# cse332-13wi-lec15-TopoSort-day2.cp3

CSE 332: Data Abstractions

Lecture 15: Topological Sort / Graph Traversals

Ruth Anderson

Winter 2013

# *Announcements*

- **Homework 4 –** due Friday Feb 15th at the BEGINNING of lecture
- **Project 2 –** Phase B due Tues Feb 19th at 11pm

# *Today*

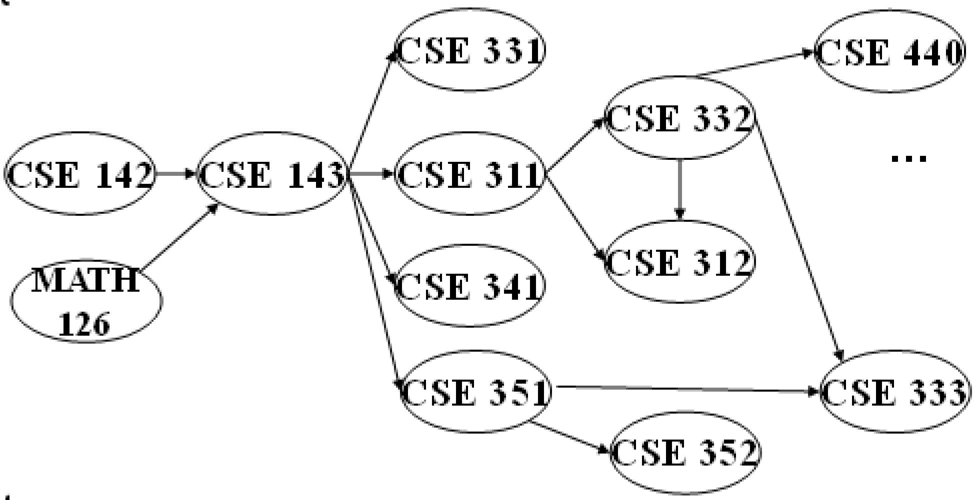
- Graphs
  - Representations
  - Topological Sort
  - Graph Traversals

Disclaimer: Do not use for official advising purposes!
(Implies that CSE 332 is a pre-req for CSE 312 – not true)

# *Topological Sort*

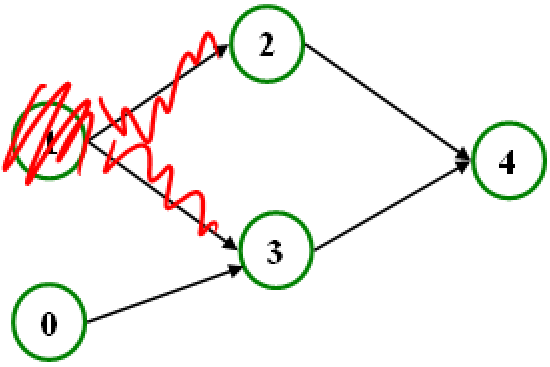
Problem: Given a DAG `G=(V,E)`, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:




Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352




**Valid Topological Sorts:**

0 1 2 3 4
1 0 2 3 4
1 0 3 2 4
0 1 3 2 4
1 2 0 3 4

# *Questions and comments*

- Why do we perform topological sorts only on DAGs?

- Is there always a unique answer?

- What DAGs have exactly 1 answer?

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

# *Questions and comments*

- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
- What DAGs have exactly 1 answer?
  - Lists
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

# *Topological Sort Uses*

- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

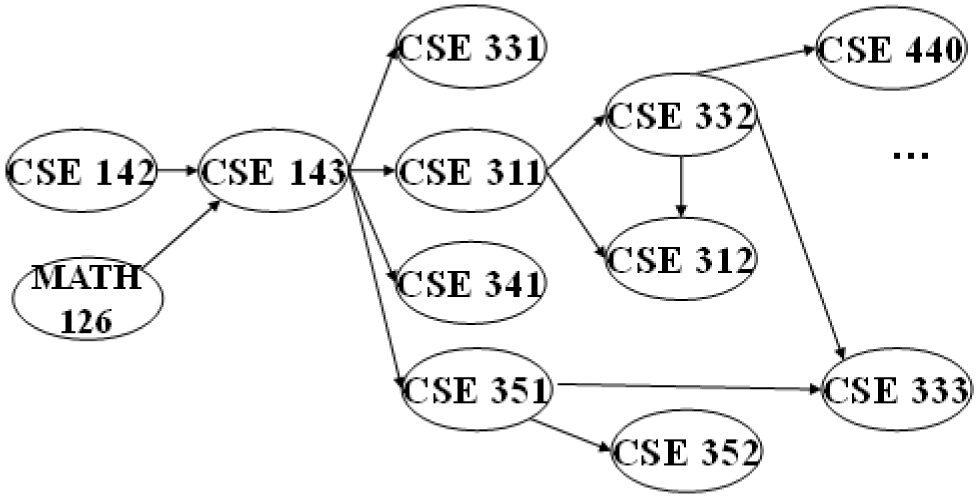
# *A First Algorithm for Topological Sort*

