statistics of random processes: I. General theory (Vol. 1). Springer Science & Business Media. (2001).

[13] G. J. Lord, C.E. Powell, T. Shardlow: An Introduction to Computational Stochastic PDEs Cambridge Texts in Applied Mathematics. (2014)

[14] McLachlan,G.J.,Krishnan,T.The EM Algorithm and Extensions. Wiley, NewYork(1997).

[15] Panik, M. J. Stochastic Differential Equations: An Introduction with Applications in Population Dynamics Modeling. John Wiley & Sons. (2017).

[16] P. Protter: Stochastic integration and differential equations, Springer-Verlag, Berlin, (2004).

[17] Suryawan, H. P. (2018). *Analytic solution of a stochastic richards equation driven by Brownian motion*. In Journal of Physics: Conference Series (Vol. 1097, No. 1, p. 012086). IOP Publishing.

[18] Wei-Cheng, M. *Estimation of diffusion parameters in diffusion processes and their asymptotic normality*, Int. J. Contemp. Math. Sci. 1, 763–776 (2006).

[19] V. Bevia, J. Calatayud, J.-C. Cortés, and M. Jornet. On the generalized logistic random differential equation: Theoretical analysis and numerical simulations with real-world data. *Communications in Nonlinear Science and Numerical Simulation*, 116:106832, 2023.

[20] Carlos A. Braumann. Growth and extinction of populations in randomly varying environments. *Computers & Mathematics with Applications*, 56(3):631–644, 2008.

[21] J.-C. Cortés, A. Navarro-Quiles, J.-V. Romero, and M.-D. Roselló. Analysis of random non-autonomous logistic-type differential equations via the Karhunen–Loève expansion and the Random Variable Transformation technique. *Communications in Nonlinear Science and Numerical Simulation*, 72:121–138, 2019.

[22] Meng Liu and Ke Wang. A note on stability of stochastic logistic equation. *Applied Mathematics Letters*, 26(6):601–606, 2013.

[23] Henri Schurz. Modeling, analysis and discretization of stochastic logistic equations. *Int. J. Numer. Anal. Model.*, 4(2):178–197, 2007.

F. Baltazar-Larios. Facultad de ciencias - Universidad Nacional Autónoma de México. Ciudad Universitaria, Ciudad de México
Email address: fernandobaltazar@ciencias.unam.mx

F. Delgado-Vences. Conacyt Research Fellow - Universidad Nacional Autónoma de México. Instituto de Matemáticas, Oaxaca, México
Email address: delgado@im.unam.mx

S. Diaz-Infante. Conacyt Research Fellow - Universidad de Sonora - México
Email address: saul.diazinfante@unison.mx

E. Lince Gómez. Facultad de ciencias - Universidad Nacional Autónoma de México. Ciudad Universitaria, Ciudad de México
Email address: lince.gomez.eduardo@ciencias.unam.mx