

$$\begin{array}{c}
 (Ax) \\
 \hline
 \begin{array}{c}
 (Ax) \quad \neg P \wedge \neg U, P \vdash \neg P \wedge \neg U \text{ (e}\neg\text{)} \\
 \neg P \wedge \neg U, P \vdash P \text{ (e}\neg\text{)} \quad \neg P \wedge \neg U, P \vdash \neg P \text{ (e}\neg\text{)} \\
 \hline
 (Ax) \quad \neg P \wedge \neg U, U \vdash \neg P \wedge \neg U \text{ (e}\neg\text{)} \\
 (Ax) \quad \neg P \wedge \neg U, U \vdash U \text{ (e}\neg\text{)} \quad \neg P \wedge \neg U, U \vdash \neg U \text{ (e}\neg\text{)} \\
 \hline
 \neg P \wedge \neg U, P \vee U \vdash P \vee U \text{ (i}\neg\text{)} \quad \neg P \wedge \neg U, P \vdash \perp \text{ (i}\neg\text{)} \quad \neg P \wedge \neg U, U \vdash \perp \text{ (i}\neg\text{)} \\
 \hline
 \neg P \wedge \neg U, P \vee U \vdash \perp \text{ (i}\neg\text{)}
 \end{array}
 \end{array}$$

$$2) (\neg P) \wedge \neg U \vdash \neg(P \vee U)$$

$$\begin{array}{c}
 (Ax) \\
 \hline
 \neg(P \vee U), U \vdash U \text{ (e}\neg\text{)} \\
 \neg(P \vee U), U \vdash \neg(P \vee U) \text{ (e}\neg\text{)} \\
 \hline
 (Ax) \quad \neg(P \vee U), U \vdash U \text{ (e}\neg\text{)} \\
 \neg(P \vee U), U \vdash \neg(P \vee U) \text{ (e}\neg\text{)} \\
 \hline
 U, \neg(P \vee U) \vdash \perp
 \end{array}$$

$$\begin{array}{c}
 (Ax) \quad \neg(P \vee U), P \vdash P \text{ (e}\neg\text{)} \\
 \neg(P \vee U), P \vdash \neg(P \vee U) \text{ (e}\neg\text{)} \quad \neg(P \vee U), P \vdash (P \vee U) \\
 \hline
 \neg(P \vee U), P \vdash \perp \text{ (i}\neg\text{)} \quad \neg(P \vee U), U \vdash \perp \text{ (i}\neg\text{)} \\
 \hline
 \neg(P \vee U) \vdash \neg P \quad \neg(P \vee U) \vdash \neg U \text{ (i}\neg\text{)} \\
 \hline
 1) \neg(P \vee U) \vdash \neg P \wedge \neg U
 \end{array}$$

$$\begin{array}{c}
 (Ax) \quad P \rightarrow U, P \wedge \neg U \vdash P \wedge \neg U \quad (Ax) \\
 \hline
 P \rightarrow U, P \wedge \neg U \vdash P \rightarrow U \quad (e \rightarrow) \quad P \rightarrow U, P \wedge \neg U \vdash P \quad (e \rightarrow) \\
 \hline
 P \rightarrow U, P \wedge \neg U \vdash U
 \end{array}$$

$$\begin{array}{c}
 \vdash P \rightarrow U, P \wedge \neg U \vdash \perp \quad (i \neg) \\
 \hline
 4) \quad P \rightarrow U \vdash \neg(P \wedge \neg U)
 \end{array}$$

