SOLVABLE OPTIMIZATION

***Risk and Asset Allocation* - Springer – `symmys.com`**

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

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programming $$\mathbf{z}^{*} \equiv \underset{\mathbf{z} \in \mathcal{C}}{\operatorname{argmin}}\, \mathcal{Q}(\mathbf{z}) \qquad (6.43)$$

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convex programming $$\mathbf{z}^{*} \equiv \underset{\substack{\mathbf{z} \in \mathcal{L} \\ \mathbf{z} \in \mathcal{V}}}{\operatorname{argmin}}\, \mathcal{Q}(\mathbf{z}) \quad (6.44)$$

$\mathcal{Q}$ is a convex function

$$\mathcal{L} \equiv \{\mathbf{z} \text{ such that } \mathbf{A}\mathbf{z} = \mathbf{a}\} \quad (6.45)$$

$$\mathcal{V} \equiv \{\mathbf{z} \text{ such that } \mathbf{F}(\mathbf{z}) \leq \mathbf{0},\ \mathbf{F} \text{ convex}\} \quad (6.46)$$

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programming $\mathbf{z}^* \equiv \underset{\mathbf{z}\in\mathcal{C}}{\operatorname{argmin}}\, \mathcal{Q}(\mathbf{z})$ (6.43)

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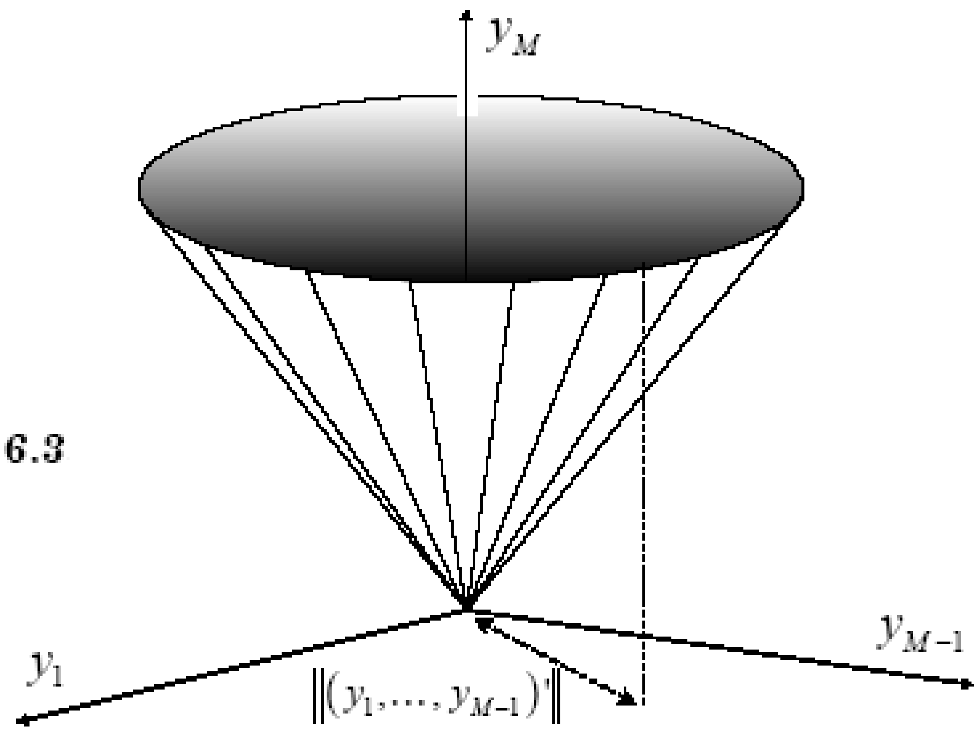
cone programming $\mathbf{z}^* \equiv \underset{\substack{\mathbf{z}\in\mathcal{L}\\ \mathbf{B}\mathbf{z}-\mathbf{b}\in\mathcal{K}}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z},$ (6.47)

$\mathbf{y} \in \mathcal{K}, \lambda \geq \mathbf{0} \Rightarrow \lambda\mathbf{y} \in \mathcal{K};$ (6.48)

$\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{K} \Rightarrow \mathbf{y} + \widetilde{\mathbf{y}} \in \mathcal{K};$ (6.49)

$\mathbf{y} \in \mathcal{K} \Rightarrow -\mathbf{y} \notin \mathcal{K}.$ (6.50)

