ARPM Bootcamp® 2015
Review Sessions

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Exercise book

A. MEUCCI, *Exercises in Advanced Risk and Portfolio Management – With Step-by-Step Solutions and Fully Documented Code*

http://symmys.com/node/170

SYMMYS
Advanced Risk and Portfolio Management®

Review Session 1

August 11, 2014

Quick introduction to Matlab

Quest for invariance
- p.27, 3.2.1: Equity
- p.28, 3.2.2: Fixed income
- p.28, 3.2.3: Derivatives

Projection and Pricing
- p.57, 5.3 Stable invariants
- p.57, 5.4.1 Random walk (linear vs compounded returns)

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3.2.1 Equity

Consider any of the daily time series P_t of the stock prices in the database DB_Equities. Consider the variables

$$X_t \equiv \frac{P_t}{P_{t-1}} \tag{122}$$

$$Y_t \equiv P_t - P_{t-1} \tag{123}$$

$$Z_t \equiv \left(\frac{P_t}{P_{t-1}}\right)^2 \tag{124}$$

$$W_t \equiv P_{t+1} - 2P_t + P_{t-1} \tag{125}$$

Determine which among X_t, Y_t, Z_t, W_t, can potentially be an invariant and which certainly cannot be an invariant, by computing the histogram from two sub-samples and by plotting the location-dispersion ellipsoid of a variable with its lagged value.

IID Analysis on

ELLIPSOID + HIST TESTS (NECESSARY CONDITIONS)

PRICES	P
TOTAL RETURNS	X
PRICE CHANGES	Y
(TOT RETS)^2	$Z = X^2$
$P_{t+1} - 2P_t + P_{t-1}$	W

- TOTAL RETURN COMPUTATION IN MATLAB

$$X_t = \frac{P_t}{P_{t-1}}$$

ENTRY BY ENTRY

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_T \end{bmatrix} \qquad X = \begin{bmatrix} P_2/P_1 \\ P_3/P_2 \\ \vdots \\ P_T/P_{T-1} \end{bmatrix} = \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_T \end{bmatrix} \odot / \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{T-1} \end{bmatrix}$$

= P(2:end) ./ P(1: end-1)

length (X) is T-1

- PRICE CHANGES $Y_t = P_t - P_{t-1}$

Y = P(2:end) - P(1: end-1)

= diff (P)

length (Y) is T-1

- (TOT RETS)^2 $\quad Z_t = X_t^2 \quad \rightarrow \quad Z = X.\hat{}2$

- $W_t = P_{t+1} - 2P_t + P_{t-1}$

$$W = \begin{bmatrix} P_3 \\ \vdots \\ P_T \end{bmatrix} - 2 * \begin{bmatrix} P_2 \\ \vdots \\ P_{T-1} \end{bmatrix} + \begin{bmatrix} P_1 \\ \vdots \\ P_{T-2} \end{bmatrix} = P(3:end) - 2*P(2:end-1) + P(1:end-2)$$

length (W) is $T-2$

3.2.2 Fixed income

Consider the time series of realizations of the yield curve in DB_FixedIncome.

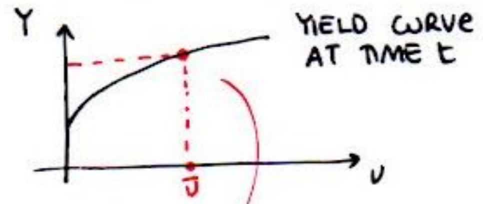
Check whether the changes in yield curve for a given time to maturity are invariants using IIDAnalysis.

Check whether the changes in the logarithm of the yield curve for a given time to maturity are invariants using IIDAnalysis.

- TIME TO MAT. (INTERVAL) $\qquad$ EXPIRY (DATE)

$$Y_t^{(u)} = -\frac{1}{u} \ln \left(\underbrace{Z_t^{(t+u)}}_{\text{ZCB price}} \right)$$

← YIELD TO MATURITY



FIX ū.
FOLLOW THIS POINT THROUGH TIME : RANDOM WALK ?

IID Analysis on the increments. ←

CHANGES IN YIELD TO MAT:

$$Y_t^{(u)} - Y_{t-1}^{(u)} = -\frac{1}{u} \ln \underbrace{\left(\frac{Z_t^{(t+u)}}{Z_{t-1}^{(t-1+u)}} \right)}$$

TOTAL "RETURN" TO MATURITY
RATIO OF PRICES OF TWO ≠ BONDS (≠ EXPIRY DATES)

- Same for $\ln Y_t^{(u)}$ $\rightarrow$ IIDAnalysis on $\Delta \ln Y_t^{(u)}$

3.2.3 Derivatives

Consider the time series of daily realizations of the implied volatility surface in `DB_Derivatives`.