$$\begin{array}{c}
 (Ax) \quad P \rightarrow U, \neg \vdash \neg \quad (Ax) \quad P \rightarrow U, P \vdash P \\
 \hline
 (Ax) \quad P \rightarrow U, \neg \vdash \neg \quad (i \vee) \quad P \rightarrow U, P \vdash U \quad (i \vee) \\
 \hline
 P \rightarrow U, \neg \vee P \vdash \neg \vee P \quad (e \vee) \quad P \rightarrow U, \neg \vdash \neg \vee U \quad (e \vee) \quad P \rightarrow U, P \vdash \neg \vee U \quad (e \vee) \\
 \hline
 P \rightarrow U, (\neg \vee P) \vdash (\neg \vee U) \quad (i \rightarrow) \\
 3) \quad P \rightarrow U \vdash (\neg \vee P) \rightarrow (\neg \vee U)
 \end{array}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg P \wedge \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg P}{(Ax)} \quad \frac{\neg P \wedge \neg q, P \vdash P}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)} \quad \frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)} \quad \frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

$$\frac{\neg P \wedge \neg q, P \vdash \neg q}{(Ax)}$$

(Ax)

(Ax)

$\neg \neg (P \wedge Q), \neg Q, P \wedge Q \vdash P \wedge Q$ (I)

$(\neg \neg (P \wedge Q), P \wedge Q \vdash P \wedge Q)$ (E) $\neg \neg (P \wedge Q), \neg Q, P \wedge Q \vdash Q$ (E)

$\neg \neg (P \wedge Q)$ (Ax)

$P \wedge Q$

(Ax)

$P \wedge Q \vdash P$ (E) $\neg \neg (P \wedge Q) \vdash P$ (E) $\neg \neg (P \wedge Q), \neg Q, P \wedge Q \vdash \neg Q$ (E)

$\neg \neg (P \wedge Q), \neg P, P \wedge Q \vdash \perp$ (E) (Ax)

$\neg \neg (P \wedge Q), \neg Q, P \wedge Q \vdash \perp$

$\neg \neg (P \wedge Q), \neg Q \vdash \neg \neg (P \wedge Q)$ (E) $\neg \neg (P \wedge Q), \neg Q \vdash \neg \neg (P \wedge Q)$ (E)

$\neg \neg (P \wedge Q), \neg P \vdash \neg \neg (P \wedge Q)$ (E) $\neg \neg (P \wedge Q), \neg P \vdash \neg \neg (P \wedge Q)$ (E)

$\neg \neg (P \wedge Q), \neg Q \vdash \neg Q$

$\neg \neg (P \wedge Q), \neg P \vdash \perp$

$\neg \neg (P \wedge Q), \neg Q \vdash \perp$ (E)

$\neg \neg (P \wedge Q) \vdash \neg \neg P$ (E)

$\neg \neg (P \wedge Q) \vdash \neg \neg Q$ (E)

9) $\neg \neg (P \wedge Q) \vdash \neg \neg P \wedge \neg \neg Q$

(Ax)

(Ax)

$\neg P, P \wedge Q \vdash P \wedge Q$ (E)

$\neg P, P \wedge Q \vdash \neg P$ (E)

$\neg P, P \wedge Q \vdash P$

$\neg P, P \wedge Q \vdash \perp$

(Ax)

$\neg Q, P \wedge Q \vdash P \wedge Q$ (E)

(Ax)

$\neg Q, P \wedge Q \vdash \neg Q$ (E)

$\neg Q, P \wedge Q \vdash \neg Q$ (E)

(Ax)

$\neg P \vee \neg Q, P \wedge Q \vdash \neg P \vee \neg Q$ (E)

$\neg P, P \wedge Q \vdash \perp$ (E)

$\neg Q, P \wedge Q \vdash \perp$ (E)

$\neg P \vee \neg Q, P \wedge Q \vdash \perp$ (E)

8) $\neg (P \vee \neg Q) \vdash \neg (P \wedge Q)$

$\Gamma, P \wedge Q, \neg(P \wedge Q), Q, P \vdash P$ (iA) $\Gamma, P \wedge Q, P, \neg(P \wedge Q), Q \vdash Q$ (iA)

$\Gamma, P \wedge Q, \neg(P \wedge Q), P, Q \vdash P \wedge Q$

(eA)

