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QUANT CLASSROOM

RISK AND CORRELATION PROPAGATION:

The Square-Root Rule, Covariances and Ellipsoids

How to analyze and visualize the propagation law of risk in a multi-dimensional market

The square-root rule of risk propagation reads: "volatility increases as the square root of the investment horizon." This rule is based on the assumption that the non-overlapping compounded returns of a security are independently and identically (i.i.d.) distributed across time (see equation 1, below) – i.e., they are market "invariants."[1]

$$R_t = \ln(P_{t+1}) - \ln(P_t) \sim \text{i.i.d.}, \qquad (1)$$

where P_t is the price of the security at time t. In particular, in the standard Black-Scholes model, the returns (1) are normal, but this assumption does not play a role in the sequel.
To derive the square root rule, we notice that the weekly returns are the sum of five daily returns, the monthly returns are the sum of 22 daily returns, and, more generally, the j-horizon return is the sum of j non-overlapping short-horizon returns, as follows:

$$\ln(P_{t+j}) - \ln(P_t) = R_t + \cdots + R_{t+j-1} \qquad (2)$$

If the short-horizon returns are invariants and if we denote by σ^2 their variance, the variance $\tilde{\sigma}^2(j)$ of the j-horizon return satisfies the following:

$$\tilde{\sigma}^2(j) = \text{Var}\{R_t + \cdots + R_{t+j-1}\} = j\,\sigma^2, \qquad (3)$$

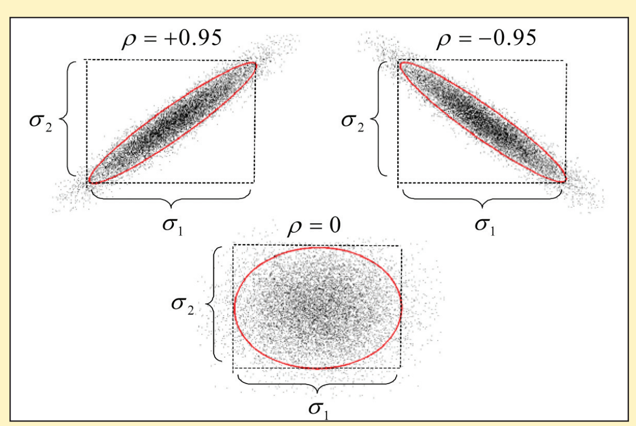
Therefore, the volatility, as represented by the square root of the variance, satisfies the square-root law. This simple result generalizes to a multivariate framework. Consider, for example, the daily returns on two stocks and their covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}, \qquad (4)$$

where ρ is the correlation and σ_1 and σ_2 are the respective volatilities.

To visualize the covariance matrix more clearly, we can use the location-dispersion ellipsoid (see Figure 1[2], below). Please note that the size of the ellipsoid represents the volatilities, whereas the shape and the orientation represent the correlation.

Figure 1: Visualizing Covariance Matrices: The Location-dispersion Ellipsoid



Now consider the covariance of the long-horizon (weekly, monthly, etc.) returns which, as in (2), are the sum of j (five, 22, respectively, etc.) non-overlapping daily returns, as follows:

$$\tilde{\Sigma}(j) = \begin{pmatrix} \tilde{\sigma}_1^2(j) & \tilde{\sigma}_1(j)\tilde{\sigma}_2(j)\tilde{\rho}(j) \\ \tilde{\sigma}_1(j)\tilde{\sigma}_2(j)\tilde{\rho}(j) & \tilde{\sigma}_2^2(j) \end{pmatrix} \qquad (5)$$

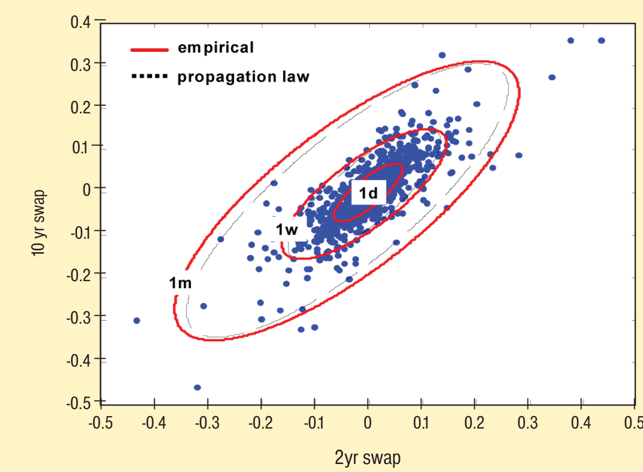
If the returns are invariants, from the same argument as (3), it follows that the covariance grows proportionally to the horizon, as depicted below:

$$\tilde{\Sigma}(j) = j\,\Sigma \qquad (6)$$

Since the variances diffuse proportionally to the horizon — i.e., $\tilde{\sigma}_1^2(j) = j\sigma_1^2$ and $\tilde{\sigma}_2^2(j) = j\sigma_2^2$ — then, if the returns are invariants, their correlation is constant at any horizon (i.e., $\tilde{\rho}(j) = \rho$).

The above results do not always, or only, apply to the compounded returns. For any variables such as (1), which (a) are invariants and (b) can be expressed as differences, volatility propagates as the square root of the horizon and correlation is constant. This relationship allows us to test whether a given set of risk factors are market invariants. For instance, the returns of near-maturity bonds are not invariants, so the square-root rule does not apply. In contrast, consider the swap market.

Figure 2: Swap Rate Changes are Approximately Invariants *



*Data Source: Bloomberg.

The square-root rule of risk propagation follows from the assumption that the returns are market 'invariants.' … To visualize the covariance matrix of the market invariants, we use the location-dispersion ellipsoid.

In Figure 2 (below, left), we scatter-plot the daily differences of the two-year versus the 10-year point of the swap curve. Then we compute the sample covariance Σ between these two series. Next, we represent geometrically Σ by means of its location dispersion ellipse, which is the smallest ellipse in Figure 2. Then we consider the empirical covariance $\tilde{\Sigma}$ at different horizons of one week and one month, respectively.

We represent all of these covariances by means of their location-dispersion ellipsoids, which we plot in the figure as solid red lines. Finally, we compare these ellipsoids with the suitable multiple $j\Sigma$ of the daily ellipsoid, which we plot as dashed ellipsoids. We see from the figure that the solid and the dashed ellipsoids are comparable, and thus the swap rate changes are approximately invariants. The volatilities increase according to the square-root rule and the correlation is approximately constant.

FOOTNOTES

1. For more details, see *Risk and Asset Allocation* (Springer, 2005) by A. Meucci.
2. Ibid

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