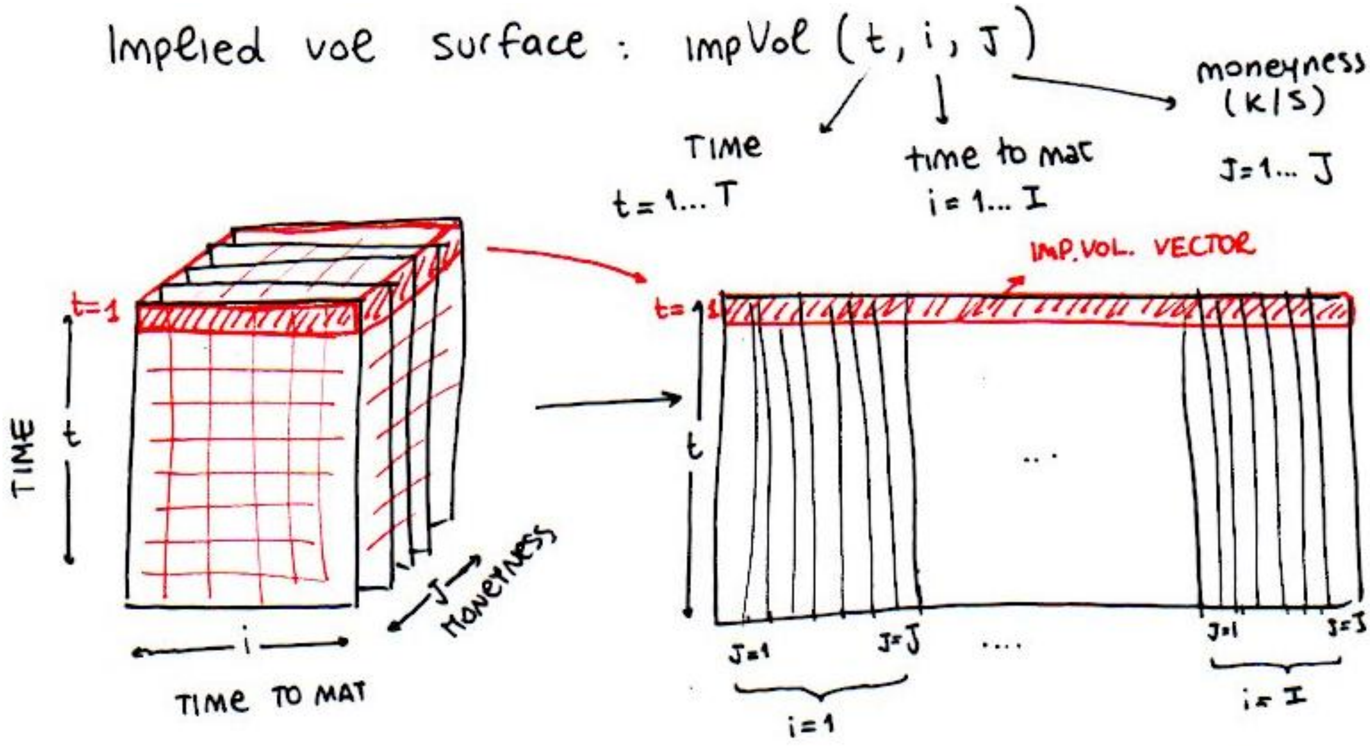
Check whether the weekly changes in implied volatility for a given level of moneyness and time to maturity are invariants using `IIDAnalysis`.

Check whether the weekly changes in the logarithm of the implied volatility for a given level of moneyness and time to maturity are invariants using `IIDAnalysis`.

Define the vector $\mathbf{Z}_t$ as the juxtaposition of all the entries of the logarithm of the implied volatility surface at time t. Fit the implied volatility data to a multivariate autoregressive process of order one:

$$\mathbf{Z}_{t+1} \equiv \widehat{\mathbf{a}} + \widehat{\mathbf{B}}\mathbf{Z}_t + \widehat{\boldsymbol{\epsilon}}_{t+1}, \tag{126}$$

where time is measured in weeks. Check whether the weekly residuals $\widehat{\boldsymbol{\epsilon}}_t$ are invariants using `IIDAnalysis`.



reshape (impVol, T, I*J)

$$Z = \text{reshape}\,(\log(\text{ImpVol}(1{:}5{:}\text{end},:,:)), T, \underbrace{I * J}_{N})$$

weekly obs.

