Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

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$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \textbf{number } \widehat{\mathbf{G}}} \quad (4.9)$$

$$i_T \equiv \left\{\mathbf{x}_1, \ldots, \mathbf{x}_T\right\} \quad (4.8)$$

$$(4.13) \quad \boxed{\widehat{\mathbf{G}}\left[i_T\right] \approx \mathbf{G}\left[f_{\mathbf{X}}\right]}$$

?

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$$G\left[f_X\right] \equiv \int_{-\infty}^{+\infty} x f_X(x)\, dx. \quad (4.7)$$

$$\widehat{G}\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T} x_t. \quad (4.10)$$

$$\widehat{G}\left[i_T\right] \equiv x_1 x_T \quad (4.11)$$

$$\widehat{G}\left[i_T\right] \equiv 3. \quad (4.12)$$

$$(4.13) \quad \boxed{\widehat{\mathbf{G}}\left[i_T\right] \approx \mathbf{G}\left[f_{\mathbf{X}}\right]} \ ?$$

$$\boxed{\mathbf{G}[f_{\mathbf{X}}] \equiv \text{"unknown truth"}} \quad (4.6)$$

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$$i_T \equiv \{\mathbf{x}_1, \ldots, \mathbf{x}_T\} \mapsto I_T \equiv \{\mathbf{X}_1, \ldots, \mathbf{X}_T\} \quad (4.14)$$

$$\widehat{\mathbf{G}}[i_T] \mapsto \widehat{\mathbf{G}}[I_T] \quad (4.15)$$

$$\boxed{\operatorname{Loss}\left(\widehat{\mathbf{G}}, \mathbf{G}\right) \equiv \left\| \widehat{\mathbf{G}}[I_T] - \mathbf{G}[f_{\mathbf{X}}] \right\|^2} \quad (4.19)$$

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$$G[f_X] = \mu. \quad (4.18)$$

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right) \quad (4.16)$$

$$\widehat{G}[I_T] \equiv \frac{1}{T}\sum_{t=1}^{T} X_t \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{T}\right) \quad (4.17)$$

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$$\mathrm{Loss}\left(\widehat{G}, G\right) \sim \mathrm{Ga}\left(1, \frac{\sigma^2}{T}\right) \quad (4.22)$$

$$\boxed{\mathrm{Loss}\left(\widehat{\mathbf{G}}, \mathbf{G}\right) \equiv \left\|\widehat{\mathbf{G}}\left[I_T\right] - \mathbf{G}\left[f_{\mathbf{X}}\right]\right\|^2} \quad (4.19)$$

$$\mathrm{Err}\left(\widehat{\mathbf{G}},\mathbf{G}\right)\equiv\sqrt{\mathrm{E}\left\{\left\|\widehat{\mathbf{G}}\left(I_T\right)-\mathbf{G}\left[f\mathbf{x}\right]\right\|^2\right\}}. \qquad (4.23)$$

$$\boxed{\mathrm{Loss}\left(\widehat{\mathbf{G}},\mathbf{G}\right)\equiv\left\|\widehat{\mathbf{G}}\left[I_T\right]-\mathbf{G}\left[f\mathbf{x}\right]\right\|^2} \quad (4.19)$$

$$\mathrm{Err}\left(\widehat{G}, G\right) = \frac{\sigma}{\sqrt{T}} \quad (4.24)$$

$$\mathrm{Err}\left(\widehat{\mathbf{G}}, \mathbf{G}\right) \equiv \sqrt{\mathrm{E}\left\{\left\|\widehat{\mathbf{G}}\left(I_T\right) - \mathbf{G}\left[f_{\mathbf{X}}\right]\right\|^2\right\}}. \quad (4.23)$$

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$$\mathrm{Inef}^2\left[\widehat{\mathbf{G}}\right] \equiv \mathrm{E}\left\{\left\|\widehat{\mathbf{G}}\left[I_T\right]-\mathrm{E}\left\{\widehat{\mathbf{G}}\left[I_T\right]\right\}\right\|^2\right\} \quad (4.26)$$

$$\mathrm{Bias}^2\left[\widehat{\mathbf{G}},\mathbf{G}\right] \equiv \left\|\mathrm{E}\left\{\widehat{\mathbf{G}}\left[I_T\right]\right\}-\mathbf{G}\left[f_{\mathbf{X}}\right]\right\|^2 \quad (4.25)$$

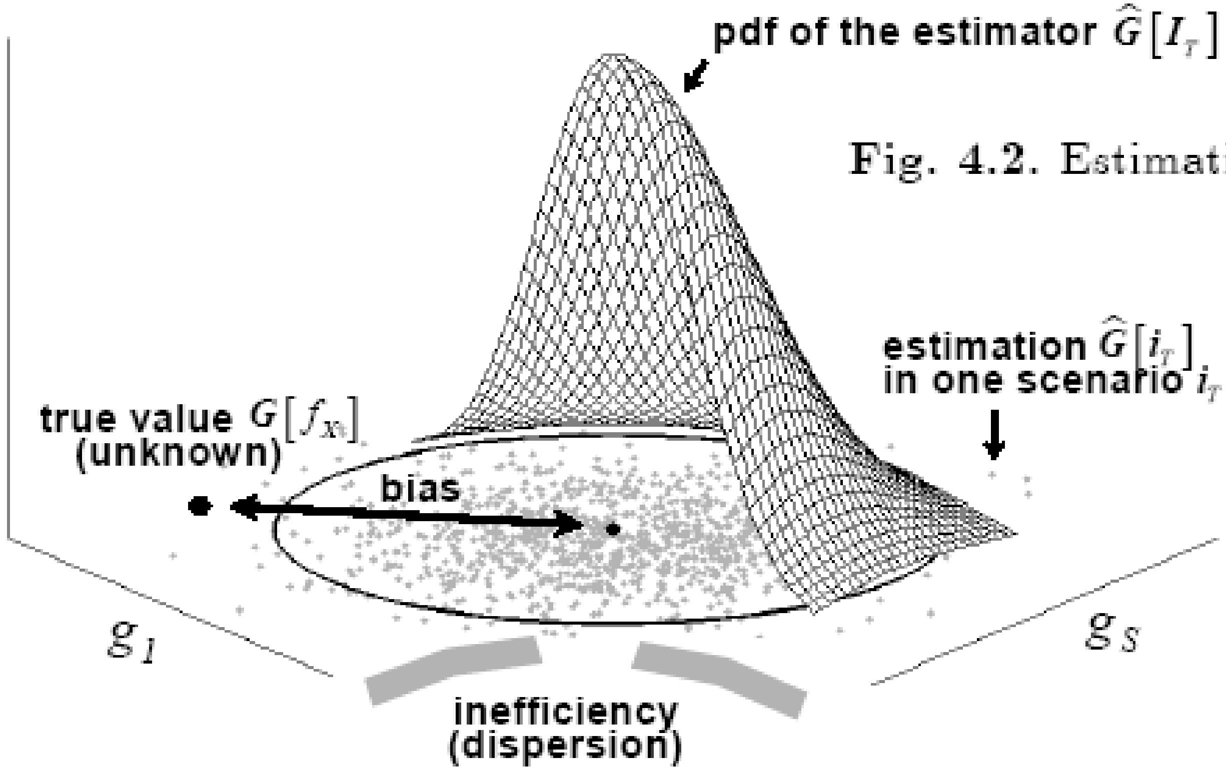

Fig. 4.2. Estimation: replicability, bias and inefficiency

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$$\operatorname{Bias}\left[\widehat{G},G\right]=0 \quad (4.28)$$

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$$\boxed{\operatorname{Err}^2\left[\widehat{\mathbf{G}},\mathbf{G}\right]=\operatorname{Bias}^2\left[\widehat{\mathbf{G}},\mathbf{G}\right]+\operatorname{Inef}^2\left[\widehat{\mathbf{G}}\right]} \quad (4.27)$$

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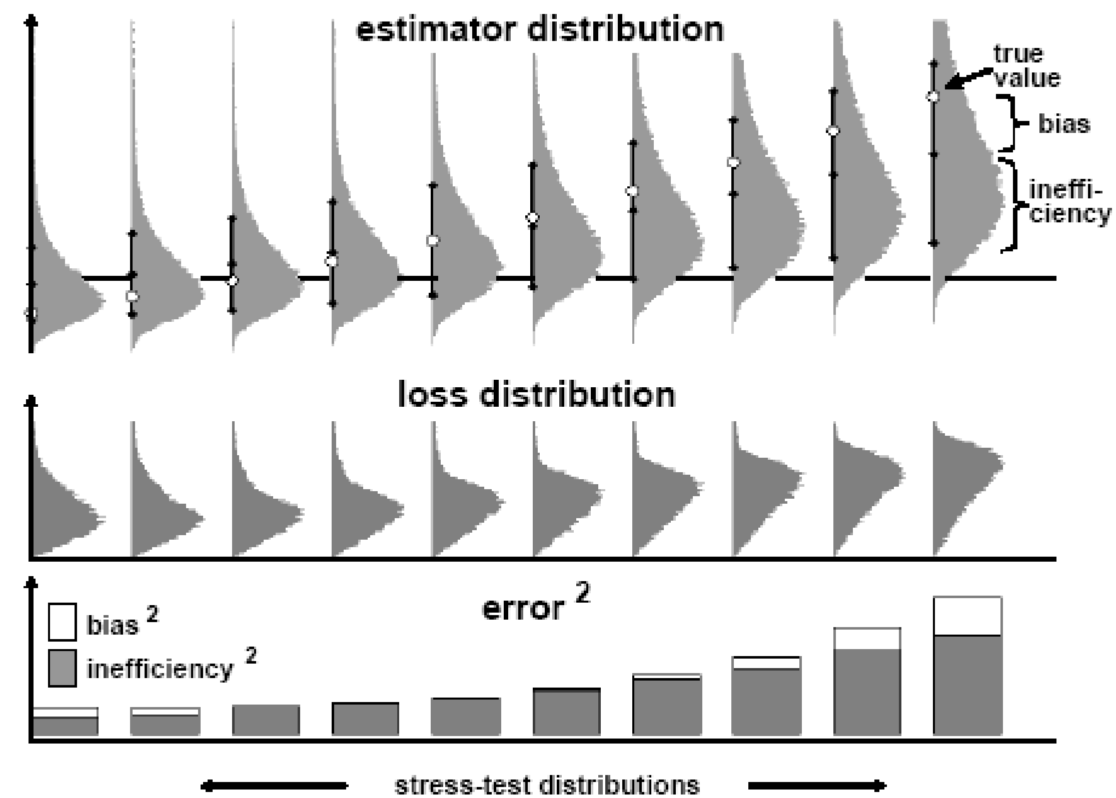

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NON-PARAMETRIC ESTIMATORS

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NON-PARAMETRIC ESTIMATORS

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$$\boxed{\textbf{information } i_T \mapsto \widehat{\mathbf{G}}\left[i_T\right] \equiv \mathbf{G}\left[f_{i_T}\right]} \quad (4.36)$$

$$f_{i_T}(\mathbf{x}) \equiv \frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)}(\mathbf{x}) \quad (4.35)$$

$$\boxed{\lim_{T\to\infty} F_{i_T}(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x})} \quad (4.34)$$

NON-PARAMETRIC ESTIMATORS

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$$\int_{-\infty}^{q_p[f_X]} f_X(x)\, dx \equiv p, \quad (4.38)$$

$$\widehat{q}_p[i_T] \equiv x_{[pT]:T} \quad (4.39)$$

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NON-PARAMETRIC ESTIMATORS – ORDINARY LEAST SQUARES

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$$\mathbf{X} = \mathbf{B}\mathbf{F} + \mathbf{U}. \quad (4.50)$$

$$\boxed{\mathbf{B}_r \equiv \mathrm{E}\left\{\mathbf{X}\mathbf{F}'\right\} \mathrm{E}\left\{\mathbf{F}\mathbf{F}'\right\}^{-1}} \quad (3.121)$$

$$\boxed{\widehat{\mathbf{B}}\left[i_T\right] \equiv \left(\sum_t \mathbf{x}_t \mathbf{f}_t'\right)\left(\sum_t \mathbf{f}_t \mathbf{f}_t'\right)^{-1}} \quad (4.52)$$

$$i_T \equiv \left\{\mathbf{x}_1, \mathbf{f}_1, \ldots, \mathbf{x}_T, \mathbf{f}_T\right\} \quad (4.51)$$