$\neg(P \wedge Q) \vdash \neg(P \wedge Q)$

(eA)

$\Gamma, P \wedge Q, \neg(P \wedge Q), P, Q \vdash \perp$ (iA)

$\Gamma, P \wedge Q \vdash \neg \neg P \wedge \neg \neg Q$ (eA)

$\Gamma, P \wedge Q, \neg(P \wedge Q), P \vdash \neg Q$

$\Gamma, P \wedge Q, (P \wedge Q), P \vdash \neg \neg Q$ (eA)

(eA)

(Ax)

$\Gamma, P \wedge Q, \neg(P \wedge Q), P \vdash \perp$ (iA)

$\Gamma, P \wedge Q, \neg(P \wedge Q) \vdash \neg \neg P \wedge \neg \neg Q$ (eA)

$\Gamma, P \wedge Q, \neg(P \wedge Q) \vdash P$ (eA)

$\Gamma, P \wedge Q, \neg(P \wedge Q) \vdash \neg \neg P$ (eA)

$\Gamma, P \wedge Q, \neg(P \wedge Q) \vdash \perp$ (iA)

$\Gamma, P \wedge Q \vdash \neg \neg(P \wedge Q)$

\vdash

$$\frac{(Ax) \quad P \vdash P}{P \vdash PV\pi} (iX)$$

$$\frac{(Ax) \quad U\Delta\pi \vdash U\Delta\pi}{U\Delta\pi \vdash \pi} (e\Delta)$$

$$\frac{U\Delta\pi \vdash \pi}{U\Delta\pi \vdash PV\pi} (iX)$$

$$\frac{(Ax) \quad P \vdash P}{P \vdash PVU} (iV)$$

$$\frac{(Ax) \quad U\Delta\pi \vdash U\Delta\pi}{U\Delta\pi \vdash U} (e\Delta)$$

$$\frac{U\Delta\pi \vdash U}{U\Delta\pi \vdash PVU} (iV)$$

$$\frac{PV(U\Delta\pi) \vdash (PVU) \wedge (PV\pi)}{PV(U\Delta\pi) \vdash (PVU) \wedge (PV\pi)} (i\Delta)$$

$$11) PV(U\Delta\pi) \vdash (PVU) \wedge (PV\pi)$$

$$\begin{array}{c}
 (Ax) \quad (Ax) \\
 (p \vee u) \wedge (p \vee h), \quad h, u \vdash u \quad (i\wedge) \quad (p \vee u) \wedge (p \vee h), \quad h, u \vdash h \quad (i\vee) \\
 \hline
 (p \vee u) \wedge (p \vee h), \quad h, u \vdash u \wedge h \quad (i\vee)
 \end{array}$$

$$\begin{array}{c}
 (Ax) \\
 (p \vee u) \wedge (p \vee h), \quad p, u \vdash p \quad (i\vee)
 \end{array}$$

$$\begin{array}{c}
 (Ax) \quad p \vee (u \wedge h) \\
 (p \vee u) \wedge (p \vee h), \quad u \vdash p \vee h \quad (p \vee u) \wedge (p \vee h), \quad p, u \vdash p \quad (i\vee) \quad (p \vee u) \wedge (p \vee h), \quad h, u \vdash p \vee (u \wedge h) \quad (i\vee) \\
 \hline
 (p \vee u) \wedge (p \vee h), \quad u \vdash p \vee (u \wedge h) \quad (i\vee)
 \end{array}$$

$$\begin{array}{c}
 (Ax) \\
 (Ax) \quad p \vee u \wedge (p \vee h), \quad p \vdash p \quad (i\vee) \\
 (p \vee u) \wedge (p \vee h), \quad p \vee u \vdash p \vee h \quad (p \vee u) \wedge (p \vee h), \quad p \vdash p \vee (u \wedge h) \quad (i\vee) \\
 \hline
 (2) \quad (p \vee u) \wedge (p \vee h) \vdash p \vee (u \wedge h)
 \end{array}$$