- FIT THE WHOLE SURFACE TO A VAR(1) MODEL

$$Z_t = \alpha + \beta Z_{t-1} + \varepsilon_t \qquad t = 1 \dots T$$

→ ESTIMATION OF A <u>REGRESSION LFM</u>

↓

$$X = \alpha + \beta F + \varepsilon$$

dim

$$\begin{cases} X : N\times 1 \\ \alpha : N\times 1 \\ \beta : N\times K \\ F : K\times 1 \\ \varepsilon : N\times 1 \end{cases}$$

- F: EXOGENOUS (full rank)
- α, β TO BE DETERMINED

→ Maximize $R^2\{\alpha + \beta F \| X\}$

under the constraint $E[\varepsilon] = 0 \Rightarrow \alpha = E[X] - \beta E[F]$

SOLUTION: $\beta = \mathrm{Cov}\{X, F\} \cdot \mathrm{Cov}\{F\}^{-1}$

$\alpha = E\{X\} - \beta E\{F\}$

OLS ESTIMATION (time series)

$$\begin{cases} \hat\beta = \hat\Sigma_{XF} \, \hat\Sigma_{FF}^{-1} \\ \hat\alpha = \hat\mu_X - \hat\beta \, \hat\mu_F \end{cases}$$

$\hat\Sigma$: sample cov
$\hat\mu$: sample mean

<u>CODE</u>: $\underset{N\times T}{X} = \underset{N\times T}{\alpha} + \underset{N\times K}{\beta} \; \underset{K\times T}{F} + \underset{N\times T}{\varepsilon}$

$\mu_X = \mathrm{mean}(X, 2)$ [N×1]

$\hat\mu_F = \mathrm{mean}(F, 2)$ [K×1]

$\tilde X = X - \mathrm{repmat}(\mu_X, 1, T)$ [N×T]

$\tilde F = F - \mathrm{repmat}(\mu_F, 1, T)$ [K×T]

In our example F is N×T (lagged X)

$$\hat{\Sigma}_{XF} = \frac{1}{T} \cdot \tilde{X} \cdot \tilde{F}'$$

$$\hat{\Sigma}_{FF} = \frac{1}{T} \tilde{F}\tilde{F}'$$

$$\hat{\beta} = \hat{\Sigma}_{XF} \; / \; \hat{\Sigma}_{FF}$$

$$\hat{\alpha} = \hat{\mu}_X - \hat{\beta}\,\hat{\mu}_F \quad [N \times 1]$$

IN MATLAB
$A / B \equiv A * \mathrm{inv}(B)$
$A \backslash B \equiv \mathrm{inv}(A) * B$

$$\hat{\varepsilon} = X - \mathrm{repmat}(\hat{\alpha}, 1, T) - \hat{\beta} F$$

↑

iID Analysis on $\hat{\varepsilon}$ (residuals)

REGRESSION LFM'S are SYSTEMATIC : $\mathrm{Cov}\{F_k, \varepsilon_n\} = 0 \quad \forall k, n$

Then, an alternative way to proceed is:

$$z_t = a + b\, z_{t-1} + \varepsilon_t \quad \rightarrow \quad z_t = [a \;\; b] \begin{bmatrix} 1 \\ z_{t-1} \end{bmatrix} + \varepsilon_t$$

$$\underset{N \times 1}{X} = \underset{N \times K}{B} \; \underset{K \times 1}{F} + \underset{N \times 1}{\varepsilon}$$

in our case ($K = N + 1$)

$X = BF + \varepsilon$

↓ multiply by F' and take the expectation

$$E[XF'] = B \underbrace{E[FF']}_{\text{INVERTIBLE}} + \underbrace{\cancel{E[\varepsilon F']}}_{=0 \text{ (SYSTEMATIC)}}$$

$$B = E[XF'] \left(E[FF']\right)^{-1}$$

REGRESSION: $\hat{B} = \frac{XF'}{T} \cdot \left(\frac{FF'}{T}\right)^{-1} = \frac{XF'}{\cancel{T}} \cdot \cancel{T}\,(FF')^{-1} = (XF')/(FF')$

$\hat{\varepsilon} = X - \hat{B} F \rightarrow$ IID Analysis

5.3 Stable invariants

Assume that the distribution of the market invariants at the estimation horizon $\widetilde{\tau}$ is multivariate Cauchy

$$\mathbf{X}_{\widetilde{\tau}} \sim \mathrm{Ca}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{298}$$

Assume that the invariants satisfy the "accordion" property (3.60) in Meucci (2005).

