# cse332-13wi-lec13-BeyondComparisonSorting-day2.cp3

CSE 332: Data Abstractions

Lecture 13: Beyond Comparison Sorting

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# *Announcements*

- **Project 2 –** Phase A due TONIGHT Wed Feb $6^{th}$ at 11pm
  - Clarifications posted, check Msg board, email cse332-staff
  - Office Hours today after class
- (No homework due Friday)
- **Midterm – Monday Feb $11^{th}$ during lecture,** info about midterm has been posted, review in section on Thurs
- **Homework 4 –** due Friday Feb $15^{th}$ at the BEGINNING of lecture

# *Today*

- Sorting
  - Comparison sorting
  - Beyond comparison sorting

# The Big Picture

**Simple algorithms: O($n^2$)**

- Insertion sort
- Selection sort
- Shell sort
- …

**Fancier algorithms: O($n \log n$)**

- Heap sort
- Merge sort
- Quick sort (avg)
- …

**Comparison lower bound: $\Omega(n \log n)$**

**Specialized algorithms: O($n$)**

- Bucket sort
- Radix sort

**Handling huge data sets**

- External sorting

# *How fast can we sort?*

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running times
- These bounds are all tight, actually $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
  - Instead: *prove* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

# *A Different View of Sorting*

- Assume we have *n* elements to sort
  - And for simplicity, none are equal (no duplicates)
- How many ***permutations*** (possible orderings) of the elements?
- Example, *n*=3,

# *A Different View of Sorting*

- Assume we have $n$ elements to sort
  - And for simplicity, none are equal (no duplicates)
- How many ***permutations*** (possible orderings) of the elements?
- Example, $n$=3, six possibilities

  a[0]<a[1]<a[2] a[0]<a[2]<a[1] a[1]<a[0]<a[2]

  a[1]<a[2]<a[0] a[2]<a[0]<a[1] a[2]<a[1]<a[0]

- In general, $n$ choices for least element, then $n$-1 for next, then $n$-2 for next, …
  - $n(n-1)(n-2)\ldots(2)(1) = n!$ possible orderings

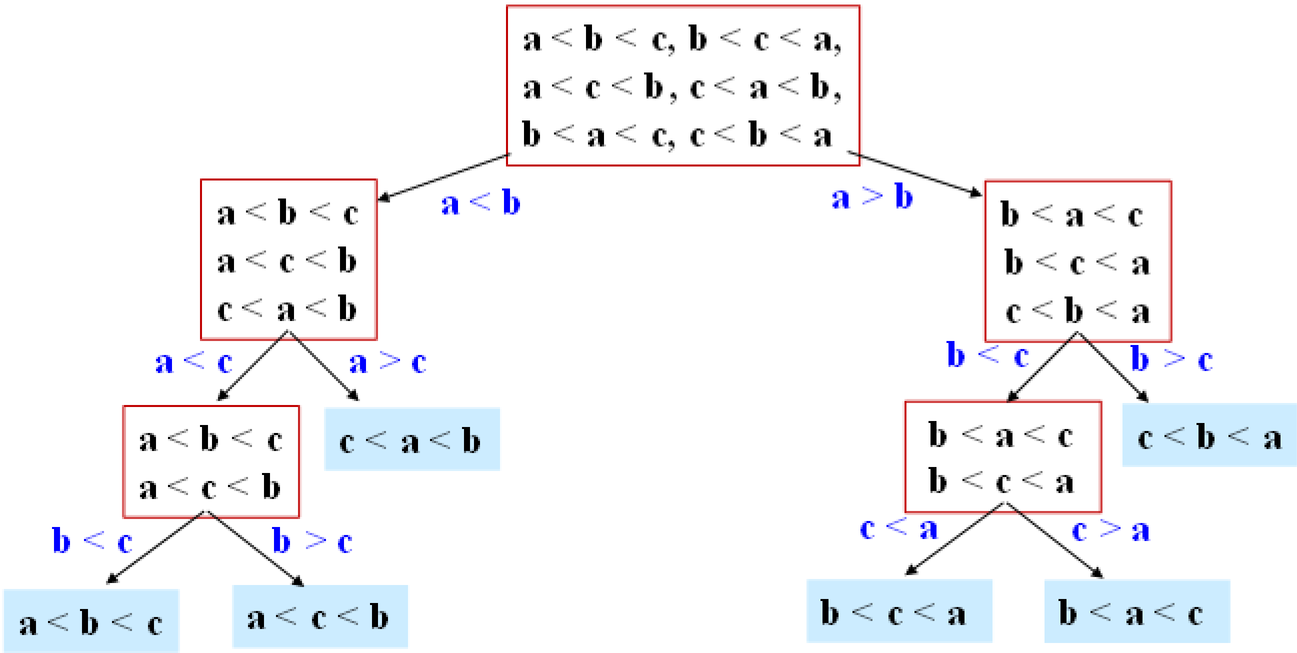
# *Describing every comparison sort*

- A different way of thinking of sorting is that the sorting algorithm has to “find” the right answer among the n! possible answers
  - Starts “knowing nothing”, “anything is possible”
  - Gains information with each comparison, eliminating some possiblities
    - Intuition: At best, each comparison can eliminate half of the remaining possibilities
  - In the end narrows down to a single possibility