1. Label ("mark") each vertex with its in-degree
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**

# *Example*

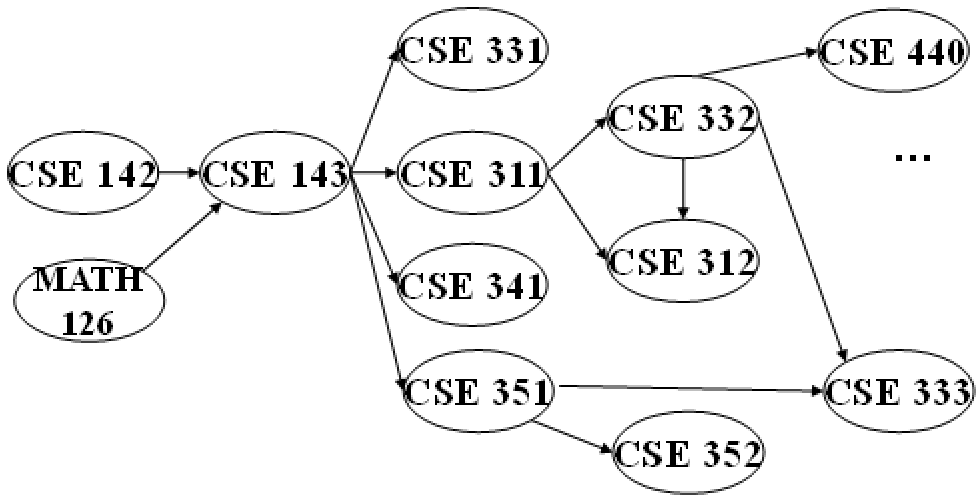
Output:




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | | | | | | | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |

# *Example*

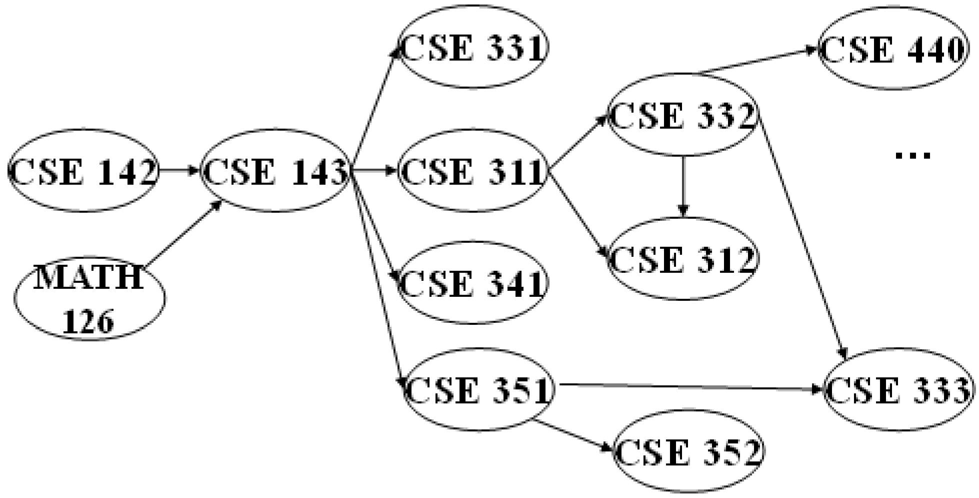
Output: 126




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | | | | | | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | | | | | | | | | |

# *Example*

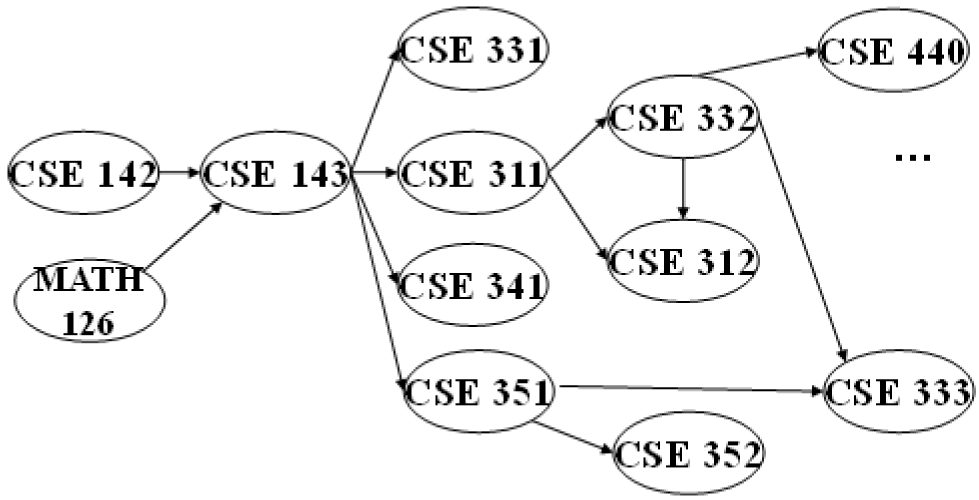
Output: 126
142




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | | | | | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | | | | | | | | | |
| | | | 0 | | | | | | | | | |

# *Example*

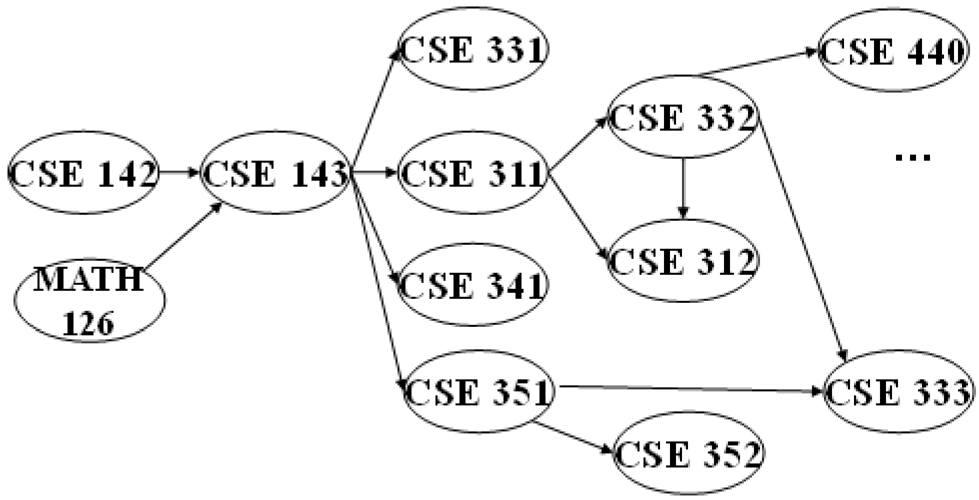
Output: 126
142
143




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | | | | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | | 0 | | | 0 | 0 | | |
| | | | 0 | | | | | | | | | |

# *Example*

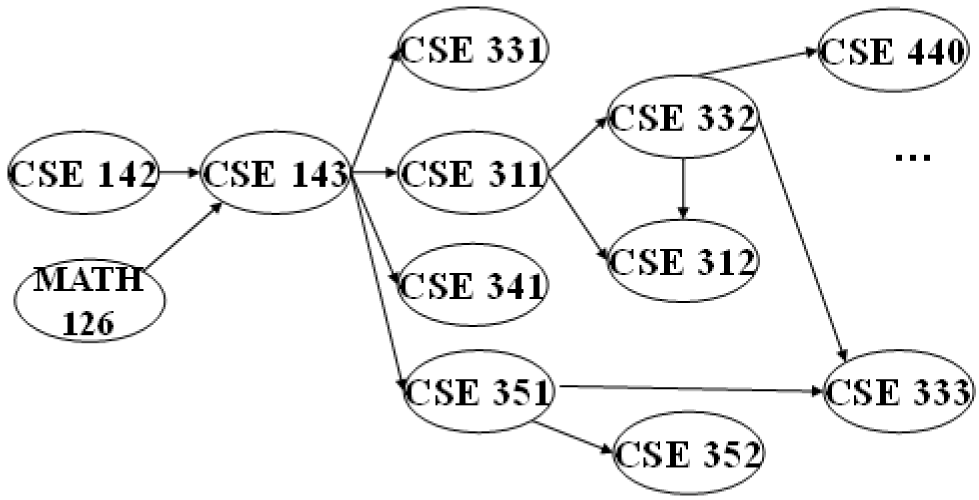
Output: 126
142
143
311




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | | | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | | 0 | 0 | | |
| | | | 0 | | | | | | | | | |