Prove that the distribution of the market invariants at *any* generic investment horizon τ is Cauchy.

Hint. Like the normal distribution, the Cauchy distribution is stable. Use the characteristic function (2.210) in Meucci (2005) to represent this distribution at any horizon.

Draw your conclusions on the propagation law of risk in terms of the modal dispersion (2.212) in Meucci (2005).

Hint. Notice that the covariance is not defined.

$X_{\tilde{\tau}} \sim Ca(\mu, \Sigma)$ multivariate Cauchy

invariants

$\tilde{\tau}$: ESTIMATION STEP

$$\phi_{X_\tau} = \left(\phi_{X_{\tilde{\tau}}}\right)^{\frac{\tau}{\tilde{\tau}}}$$

CF Cauchy:

$$\phi_{X_{\tilde{\tau}}}(\omega) = e^{i\omega'\mu - \sqrt{\omega'\Sigma\omega}}$$

$$(\cdot)^{\frac{\tau}{\tilde{\tau}}}$$

$$\phi_{X_\tau} = e^{\left(i\omega'\mu\frac{\tau}{\tilde{\tau}} - \sqrt{\omega'\Sigma\left(\frac{\tau}{\tilde{\tau}}\right)^2\omega}\right)}$$

$$\rightarrow X_\tau \sim Ca\left(\mu \cdot \frac{\tau}{\tilde{\tau}}, \Sigma\left(\frac{\tau}{\tilde{\tau}}\right)^2\right)$$

$$\mathrm{MDis}(X_{\tilde{\tau}}) = \frac{1}{N+1}\Sigma$$

$$\mathrm{MDis}(X_{\tau}) = \frac{1}{N+1}\left(\frac{\tau}{\tilde{\tau}}\right)^2\Sigma = \left(\frac{\tau}{\tilde{\tau}}\right)^2 \cdot \mathrm{MDis}(X_{\tilde{\tau}})$$

$\downarrow$

THE PROPAGATION OF RISK IS LINEAR

NORMAL CASE

$$Y_{\tilde{\tau}} \sim N(\mu, \Sigma) \longrightarrow \phi_{Y_{\tilde{\tau}}}(\omega) = e^{+i\omega'\mu - \frac{1}{2}\omega'\Sigma\omega}$$

$\downarrow$

$$Y_{\tau} \sim N\left(\mu\frac{\tau}{\tilde{\tau}}, \Sigma\cdot\frac{\tau}{\tilde{\tau}}\right) \longleftarrow \phi_{Y_{\tau}}(\omega) = \left(\phi_{Y_{\tilde{\tau}}}\right)^{\frac{\tau}{\tilde{\tau}}} = e^{i\omega'\mu\frac{\tau}{\tilde{\tau}} - \frac{1}{2}\omega'\Sigma\frac{\tau}{\tilde{\tau}}\omega}$$

univariate case $\rightarrow$ $Y_{\tau} \sim N\left(\mu\frac{\tau}{\tilde{\tau}}, \sigma^2\frac{\tau}{\tilde{\tau}}\right)$

$\downarrow$

$$\mathrm{Sd}\{Y_{\tau}\} = \sigma\cdot\sqrt{\frac{\tau}{\tilde{\tau}}}$$

SQUARE ROOT RULE PROPAGATION OF RISK

C.L.T. APPLIES TO RANDOM WALKS WITH FINITE VARIANCE

(NOT THE CASE FOR THE CAUCHY DIST.)

5.4.1 Random walk (linear vs. compounded returns)

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010d), freely available online at ssrn.com.

Assume that the compounded returns (3.11) in Meucci (2005) of a given stock are market invariants, i.e. they are i.i.d. across time. Consider an estimation interval of one week $\widetilde{\tau} \equiv 1/52$ (time is measured in years). Assume that the distribution of the returns is normal:

$$C_{t,\widetilde{\tau}} \sim \mathrm{N}\left(0, \sigma^2 \widetilde{\tau}\right), \tag{302}$$

where $\sqrt{\sigma^2} \equiv 0.4$. Assume that the stock currently trades at the price $P_T \equiv 1$. Fix a generic horizon τ.

Compute and plot the analytical pdf of the price $P_{T+\tau}$.

