Data Reduction

- VC

(1) Degree 0

(2) Degree 1 — fix w, remove v, $e(v,w)$.

(3) Degree 2

① $(w,z) \in E$: $\exists$ opt VC. $v \notin C$, $w \in C$, $z \in C$.

② $(w,z) \notin E$ — folding G', v', C'

if $v' \in C' \Rightarrow w \in C, z \in C, v \notin C$

if $v' \notin C' \Rightarrow v \in C, w \notin C, z \notin C$.

(4) Dominance Rule

v doms u: $N[v] \supseteq N[u]$

if a vertex v doms some vertex, then $\exists$ opt C, $v \in C$.

Proof: if $v \notin C$

$\Rightarrow N(v) \subseteq C$

$\Rightarrow N[u] \setminus \{v\} \subseteq N(v) \subseteq C$

Remove $u \longrightarrow$ uncovered edge (u,v)

can be covered by v.

$C \setminus \{u\} \cup \{v\}$ is a VC.

(5) High degree rule. (G, k)

$deg(v) \geq k+1 \Rightarrow v \in C$.

$\forall u \in V(G')$, $deg(u) \leq k$

$E(G') \leq k^2$

(if $E(G') > k^2$, k vertices cover at most k^2 edges $\rightarrow$ can not be a VC.

$\sum deg(u) = 2m' \leq 2k^2$

We have a kernel with at most $2k^2$ vertices.

(6) Crown rule

(1) $H = N(I)$

(2) I is an indepent set

(3) $\exists$ matching M, H is all matched

$\Rightarrow \exists$ opt C, $H \subseteq C$, $I \cap C = \phi$

$\geq |H|$ vertices cover M.

- Max weight clique

Lower bound. $w(C)$: C is a clique.

$UB(v) \geq \max\{w(C_v) \mid C_v \text{ is a clique containing } v\}$

if $UB(v) \leq w(C) \Rightarrow G' = G - v$.

(1) $UB_0(v) = w(N[v])$

(2) n^*: v's largest weight neighbor.

① $n^* \in C_v$. $UB_{1a}(v) = w(n^*) + w(v) + w(N(v) \cap N(n^*))$

② $n^* \notin C_v$. $UB_{1b}(v) = w(N[v]) - w(n^*)$

$UB_1(v) = \max\{UB_{1a}(v), UB_{1b}(v)\}$

n'

(3) $UB_2(v) = w(v) + \sum_{v^*} w(v^*)$

at most one vertex each color.

coloring $G(N(v))$

- Color

G. $\exists \ell$ colors

clique

ℓ-clique

$\chi(G)$ lower bound.

$|N(v)| < \ell$ $\longrightarrow$ has some color for v.

G allow $\geq \ell$ colors.

Extend: I — $\forall v \in I$, $deg(v) < \ell$.

$G' = G - I$

$2(I, k)$

FPT

fixed-parameter tractable: $f(k) \cdot n^{O(1)}$ time.

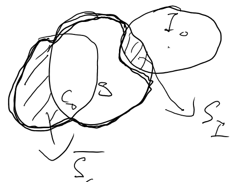
kernelization based on LP.

$$\left.\begin{aligned} &\text{Min} \sum_{v \in V} x_v \\ &\text{s.t. } x_u + x_v \geq 1, \ \forall \{u,v\} \in E \\ &\qquad x_v \in [0,1] \quad (\text{relaxed}) \end{aligned}\right\} \text{ is P}$$

Solve LP, get an opt solution of LP

$C_0 = \{x_v > 0.5\}$, $V_0 = \{x_v = 0.5\}$. $I_0 = \{x_v < 0.5\}$

Theorem: For (G, k) parameter VC problem, $\exists$ opt S s.t. $C_0 \subseteq S$, $S \cap I_0 = \phi$. kernel $(G[V_0], k - |C_0|)$ with $|V_0| \leq 2k$.



We'll show $\exists$ opt $S' := (S \setminus S_I) \cup \bar{S}_c$

① S' is a VC

$\forall \{u,v\} \in E$: $x_u + x_v \geq 1 \Rightarrow N(I_0) \subseteq C_0$

Remove S_I, $\rightarrow$ for all uncovered edges e, e has one endpoint in C_0.

Add C_0 can make it a VC.

② $|S_I| = |\bar{S}_c|$ $\rightarrow$ get a better VC than opt. Contradict.

a. if $|\bar{S}_c| < |S_I|$ ✗

b. if $|\bar{S}_c| > |S_I|$ ✗

let $\epsilon := \min\{x_v - 0.5 \mid v \in C_0\}$. Modify LP's opt sol: α $\quad f: \sum x_v$

$\forall u \in S_I$: $x_u' := x_u + \epsilon$

$\forall v \in \bar{S}_c$: $x_v' := x_v - \epsilon$ $\rightarrow$ get LP's sol. β

$f(\beta) = f(\alpha) + |S_I| \cdot \epsilon - |\bar{S}_c| \cdot \epsilon < f(\alpha) \rightarrow$ get β better than α. contradict.

③ $|V_0| \leq 2k$

Suppose $|V_0| > 2k$

$\forall v \in V_0$: $x_v = 0.5 \Rightarrow \sum_{v \in V_0} x_v = 0.5|V_0| > k$

$\therefore$ LP's opt $> k$

$\therefore$ MVC's opt $\geq$ LP's opt $> k$ So No solution.