# Example

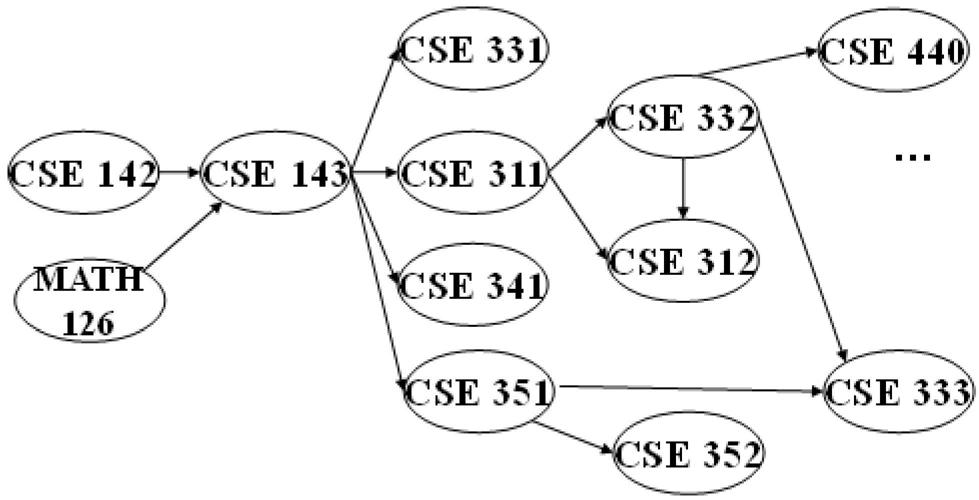
Output: 126
142
143
311
331




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | | x | | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | | 0 | 0 | | |
| | | | 0 | | | | | | | | | |

# Example

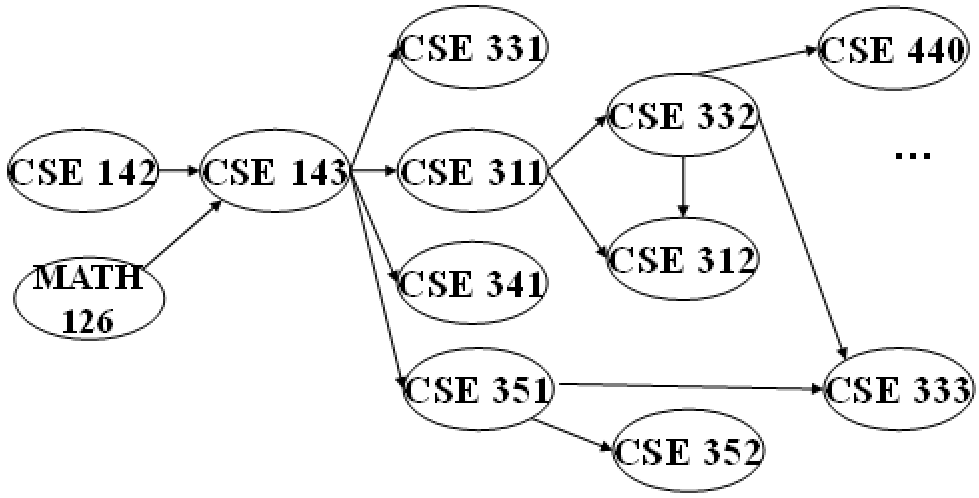
Output: 126
142
143
311
331
332




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | | x | x | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | | 0 |
| | | | 0 | | 0 | | | | | | | |