Simulate the compounded return at the investment horizon and map these simulations into simulations of the price $P_{T+\tau}$ at the generic horizon τ.

Superimpose the rescaled histogram from the simulations of $P_{T+\tau}$ to show that they coincide.

Hint. Use (T1.43) in the technical appendix at `symmys.com` > Book > `Downloads` for the rescaling.

Compute analytically the distribution of the first-order Taylor approximation of the pricing function around zero and superimpose this pdf to the above plots. Notice how the approximation is good for short horizons and bad for long horizons.

$C_{\tilde{\tau}} \sim N(0, \sigma^2 \tilde{\tau})$ Invariants : $C_{\tilde{\tau}} = \ln\left(\frac{P_{t+\tilde{\tau}}}{P_t}\right)$

$t = 1 \dots T$

$\tilde{\tau} = \frac{1}{52}$ (years)

$\sigma = 0.4$

$P_T = 1$

$\tau \rightarrow$ GENERIC INVESTMENT HORIZON

- ANALYTICAL PDF of $P_{T+\tau}$

$$P_{T+\tau} = P_T \cdot e^{C_\tau}$$

$$= e^{\log P_T + C_\tau}$$

$$C_{\tilde\tau} \sim N(0, \sigma^2\tilde\tau) \to C_\tau \sim \underbrace{N\left(0\cdot\frac{\tau}{\tilde\tau}, \sigma^2 \cancel{\tilde\tau}\cdot\frac{\tau}{\cancel{\tilde\tau}}\right)}_{N(0,\,\sigma^2\tau)}$$

$$\to P_{T+\tau} \sim \underbrace{\log N(\log \overset{=1}{P_T} + 0,\ \sigma^2\tau)}_{\log N(0,\,\sigma^2\tau)}$$

$$f^{\log N}_{(0,\sigma^2\tau)}(x) = \frac{1}{x\sqrt{2\pi\sigma^2\tau}} \cdot e^{-\frac{1}{2}\frac{\ln(x)^2}{\sigma^2\tau}}$$

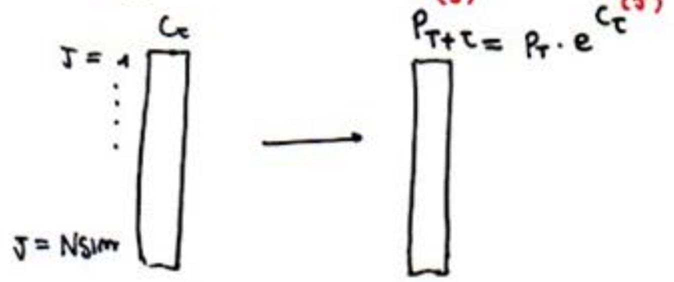
MATLAB:

→ f = lognpdf $(x, 0, \sigma\sqrt{\tau})$

grid

- SIMULATION

C_τ = normrnd $(0, \sigma\sqrt{\tau}, [\text{Nsim}, 1])$



hist $(P_{T+\tau}, \text{Nbins})$

→ NORMALIZE TO MAKE IT COMPARABLE WITH THE PDF

- TAYLOR APPROX (1st ORDER)

$$P_{T+\tau} = P_T \cdot e^{C_\tau} = P_T \cdot \left(\underbrace{1 + C_\tau}_{1^{st}\text{ ORDER}} + \frac{C_\tau^2}{2} + \frac{C_\tau^3}{3!} + \cdots \right)$$

$$\approx P_T \cdot (1 + C_\tau) \sim N(P_T,\ P_T^2 \cdot \sigma^2\tau)$$

MATLAB: f = normpdf (x, μ, σ)