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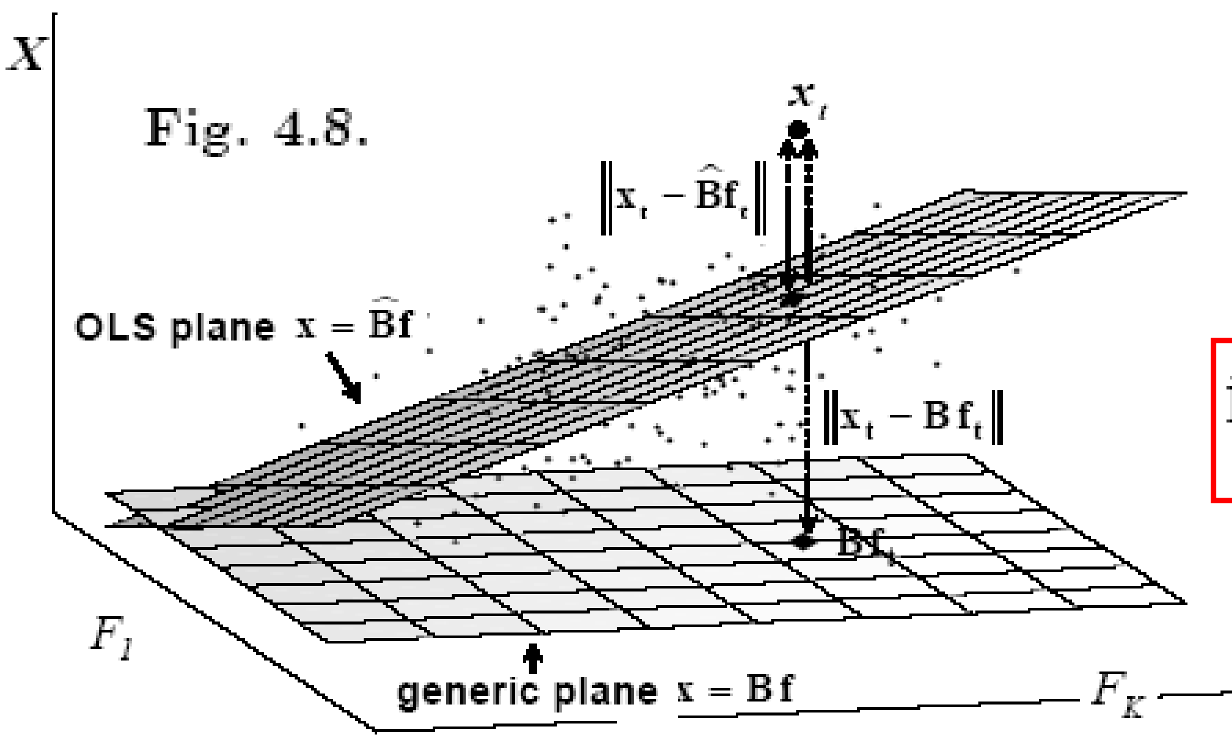
Fig. 4.8.



$$\boxed{\widehat{\mathbf{B}} = \underset{\mathbf{B}}{\operatorname{argmin}} \sum_t \left\|\mathbf{x}_t - \mathbf{B}\mathbf{f}_t\right\|^2} \quad (4.53)$$

NON-PARAMETRIC ESTIMATORS – SAMPLE MEAN/COVARIANCE

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$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\mathrm{E}\left\{\mathbf{X}\right\}} \quad (2.54)$$

$$\boxed{\mathrm{Cov}\left\{\mathbf{X}\right\}} \quad (2.67)$$

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$$\boxed{\widehat{\mathrm{Cov}}\left[i_T\right] = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}_t - \widehat{\mathrm{E}}\left[i_T\right]\right)\left(\mathbf{x}_t - \widehat{\mathrm{E}}\left[i_T\right]\right)'} \quad (4.42)$$

NON-PARAMETRIC ESTIMATORS – SAMPLE MEAN/COVARIANCE

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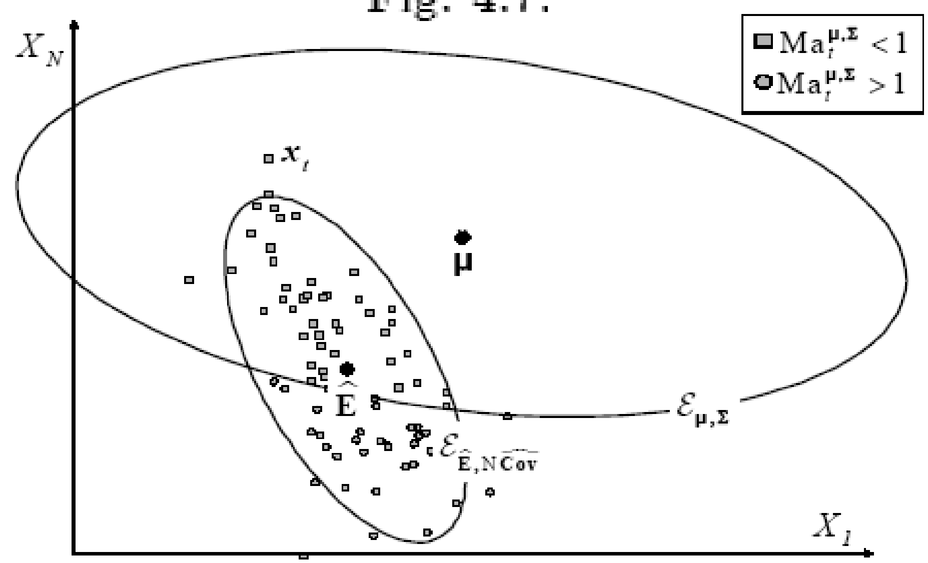
$$\mathcal{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \left\{\mathbf{x} \in \mathbb{R}^N \text{ such that } (\mathbf{x}-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \leq 1\right\} \quad (4.45)$$

$$\mathrm{Ma}_t^{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \mathrm{Ma}\left(\mathbf{x}_t, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{\left(\mathbf{x}_t-\boldsymbol{\mu}\right)'\boldsymbol{\Sigma}^{-1}\left(\mathbf{x}_t-\boldsymbol{\mu}\right)} \quad (4.46)$$

Fig. 4.7.



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$$\boxed{\mathrm{E}\left\{\mathbf{X}\right\}} \quad (2.54)$$

$$\boxed{\widehat{\mathrm{E}}\left[i_T\right] = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t} \quad (4.41)$$

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$$\boxed{\widehat{\mathrm{Cov}}\left[i_T\right] = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}_t - \widehat{\mathrm{E}}\left[i_T\right]\right)\left(\mathbf{x}_t - \widehat{\mathrm{E}}\left[i_T\right]\right)'} \quad (4.42)$$

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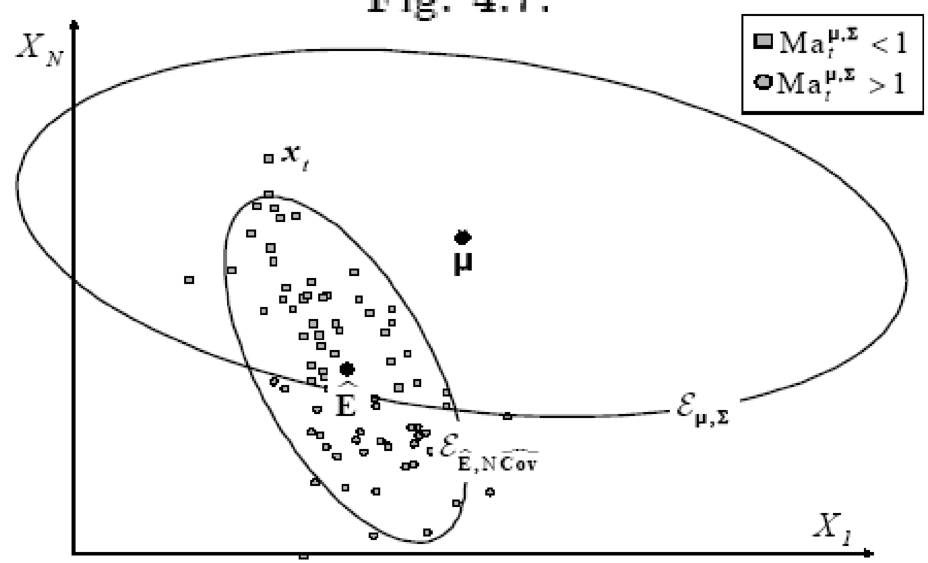
$$\mathrm{Ma}_t^{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \mathrm{Ma}\left(\mathbf{x}_t, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{(\mathbf{x}_t-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}_t-\boldsymbol{\mu})} \quad (4.46)$$

$$\overline{r^2}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \frac{1}{T}\sum_{t=1}^{T}\left(\mathrm{Ma}_t^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\right)^2 \quad (4.47)$$

$$\boxed{\begin{gathered}\left(\widehat{\mathrm{E}}, N\widehat{\mathrm{Cov}}\right) = \underset{(\boldsymbol{\mu},\boldsymbol{\Sigma})\in\mathcal{C}}{\operatorname{argmin}}\left[\mathrm{Vol}\left\{\mathcal{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}\right\}\right] \\ \overline{r^2}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv 1\end{gathered}} \quad (4.48)$$

Fig. 4.7.



NON-PARAMETRIC ESTIMATORS – KERNELS

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \widehat{\mathbf{G}}\left[i_T\right] \equiv \mathbf{G}\left[f_{i_T}\right]} \quad (4.36)$$

$$f_{i_T}(\mathbf{x}) \equiv \frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)}(\mathbf{x}) \quad (4.35)$$

$$\boxed{\textbf{information } i_T \mapsto \widehat{\mathbf{G}}\left[i_T\right] \equiv \mathbf{G}\left[f_{i_T;\epsilon}\right]} \quad (4.56)$$

$$f_{i_T} \mapsto f_{i_T;\epsilon} \equiv \frac{1}{T}\sum_{t=1}^{T} \frac{1}{(2\pi)^{\frac{N}{2}}\epsilon^N} e^{-\frac{1}{2\epsilon^2}(\mathbf{x}-\mathbf{x}_t)'(\mathbf{x}-\mathbf{x}_t)}. \quad (4.55)$$

MAXIMUM LIKELIHOOD ESTIMATORS

Risk and Asset Allocation **- Springer –** ***`symmys.com`***

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

MAXIMUM LIKELIHOOD ESTIMATORS

$$\boxed{\mathbf{G}[f_{\mathbf{X}}] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \textbf{number } \widehat{\mathbf{G}}} \quad (4.9)$$

$$i_T \equiv \{\mathbf{x}_1, \ldots, \mathbf{x}_T\} \quad (4.8)$$

MAXIMUM LIKELIHOOD ESTIMATORS

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \textbf{number } \widehat{\mathbf{G}}} \quad (4.9)$$

$$f_{\boldsymbol{\theta}}\left(i_T\right) \equiv f_{\boldsymbol{\theta}}\left(\mathbf{x}_1\right) \cdots f_{\boldsymbol{\theta}}\left(\mathbf{x}_T\right). \quad (4.65)$$

$$\widehat{\boldsymbol{\theta}}\left[i_T\right] \equiv \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmax}}\, f_{\boldsymbol{\theta}}\left(i_T\right) \quad (4.66)$$

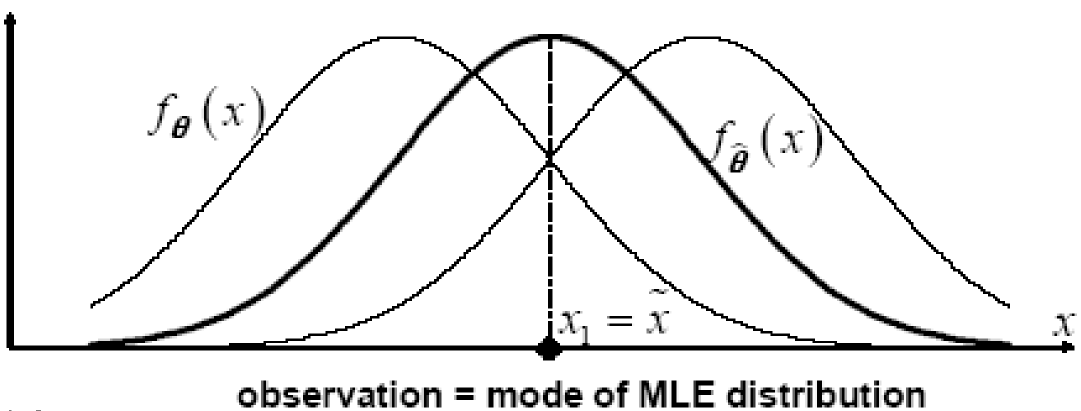


Fig. 4.10

MAXIMUM LIKELIHOOD ESTIMATORS

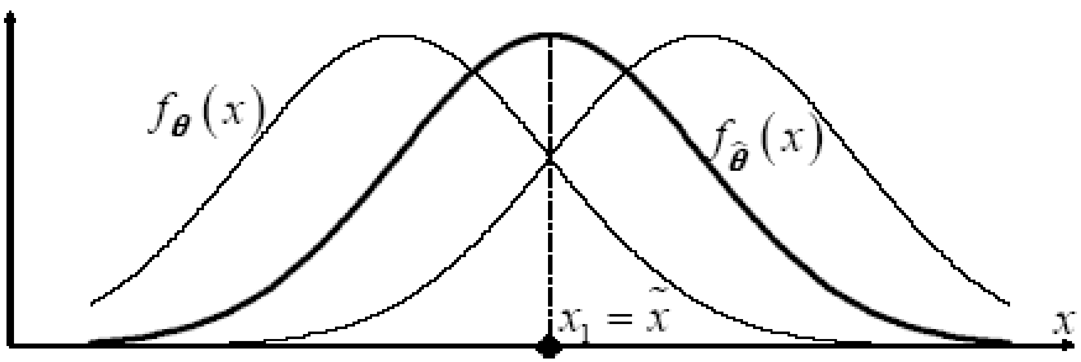
***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \textbf{number } \widehat{\mathbf{G}}} \quad (4.9)$$

$$f_{\boldsymbol{\theta}}\left(i_T\right) \equiv f_{\boldsymbol{\theta}}\left(\mathbf{x}_1\right) \cdots f_{\boldsymbol{\theta}}\left(\mathbf{x}_T\right). \quad (4.65)$$

$$\widehat{\boldsymbol{\theta}}\left[i_T\right] \equiv \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmax}} f_{\boldsymbol{\theta}}\left(i_T\right) \quad (4.66)$$