$[Ax]$

$[Ax]$

$$\frac{P \wedge (U \vee Y) \vdash P \wedge (U \vee Y) \text{ (e}\wedge\text{)}}{P \wedge (U \vee Y) \vdash P} \text{ (e}\wedge\text{)}$$

$$\frac{P \wedge (U \vee Y) \vdash P \quad P \wedge (U \vee Y) \vdash P}{P \wedge (U \vee Y) \vdash P} \text{ (e}\wedge\text{)}$$

$$\frac{P \wedge (U \vee Y) \vdash P \vee U \text{ (i}\wedge\text{)}}{P \wedge (U \vee Y) \vdash P \vee Y \text{ (i}\wedge\text{)}}$$

$$13) P \wedge (U \vee Y) \vdash (P \vee U) \wedge (P \vee Y)$$

	(Ax)	(Ax)
	$\frac{P \wedge Q \vdash \neg(P \wedge Q)}{P \wedge Q \vdash \neg Q} (e\wedge)$	$\frac{P \wedge \neg P \vdash P \wedge \neg P}{P \wedge \neg P \vdash \neg P} (e\wedge)$
	$\frac{P \wedge Q \vdash \neg Q}{P \wedge Q \vdash \neg Q \vee \neg P} (i\vee)$	$\frac{P \wedge \neg P \vdash \neg P}{P \wedge \neg P \vdash \neg P \vee \neg P} (i\vee)$
$\frac{P \wedge Q \vdash P \wedge Q}{P \wedge Q \vdash P \wedge Q} (i\vee)$	$\frac{P \wedge Q \vdash \neg Q \vee \neg P}{(P \wedge Q \vee (P \wedge \neg P)) \vdash \neg Q \vee \neg P} (i\vee)$	
	(Ax)	(Ax)
	$\frac{(P \wedge Q) \vee (P \wedge \neg P) \vdash (P \wedge Q) \vee (P \wedge \neg P)}{(P \wedge Q) \vee (P \wedge \neg P) \vdash P} (i\wedge)$	$\frac{(P \wedge Q) \vee (P \wedge \neg P) \vdash (P \wedge Q) \vee (P \wedge \neg P)}{(P \wedge Q) \vee (P \wedge \neg P) \vdash P} (i\wedge)$
	$\frac{(P \wedge Q) \vee (P \wedge \neg P) \vdash P}{(P \wedge Q) \vee (P \wedge \neg P) \vdash P \wedge (Q \vee \neg P)} (i\wedge)$	
44)	$(P \wedge Q) \vee (P \wedge \neg P) \vdash P \wedge (Q \vee \neg P)$	

$$\vdash P \wedge \neg P \quad (e \wedge)$$

$$\vdash P \wedge \neg P \quad (e \wedge)$$

$$P \wedge \neg P \vdash \neg P \quad (e \neg)$$

$$P \wedge \neg P \vdash P \quad (e \neg)$$

$$P \wedge \neg P \vdash \perp \quad (i \neg)$$

$$16) \vdash \neg(P \wedge \neg P)$$

(Ax)

$$\neg(P \vee \neg P), P \vdash P \quad (e \vee)$$

(Ax)

$$\neg(P \vee \neg P), P \vdash P \vee \neg P \quad (e \vee)$$

$$\neg(P \vee \neg P), P \vdash \neg(P \vee \neg P) \quad (e \neg)$$

$$\neg(P \vee \neg P), P \vdash \perp \quad (e \neg)$$

$$\neg(P \vee \neg P) \vdash \neg P \quad (i \neg)$$

$$\neg(P \vee \neg P) \vdash \neg(P \vee \neg P) \quad (e \neg)$$

$$\neg(P \vee \neg P) \vdash \perp \quad (i \neg)$$

$$17) \quad 15)$$

$$\vdash \neg \neg(P \vee \neg P)$$