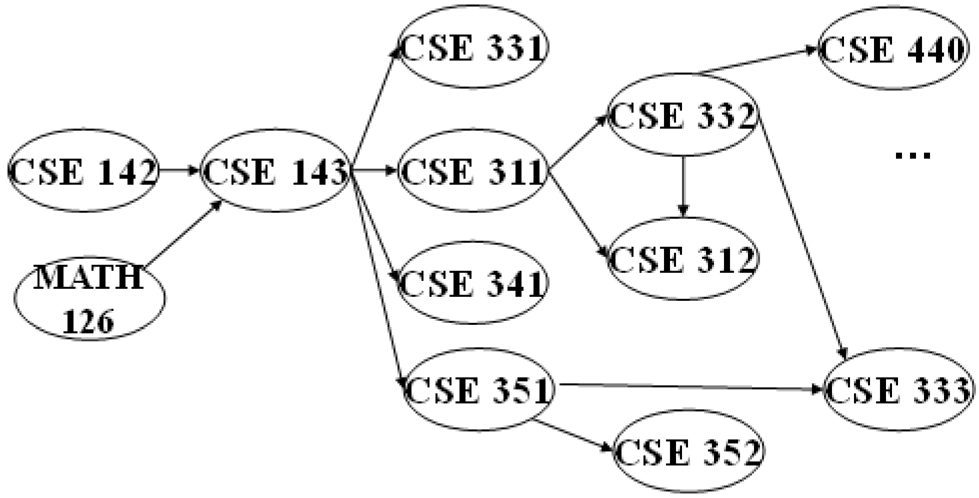
# *Example*

Output: 126
142
143
311
331
332
312




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | x | x | x | | | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | | 0 |
| | | | 0 | | 0 | | | | | | | |

# Example

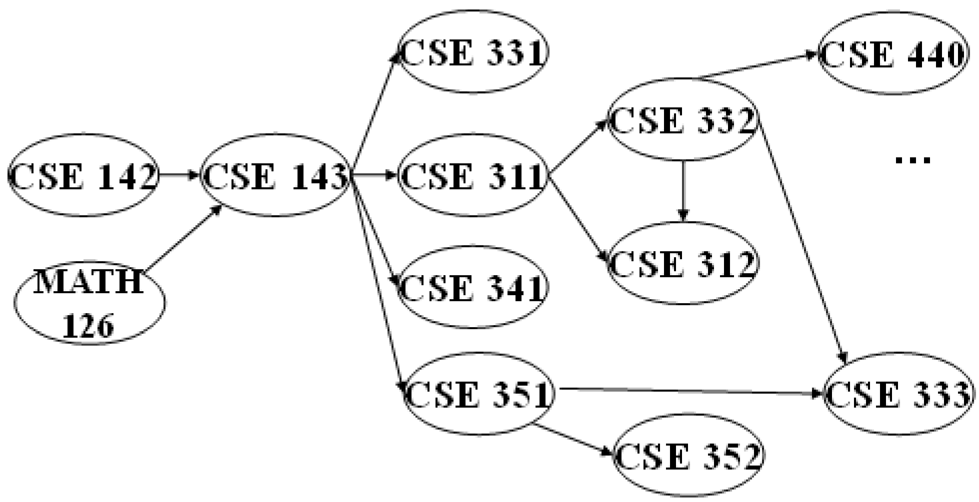



Output: 126
142
143
311
331
332
312
341

| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | x | x | x | | x | | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | | 0 |
| | | | 0 | | 0 | | | | | | | |

# *Example*

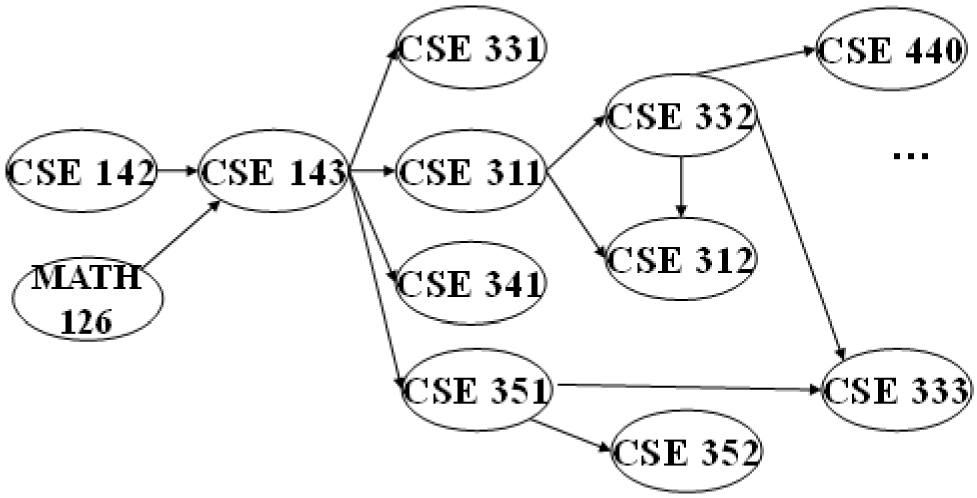
Output: 126
142
143
311
331
332
312
341
351