observation = mode of MLE distribution

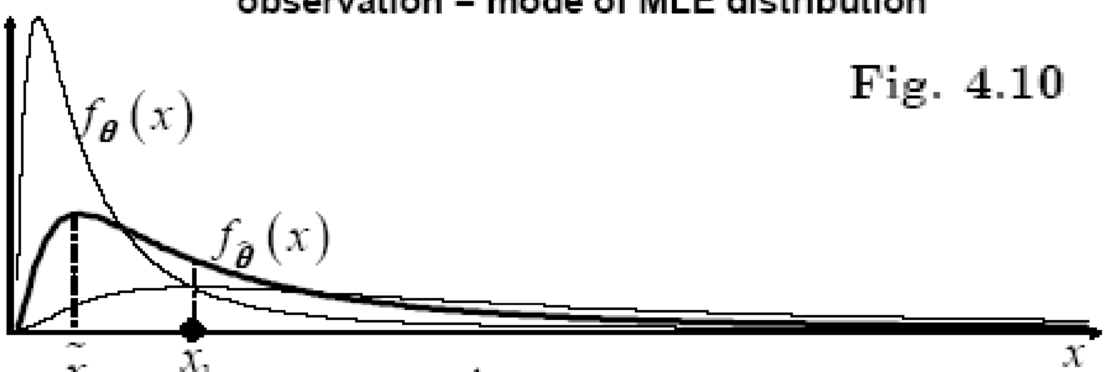


Fig. 4.10

observation ≠ mode of MLE distribution

MAXIMUM LIKELIHOOD ESTIMATORS

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\boxed{\mathbf{G}\left[f_{\mathbf{X}}\right] \equiv \text{"unknown truth"}} \quad (4.6)$$

$$\boxed{\textbf{information } i_T \mapsto \textbf{number } \widehat{\mathbf{G}}} \quad (4.9)$$

$$f_{\boldsymbol{\theta}}\left(i_T\right) \equiv f_{\boldsymbol{\theta}}\left(\mathbf{x}_1\right) \cdots f_{\boldsymbol{\theta}}\left(\mathbf{x}_T\right). \quad (4.65)$$

$$\widehat{\boldsymbol{\theta}}\left[i_T\right] \equiv \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmax}}\, f_{\boldsymbol{\theta}}\left(i_T\right) \quad (4.66)$$

- $\widehat{g\left(\boldsymbol{\theta}\right)} = g\left(\widehat{\boldsymbol{\theta}}\right) \quad (4.70)$

- $\widehat{\boldsymbol{\theta}}\left[i_T\right] \sim \mathrm{N}\left(\boldsymbol{\theta}, \frac{\boldsymbol{\Gamma}}{T}\right) \quad (4.71)$

$$\boldsymbol{\Gamma} \equiv \operatorname{Cov}\left\{\frac{\partial \ln\left(f_{\boldsymbol{\theta}}\left(\mathbf{X}\right)\right)}{\partial \boldsymbol{\theta}}\right\} \quad (4.72)$$

MAXIMUM LIKELIHOOD ESTIMATORS – ELLIPTICAL

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\mathbf{X} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g\right) \qquad (4.73)$$

$$f_{\boldsymbol{\theta}}\left(\mathbf{x}\right) \equiv \frac{1}{\sqrt{\left|\boldsymbol{\Sigma}\right|}} g\left(\mathrm{Ma}^2\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right) \qquad (4.74)$$

$$\mathrm{Ma}\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{\left(\mathbf{x}-\boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)} \qquad (4.75)$$

MAXIMUM LIKELIHOOD ESTIMATORS – ELLIPTICAL

Risk and Asset Allocation - Springer – `symmys.com`

$$\mathbf{X} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g\right) \quad (4.73)$$

$$f_{\boldsymbol{\theta}}\left(\mathbf{x}\right) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g\left(\mathrm{Ma}^2\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right) \quad (4.74)$$

$$\mathrm{Ma}\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{\left(\mathbf{x} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}\right)} \quad (4.75)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w_t}{\sum_{s=1}^{T} w_s} \mathbf{x}_t \quad (4.81)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} w_t \left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right) \left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)' . \quad (4.82)$$

$$w_t \equiv w\left(\mathrm{Ma}^2\left(\mathbf{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.80)$$

$$w\left(z\right) \equiv -2\frac{g'\left(z\right)}{g\left(z\right)} \quad (4.79)$$

MAXIMUM LIKELIHOOD ESTIMATORS – ELLIPTICAL

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\mathbf{X} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g\right) \qquad (4.73)$$

$$f_{\boldsymbol{\theta}}\left(\mathrm{x}\right) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g\left(\mathrm{Ma}^2\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right) \qquad (4.74)$$

$$\mathrm{Ma}\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{\left(\mathrm{x} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\mathrm{x} - \boldsymbol{\mu}\right)} \qquad (4.75)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w_t}{\sum_{s=1}^{T} w_s} \mathrm{x}_t \qquad (4.81)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} w_t \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right) \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)' . \qquad (4.82)$$

$$w_t \equiv w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \qquad (4.80)$$

$$w\left(z\right) \equiv -2 \frac{g'\left(z\right)}{g\left(z\right)} \qquad (4.79)$$

$$g^{\mathrm{N}}\left(z\right) \equiv \frac{e^{-\frac{z}{2}}}{\left(2\pi\right)^{\frac{N}{2}}}, \qquad (4.96)$$

$$w\left(z\right) \equiv 1 \qquad (4.97)$$

MAXIMUM LIKELIHOOD ESTIMATORS – ELLIPTICAL

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\mathbf{X} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g\right) \quad (4.73)$$

$$f_{\boldsymbol{\theta}}\left(\mathbf{x}\right) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g\left(\mathrm{Ma}^2\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right) \quad (4.74)$$

$$\mathrm{Ma}\left(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \sqrt{\left(\mathbf{x}-\boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\mathbf{x}-\boldsymbol{\mu}\right)} \quad (4.75)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w_t}{\sum_{s=1}^{T} w_s} \mathbf{x}_t \quad (4.81)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} w_t \left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)' . \quad (4.82)$$

$$w_t \equiv w\left(\mathrm{Ma}^2\left(\mathbf{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.80)$$

$$w\left(z\right) \equiv -2\frac{g'\left(z\right)}{g\left(z\right)} \quad (4.79)$$

$$g^{\mathrm{N}}\left(z\right) \equiv \frac{e^{-\frac{z}{2}}}{\left(2\pi\right)^{\frac{N}{2}}}, \quad (4.96)$$

$$w\left(z\right) \equiv 1 \quad (4.97)$$

$$g^{\mathrm{Ca}}\left(z\right) = \frac{\Gamma\left(\frac{1+N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\pi\right)^{\frac{N}{2}}}\left(1+z\right)^{-\frac{1+N}{2}} \quad (4.83)$$

$$w_t = \frac{N+1}{1+\mathrm{Ma}^2\left(\mathbf{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)} \quad (4.84)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOCATION)

Risk and Asset Allocation **- Springer –** `symmys.com`

normal assumption $$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.95)$$

estimator definition $$\widehat{\boldsymbol{\mu}}[i_T] = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t . \quad (4.98)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOCATION)

Risk and Asset Allocation - Springer – `symmys.com`

normal assumption

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.95)$$

estimator definition

$$\widehat{\boldsymbol{\mu}}[i_T] = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t. \quad (4.98)$$

estimator distribution
(replicability)

$$\widehat{\boldsymbol{\mu}}[I_T] \sim \mathrm{N}\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{T}\right) \quad (4.102)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOCATION)

Risk and Asset Allocation **- Springer –** `symmys.com`

normal assumption $$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.95)$$

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estimator distribution (replicability) $$\widehat{\boldsymbol{\mu}}[I_T] \sim \mathrm{N}\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{T}\right) \quad (4.102)$$

Generalized t-tests…

Generalized p-values…

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOCATION)

***Risk and Asset Allocation* - Springer – `symmys.com`**

normal assumption
$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.95)$$

estimator definition
$$\widehat{\boldsymbol{\mu}}[i_T] = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t. \quad (4.98)$$

estimator distribution (replicability)
$$\widehat{\boldsymbol{\mu}}[I_T] \sim \mathrm{N}\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{T}\right) \quad (4.102)$$

estimator evaluation (global)
$$\mathrm{Loss}(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) \equiv [\widehat{\boldsymbol{\mu}}[I_T] - \boldsymbol{\mu}]' [\widehat{\boldsymbol{\mu}}[I_T] - \boldsymbol{\mu}] \quad (4.108)$$

estimator evaluation (summary)
$$\begin{cases} \mathrm{Err}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = \dfrac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma}). & (4.109) \\ \mathrm{Inef}^2(\widehat{\boldsymbol{\mu}}) = \dfrac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma}) & (4.110) \\ \mathrm{Bias}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = 0. & (4.111) \end{cases}$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOCATION)

***Risk and Asset Allocation* - Springer – `symmys.com`**

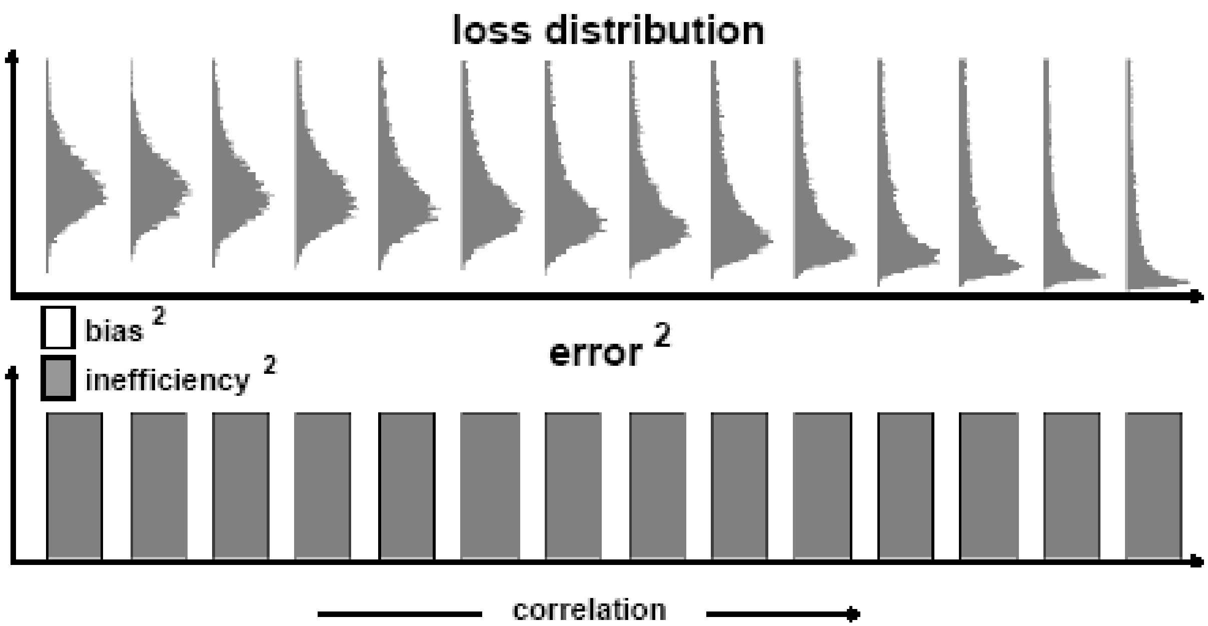

Fig. 4.11.

estimator evaluation (global)

$$\operatorname{Loss}(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) \equiv [\widehat{\boldsymbol{\mu}}[I_T] - \boldsymbol{\mu}]' [\widehat{\boldsymbol{\mu}}[I_T] - \boldsymbol{\mu}] \quad (4.108)$$

estimator evaluation (summary)

$$\operatorname{Err}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = \frac{1}{T} \operatorname{tr}(\boldsymbol{\Sigma}) . \quad (4.109)$$

$$\operatorname{Inef}^2(\widehat{\boldsymbol{\mu}}) = \frac{1}{T} \operatorname{tr}(\boldsymbol{\Sigma}) \quad (4.110)$$

$$\operatorname{Bias}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = 0. \quad (4.111)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (SCATTER)

***Risk and Asset Allocation* - Springer – `symmys.com`**

normal assumption $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (4.95)

estimator definition $\widehat{\boldsymbol{\Sigma}}[i_T] = \frac{1}{T} \sum_{t=1}^{T} (\mathbf{x}_t - \widehat{\boldsymbol{\mu}})(\mathbf{x}_t - \widehat{\boldsymbol{\mu}})'$ (4.99)

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (SCATTER)

Risk and Asset Allocation - Springer – `symmys.com`

normal assumption
$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (4.95)$$

estimator definition
$$\widehat{\boldsymbol{\Sigma}}\left[i_T\right] = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)' \quad (4.99)$$

estimator distribution
(replicability)
$$T\widehat{\boldsymbol{\Sigma}}\left[I_T\right] \sim \mathrm{W}\left(T-1, \boldsymbol{\Sigma}\right) \quad (4.103)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (SCATTER)

Risk and Asset Allocation **- Springer –** `symmys.com`

normal assumption $$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (4.95)$$

estimator definition $$\widehat{\boldsymbol{\Sigma}}\left[i_T\right] = \frac{1}{T} \sum_{t=1}^{T} \left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\right)' \quad (4.99)$$

estimator distribution (replicability) $$T\widehat{\boldsymbol{\Sigma}}\left[I_T\right] \sim \mathrm{W}\left(T-1, \boldsymbol{\Sigma}\right) \quad (4.103)$$

Generalized t-tests…

Generalized p-values…

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (SCATTER)