Fig. 6.3



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linear programming (LP) $$\mathbf{z}^* \equiv \underset{\substack{\mathbf{A}\mathbf{z}=\mathbf{a}\\ \mathbf{B}\mathbf{z}\geq\mathbf{b}}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z}. \quad (6.52)$$

$$\mathbb{R}^M_+ \equiv \left\{\mathbf{y} \in \mathbb{R}^M \text{ such that } y_1 \geq 0, \ldots, y_M \geq 0\right\} \quad (6.51)$$

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second-order cone programming (SOCP)

$$\mathbf{z}^* \equiv \operatorname{argmin} \mathbf{c}'\mathbf{z} \quad (6.55)$$
$$\mathbf{A}\mathbf{z} = \mathbf{a}$$
$$\left\|\mathbf{D}_{(1)}\mathbf{z} - \mathbf{q}_{(1)}\right\| \leq \mathbf{p}_{(1)}'\mathbf{z} - r_{(1)}$$

$$\mathbb{K}^M \equiv \left\{\mathbf{y} \in \mathbb{R}^M \text{ such that } \left\|(y_1, \ldots, y_{M-1})'\right\| \leq y_M\right\} \quad (6.53)$$

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$$\mathrm{S}_{(j)} \equiv \mathbf{E}_{(j)} \mathbf{\Lambda}_{(j)} \mathbf{E}'_{(j)}$$

$$\begin{aligned}
(\mathbf{z}^*, t^*) \equiv \operatorname{argmin} t \\
\mathbf{A}\mathbf{z} &= \mathbf{a} \\
\left\| \mathbf{\Lambda}_{(0)}^{1/2} \mathbf{E}'_{(0)} \mathbf{z} + \mathbf{\Lambda}_{(0)}^{-1/2} \mathbf{E}'_{(0)} \mathbf{u}_{(0)} \right\| &\leq t \\
\left\| \mathbf{\Lambda}_{(1)}^{1/2} \mathbf{E}'_{(1)} \mathbf{z} + \mathbf{\Lambda}_{(1)}^{-1/2} \mathbf{E}'_{(1)} \mathbf{u}_{(1)} \right\| &\leq \sqrt{\mathbf{u}_{(1)} \mathrm{S}_{(1)}^{-1} \mathbf{u}_{(1)} - v_{(1)}}
\end{aligned}$$

$$\Updownarrow$$

quadratically constrained quadratic programming (QCQP)

$$\begin{aligned}
\mathbf{z}^* \equiv \operatorname{argmin} \left\{ \mathbf{z}' \mathrm{S}_{(0)} \mathbf{z} + 2\mathbf{u}'_{(0)} \mathbf{z} + v_{(0)} \right\} \\
\mathbf{A}\mathbf{z} = \mathbf{a} \\
\mathbf{z}' \mathrm{S}_{(1)} \mathbf{z} + 2\mathbf{u}'_{(1)} \mathbf{z} + v_{(1)} \leq 0
\end{aligned} \tag{6.57}$$

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second-order cone programming (SOCP)

$$\begin{aligned} \mathbf{z}^* \equiv \operatorname{argmin} \mathbf{c}'\mathbf{z} & \\ & \mathbf{A}\mathbf{z} = \mathbf{a} \\ & \left\| \mathbf{D}_{(1)} \mathbf{z} - \mathbf{q}_{(1)} \right\| \leq \mathbf{p}'_{(1)} \mathbf{z} - r_{(1)} \end{aligned} \quad (6.55)$$

$$\mathbb{K}^M \equiv \left\{ \mathbf{y} \in \mathbb{R}^M \text{ such that } \left\| (y_1, \ldots, y_{M-1})' \right\| \leq y_M \right\} \quad (6.53)$$

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semidefinite programming

(SDP)

$$\mathbf{z}^* \equiv \operatorname{argmin}\, \mathbf{c}'\mathbf{z} \tag{6.61}$$

$$\mathbf{A}\mathbf{z}=\mathbf{a}$$

$$\mathbf{B}_{(1)}z_1+\cdots+\mathbf{B}_{(N)}z_N-\mathbf{B}_{(0)}\succeq\mathbf{0}.$$

$$\mathbb{S}^M_+ \equiv \{\mathbf{S}\succeq\mathbf{0}\} \tag{6.60}$$

$\bigcup$

second-order cone programming

(SOCP)

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