# *Counting Comparisons*

- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison "is $a < b$ ?"
  - Can use the result to decide what second comparison to do
  - Etc.: comparison $k$ can be chosen based on first $k$-*1* results
- Can represent this process as a *decision tree*
  - Nodes contain "set of remaining possibilities"
  - At root, anything is possible; no option eliminated
  - Edges are "answers from a comparison"
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent "the most the algorithm could know so far" as the algorithm progresses

# One Decision Tree for n=3




- The leaves contain all the possible orderings of a, b, c
- A different algorithm would lead to a different tree

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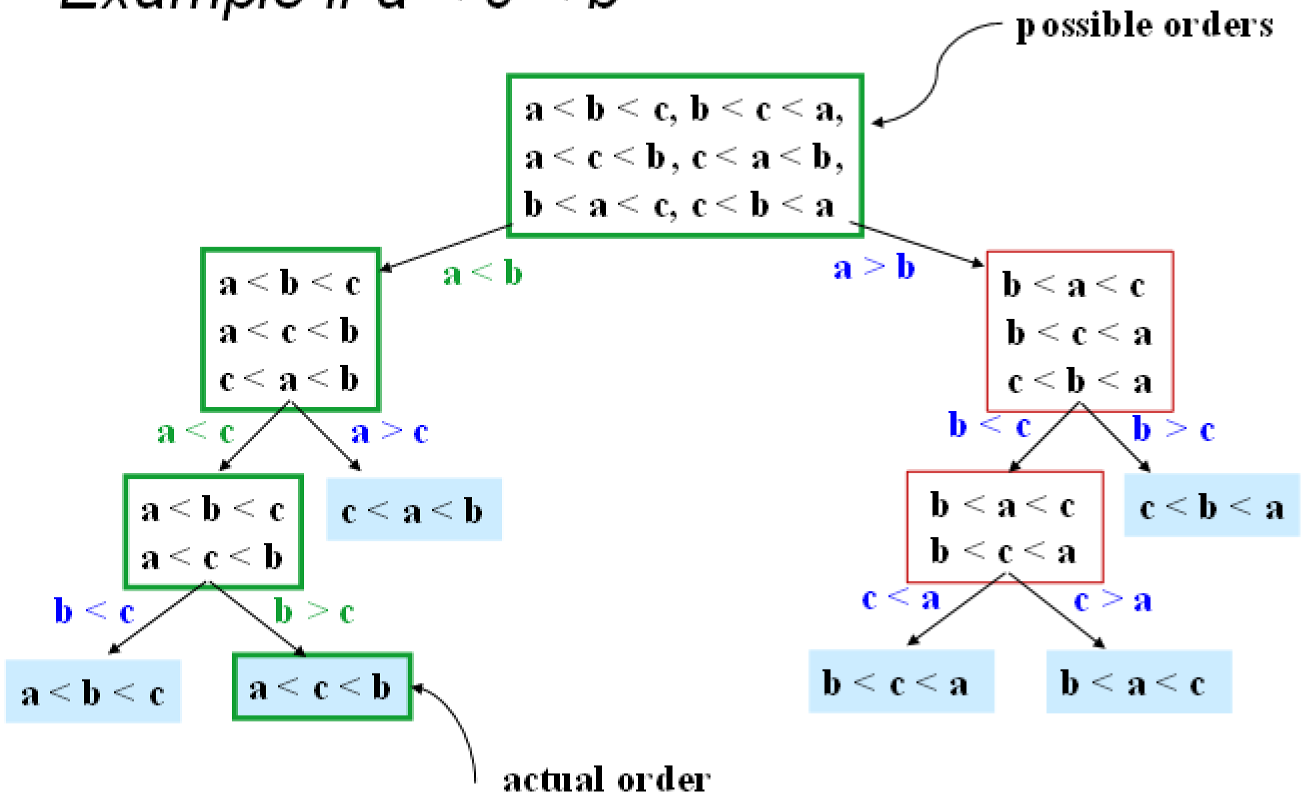
# Example if a < c < b