Risk and Asset Allocation **- Springer –** `symmys.com`

normal assumption $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (4.95)

estimator definition $\widehat{\boldsymbol{\Sigma}}[i_T] = \frac{1}{T} \sum_{t=1}^{T} (\mathbf{x}_t - \widehat{\boldsymbol{\mu}})(\mathbf{x}_t - \widehat{\boldsymbol{\mu}})'$ (4.99)

estimator distribution (replicability) $T\widehat{\boldsymbol{\Sigma}}[I_T] \sim \mathrm{W}(T-1, \boldsymbol{\Sigma})$ (4.103)

estimator evaluation (global) $\mathrm{Loss}(\widehat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) \equiv \mathrm{tr}\left[\left(\widehat{\boldsymbol{\Sigma}}[I_T] - \boldsymbol{\Sigma}\right)^2\right]$ (4.118)

estimator evaluation (summary)

$$\mathrm{Err}^2(\widehat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) = \frac{1}{T}\left[\mathrm{tr}(\boldsymbol{\Sigma}^2) + \left(1 - \frac{1}{T}\right)[\mathrm{tr}(\boldsymbol{\Sigma})]^2\right] \quad (4.119)$$

$$\mathrm{Inef}^2(\widehat{\boldsymbol{\Sigma}}) = \frac{1}{T}\left(1 - \frac{1}{T}\right)\left[\mathrm{tr}(\boldsymbol{\Sigma}^2) + [\mathrm{tr}(\boldsymbol{\Sigma})]^2\right] \quad (4.120)$$

$$\mathrm{Bias}^2(\widehat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) = \frac{1}{T^2}\mathrm{tr}(\boldsymbol{\Sigma}^2) \quad (4.121)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (SCATTER)

Risk and Asset Allocation **- Springer –** `symmys.com`

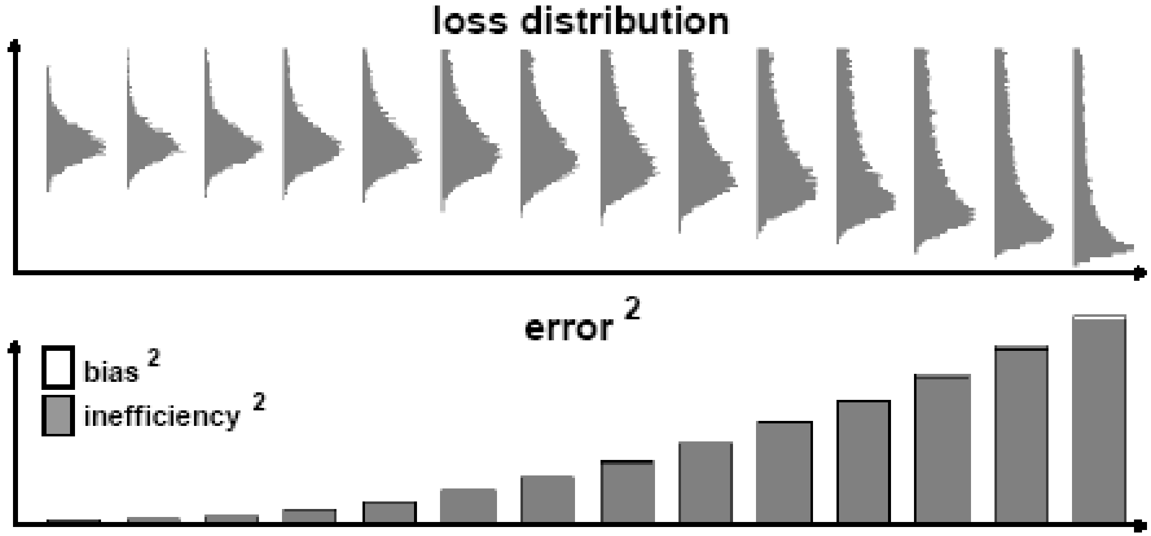

Fig. 4.12

estimator evaluation (global)

$$\text{Loss}\left(\widehat{\Sigma}, \Sigma\right) \equiv \text{tr}\left[\left(\widehat{\Sigma}\left[I_T\right] - \Sigma\right)^2\right] \qquad (4.118)$$

estimator evaluation (summary)

$$\text{Err}^2\left(\widehat{\Sigma}, \Sigma\right) = \frac{1}{T}\left[\text{tr}\left(\Sigma^2\right) + \left(1 - \frac{1}{T}\right)\left[\text{tr}\left(\Sigma\right)\right]^2\right] \qquad (4.119)$$

$$\text{Inef}^2\left(\widehat{\Sigma}\right) = \frac{1}{T}\left(1 - \frac{1}{T}\right)\left[\text{tr}\left(\Sigma^2\right) + \left[\text{tr}\left(\Sigma\right)\right]^2\right] \qquad (4.120)$$

$$\text{Bias}^2\left(\widehat{\Sigma}, \Sigma\right) = \frac{1}{T^2}\,\text{tr}\left(\Sigma^2\right) \qquad (4.121)$$

MAXIMUM LIKELIHOOD ESTIMATORS – NORMAL (LOADINGS)

***Risk and Asset Allocation* - Springer – `symmys.com`**

normal assumption

$$\mathbf{X}|\mathbf{f} = \mathbf{B}\mathbf{f} + \mathbf{U}|\mathbf{f} \quad (4.88)$$

$$\mathbf{U}_t|\mathbf{f}_t \sim \mathrm{N}\left(\mathbf{0}, \mathbf{\Sigma}\right) \quad (4.123)$$

estimator definition

$$\widehat{\mathbf{B}}\left[i_T\right] = \widehat{\mathbf{\Sigma}}_{XF}\left[i_T\right]\widehat{\mathbf{\Sigma}}_F^{-1}\left[i_T\right] \quad (4.126)$$

$$\widehat{\mathbf{\Sigma}}_{XF}\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{f}_t', \quad \widehat{\mathbf{\Sigma}}_F\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{f}_t\mathbf{f}_t'. \quad (4.127)$$

$$\widehat{\mathbf{\Sigma}}\left[i_T\right] = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}_t - \widehat{\mathbf{B}}\left[i_T\right]\mathbf{f}_t\right)\left(\mathbf{x}_t - \widehat{\mathbf{B}}\left[i_T\right]\mathbf{f}_t\right)' \quad (4.128)$$

estimator distribution
(replicability)

$$\widehat{\mathbf{B}}\left[I_T|\mathbf{f}_1,\ldots,\mathbf{f}_T\right] \sim \mathrm{N}\left(\mathbf{B}, \frac{\mathbf{\Sigma}}{T}, \widehat{\mathbf{\Sigma}}_F^{-1}\right) \quad (4.129)$$

$$T\widehat{\mathbf{\Sigma}}\left[I_T|\mathbf{f}_1,\ldots,\mathbf{f}_T\right] \sim \mathrm{W}\left(T-K, \mathbf{\Sigma}\right) \quad (4.130)$$

Generalized t-tests, generalized p-values, estimator evaluation

SHRINKAGE ESTIMATORS – LOCATION PARAMETER

***Risk and Asset Allocation* - Springer – `symmys.com`**

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

SHRINKAGE ESTIMATORS – LOCATION PARAMETER

Risk and Asset Allocation **- Springer –** `symmys.com`

$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \qquad (4.132)$$

$$\widehat{\boldsymbol{\mu}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t. \qquad (4.135)$$

$$\widehat{\boldsymbol{\mu}}[I_T] \sim \mathrm{N}\left(\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{T}\right) \qquad (4.102)$$

$$\mathrm{Err}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = \frac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma}) \qquad (4.136)$$

SHRINKAGE ESTIMATORS – LOCATION PARAMETER

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.132)$$

$$\widehat{\boldsymbol{\mu}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t. \quad (4.135)$$

$$\boxed{\mathrm{Err}^2\left(\widehat{\boldsymbol{\mu}}^S, \boldsymbol{\mu}\right) < \frac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma})} \quad (4.137)$$

?

$$\mathrm{Err}^2(\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = \frac{1}{T}\mathrm{tr}(\boldsymbol{\Sigma}) \quad (4.136)$$

SHRINKAGE ESTIMATORS – LOCATION PARAMETER

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \qquad (4.132)$$

$$\widehat{\boldsymbol{\mu}}[i_T] \equiv \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t. \qquad (4.135)$$

$$\mathrm{Err}^2\left(\widehat{\boldsymbol{\mu}}^S, \boldsymbol{\mu}\right) < \frac{1}{T} \mathrm{tr}(\boldsymbol{\Sigma}) \qquad (4.137)$$

$$\boxed{\widehat{\boldsymbol{\mu}}^S \equiv (1-\alpha)\, \widehat{\boldsymbol{\mu}} + \alpha \mathbf{b}.} \qquad (4.138)$$

$$\alpha \equiv \frac{1}{T} \frac{N\overline{\lambda} - 2\lambda_1}{(\widehat{\boldsymbol{\mu}} - \mathbf{b})' (\widehat{\boldsymbol{\mu}} - \mathbf{b})} \qquad (4.139)$$

SHRINKAGE ESTIMATORS – LOCATION PARAMETER

***Risk and Asset Allocation* - Springer – `symmys.com`**

$$\mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \qquad (4.132)$$

$$\widehat{\boldsymbol{\mu}}\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t. \qquad (4.135)$$

$$\mathrm{Err}^2\left(\widehat{\boldsymbol{\mu}}^S, \boldsymbol{\mu}\right) < \frac{1}{T}\,\mathrm{tr}\left(\boldsymbol{\Sigma}\right) \qquad (4.137)$$

$$\boxed{\widehat{\boldsymbol{\mu}}^S \equiv \left(1-\alpha\right)\widehat{\boldsymbol{\mu}} + \alpha\mathbf{b}.} \qquad (4.138)$$

$$\alpha \equiv \frac{1}{T}\frac{N\overline{\lambda} - 2\lambda_1}{\left(\widehat{\boldsymbol{\mu}}-\mathbf{b}\right)'\left(\widehat{\boldsymbol{\mu}}-\mathbf{b}\right)} \qquad (4.139)$$

$$\boldsymbol{\Sigma} \mapsto \widehat{\boldsymbol{\Sigma}}.$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\mu}}}{N}\mathbf{1} \qquad (4.141)$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\mu}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\mathbf{1}. \qquad (4.142)$$

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$$\mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \qquad (4.132)$$

$$\widehat{\boldsymbol{\mu}}\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t. \qquad (4.135)$$

$$\mathrm{Err}^2\left(\widehat{\boldsymbol{\mu}}^S, \boldsymbol{\mu}\right) < \frac{1}{T}\mathrm{tr}\left(\boldsymbol{\Sigma}\right) \qquad (4.137)$$

$$\boxed{\widehat{\boldsymbol{\mu}}^S \equiv (1-\alpha)\,\widehat{\boldsymbol{\mu}} + \alpha\mathbf{b}.} \qquad (4.138)$$

$$\alpha \equiv \frac{1}{T}\frac{N\overline{\lambda} - 2\lambda_1}{\left(\widehat{\boldsymbol{\mu}}-\mathbf{b}\right)'\left(\widehat{\boldsymbol{\mu}}-\mathbf{b}\right)} \qquad (4.139)$$

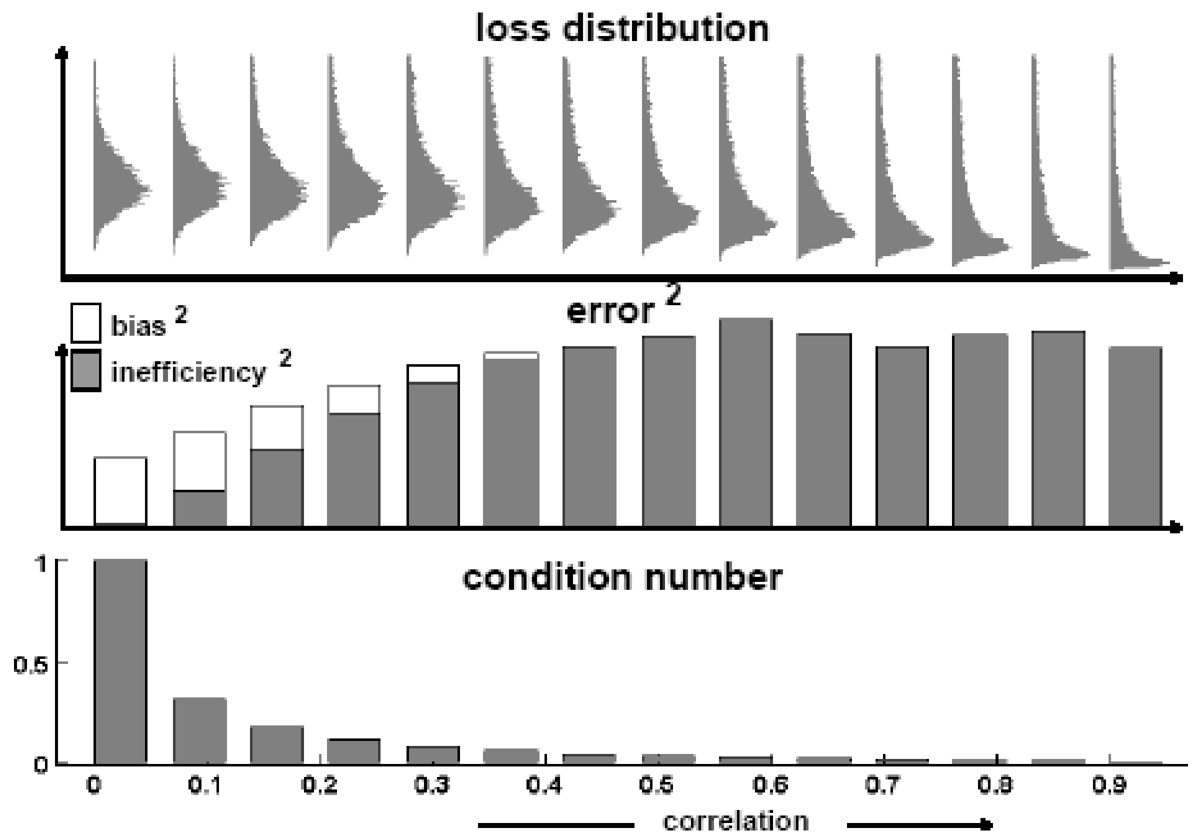

Fig. 4.13. Shrinkage estimator of mean: evaluation

$$\boldsymbol{\Sigma} \mapsto \widehat{\boldsymbol{\Sigma}}.$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\mu}}}{N}\mathbf{1} \qquad (4.141)$$

$$\mathbf{b} \mapsto \frac{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\mu}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\mathbf{1}. \qquad (4.142)$$

