Stress-Testing with
Fully Flexible Causal Inputs

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Latest version and code at `http://symmys.com/node/152`

Abstract

Propagating causal stress-tests or contagion on selected risk factors to all the risk drivers is a challenging task. We use Entropy Pooling by Meucci (2008) to address this issue. Our causal stress-tests comprise, but are not restricted to, stress-testing Bayesian networks. We detail the theory and we present a case study: stress-testing a market driven by swap curve, credit spreads, stock market return, stock market volatility, currency strength, inflation, and commodities. Fully commented code supporting the empirical analysis is available at http://symmys.com/node/152.

JEL Classification: C1, G11

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1 Introduction

The distribution of the key risk drivers in any given market can never be estimated correctly. Therefore, stress-testing becomes the only effective tool to handle estimation risk both in risk management and in portfolio management: the base-case risk model is modified manually and the ensuing p&l distribution is computed and evaluated.

Stress-testing has earned an even more prominent place in the industry after the financial crises of the late 2000's. Basel III, the global regulatory standard for banks, advocates for tighter limits on risk-based capital. The US Federal Reserve declared in December 2011 that it will conform to the Basel III standards. More precisely, "...a wide range of measures addressing issues such as capital, liquidity, credit exposure, stress testing, risk management, and early remediation requirements, is mandated by the Dodd-Frank Wall Street Reform and Consumer Protection Act...". Specifically on stress-testing, the US Federal Reserve declared that "...stress tests of the companies would be conducted annually by the Board using three economic and financial market scenarios...", see Fed (2011).

However, stress-testing presents two difficulties. First, it is hard to specify reasonable modifications of a base-case risk model. Second, it is challenging to embed such modifications coherently in the base-case risk model.

To cope with the first problem, namely the difficulty to specify reasonable stress-tests, Rebonato (2010) proposes to use Boolean Bayesian networks to greatly simplify the structure of the risk model, in two directions. First, the risk drivers are modeled as Boolean variables, which can only take on the values "true" or "false". Second, such risk drivers are assumed to be connected by a parsimonious causal Bayesian network: this network defines the risk drivers distribution in terms of a limited number of conditional probabilities. This double layer of simplification, namely true/false events and sparse causal network, reduces stress-testing to re-assessing a manageable number of conditional probabilities, of which the practitioner is supposed to have a deep understanding. Potential disadvantages of the above Boolean Bayes network approach include that the true/false discretization of the risk space can prove too coarse to model realistic risk scenarios; and that the causal structure imposed by the Bayesian network might be too rigid and prevent interesting counter-causal stress-tests.

For the second problem, namely embedding stress-tests coherently in the base-case risk model, in theory the Entropy Pooling approach in Meucci (2008) addresses the issue in full generality: under arbitrary distributional assumptions for the base-case risk model, the stressed model is defined as the one that displays the least distortion from the base-case risk model and yet satisfies the stress-tests, where the distortion is measured by the relative entropy. In practice, Meucci (2008) discusses how to compute the stressed distribution in two cases: when the markets are normal, or when the market distribution is represented by historical or Monte Carlo scenarios.

Here we discuss the implementation of Entropy Pooling when the market is represented by Boolean distributions, or more general discrete-space distributions. Then we use our results to extend the Boolean Bayes network approach to stress-testing. More precisely, any situation covered by the Boolean Bayes network approach is also covered by the current approach as a special case, but extensions are provided in four directions.

First, we do not impose that the risk drivers only assume "true" or "false" values. Instead, we let them take on more realistic arbitrary sets of discrete outcomes.

Second, we do not need to assume that the risk model is a Bayesian network. Instead, we stress-test arbitrarily complex causal conditional probabilities, and we recover any unstressed feature by minimum-entropy arguments.

Third, we allow for additional direct stress-testing of correlations, volatilities, expectations, quantiles, and other features of the risk drivers.