grid

```matlab
% SCRIPTS used during the class

%S_QuestForInvariance
%% EQUITY

clear; clc;
load DB_Equities

pick=20; %consider the 20th stock

%Prices
P=Prices(end-599:end,pick);
IIDAnalysis(P)

%Total returns
X=P(2:end)./P(1:end-1);
IIDAnalysis(X)

%Price increments
Y=P(2:end)-P(1:end-1); %or Y=diff(P)
IIDAnalysis(Y)

%Squared total returns
Z=X.^2;
IIDAnalysis(Z)

%W
W=P(3:end)-2*P(2:end-1)+P(1:end-2);
IIDAnalysis(W)

%%FIXED INCOME

clear;close all; clc;
load DB_FixedIncome

t2m=ycMaturityYrs;
yields=ycYieldPercent/100;
clear ycMaturityYrs ycYieldPercent

line(t2m,yields(1,:))
line(t2m,yields(2,:),'color','r')
line(t2m,yields(end,:),'color','g')
xlabel('time to maturity')
ylabel('yield to maturity')

%changes in the yield for a specific time to mat
Y=yields(:,t2m==5);
dY=diff(Y);
IIDAnalysis(dY)

%changes in the log-yield for a specific time to mat
X=log(Y);
dX=diff(X);
IIDAnalysis(dX)

perm_idx=randperm(length(dX));
dX1=dX(perm_idx);

sample1=dX1(1:length(dX1)/2);
sample2=dX1(length(dX1)/2+1:end);

subplot(1,2,1)
hist(sample1)
subplot(1,2,2)
hist(sample2)

figure()
```

```
[f,x]=ecdf(sample1);
plot(x,f)
[f,x]=ecdf(sample2);
hold on
plot(x,f,'r')

%% DERIVATIVES

clear;close all; clc;
load DB_Derivatives

colormap(bone)
surf(moneyness,days2Maturity,squeeze(impVol(1,:,:)))
xlabel('moneyness (K/S)')
ylabel('time to mat (days)')
zlabel('implied vol.')

pick_t2m=1;
pick_mon=find(moneyness==1);

%weekly changes in implied volatility
X_daily=impVol(:,pick_t2m,pick_mon); %daily time series of imp. vol. for fixed time to
mat and moneyness
X_weekly=X_daily(1:5:end);%weekly time series of imp. vol. for fixed time to mat and
moneyness
dX_weekly=diff(X_weekly);
IIDAnalysis(dX_weekly)

%weekly changes in log-implied volatility
Y=log(X_weekly);
dY=diff(Y);
IIDAnalysis(dY)


%define variable Z
[T,Mat,Mon]=size(impVol(1:5:end,:,:));
Z=reshape(log(impVol(1:5:end,:,:)),T,Mat*Mon);

% VAR(1) model estimation
X=Z(2:end,:)'; %dependent variables time series (N x T)
F=Z(1:end-1,:)'; %factors time series (K x T) --> VAR(1): N=K

T=size(X,2);

mean_X=mean(X,2); %
mean_F=mean(F,2);

Xcent=X-repmat(mean_X,1,size(X,2));
Fcent=F-repmat(mean_F,1,size(F,2));

cov_XF=(1/T)*(Xcent)*(Fcent)';
cov_F=(1/T)*(Fcent)*(Fcent)';

b=cov_XF/cov_F; % loadings
a=mean_X-b*mean_F; %shift (set such that residuals have zero mean)

% residuals
eps=X-repmat(a,1,T)-b*F; %residuals (N x T)

pick=Mon*(pick_t2m-1)+pick_mon;
IIDAnalysis(eps(pick,:)')
```

```
% S_EquityProjectionPricing

clc; clear; close all;

% inputs
tau_tilde=1/52;  % estimation period expressed in years
sig=.4;
P_T=1;

Nsim=10^5;

%tau=1/252; tauName = '1 day';           % times to horizon expressed in years
% tau=1/52;  tauName = '1 week';
%tau= 1/12;  tauName = '1 month';
% tau=1; tauName = '1 year';
 tau=2;   tauName = '2 years';


% exact simulation of horizon prices
C_tau=normrnd(0,sig*sqrt(tau),Nsim,1);
P_Ttau=P_T*exp(C_tau);

% compute analytical pdf
p_lo=min(P_Ttau);
p_hi=max(P_Ttau);

p=p_lo : (p_hi-p_lo)/1000 : p_hi;
%p=linspace(p_lo,p_hi,1000);

m=log(P_T);
s=sig*sqrt(tau);
f=lognpdf(p,m,s);


% compute approximate pdf
f_approx=normpdf(p,P_T,P_T*sig*sqrt(tau));

% plots
figure
Nbins=round(10*log(Nsim));
[f_hist,p_hist]=hist(P_Ttau,Nbins);
f_hist=f_hist/sum((p_hist(2)-p_hist(1))*f_hist); %normalize area under the hist to 1,
to make it comparable with a pdf
bar(p_hist,f_hist,'Facecolor',[.7 .7 .7],'Edgecolor',[.5 .5 .5])

hold on
plot(p,f,'r','linewidth',1.5)

hold on
plot(p,f_approx,'b','linewidth',1.5)
xlabel('Price at the horizon')
ylabel('pdf')
title(['Time to horizon \tau = ' tauName])
legend('full Monte Carlo','analytical','Taylor approx')
```