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$$\mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (4.143)$$

$$\widehat{\boldsymbol{\Sigma}}\left[i_T\right] \equiv \frac{1}{T} \sum_{t=1}^{T} \left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\left[i_T\right]\right]\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\left[i_T\right]\right]' . \quad (4.146)$$

$$T\widehat{\boldsymbol{\Sigma}}\left[I_T\right] \sim \mathrm{W}\left(T-1, \boldsymbol{\Sigma}\right) \quad (4.103)$$

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$$\mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (4.143)$$

$$\widehat{\boldsymbol{\Sigma}}\left[i_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\left[i_T\right]\right]\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}\left[i_T\right]\right]' . \quad (4.146)$$

$$\widehat{\mathrm{CN}}\left\{\mathbf{X}\right\} \equiv \frac{\widehat{\lambda}_N}{\widehat{\lambda}_1} < \frac{\lambda_N}{\lambda_1} \equiv \mathrm{CN}\left\{\mathbf{X}\right\} \quad (4.156)$$

$$T\widehat{\boldsymbol{\Sigma}}\left[I_T\right] \sim \mathrm{W}\left(T-1, \boldsymbol{\Sigma}\right) \quad (4.103)$$

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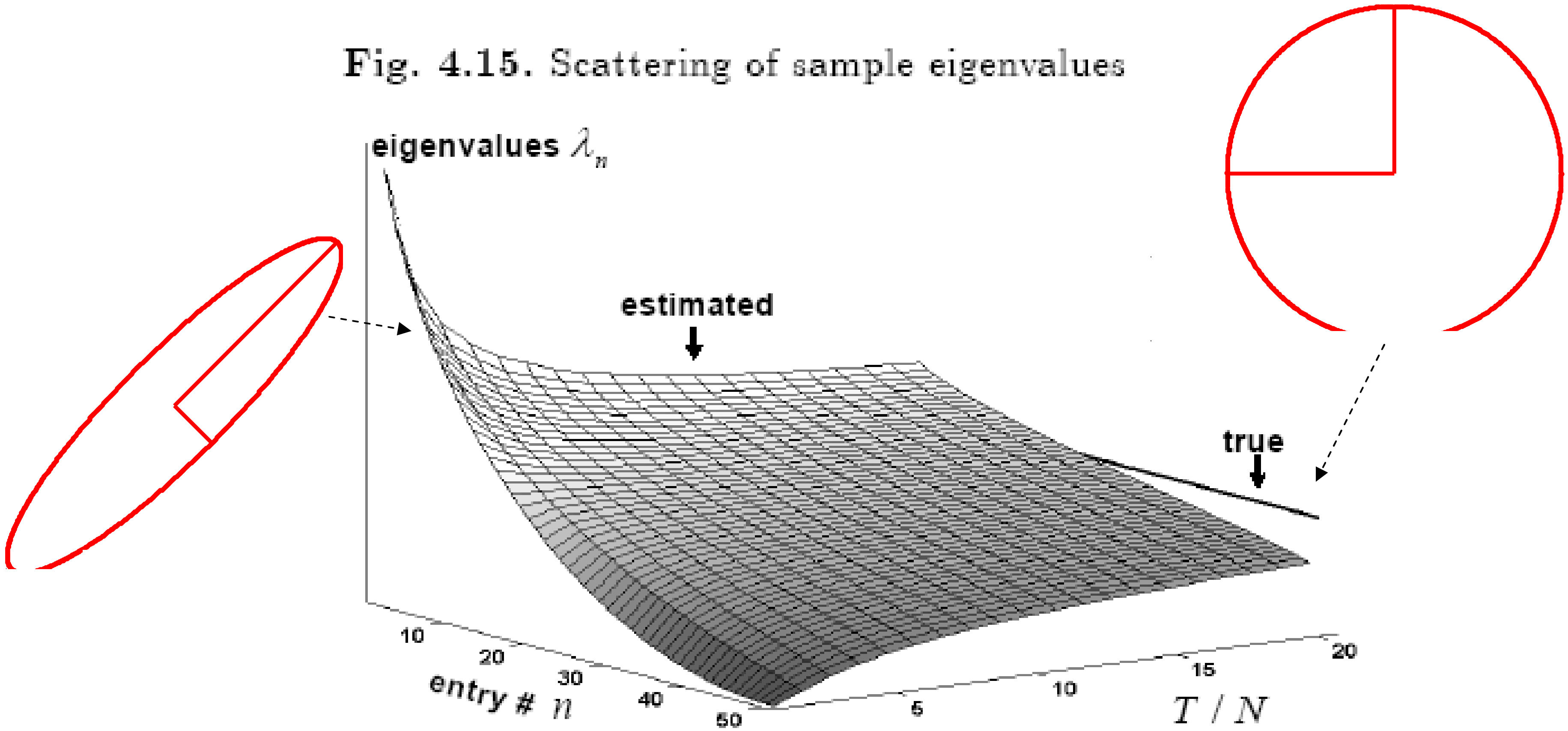
$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

$$\widehat{\boldsymbol{\Sigma}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} [\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]][\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]]' . \quad (4.146)$$

$$\widehat{\mathrm{CN}}\{\mathbf{X}\} \equiv \frac{\widehat{\lambda}_N}{\widehat{\lambda}_1} < \frac{\lambda_N}{\lambda_1} \equiv \mathrm{CN}\{\mathbf{X}\} \quad (4.156)$$

Fig. 4.15. Scattering of sample eigenvalues



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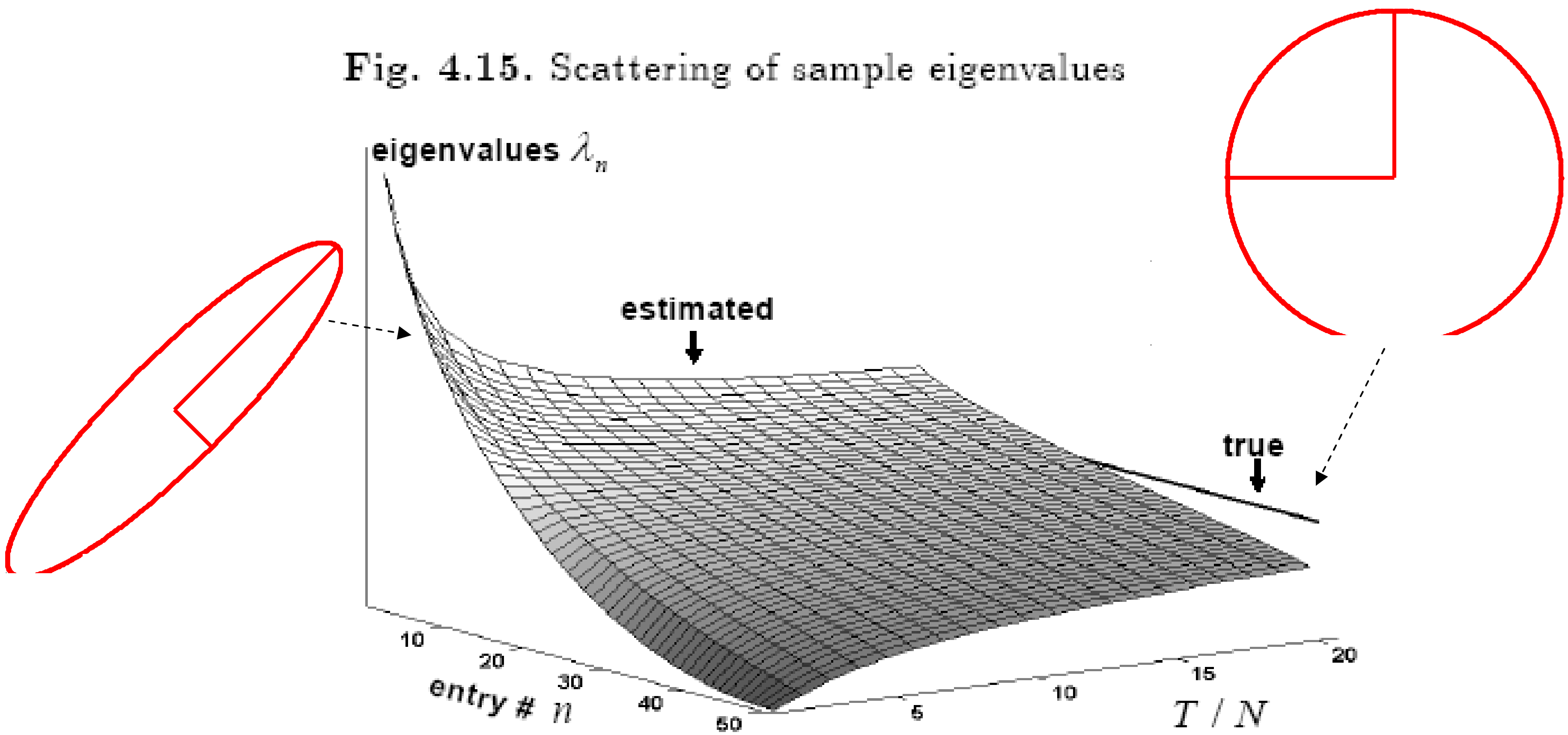
$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

$$\widehat{\boldsymbol{\Sigma}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]\right]\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]\right]'. \quad (4.146)$$

$$\boxed{\widehat{\boldsymbol{\Sigma}}^S \equiv (1-\alpha)\,\widehat{\boldsymbol{\Sigma}} + \alpha\widehat{\mathbf{C}}.} \quad (4.160)$$

$$\widehat{\mathbf{C}} \equiv \frac{\sum_{n=1}^{N}\widehat{\lambda}_n}{N}\mathbf{I}. \quad (4.159)$$

Fig. 4.15. Scattering of sample eigenvalues



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$$\mathbf{X}_t \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4.143)$$

$$\boxed{\widehat{\boldsymbol{\Sigma}}^S \equiv (1-\alpha)\,\widehat{\boldsymbol{\Sigma}} + \alpha \widehat{\mathbf{C}}} \quad (4.160)$$

$$\alpha \equiv \frac{1}{T} \frac{\frac{1}{T}\sum_{t=1}^{T} \mathrm{tr}\left\{\left(\mathbf{x}_t \mathbf{x}_t' - \widehat{\boldsymbol{\Sigma}}\right)^2\right\}}{\mathrm{tr}\left\{\left(\widehat{\boldsymbol{\Sigma}} - \widehat{\mathbf{C}}\right)^2\right\}} \quad (4.161)$$

$$\widehat{\boldsymbol{\Sigma}}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]\right]\left[\mathbf{x}_t - \widehat{\boldsymbol{\mu}}[i_T]\right]' . \quad (4.146)$$

$$\widehat{\mathbf{C}} \equiv \frac{\sum_{n=1}^{N} \widehat{\lambda}_n}{N}\mathbf{I}. \quad (4.159)$$

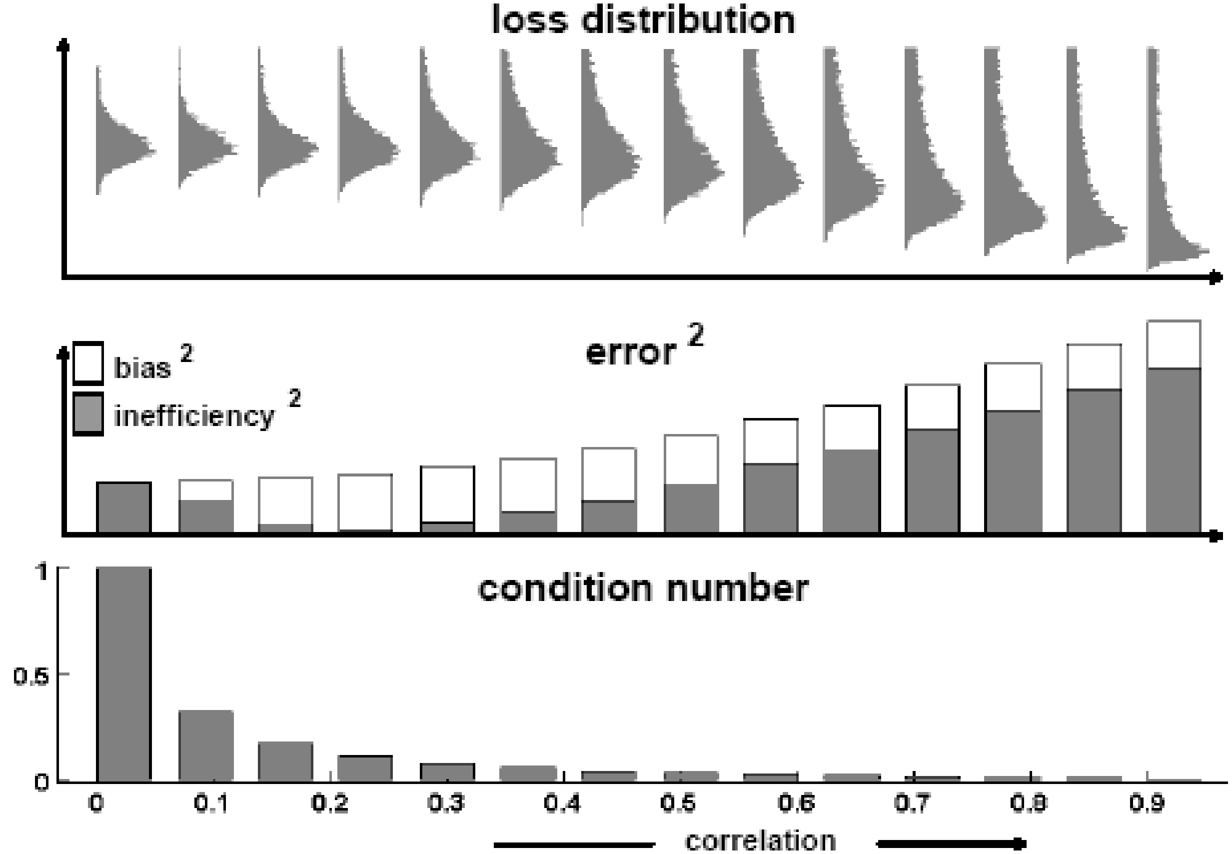

Fig. 4.16. Shrinkage estimator of covariance: evaluation

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Attilio Meucci

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Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