# *What the decision tree tells us*

- A *binary* tree because each comparison has 2 outcomes
  - Perform only comparisons between 2 elements; binary result
    - Ex: Is a<b? Yes or no?
  - We assume no duplicate elements
  - Assume algorithm doesn't ask redundant questions
- Because any data is possible, any algorithm needs to ask enough questions to produce all *n*! answers
  - Each answer is a different leaf
  - So the tree must be big enough to have *n*! leaves
  - Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with *n*! leaves
  - So no algorithm can have worst-case running time better than the height of a tree with *n*! leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

# Where are we

**Proven**: No comparison sort can have worst-case running time better than: the height of a binary tree with $n$! leaves

- Turns out average-case is same asymptotically
- A comparison sort could be worse than this height, but it cannot be better
- Fine, *how tall is a binary tree with n! leaves?*

**Now**: Show that a binary tree with n! leaves has height $\Omega(n \log n)$

- That is, n log n is the lower bound, the height must be at least this, could be more, (in other words your comparison sorting algorithm could take longer than this, but it won't be faster)
- Factorial function grows very quickly

Then we'll conclude that: (Comparison) Sorting is $\Omega(n \log n)$

- This is an amazing computer-science result: proves all the clever programming in the world can't sort in linear time!

## *Lower bound on Height*

- A binary tree of height h has **at most** *how many* leaves?

  L ≤ ________________

- A binary tree with L leaves has height **at least**:

  h ≥ ________________

- The decision tree has how many leaves: ________
- So the decision tree has height:

  h ≥ ________________

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# *Lower bound on Height*

- A binary tree of height h has **at most** *how many* leaves?

  $L \leq 2^h$

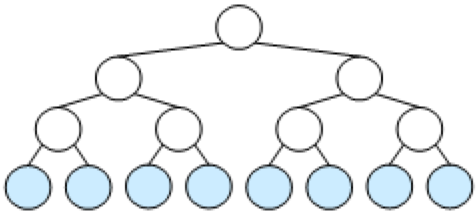
- A binary tree with L leaves has height **at least**:

  $h \geq \log_2 L$

- The decision tree has how many leaves: N!
- So the decision tree has height:

  $h \geq \log_2 N!$

# *Lower bound on height*



- The height of a binary tree with $L$ leaves is at least $\log_2 L$
- So the height of our decision tree, $h$:

| | |
|---|---|
| $h \geq \log_2 (n!)$ | property of binary trees |
| $= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$ | definition of factorial |
| $= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$ | property of logarithms |
| $\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$ | keep first n/2 terms |
| $\geq (n/2) \log_2 (n/2)$ | each of the n/2 terms left is $\geq \log_2 (n/2)$ |
| $= (n/2)(\log_2 n - \log_2 2)$ | property of logarithms |
| $= (1/2)n\log_2 n - (1/2)n$ | arithmetic |
| "=" $\Omega(n \log n)$ | |

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# *The Big Picture*

| Simple algorithms: $O(n^2)$ | Fancier algorithms: $O(n \log n)$ | Comparison lower bound: $\Omega(n \log n)$ | Specialized algorithms: $O(n)$ | Handling huge data sets |
|---|---|---|---|---|
| Insertion sort<br>Selection sort<br>Shell sort<br>… | Heap sort<br>Merge sort<br>Quick sort (avg)<br>… | | Bucket sort<br>Radix sort | External sorting |

How???

- Change the model – assume more than 'compare(a,b)'

# BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
  - Create an array of size *K*, and put each element in its proper bucket (a.ka. bin)
  - *If* data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

| count array | |
|---|---|
| 1 | III |
| 2 | I |
| 3 | II |
| 4 | II |
| 5 | III |

- Example:

  K=5

  Input: (5,1,3,4,3,2,1,1,5,4,5)

  output: 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 5

$O(N + (N + k))$

1st pass

# *BucketSort (a.k.a. BinSort)*

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
  - Create an array of size *K*, and put each element in its proper bucket (a.ka. bin)
  - *If* data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

| `count` array | |
|---|---|
| 1 | 3 |
| 2 | 1 |
| 3 | 2 |
| 4 | 2 |
| 5 | 3 |

- Example:
  K=5
  input (5,1,3,4,3,2,1,1,5,4,5)
  output: 1,1,1,2,3,3,4,4,5,5,5

What is the running time?