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | x | x | x | | x | x | | |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| | | | 0 | | 0 | | | 0 | | | | |

# *Example*

Output: 126
142
143
311
331
332
312
341
351
333
352
440




| Node: | 126 | 142 | 143 | 311 | 312 | 331 | 332 | 333 | 341 | 351 | 352 | 440 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Removed? | x | x | x | x | x | x | x | x | x | x | x | x |
| In-degree: | 0 | 0 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| | | | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| | | | 0 | | 0 | | | 0 | | | | |

# *A couple of things to note*

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

# Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

O(V+E)
V times
O(V)
O(1)
d times
O(1)

# Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

← E times total

- What is the worst-case running time?
  - Initialization $O(|V| + |E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

# *Doing better*

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) v = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**, if new degree is 0, enqueue it

# Running time?

$O(E+V)$

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(v);
  }
}
```

V times

O(1)

O(1)

E times total

O(1)

O(1)

O(1)

$$O(E+V)$$

# Running time?

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(v);
  }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacenty list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

# *Graph Traversals*

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* (i.e., there exists a path) from **v**

- Possibly "do something" for each node (an iterator!)
  - E.g. Print to output, set some field, etc.

Related:

- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

# Abstract Idea

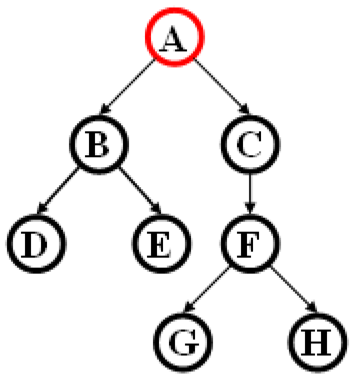
```
traverseGraph(Node start) {
   Set pending = emptySet();
   pending.add(start)
   mark start as visited
   while(pending is not empty) {
     next = pending.remove()
     for each node u adjacent to next
        if(u is not marked) {
          mark u
          pending.add(u)
        }
   }
}
```

# *Running time and options*

- Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation
- The order we traverse depends entirely on how add and remove work/are implemented
  - Depth-first graph search (DFS): a stack
  - Breadth-first graph search (BFS): a queue
- DFS and BFS are "big ideas" in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: Explore areas closer to the start node first

# Recursive DFS, Example : trees

- A tree is a graph and DFS and BFS are particularly easy to "see"

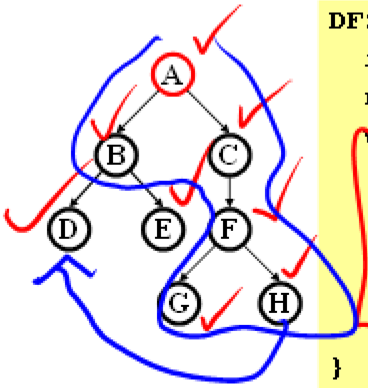



```
DFS(Node start) {
  mark and "process"(e.g. print) start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

- Exactly what we called a "pre-order traversal" for trees
- The marking is not needed here, but we need it to support arbitrary graphs , we need a way to process each node exactly once