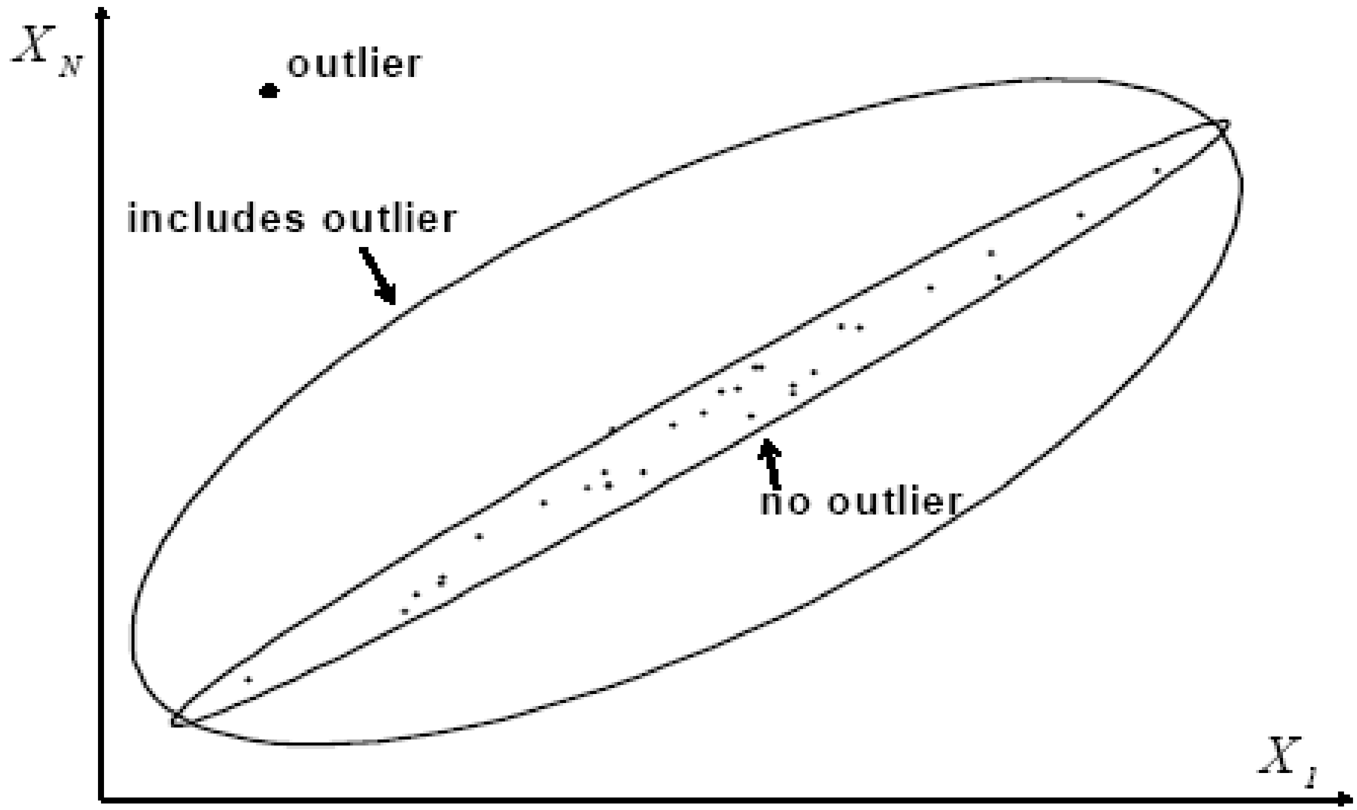

Fig. 4.18. Sample estimators: lack of robustness

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$$\boxed{\mathrm{SC}\left(\mathbf{x},\widehat{\mathbf{G}}\right)\equiv T\widehat{\mathbf{G}}\left(\mathbf{x}_1,\ldots,\mathbf{x}_T,\mathbf{x}\right)-T\widehat{\mathbf{G}}\left(\mathbf{x}_1,\ldots,\mathbf{x}_T\right)} \qquad (4.166)$$

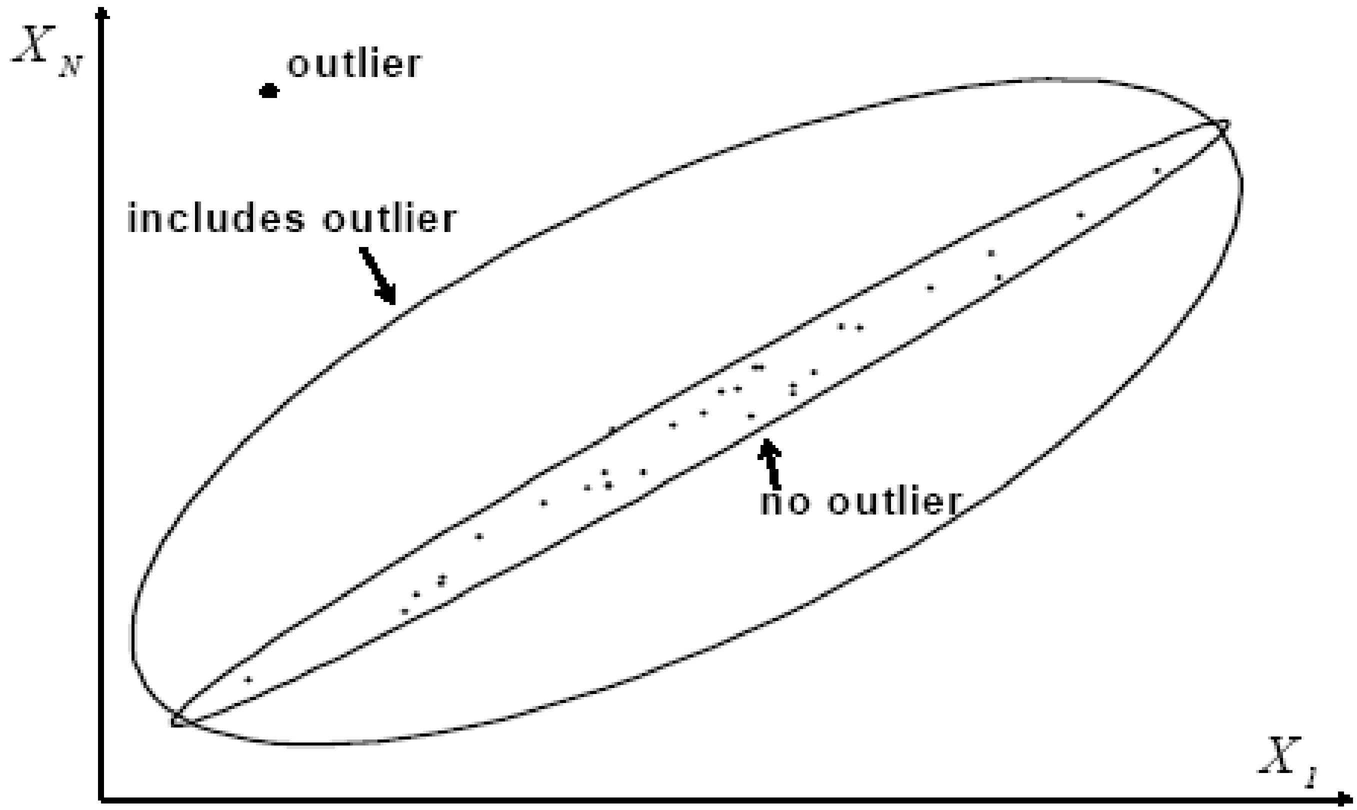

Fig. 4.18. Sample estimators: lack of robustness

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- $$\boxed{\mathrm{SC}\left(\mathbf{x}, \widehat{\mathbf{G}}\right) \equiv T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T, \mathbf{x}\right) - T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T\right)} \qquad (4.166)$$

$$f_{i_T} \mapsto (1-\epsilon)\, f_{i_T} + \epsilon\delta^{(\mathbf{x})}. \qquad (4.183)$$

$$f_{i_T} \equiv \frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)} \qquad \epsilon \equiv 1/\left(T+1\right)$$

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- $$\boxed{\mathrm{SC}\left(\mathbf{x}, \widehat{\mathbf{G}}\right) \equiv T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T, \mathbf{x}\right) - T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T\right)} \quad (4.166)$$

$$f_{i_T} \mapsto (1-\epsilon) f_{i_T} + \epsilon \delta^{(\mathbf{x})}. \quad (4.183)$$

$$f_{i_T} \equiv \frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)} \qquad \epsilon \equiv 1/(T+1)$$

- $$\boxed{\widehat{\mathbf{G}} \equiv \widetilde{\mathbf{G}}\left[f_{i_T}\right]} \quad (4.167)$$

- **non-parametric**

$$\boxed{\widehat{\mathbf{G}} \equiv \mathbf{G}\left[f_{i_T}\right]} \quad (4.169)$$

- **maximum likelihood**

$$\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}) \equiv \frac{\partial}{\partial \boldsymbol{\theta}} \ln\left(f_{\boldsymbol{\theta}}(\mathbf{x})\right) \quad (4.176)$$

$$\widetilde{\boldsymbol{\theta}}[h]: \quad \int_{\mathbb{R}^N} \boldsymbol{\psi}\left(\mathbf{x}, \widetilde{\boldsymbol{\theta}}\right) h(\mathbf{x})\, d\mathbf{x} \equiv \mathbf{0}. \quad (4.175)$$

$$\boxed{\widehat{\boldsymbol{\theta}} \equiv \widetilde{\boldsymbol{\theta}}\left[f_{i_T}\right]} \quad (4.177)$$

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- $$\boxed{\mathrm{SC}\left(\mathbf{x},\widehat{\mathbf{G}}\right)\equiv T\widehat{\mathbf{G}}\left(\mathbf{x}_1,\ldots,\mathbf{x}_T,\mathbf{x}\right)-T\widehat{\mathbf{G}}\left(\mathbf{x}_1,\ldots,\mathbf{x}_T\right)} \quad (4.166)$$

$$f_{i_T}\mapsto\left(1-\epsilon\right)f_{i_T}+\epsilon\delta^{(\mathbf{x})}. \quad (4.183)$$

$$f_{i_T}\equiv\frac{1}{T}\sum_{t=1}^{T}\delta^{(\mathbf{x}_t)} \qquad \epsilon\equiv 1/\left(T+1\right)$$

- $$\boxed{\widehat{\mathbf{G}}\equiv\widetilde{\mathbf{G}}\left[f_{i_T}\right]} \quad (4.167)$$

$$\mathrm{SC}\left(\mathbf{x},\widehat{\mathbf{G}}\right)\equiv\frac{1-\epsilon}{\epsilon}\left\{\widetilde{\mathbf{G}}\left[\left(1-\epsilon\right)f_{i_T}+\epsilon\delta^{(\mathbf{x})}\right]-\widetilde{\mathbf{G}}\left[f_{i_T}\right]\right\} \quad (4.184)$$

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- $$\boxed{\mathrm{SC}\left(\mathbf{x}, \widehat{\mathbf{G}}\right) \equiv T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T, \mathbf{x}\right) - T\widehat{\mathbf{G}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_T\right)} \qquad (4.166)$$

$$f_{i_T} \mapsto (1-\epsilon)\, f_{i_T} + \epsilon \delta^{(\mathbf{x})}: \qquad (4.183)$$

$$f_{i_T} \equiv \frac{1}{T}\sum_{t=1}^{T} \delta^{(\mathbf{x}_t)} \qquad \epsilon \equiv 1/(T+1)$$