# *Analyzing bucket sort*

- Overall: $O(n+K)$
  - Linear in $n$, but also linear in $K$
  - $\Omega(n \log n)$ lower bound does not apply because **this is not a comparison sort**
- Good when range, $K$, is smaller (or not much larger) than $n$
  - (We don't spend time doing lots of comparisons of duplicates!)
- Bad when $K$ is much larger than $n$
  - Wasted space; wasted time during final linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

# *Bucket Sort with Data*

Bucket sort illustrates a more general trick: Imagine a heap for a small range of integer priorities

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end O(1) (keep pointer to last element)

| count array | |
|---|---|
| 1 | → **Rocky V** |
| 2 | |
| 3 | → **Harry Potter** |
| 4 | |
| 5 | → **Casablanca** → **Star Wars** |

- Example: Movie ratings: 1=bad,… 5=excellent
- Input=
  - 5: Casablanca
  - 3: Harry Potter movies
  - 1: Rocky V
  - 5: Star Wars

**Result**: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars

This result is stable; Casablanca still before Star Wars

# *Radix sort*

- Radix = "the base of a number system"
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit, sort with Bucket Sort
    - Keeping sort *stable*
  - Do one pass per digit
- **Invariant**: After *k* passes, the last *k* digits are sorted

- Aside: Origins go back to the 1890 U.S. census

# *Example*

Radix = 10

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| | 721 | | 3<br>143 | | | | 537<br>67 | 478<br>38 | 9 |

**Input:** 478
537
9
721
3
38
143
67

**First pass:**

1. bucket sort by ones digit
2. Iterate thru and collect into a list

• List is sorted by first digit

**Order now:** 721
3
143
537
67
478
38
9

# Example

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| | 721 | | 3<br>143 | | | | 537<br>67 | 478<br>38 | 9 |

Radix = 10

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 3<br>9 | | 721 | 537<br>38 | 143 | | 67 | 478 | | |

Order was: 721
3
143
537
67
478
38
9

Second pass:

stable bucket sort by tens digit

If we chop off the 100's place, these #s are sorted

Order now: 3
9
721
537
38
143
67
478

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# *Example*

Radix = 10

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 3<br>9 | | 721 | 537<br>38 | 143 | | 67 | 478 | | |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 3<br>9<br>38<br>67 | 143 | | | 478 | 537 | | 721 | | |

Order was: 3, 9, 721, 537, 38, 143, 67, 478

Third pass:

stable bucket sort by 100s digit

Only 3 digits: We're done!

Order now: 3, 9, 38, 67, 143, 478, 537, 721

**Student Activity**

# RadixSort

- Input:126, 328, 636, 341, 416, 131, 328

**BucketSort on lsd:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**BucketSort on next-higher digit:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**BucketSort on msd:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# *Analysis of Radix Sort*

Performance depends on:

- Input size: *n*
- Number of buckets = Radix: *B*
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = "Digits": *P*
  - e.g. Ages of people: 3; Phone #: 10; Person's name: ?

- Work per pass is 1 bucket sort: ___________
  - Each pass is a Bucket Sort
- Total work is _____________
  - We do 'P' passes, each of which is a Bucket Sort

# *Analysis of Radix Sort*

Performance depends on:

- Input size: *n*
- Number of buckets = Radix: *B*
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = "Digits": *P*
  - e.g. Ages of people: 3; Phone #: 10; Person's name: ?

- Work per pass is 1 bucket sort: $O(B+n)$
  - Each pass is a Bucket Sort
- Total work is $O(P(B+n))$
  - We do 'P' passes, each of which is a Bucket Sort

$\log_B MaxVal$

# *Comparison to Comparison Sorts*

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Approximate run-time: 15*(52 + *n*)
  - This is less than *n* log n only if *n* > 33,000
  - Of course, cross-over point depends on constant factors of the implementations plus *P* and *B*
    - And radix sort can have poor locality properties
- Not really practical for many classes of keys
  - Strings: Lots of buckets

# *Recap: Features of Sorting Algorithms*

**In-place**

– Sorted items occupy the same space as the original items. (No copying required, only O(1) extra space if any.)

**Stable**

– Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place, stable
- Quick Sort - in place, not stable

swap Pivot=4

Orig: $6_A$ 5 3 7 $6_B$ 2 $6_C$

Stable Sorted: 2 3 5 $6_A$ $6_B$ $6_C$ 7

# *Sorting massive data: External Sorting*

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)

- Mergesort can leverage multiple disks
- Weiss gives some examples

## Sorting Summary

- Simple $O(n^2)$ sorts can be fastest for small $n$
  - selection sort, insertion sort (latter linear for mostly-sorted)
  - good for "below a cut-off" to help divide-and-conquer sorts
- $O(n \log n)$ sorts
  - heap sort, in-place but not stable nor parallelizable
  - merge sort, not in place but stable and works as external sort
  - quick sort, in place but not stable and $O(n^2)$ in worst-case
    - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!

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Student Activity

## RadixSort

- Input: 126, 328A, 636, 341, 416, 131, 328B

BucketSort on lsd:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 341<br>131 | | | | | 126<br>636<br>416 | | 328A<br>328B | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

BucketSort on next-higher digit:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 416 | 126<br>328A<br>328B | 131<br>636 | 341 | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

BucketSort on msd:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 126<br>131 | | 328A<br>328B<br>341 | 416 | | 636 | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

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→ 126, 131, 328A, 328B, 341, 416, 636