Finally, we provide a fully general algorithm to ensure that the stress-tests are consistent with each other, i.e. we do not obtain negative probabilities, negative volatilities, or correlations larger than one. Our algorithm applies to all possible assessments of the probabilities of the scenarios, conditioned on any number of statements.

In Section 2 we present our theoretical framework. First, we review entropy pooling, then we discuss how it applies to generalized stress-testing of conditional probabilities under discrete scenarios. Then we introduce our consistency algorithm for the stress-tests. Throughout the discussion we illustrate the theory by means of a simple example.

In Section 3 we present a real-life case study. We consider a market driven by swap curve, credit spreads, stock market return, stock market volatility, currency strength, inflation, and commodities. We stress-test this market by assessing a few key conditional and marginal probabilities as well as correlations. Then we compare the prior and the posterior distribution that follow from those stress-tests.

Fully commented code supporting the empirical analysis is available for download at http://symmys.com/node/152.

2 Entropy pooling on conditional distributions

The Entropy Pooling approach in Meucci (2008) introduces a theoretical framework to perform under the same umbrella stress-testing for risk management and signal processing for portfolio management: a base-case market distribution is perturbed by stress-tests or signals into a new distribution, which is close to the base-case, but is consistent with the stress-tests and the signals.

In full generality, the market distribution can be represented in three ways. First, the parametric approach, where the distribution is fully determined by a set of parameters, such as the normal distribution, or the Student t distribution, or the marginal-copula decomposition with parametric marginals and copula. The implementation of Entropy Pooling with parametric distributions is explored in Meucci, Ardia, and Keel (2011).

The second way to represent the market distribution is by means of random scenarios, such as those generated via Monte Carlo simulations, or those provided by the historical realizations of the risk drivers. The implementation of Entropy Pooling with random scenarios is explored in Meucci (2008), Meucci (2010) and Meucci (2012).

The third way to represent a distribution is by histograms. In the univariate case, a histogram represents a distribution by a discrete set of deterministic values, typically an equally spaced grid and the associated probabilities, or "bins". A Boolean true/false event is the simplest case of histogram, with only two bins. In the more general case a market represented by a histogram is multivariate, and in each dimension there is a discrete set of more than two possible outcomes.

In this article, we discuss the implementation of Entropy Pooling for multivariate histograms and we apply it to the generalization of the Boolean Bayes network approach to stress-testing. To do so, first, we define the base-case distribution for the risk drivers in terms of a histogram, which we call the "prior". Then we formulate the stress-tests of the prior histogram in terms of linear constraints on the probabilities of the histogram's bins. Next, we check for the consistency of such constraints. Finally, we define the stressed distribution, which we call the "posterior", as the new probabilities of the histogram's bins that display the least relative entropy from the prior and at the same time satisfy the stress-test constraints.

The prior

We consider N risk drivers $\mathbf{X} \equiv (X_1, \ldots, X_N)'$, namely random variables that determine the p&l of the portfolio under scrutiny. We assume that such drivers are discrete, i.e. for each $n = 1, \ldots, N$ the driver X_n can only result in a finite set of K_n outcomes.

$$X_n \in \{x_{n,1}, \ldots, x_{n,K_n}\}. \tag{1}$$

For instance, consider the case of $N \equiv 3$ drivers X_1, X_2, X_3. Say X_1 describes the widening of spreads, which can take values "low", "medium", and "high"; X_2 describes the default of a given country, which can take on values "default" or "survival"; and X_3 describes the central bank intervention, which can either "cut" or "raise" interest rates. Therefore in the formalism (1) we obtain

$$X_1 \in \{L, M, H\} \tag{2}$$

$$X_2 \in \{D, S\} \tag{3}$$

$$X_3 \in \{C, R\}. \tag{4}$$

The total number of joint scenarios is $J \equiv \prod_{n=1}^{N} K_n$. Rebonato (2010) discusses a special case of this framework, where each driver is Boolean, i.e. $X_n \in \{T, F\}$ and thus there are $J \equiv 2^N$ possible joint scenarios.

We collect all the J joint scenarios of the N risk drivers $\mathbf{X}$ in a $J \times N$ panel $\mathcal{X}$. The stochastic properties of these events are fully described by the

J-dimensional vector $\mathbf{p}$ of the probabilities of each joint scenario. We call $(\mathcal{X}, \mathbf{p})$ the "prior" distribution of the market.

In our example, there are $J = 3 \times 2 \times 2 = 12$ joint scenarios. Assume that all the joint events are equally probable. Then the prior becomes

$$\begin{array}{ccc|c} & \mathcal{X} & & \mathbf{p} \\ \hline L & D & C & 1/12 \\ L & D & R & 1/12 \\ L & S & C & 1/12 \\ L & S & R & 1/12 \\ M & D & C & 1/12 \\ M & D & R & 1/12 \\ M & S & C & 1/12 \\ M & S & R & 1/12 \\ H & D & C & 1/12 \\ H & D & R & 1/12 \\ H & S & C & 1/12 \\ H & S & R & 1/12 \end{array} \tag{5}$$

Rebonato (2010) determines the prior probabilities by imposing a parsimonious Bayesian network structure. Mathematically, this amounts to replacing the full specification of the joint probability $\mathbb{P}\{x_1, \ldots, x_N\}$ with a parsimonious conditional representation

$$\mathbb{P}\{x_1, \ldots, x_N\} \equiv \prod_{n=1}^{N} \mathbb{P}\left\{x_n | \mathbf{x}_{c(n)}\right\}, \tag{6}$$

where $\mathbf{x}_{c(n)}$ denotes the set of variables among $(x_1, \ldots, x_N)$ that have a causal effect on the driver X_n. For more details on Bayesian networks, refer e.g. to Williamson (2005).

For instance assume that in our example the spread changes X_1 are caused by the central bank action X_3 and nothing else. Then the following Bayesian network is defined: $\{X_2; X_1 \leftarrow X_3\}$. Therefore the entries of the probability vector $\mathbf{p}$ in (5) can be generated as follows

$$\mathbb{P}\{x_1, x_2, x_3\} = \mathbb{P}\{x_1 | x_3\} \mathbb{P}\{x_2\} \mathbb{P}\{x_3\}. \tag{7}$$

To generate all the $J = 12$ probabilities, only 6 numbers must be specified, namely $\mathbb{P}\{X_2 = D\}$, $\mathbb{P}\{X_3 = C\}$, $\mathbb{P}\{X_1 = L | X_3 = C\}$, $\mathbb{P}\{X_1 = M | X_3 = C\}$, $\mathbb{P}\{X_1 = L | X_3 = R\}$, and $\mathbb{P}\{X_1 = M | X_3 = R\}$.

More in general, we can assign the prior distribution using classical frequentist analysis, as in the case study below. Alternative approaches are also possible, we refer the reader to the vast literature on estimation theory.

The stress-tests

A stress-test is a subjective statement on features of the distribution of the risk drivers $\mathbf{X}$. Typically, risk managers stress-test expectations, correlations and volatilities of the risk drivers. However, Rebonato (2010) proposes to stress-test the conditional probabilities that define the Bayesian network, under the rationale that practitioners have a better grasp on their market in relative-causal terms than in absolute terms.

More in general, we want to stress-test any generic conditional probability. Denoting by $\widetilde{\mathbb{P}}$ the subjective probability of the practitioner, the generic k-th view is in the format

$$\widetilde{\mathbb{P}}\left\{X_{n_1} \in \mathbf{x}_{n_1}, X_{n_2} \in \mathbf{x}_{n_2}, \ldots | X_{m_1} \in \mathbf{x}_{m_1}, X_{m_2} \in \mathbf{x}_{m_2}, \ldots\right\} \gtreqless \widetilde{v}_k, \tag{8}$$

where $\mathbf{x}_n$ denotes a subset of the potential outcomes (1) of the risk driver X_n, i.e. $\mathbf{x}_n \subset \{x_{n,1}, \ldots, x_{n,K_n}\}$; $\gtreqless$ denotes any (in)equality; and $\widetilde{v}_k$ is a subjective probability threshold.

We emphasize that our framework (8) allows for multiple outcomes in the conditional probabilities. Furthermore, we do not require the existence of a Bayesian network.

For instance, consider the following stress-test: the probability of the foreign country defaulting is at least 30%, whereas the probability of spreads widening to a moderate or large extent if the foreign country defaults is at least 70%. In formulas, this reads

$$\widetilde{\mathbb{P}}\left\{X_1 \in \{M, H\} | X_2 = D\right\} \geq 0.7. \tag{9}$$

$$\widetilde{\mathbb{P}}\left\{X_2 = D\right\} \geq 0.3 \tag{10}$$

Notice that in the first stress-test we are letting the variable X_1 take on multiple values, namely M and H, and we are not stress-testing the probabilities $\widetilde{\mathbb{P}}\{x_1|x_3\}$, and $\widetilde{\mathbb{P}}\{x_3\}$that define the Bayesian network (7).

Each view/stress-test (8) can be rephrased in terms of linear constraints on the vector $\widetilde{\mathbf{p}}$ of subjective probabilities which are associated with each scenario. Indeed, denote by $\mathbf{I}_{jnt(k)}$ the indicator of the scenarios where the joint conditions in the stress-test (8) are satisfied and denote by $\mathbf{I}_{cnd(k)}$ the indicator of the scenarios where the conditioning events in the stress-test (8) are satisfied. Then using the identity $\mathbb{P}\{A|B\} = \mathbb{P}\{A \cap B\} / \mathbb{P}\{B\}$ we can reformulate (8) as

$$\left(\mathbf{I}_{jnt(k)} - \widetilde{v}_k \mathbf{I}_{cnd(k)}\right)' \widetilde{\mathbf{p}} \gtreqless 0. \tag{11}$$

Also, In the special case where there is no conditioning statement, (8) becomes

$$\mathbf{I}'_{jnt(k)} \widetilde{\mathbf{p}} \gtreqless \widetilde{v}_k. \tag{12}$$

Both (11) and (12) are linear constraints on $\widetilde{\mathbf{p}}$.

Consider our example (9), where

$$\begin{aligned} \mathbf{I}_{jnt} &\equiv \mathbf{I}_{X_1 \in \{M,H\} \cap X_2 = D} && (13) \\ \mathbf{I}_{cnd} &\equiv \mathbf{I}_{X_2 = D}. && (14) \end{aligned}$$

From (5) we obtain

$$\begin{aligned} \mathbf{I}_{jnt} &= (0,0,0,0,1,1,0,0,1,1,0,0)' && (15) \\ \mathbf{I}_{cnd} &= (1,1,0,0,1,1,0,0,1,1,0,0)'. && (16) \end{aligned}$$

Therefore, according to (11), the stress-test (9) reads

$$-0.7\widetilde{p}_1 - 0.7\widetilde{p}_2 + 0.3\widetilde{p}_5 + 0.3\widetilde{p}_6 + 0.3\widetilde{p}_9 + 0.3\widetilde{p}_{10} \geq 0. \tag{17}$$

Similarly, for (10) we obtain

$$\widetilde{p}_5 + \widetilde{p}_6 + \widetilde{p}_7 + \widetilde{p}_8 + \widetilde{p}_9 + \widetilde{p}_{10} \geq 0.3. \tag{18}$$

Furthermore, as we show in Meucci (2008), stress-tests on expectations, volatilities, and correlations can also be expressed as linear constraints on the probabilities. Therefore, all the views/stress-tests can be summarized as

$$\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b}, \tag{19}$$

where $\mathbf{A}$ and $\mathbf{b}$ are a suitable conformable matrix and vector respectively.

Consistency check

The views/stress-tests might not be consistent, i.e. there might not exist a vector $\mathbf{p}$ that satisfies them and is also a probability vector

$$\mathbf{1}'\mathbf{p} \equiv 1, \quad \mathbf{p} \geq \mathbf{0}. \tag{20}$$

In Rebonato (2010) the problem is solved under specific circumstances, namely when the stress tests are performed on conditional distributions that do not involve more than a few conditioning variables. Here, we propose an algorithm that works in full generality.

In our approach, in order to guarantee the consistency of the stress-tests, we relax them

$$\mathbf{b} \mapsto \mathbf{b} + \delta\mathbf{b}, \tag{21}$$

where $\delta\mathbf{b} \geq \mathbf{0}$ is a minimal perturbation that in general is null $\delta\mathbf{b} \equiv \mathbf{0}$, but that could become strictly positive in some or all of the entries. To compute $\delta\mathbf{b}$ we solve the following problem

$$\delta\mathbf{b} \equiv \underset{\mathbf{x}}{\operatorname{argmin}} \{\boldsymbol{\gamma}'\mathbf{x}\} \tag{22}$$

where $\mathbf{x}$ satisfies

$$\begin{aligned} \mathbf{x} &\geq \mathbf{0} && (23) \\ \mathbf{A}\mathbf{p} &\leq \mathbf{b} + \mathbf{x} && (24) \end{aligned}$$

and $\mathbf{p}$ satisfies (20). The constant vector $\boldsymbol{\gamma} \equiv -\ln(\mathbf{1}-\mathbf{c})$ in the target (22) is a function of the relative confidence in each view, as summarized by the vector $\mathbf{c} \in (0,1)$. The entries in $\boldsymbol{\gamma}$ are always positive: they are close to zero for low-confidence views and become arbitrarily large for high-confidence views. Therefore, the target (22) perturbs low-confidence views, while leaving high-confidence views untouched. In particular, notice that if the original stress-test constraints (19) are consistent, then, as expected, $\delta\mathbf{b} = \mathbf{0}$.

The optimization (20)-(22)-(23)-(24) is an instance of linear programming in the variables $(\mathbf{x}, \mathbf{p})$, whose dimension equals the number of stress-tests plus the number of joint scenarios. This problem can be easily solved numerically even when such dimension is large.

In our example the adjustment is not required, i.e. $\delta\mathbf{b} = \mathbf{0}$, because the stress-tests are fully consistent with a probability set, please refer to the MATLAB code at http://symmys.com/node/152.

The posterior

Consider the prior distribution, represented by the multivariate histogram $(\mathcal{X}, \mathbf{p})$. Consider an alternative distribution on the same bins $(\mathcal{X}, \mathbf{q})$, where $\mathbf{q}$ is a new vector of probabilities. In this context, the distance of $(\mathcal{X}, \mathbf{q})$ from $(\mathcal{X}, \mathbf{p})$ used in Meucci (2008), i.e. the relative entropy, becomes

$$\mathcal{E}(\mathbf{q}, \mathbf{p}) \equiv \sum_{j=1}^{J} q_j \ln \frac{q_j}{p_j}. \tag{25}$$

The relative entropy is null only when $\mathbf{q} = \mathbf{p}$ and otherwise it is positive, growing larger as $\mathbf{q}$ diverges from $\mathbf{p}$.

The posterior distribution is represented by new probabilities $\widetilde{\mathbf{p}}$ that are as close as possible to the prior probabilities in terms of the relative entropy (25), but that reflect the stress-tests, i.e. they satisfy (19)-(21). Therefore, the full-confidence posterior is defined as

$$\widetilde{\mathbf{p}} = \underset{\mathbf{A}\mathbf{q} \leq \mathbf{b} + \delta\mathbf{b}}{\operatorname{argmin}} \mathcal{E}(\mathbf{q}, \mathbf{p}). \tag{26}$$

We emphasize that one could minimize different distances between two distributions, instead of the relative entropy (26), such as for instance the standard Euclid distance $\|\mathbf{q} - \mathbf{p}\|$. However, our choice is particularly appealing for both theoretical and practical reasons.

From a theoretical perspective, the Entropy Pooling optimization (26) generalizes the well-founded Bayesian updating principle, see Caticha and Giffin (2006).

From a practical perspective, the Entropy Pooling optimization (26) is surprisingly simple to solve. The number of the variables $\mathbf{q}$ in (26) is J, i.e. the number of joint scenarios, which can be prohibitively large for practical purposes: for instance, for $N = 10$ risk drivers and $K = 3$ discrete outcomes each, we have $J = 3^{10} = 59,049$ joint scenarios, see also the case study in Section 3. However, through its dual formulation, (26) becomes a simple convex problem in a number of variables equal to the always parsimonious number of stress-tests, and thus it can be solved numerically very efficiently, refer to Meucci (2008) for the proof.

In our example, we compute the probabilities (26) that minimize the relative entropy with the uniform prior (5) under the constraint (17). The result is

$$\widetilde{\mathbf{p}} = \frac{1}{12}\left(0.9, 0.9, 1, 1, 1.05, 1.05, 1, 1, 1.05, 1.05, 1, 1\right), \tag{27}$$

please refer to the MATLAB code at http://symmys.com/node/152.

3 Case study: stress-testing a global market

In this section we consider a realistic case of stress-testing. Please refer to the MATLAB code http://symmys.com/node/152 for all the details and to replicate these result.

We consider a market driven by $N \equiv 9$ risk factors: Z_1 is the two-year and Z_2 the ten-year points of the swap curve; Z_3 is the CDX credit default swap index; Z_4 is the S&P 500 stock market index; Z_5 is the VIX index of the implied volatility in the market; Z_6 is the dollar currency strength index; Z_7 is the crude oil price; Z_8 is the gold price; and Z_9 is the ten-year inflation swap rate.

We discretise our drivers into three buckets: the risk factors stay within a given range, or widen above a given threshold, or widen below another threshold. More precisely, we define

$$X_n \equiv \begin{cases} 1 & \text{if } \Delta Z_n > \overline{q}_n \\ 0 & \text{if } \underline{q}_n \leq \Delta Z_n \leq \overline{q}_n \\ -1 & \text{if } \Delta Z_n < \underline{q}_n \end{cases}, \quad n = 1, \ldots, N, \tag{28}$$

where ΔZ_n is the change of a risk factor over a day, and the thresholds $\underline{q}_n$ and $\overline{q}_n$ are the lower and upper historical terciles.

We collect all the joint scenarios for $\mathbf{X}$ in a $J \times N$ panel $\mathcal{X}$, where $J = 3^9 = 19,683$ and $N = 9$.

The prior

To define the prior, we must assign a probability p_j to each of the 19,683 joint scenarios for the risk drivers, thereby obtaining the probability vector $\mathbf{p}$.

We do so with a frequentist estimate. First we collect the daily time series of the risk factors $\mathbf{Z}$ from September 1, 2005 to March 30, 2010. Then we compute the daily changes of $\mathbf{Z}$ over the above time period and we calculate the lower and upper terciles that appear in (28).

Then we can count how many times each row in the panel $\mathcal{X}$ was realized historically. For instance, the first row is a set of -1, corresponding to the

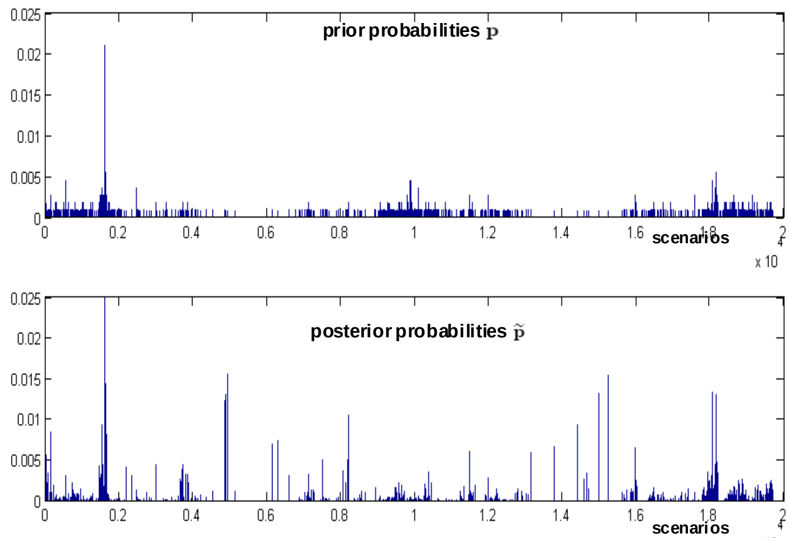

Figure 1: Prior (shrunk historical) versus posterior (stress-tested) probabilities

scenario where all the risk drivers jointly trespassed their lower tercile: such scenario never materialized and thus the frequentist probability is null.

The above process yields the purely frequentist J-dimensional vector $\mathbf{p}$ of prior probabilities. To ensure that no scenario is strictly impossible under the prior distribution, we shrink the frequentist estimation to the uniform distribution

$$\mathbf{p} \mapsto (1-\epsilon)\,\mathbf{p} + \epsilon \frac{\mathbf{1}}{J}, \tag{29}$$

where we set the shrinkage factor as $\epsilon \equiv 0.01$.

In the top plot in Figure 1 we report the bar plot of the prior probabilities (29). To gain a better understanding of such probabilities, in Table 30 we report the correlations (in %) among the risk drivers as implied by the prior distribution

$$\begin{array}{ccccccccc} & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\ X_1 & 77 & -15 & 24 & 20 & 4 & -8 & -19 & 29 \\ X_2 & \cdot & -15 & 21 & 12 & 5 & -5 & -19 & 31 \\ X_3 & \cdot & \cdot & -44 & 10 & -11 & -4 & 38 & -15 \\ X_4 & \cdot & \cdot & \cdot & -9 & 14 & 5 & -71 & 15 \\ X_5 & \cdot & \cdot & \cdot & \cdot & -29 & -41 & 5 & -16 \\ X_6 & \cdot & \cdot & \cdot & \cdot & \cdot & 31 & -8 & 24 \\ X_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & 11 \\ X_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -13 \end{array} \tag{30}$$

The stress-tests

We perform a stress test. First, we impose fatter tails on the scenarios, by increasing the probability of extreme events:

$$\widetilde{\mathbb{P}}\{X_n = -1\} \geq 40\%, \quad \widetilde{\mathbb{P}}\{X_n = 1\} \geq 40\%, \quad n = 1, \ldots N. \tag{31}$$

In other words, the probability of falling in the upper or lower tercile of all the indicators becomes at least 40%, whereas by construction historically it was 33%.

Second, we state that the probability of swap rates falling or remaining stable and gold rising, conditioned on the stock market rising is at least 90%

$$\widetilde{\mathbb{P}}\{X_1 \in \{-1, 0\} \cap X_2 \in \{-1, 0\} \cap X_8 = 1 | X_4 = 1\} \geq 90\% \tag{32}$$

Using the rules (11)-(12) we convert such statements in linear constraints in the format $\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b}$ as in (19).

Finally, we assess that the correlation between the two-year and the ten-year points of the curve will not exceed 60% over the next day, due to some awaited announcement.

$$\widetilde{\mathbb{C}}or\{X_1, X_2\} \leq 60\%. \tag{33}$$

This statement can be added to the above in the form of a linear constraint on the posterior probabilities as in Meucci (2008), giving rise to a global set of linear constraints $\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b}$.

Consistency check

The marginal statements (31) are fully consistent. Indeed, if we only stress-tested the system by imposing (31), the algorithm (22), would yield a null adjustment vector $\delta\mathbf{b} = (0, \ldots, 0,)'$ for the upper boundary $\mathbf{b}$ in the linear constraints $\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b}$ that represent the stress-tests.

On the other hand, the conditional statement (32) is inconsistent, although only minorly, with any probability distribution. Therefore, we relax the linear constraints $\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b} + \delta\mathbf{b}$, where

$$\delta\mathbf{b} \approx \left(0, \ldots, 0, 10^{-6}\right)' \tag{34}$$

follows from (22). Notice that, as expected, only the last constraint is relaxed, namely the one corresponding to the conditional stress-test.

The statement on correlation does not violate any probability distribution, and therefore it does not elicit a perturbation of the respective upper boundary.

The posterior

With the prior probabilities (29) and the stress-tests (31)-(32) in the format of constraints $\mathbf{A}\widetilde{\mathbf{p}} \leq \mathbf{b}$ we can now compute the posterior distribution as in (26). In the bottom plot in Figure 1 we report the bar plot of the posterior probabilities $\widetilde{\mathbf{p}}$. To gain a better understanding of such probabilities, in Table 35 we report the correlations (in %) among the risk drivers as implied by the posterior distribution

$$
\begin{array}{ccccccccc}
 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\
X_1 & 85 & -34 & 57 & 13 & 10 & -5 & -52 & 38 \\
X_2 & \cdot & -33 & 55 & 4 & 14 & -3 & -52 & 42 \\
X_3 & \cdot & \cdot & -54 & 13 & -16 & -4 & 50 & -26 \\
X_4 & \cdot & \cdot & \cdot & -7 & 19 & 4 & -86 & 34 \\
X_5 & \cdot & \cdot & \cdot & \cdot & -40 & -46 & 3 & -24 \\
X_6 & \cdot & \cdot & \cdot & \cdot & \cdot & 35 & -14 & 32 \\
X_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 12 \\
X_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -32
\end{array} \tag{35}
$$

When the stress-test on the correlation (33) is added to the picture we obtain a new set of posterior probabilities $\widetilde{\mathbf{p}}$ and a different global correlation matrix

$$
\begin{array}{ccccccccc}
 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\
X_1 & 50 & -26 & 52 & 16 & 8 & -4 & -45 & 27 \\
X_2 & \cdot & -38 & 54 & -6 & 14 & -1 & -55 & 41 \\
X_3 & \cdot & \cdot & -53 & 16 & -7 & -6 & 50 & -28 \\
X_4 & \cdot & \cdot & \cdot & -9 & 16 & 7 & -84 & 38 \\
X_5 & \cdot & \cdot & \cdot & \cdot & -35 & -46 & 6 & -28 \\
X_6 & \cdot & \cdot & \cdot & \cdot & \cdot & 34 & -14 & 30 \\
X_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -5 & 15 \\
X_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -38
\end{array} \tag{36}
$$

Notice how the correlations in the prior (30) change to (35) as a consequence of the conditional and marginal views (31)-(32) and how they change even further to (36) due to one single statement on one correlation.

4 Conclusions

We discussed the implementation of the Entropy Pooling approach, whose general theory was laid out in Meucci (2008), in the case where distributions are represented by multivariate histograms, i.e. deterministic bins and their respective probabilities. Then we showed how to apply our results to the stress-testing of global markets, thereby generalizing the Boolean Bayes network approach in Rebonato (2010).

With our approach we can stress-test markets with arbitrary discrete outcomes; we can have views on arbitrary conditional probabilities that need not be consistent with a Bayesian network; and we can stress not only conditional probabilities, but any feature of the market, such as correlations, volatilities, expectations, etc. We illustrated the above features of our approach to stress-testing in a real-life case study.

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