- $$\boxed{\widehat{\mathbf{G}} \equiv \widetilde{\mathbf{G}}\left[f_{i_T}\right]} \qquad (4.167)$$

$$\mathrm{SC}\left(\mathbf{x}, \widehat{\mathbf{G}}\right) \equiv \frac{1-\epsilon}{\epsilon}\left\{\widetilde{\mathbf{G}}\left[(1-\epsilon)\, f_{i_T} + \epsilon\delta^{(\mathbf{x})}\right] - \widetilde{\mathbf{G}}\left[f_{i_T}\right]\right\} \qquad (4.184)$$

$$\boxed{\mathrm{IF}\left(\mathbf{x}, f_{\mathbf{X}}, \widehat{\mathbf{G}}\right) \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\widetilde{\mathbf{G}}\left[(1-\epsilon)\, f_{\mathbf{X}} + \epsilon\delta^{(\mathbf{x})}\right] - \widetilde{\mathbf{G}}\left[f_{\mathbf{X}}\right]\right)} \qquad (4.185)$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T} \sum_{t=1}^{T} \mathrm{x}_t \quad (4.196)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T} \sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right) \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\} \quad (4.198)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = \left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right) \left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right)' - \mathrm{Cov}\left\{\mathbf{X}\right\} \quad (4.199)$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t \quad (4.196)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)\left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\} \quad (4.198)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = \left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right)\left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right)' - \mathrm{Cov}\left\{\mathbf{X}\right\} \quad (4.199)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)}{\sum_{s=1}^{T} w\left(\mathrm{Ma}^2\left(\mathrm{x}_s, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)} \mathrm{x}_t \quad (4.203)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)' w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.204)$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t \quad (4.196)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)\left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathrm{x} - \mathrm{E}\{\mathbf{X}\} \quad (4.198)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = \left(\mathrm{x} - \mathrm{E}\{\mathbf{X}\}\right)\left(\mathrm{x} - \mathrm{E}\{\mathbf{X}\}\right)' - \mathrm{Cov}\{\mathbf{X}\} \quad (4.199)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)}{\sum_{s=1}^{T} w\left(\mathrm{Ma}^2\left(\mathrm{x}_s, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)} \mathrm{x}_t \quad (4.203)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)' w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.204)$$

$$\boxed{w(z) \equiv -2\frac{g'(z)}{g(z)}.} \quad (4.205)$$

$$f_{\boldsymbol{\theta}}(\mathrm{x}) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g\left(\mathrm{Ma}^2(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})\right) \quad (4.201)$$

$$\mathrm{Ma}^2(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) \equiv (\mathrm{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathrm{x} - \boldsymbol{\mu}) \quad (4.202)$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t \quad (4.196)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)\left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\} \quad (4.198)$$

$$\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = \left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right)\left(\mathrm{x} - \mathrm{E}\left\{\mathbf{X}\right\}\right)' - \mathrm{Cov}\left\{\mathbf{X}\right\} \quad (4.199)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)}{\sum_{s=1}^{T} w\left(\mathrm{Ma}^2\left(\mathrm{x}_s, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)} \mathrm{x}_t \quad (4.203)$$

$$\boxed{\left\| \mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \left(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)\right\| \propto \left\|\boldsymbol{\psi}\right\|} \quad (4.208)$$

$$\boldsymbol{\psi}\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \begin{pmatrix} w\left(\mathrm{Ma}^2\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right)\left(\mathrm{x} - \boldsymbol{\mu}\right) \\ w\left(\mathrm{Ma}^2_{\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}}\right) \mathrm{vec}\left[\left(\mathrm{x} - \boldsymbol{\mu}\right)\left(\mathrm{x} - \boldsymbol{\mu}\right)'\right] - \mathrm{vec}\left[\boldsymbol{\Sigma}\right] \end{pmatrix} \quad (4.207)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)' w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.204)$$

$$\boxed{w\left(z\right) \equiv -2\frac{g'\left(z\right)}{g\left(z\right)}.} \quad (4.205)$$

$$f_{\boldsymbol{\theta}}\left(\mathrm{x}\right) \equiv \frac{1}{\sqrt{\left|\boldsymbol{\Sigma}\right|}} g\left(\mathrm{Ma}^2\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right) \quad (4.201)$$

$$\mathrm{Ma}^2\left(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \equiv \left(\mathrm{x} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1}\left(\mathrm{x} - \boldsymbol{\mu}\right) \quad (4.202)$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t \quad (4.196)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)\left(\mathrm{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\boxed{\begin{aligned} &\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathrm{x} - \mathrm{E}\{\mathbf{X}\} \quad (4.198) \\ &\mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = \left(\mathrm{x} - \mathrm{E}\{\mathbf{X}\}\right)\left(\mathrm{x} - \mathrm{E}\{\mathbf{X}\}\right)' - \mathrm{Cov}\{\mathbf{X}\} \quad (4.199) \end{aligned}}$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)}{\sum_{s=1}^{T} w\left(\mathrm{Ma}^2\left(\mathrm{x}_s, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)} \mathrm{x}_t \quad (4.203)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)\left(\mathrm{x}_t - \widehat{\boldsymbol{\mu}}\right)' w\left(\mathrm{Ma}^2\left(\mathrm{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.204)$$

$$\boxed{\left\| \mathrm{IF}\left(\mathrm{x}, f_{\mathbf{X}}, \left(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)\right\| \propto \|\boldsymbol{\psi}\|} \quad (4.208)$$

$$\boldsymbol{\psi}(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) \equiv \boxed{\begin{pmatrix} w\left(\mathrm{Ma}^2(\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})\right)(\mathrm{x} - \boldsymbol{\mu}) \\ w\left(\mathrm{Ma}^2_{\mathrm{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}}\right) \mathrm{vec}\left[(\mathrm{x} - \boldsymbol{\mu})(\mathrm{x} - \boldsymbol{\mu})'\right] - \mathrm{vec}\left[\boldsymbol{\Sigma}\right] \end{pmatrix}} \quad (4.207)$$

$$w(z) \equiv -2\frac{g'(z)}{g(z)}. \quad (4.205)$$

$$\boxed{\begin{array}{c} \text{normal} \\ w \equiv 1 \end{array}}$$

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$$\widehat{\mathrm{E}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t \quad (4.196)$$

$$\mathrm{IF}\left(\mathbf{x}, f_{\mathbf{X}}, \widehat{\mathrm{E}}\right) = \mathbf{x} - \mathrm{E}\{\mathbf{X}\} \quad (4.198)$$

$$\widehat{\mathrm{Cov}} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{x}_t - \widehat{\mathrm{E}}\right)\left(\mathbf{x}_t - \widehat{\mathrm{E}}\right)' \quad (4.197)$$

$$\mathrm{IF}\left(\mathbf{x}, f_{\mathbf{X}}, \widehat{\mathrm{Cov}}\right) = (\mathbf{x} - \mathrm{E}\{\mathbf{X}\})(\mathbf{x} - \mathrm{E}\{\mathbf{X}\})' - \mathrm{Cov}\{\mathbf{X}\} \quad (4.199)$$

$$\widehat{\boldsymbol{\mu}} = \sum_{t=1}^{T} \frac{w\left(\mathrm{Ma}^2\left(\mathbf{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)}{\sum_{s=1}^{T} w\left(\mathrm{Ma}^2\left(\mathbf{x}_s, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)} \mathbf{x}_t \quad (4.203)$$

$$\boxed{\left|\mathrm{IF}\left(\mathbf{x}, f_{\mathbf{X}}, \left(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right)\right|} \propto \|\boldsymbol{\psi}\| \quad (4.208)$$

$$\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) \equiv \boxed{\begin{pmatrix} w\left(\mathrm{Ma}^2(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})\right)(\mathbf{x} - \boldsymbol{\mu}) \\ w\left(\mathrm{Ma}^2_{\mathbf{x},\boldsymbol{\mu},\boldsymbol{\Sigma}}\right) \mathrm{vec}\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\right] - \mathrm{vec}[\boldsymbol{\Sigma}] \end{pmatrix}} \quad (4.207)$$

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{x}_t - \widehat{\boldsymbol{\mu}})(\mathbf{x}_t - \widehat{\boldsymbol{\mu}})' \, w\left(\mathrm{Ma}^2\left(\mathbf{x}_t, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right)\right) \quad (4.204)$$

$$w(z) \equiv -2\frac{g'(z)}{g(z)}. \quad (4.205)$$

normal

$$w \equiv 1$$

$$\boxed{\begin{array}{l}\text{Cauchy} \\ w(z) = \dfrac{N+1}{1+z}\end{array}} \quad (4.209)$$

ROBUST ESTIMATORS – HIGH BREAKDOWN

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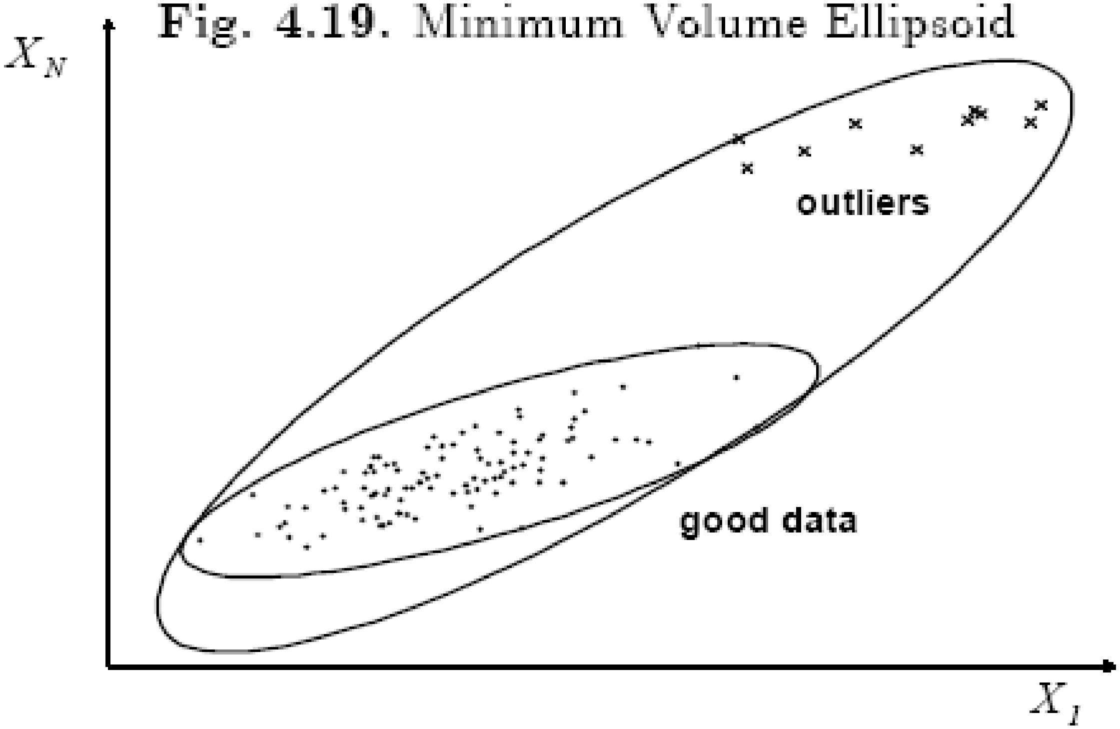
$$\mathcal{E}^{q}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \left\{ \mathrm{x} \in \mathbb{R}^{N} \text{ such that } \left(\mathrm{x} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\mathrm{x} - \boldsymbol{\mu}\right) \leq q^{2} \right\} \quad (4.231)$$

Fig. 4.19. Minimum Volume Ellipsoid



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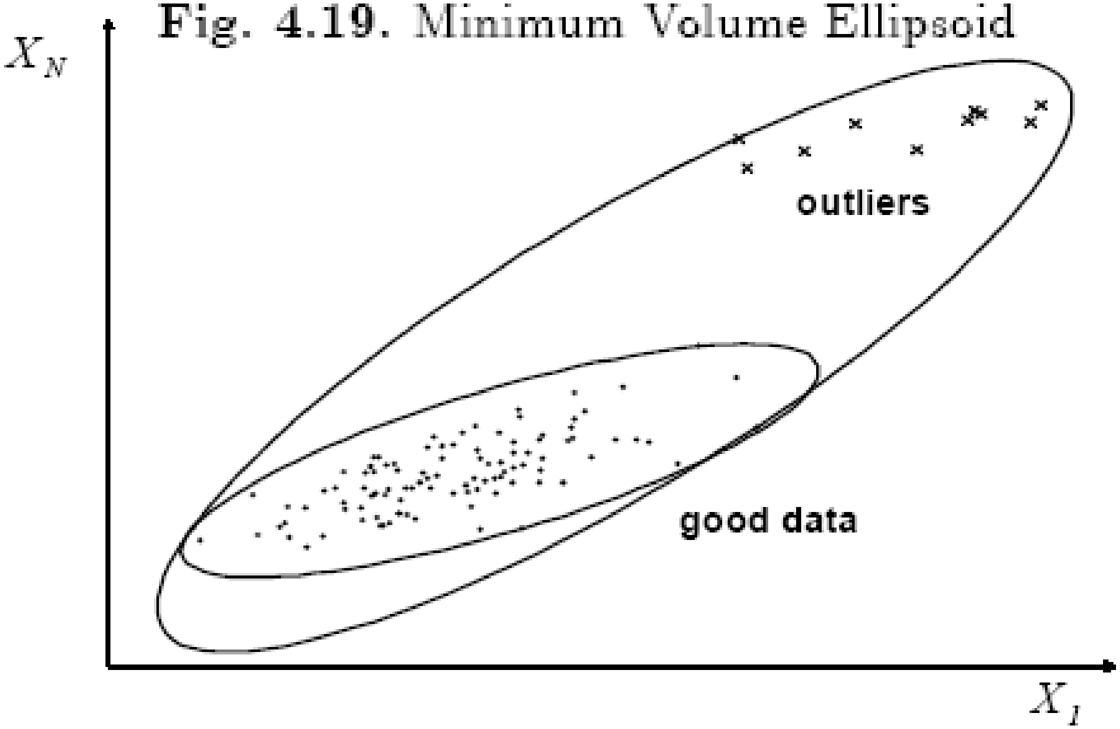
$$\mathcal{E}^{q}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \left\{ \mathrm{x} \in \mathbb{R}^N \text{ such that } (\mathrm{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathrm{x}-\boldsymbol{\mu}) \leq q^2 \right\} \quad (4.231)$$

$$\mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_t \equiv \sqrt{(\mathrm{x}_t-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathrm{x}_t-\boldsymbol{\mu})}. \quad (4.234)$$

$$\mathrm{Vol}\left\{ \mathcal{E}^{q_{T_G}}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \right\} = \gamma_N \left(\mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{T_G:T} \right)^N \sqrt{|\boldsymbol{\Sigma}|}. \quad (4.236)$$

$$q_{T_G} \equiv \mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{T_G:T} \quad (4.235)$$

Fig. 4.19. Minimum Volume Ellipsoid



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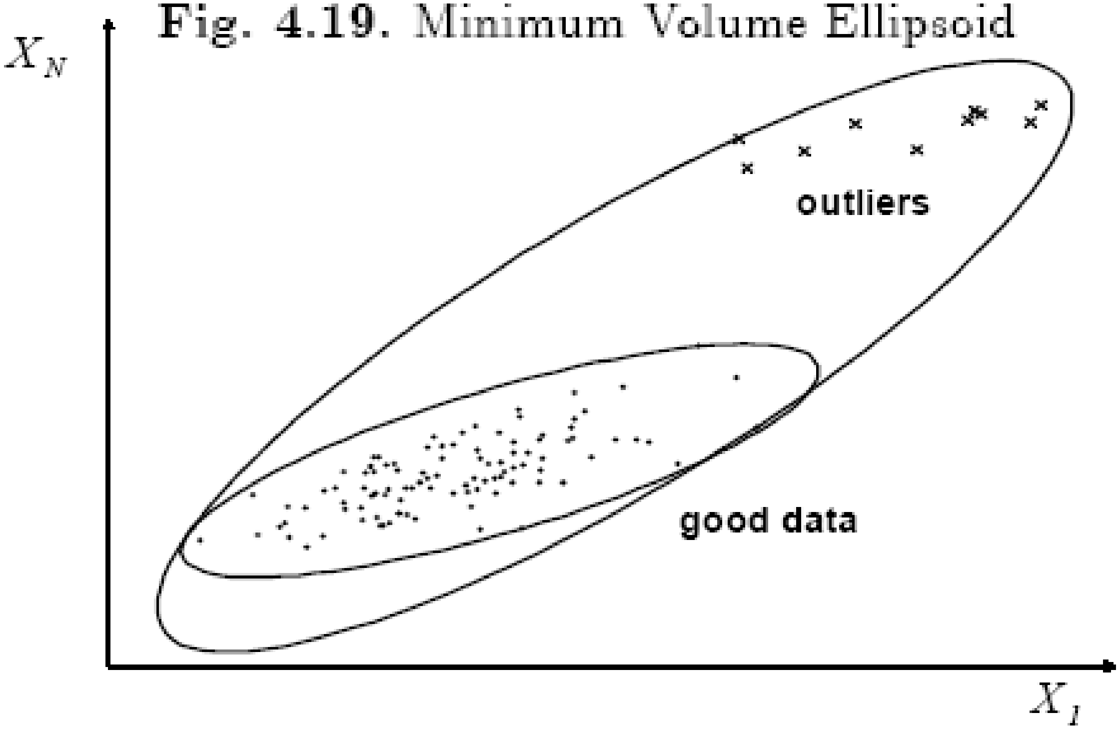
$$\mathcal{E}^{q}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \equiv \left\{ \mathrm{x} \in \mathbb{R}^N \text{ such that } (\mathrm{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathrm{x} - \boldsymbol{\mu}) \leq q^2 \right\} \quad (4.231)$$

$$\mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{t} \equiv \sqrt{(\mathrm{x}_t - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathrm{x}_t - \boldsymbol{\mu})}. \quad (4.234)$$

$$\mathrm{Vol}\left\{ \mathcal{E}^{q_{T_G}}_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \right\} = \gamma_N \left(\mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{T_G:T} \right)^N \sqrt{|\boldsymbol{\Sigma}|}. \quad (4.236)$$

$$q_{T_G} \equiv \mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{T_G:T} \quad (4.235)$$

Fig. 4.19. Minimum Volume Ellipsoid



$$\boxed{\left(\widehat{\boldsymbol{\mu}}_{T_G}, \widehat{\boldsymbol{\Sigma}}_{T_G} \right) = \underset{\boldsymbol{\mu},\boldsymbol{\Sigma}\succeq \mathbf{0},|\boldsymbol{\Sigma}|=1}{\operatorname{argmin}} \left\{ \mathrm{Ma}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{T_G:T} \right\}} \quad (4.237)$$

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Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

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$$\mathbf{X} \equiv \begin{pmatrix} \mathbf{X}_A \\ \mathbf{X}_B \end{pmatrix} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (2.160)$$

$$\boldsymbol{\mu} \equiv \begin{pmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{pmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} \boldsymbol{\Sigma}_{AA} & \boldsymbol{\Sigma}_{AB} \\ \boldsymbol{\Sigma}_{BA} & \boldsymbol{\Sigma}_{BB} \end{pmatrix} \quad (2.161)$$

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$$\mathbf{X} \equiv \begin{pmatrix} \mathbf{X}_A \\ \mathbf{X}_B \end{pmatrix} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \quad (2.160)$$

$$\boldsymbol{\mu} \equiv \begin{pmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{pmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} \boldsymbol{\Sigma}_{AA} & \boldsymbol{\Sigma}_{AB} \\ \boldsymbol{\Sigma}_{BA} & \boldsymbol{\Sigma}_{BB} \end{pmatrix} \quad (2.161)$$

$$\boldsymbol{\mu}_B | \mathbf{x}_A \equiv \boldsymbol{\mu}_B + \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}_{AA}^{-1} \left(\mathbf{x}_A - \boldsymbol{\mu}_A\right) \quad (2.165)$$

$$\mathbf{X}_B | \mathbf{x}_A \sim \mathrm{N}\left(\boldsymbol{\mu}_B | \mathbf{x}_A, \boldsymbol{\Sigma}_B | \mathbf{x}_A\right) \quad (2.164)$$

$$\boldsymbol{\Sigma}_B | \mathbf{x}_A \equiv \boldsymbol{\Sigma}_{BB} - \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}_{AA}^{-1} \boldsymbol{\Sigma}_{AB} \quad (2.166)$$

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$$\begin{pmatrix} \mathbf{X}_{t,\mathrm{mis}(t)} \\ \mathbf{X}_{t,\mathrm{obs}(t)} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathrm{mis}(t)} \\ \boldsymbol{\mu}_{\mathrm{obs}(t)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{obs}(t)} \\ \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{obs}(t)} \end{pmatrix} \right) \tag{4.257}$$

$$\boldsymbol{\mu}_B | \mathbf{x}_A \equiv \boldsymbol{\mu}_B + \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}_{AA}^{-1} \left(\mathbf{x}_A - \boldsymbol{\mu}_A \right) \tag{2.165}$$

$$\boldsymbol{\Sigma}_B | \mathbf{x}_A \equiv \boldsymbol{\Sigma}_{BB} - \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}_{AA}^{-1} \boldsymbol{\Sigma}_{AB} \tag{2.166}$$

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$$\begin{pmatrix} \mathbf{X}_{t,\mathrm{mis}(t)} \\ \mathbf{X}_{t,\mathrm{obs}(t)} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathrm{mis}(t)} \\ \boldsymbol{\mu}_{\mathrm{obs}(t)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{obs}(t)} \\ \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{obs}(t)} \end{pmatrix} \right) \quad (4.257)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{obs}(t)} \equiv \mathbf{x}_{t,\mathrm{obs}(t)} \quad (4.261)$$

$$\boldsymbol{\mu}_B | \mathbf{x}_A \equiv \boldsymbol{\mu}_B + \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} \left(\mathbf{x}_A - \boldsymbol{\mu}_A \right) \quad (2.165)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{mis}(t)} \equiv \boldsymbol{\mu}^{(u)}_{\mathrm{mis}(t)} \quad (4.262)$$
$$+ \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{obs}(t)} \left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{obs}(t)} \right)^{-1} \left(\mathbf{x}_{t,\mathrm{obs}(t)} - \boldsymbol{\mu}^{(u)}_{\mathrm{obs}(t)} \right)$$

$$\boldsymbol{\Sigma}_B | \mathbf{x}_A \equiv \boldsymbol{\Sigma}_{BB} - \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} \boldsymbol{\Sigma}_{AB} \quad (2.166)$$

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$$\begin{pmatrix} \mathbf{X}_{t,\mathrm{mis}(t)} \\ \mathbf{X}_{t,\mathrm{obs}(t)} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathrm{mis}(t)} \\ \boldsymbol{\mu}_{\mathrm{obs}(t)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{obs}(t)} \\ \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{obs}(t)} \end{pmatrix} \right) \quad (4.257)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{obs}(t)} \equiv \mathbf{x}_{t,\mathrm{obs}(t)} \quad (4.261)$$

$$\boldsymbol{\mu}_B | \mathbf{x}_A \equiv \boldsymbol{\mu}_B + \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} (\mathbf{x}_A - \boldsymbol{\mu}_A) \quad (2.165)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{mis}(t)} \equiv \boldsymbol{\mu}^{(u)}_{\mathrm{mis}(t)} + \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{obs}(t)} \left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{obs}(t)} \right)^{-1} \left(\mathbf{x}_{t,\mathrm{obs}(t)} - \boldsymbol{\mu}^{(u)}_{\mathrm{obs}(t)} \right) \quad (4.262)$$

$$\mathbf{C}^{(u)}_{t,\mathrm{obs}(t),\mathrm{mis}(t)} \equiv \mathbf{0}, \quad \mathbf{C}^{(u)}_{t,\mathrm{obs}(t),\mathrm{obs}(t)} \equiv \mathbf{0}; \quad (4.263)$$

$$\boldsymbol{\Sigma}_B | \mathbf{x}_A \equiv \boldsymbol{\Sigma}_{BB} - \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} \boldsymbol{\Sigma}_{AB} \quad (2.166)$$

$$\mathbf{C}^{(u)}_{t,\mathrm{mis}(t),\mathrm{mis}(t)} \equiv \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{mis}(t)} - \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{obs}(t)} \left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{obs}(t)} \right)^{-1} \boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{mis}(t)}. \quad (4.264)$$

estimate

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$$\begin{pmatrix} \mathbf{X}_{t,\mathrm{mis}(t)} \\ \mathbf{X}_{t,\mathrm{obs}(t)} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathrm{mis}(t)} \\ \boldsymbol{\mu}_{\mathrm{obs}(t)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{mis}(t),\mathrm{obs}(t)} \\ \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{mis}(t)} & \boldsymbol{\Sigma}_{\mathrm{obs}(t),\mathrm{obs}(t)} \end{pmatrix} \right) \quad (4.257)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{obs}(t)} \equiv \mathbf{x}_{t,\mathrm{obs}(t)} \quad (4.261)$$

$$\boldsymbol{\mu}_B | \mathbf{x}_A \equiv \boldsymbol{\mu}_B + \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} (\mathbf{x}_A - \boldsymbol{\mu}_A) \quad (2.165)$$

$$\mathbf{x}^{(u)}_{t,\mathrm{mis}(t)} \equiv \boldsymbol{\mu}^{(u)}_{\mathrm{mis}(t)} + \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{obs}(t)} \left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{obs}(t)} \right)^{-1} \left(\mathbf{x}_{t,\mathrm{obs}(t)} - \boldsymbol{\mu}^{(u)}_{\mathrm{obs}(t)} \right) \quad (4.262)$$

$$\mathbf{C}^{(u)}_{t,\mathrm{obs}(t),\mathrm{mis}(t)} \equiv \mathbf{0}, \quad \mathbf{C}^{(u)}_{t,\mathrm{obs}(t),\mathrm{obs}(t)} \equiv \mathbf{0}. \quad (4.263)$$

$$\boldsymbol{\Sigma}_B | \mathbf{x}_A \equiv \boldsymbol{\Sigma}_{BB} - \boldsymbol{\Sigma}_{BA} \boldsymbol{\Sigma}^{-1}_{AA} \boldsymbol{\Sigma}_{AB} \quad (2.166)$$

$$\mathbf{C}^{(u)}_{t,\mathrm{mis}(t),\mathrm{mis}(t)} \equiv \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{mis}(t)} - \boldsymbol{\Sigma}^{(u)}_{\mathrm{mis}(t),\mathrm{obs}(t)} \left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{obs}(t)} \right)^{-1} \boldsymbol{\Sigma}^{(u)}_{\mathrm{obs}(t),\mathrm{mis}(t)}. \quad (4.264)$$

estimate

$$\boldsymbol{\mu}^{(u+1)} \equiv \frac{1}{T} \sum_t \mathbf{x}^{(u)}_t \quad (4.265)$$

$$\boldsymbol{\Sigma}^{(u+1)} \equiv \frac{1}{T} \sum_t \left[\mathbf{C}^{(u)}_t + \left(\mathbf{x}^{(u)}_t - \boldsymbol{\mu}^{(u)} \right) \left(\mathbf{x}^{(u)}_t - \boldsymbol{\mu}^{(u)} \right)' \right] \quad (4.266)$$

update