# DFS with a stack, Example: trees




```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

E
D
~~H~~
~~G~~
~~F~~
~~C~~
B
~~A~~

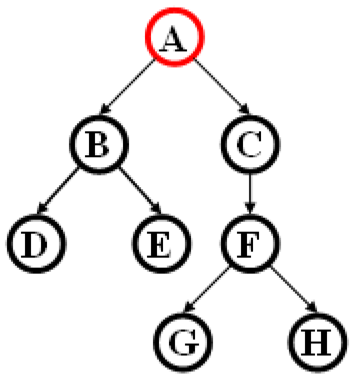
Stack

Order processed:

- A different but perfectly fine traversal

A C F H G B E D

# *DFS with a stack, Example: trees*

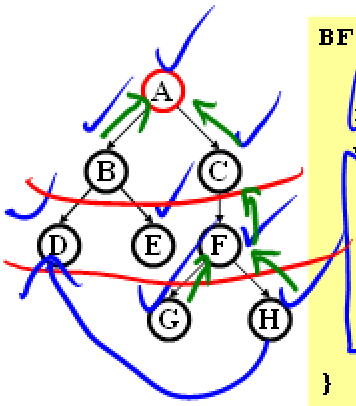



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

Order processed: A, C, F, H, G, B, E, D

- A different but perfectly fine traversal

# *BFS with a queue, Example: trees*




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue()// and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

Order processed: A B C D E F G H

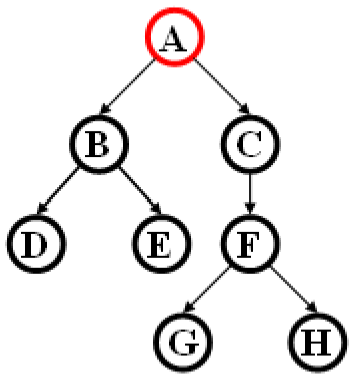
- A "level-order" traversal

2/13/2013 32

Queue →

H G ~~F~~ ~~E~~ ~~D~~ ~~C~~ ~~B~~ ~~A~~

# BFS with a queue, Example: trees




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue()// and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

Order processed: A, B, C, D, E, F, G, H

- A "level-order" traversal

# *DFS/BFS Comparison*

Breadth-first search:

- Always finds shortest paths, i.e., "optimal solutions
  - Better for "what is the shortest path from **x** to **y**"
- Queue may hold $O(|V|)$ nodes (e.g. at the bottom level of binary tree of height h, $2^h$ nodes in queue)

Depth-first search:

- Can use less space in finding a path
  - If *longest path* in the graph is p and highest out-degree is d then DFS stack never has more than d*p elements

A third approach: *Iterative deepening (IDFS)*:

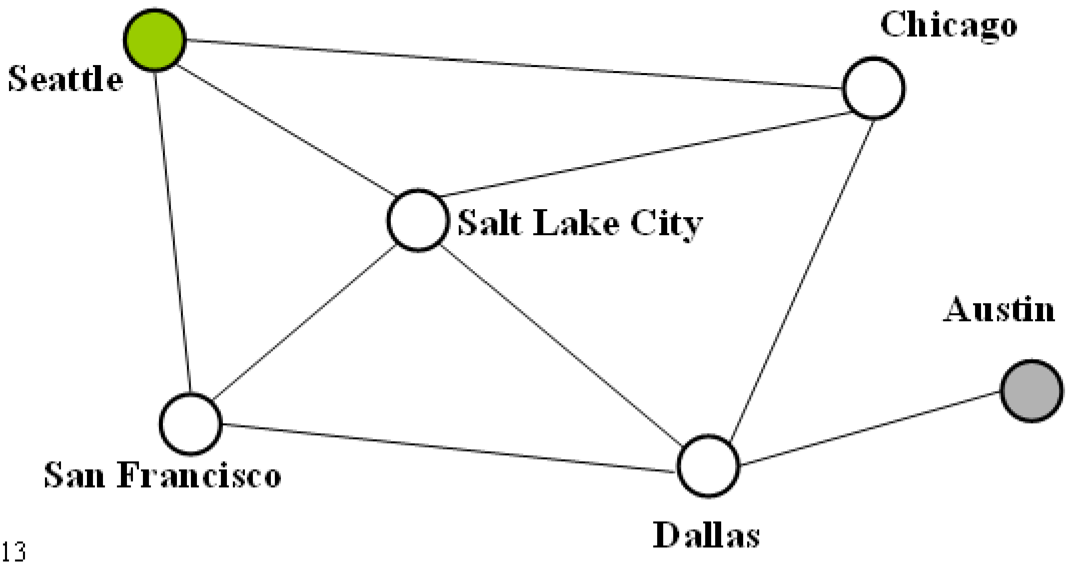
  - Try DFS but don't allow recursion more than K levels deep.
  - If that fails, increment K and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

# *Saving the path*

- Our graph traversals can answer the "reachability question":
  - "***Is there*** a path from node x to node y?"
- Q: But what if we want to ***output the actual path***?
  - Like getting driving directions rather than just knowing it's possible to get there!
- A: Like this:
  - Instead of just "marking" a node, store the **previous node** along the path (when processing **u** causes us to add **v** to the search, set `v.path` field to be **u**)
  - When you reach the goal, follow `path` fields backwards to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

# Example using BFS

What is a path from Seattle to Austin
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



# Example using BFS

What is a path from Seattle to Austin
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique