Attilio Meucci

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Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

BAYESIAN ESTIMATION – THEORY

$$\textbf{classical estimation: } i_T \mapsto \widehat{\boldsymbol{\theta}} \qquad (7.2)$$

$$\textbf{Bayesian estimation: } i_T, e_C \mapsto f_{\mathbf{po}}(\boldsymbol{\theta}) \qquad (7.3)$$

BAYESIAN ESTIMATION – THEORY

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$$\text{Bayesian estimation: } i_T, e_C \mapsto f_{\text{po}}(\boldsymbol{\theta}) \tag{7.3}$$

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{\text{ce}}[i_T, e_C] &\equiv \mathrm{E}_{i_T, e_C}\{\boldsymbol{\theta}\} \\ &\equiv \int_{\boldsymbol{\Theta}} \boldsymbol{\theta} f_{\text{po}}(\boldsymbol{\theta}; i_T, e_C)\, d\boldsymbol{\theta}.\end{aligned} \tag{7.5}$$

$$\begin{aligned}\mathrm{S}_{\boldsymbol{\theta}}[i_T, e_C] &\equiv \mathrm{Cov}_{i_T, e_C}\{\boldsymbol{\theta}\} \\ &\equiv \int_{\boldsymbol{\Theta}} (\boldsymbol{\theta} - \mathrm{E}\{\boldsymbol{\theta}\})(\boldsymbol{\theta} - \mathrm{E}\{\boldsymbol{\theta}\})' f_{\text{po}}(\boldsymbol{\theta}; i_T, e_C)\, d\boldsymbol{\theta}\end{aligned} \tag{7.7}$$

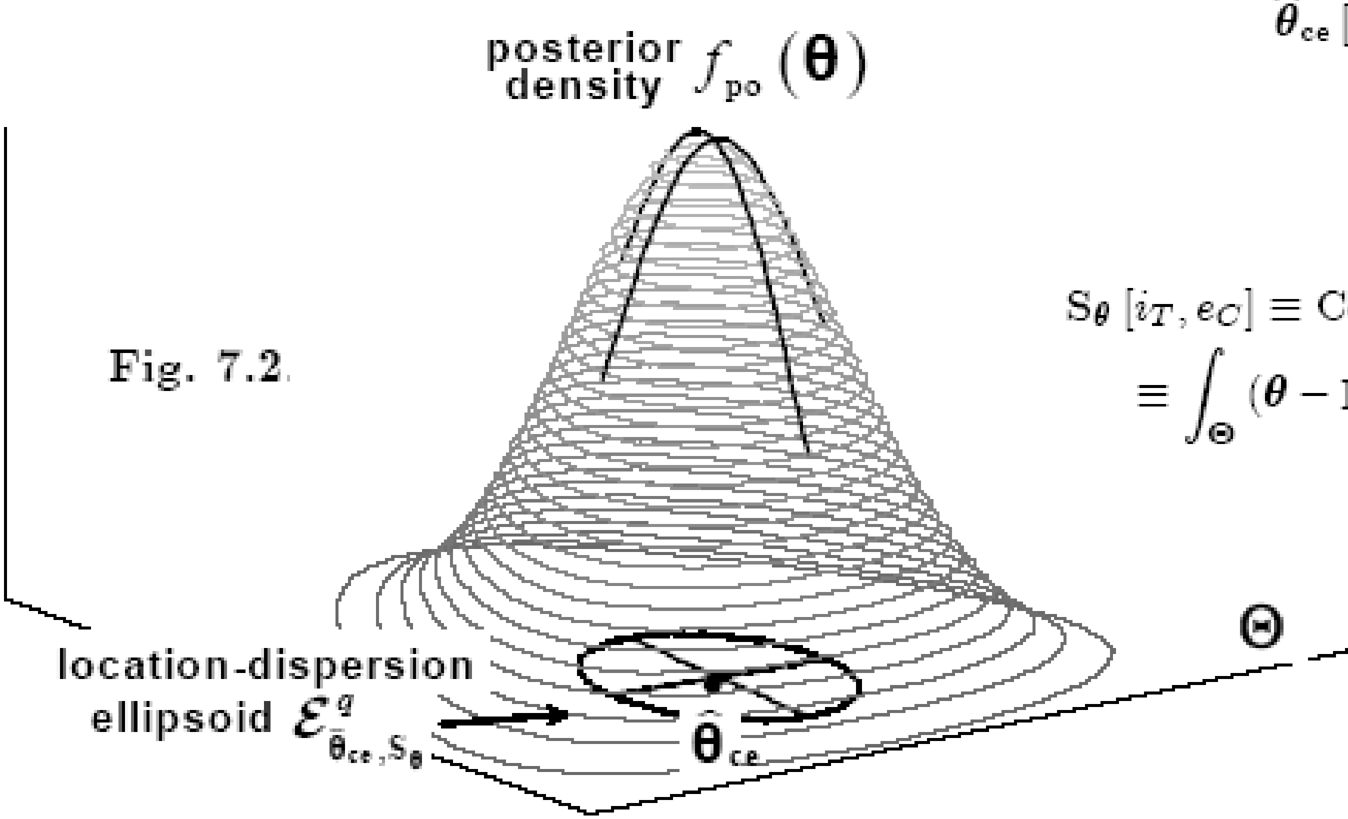

Fig. 7.2

BAYESIAN ESTIMATION – THEORY

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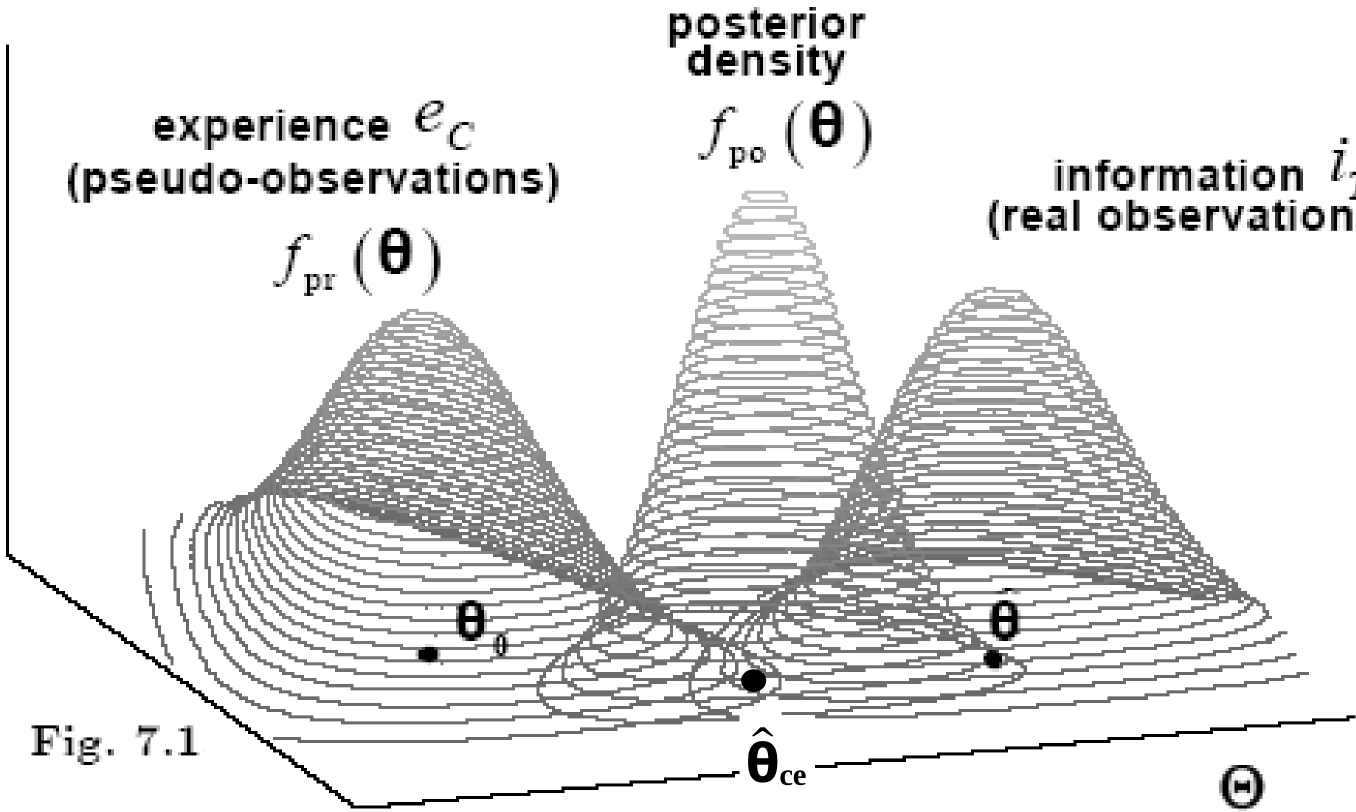

Fig. 7.1

$$f_{\text{po}}(\boldsymbol{\theta}) \begin{array}{l} \nearrow \ \boldsymbol{\theta}_0 \ (C \to \infty) \\ \\ \searrow \ \widehat{\boldsymbol{\theta}} \ (T \to \infty). \end{array} \tag{7.4}$$

$$f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right)=\boxed{f\left(\mathrm{x}_1|\boldsymbol{\theta}\right)}\cdots f\left(\mathrm{x}_T|\boldsymbol{\theta}\right) \qquad (7.13)$$

$$f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right)=f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right)\boxed{f_{\mathrm{pr}}\left(\boldsymbol{\theta}\right)} \qquad (7.15)$$

$$f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right) = \boxed{f\left(\mathrm{x}_1|\boldsymbol{\theta}\right)} \cdots f\left(\mathrm{x}_T|\boldsymbol{\theta}\right) \qquad (7.13)$$

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$$f_{\mathrm{po}}\left(\boldsymbol{\theta}; i_T, e_C\right) \equiv \boxed{f\left(\boldsymbol{\theta}|i_T\right)} = \frac{f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right)}{\int_{\boldsymbol{\Theta}} f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right) d\boldsymbol{\theta}}. \qquad (7.14)$$

BAYESIAN ESTIMATION – NORMAL INVERSE WISHART

$$f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right) = \boxed{f\left(\mathrm{x}_1|\boldsymbol{\theta}\right)} \cdots f\left(\mathrm{x}_T|\boldsymbol{\theta}\right) \tag{7.13}$$

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$$\mathbf{X}_t|\boldsymbol{\mu}, \boldsymbol{\Sigma} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{7.16}$$

$$\boldsymbol{\mu}|\boldsymbol{\Sigma} \sim \mathrm{N}\left(\boldsymbol{\mu}_0, \frac{\boldsymbol{\Sigma}}{T_0}\right) \tag{7.20}$$

$$\boldsymbol{\Sigma}^{-1} \sim \mathrm{W}\left(\nu_0, \frac{\boldsymbol{\Sigma}_0^{-1}}{\nu_0}\right) \tag{7.21}$$

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$$\mathrm{E}\{\boldsymbol{\mu}\} = \boldsymbol{\mu}_0 \qquad (7.23)$$

$$\mathrm{Cov}\{\boldsymbol{\mu}\} = \frac{\nu_0}{\nu_0 - 2}\frac{\boldsymbol{\Sigma}_0}{T_0} \qquad (7.24)$$

$$\boldsymbol{\mu} \sim \mathrm{St}\left(\nu_0, \boldsymbol{\mu}_0, \frac{\boldsymbol{\Sigma}_0}{T_0}\right) \qquad (7.22)$$

$$\boldsymbol{\mu}|\boldsymbol{\Sigma} \sim \mathrm{N}\left(\boldsymbol{\mu}_0, \frac{\boldsymbol{\Sigma}}{T_0}\right) \qquad (7.20)$$

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BAYESIAN ESTIMATION – NORMAL INVERSE WISHART

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$$\boldsymbol{\Sigma}^{-1} \sim \mathrm{W}\left(\nu_0, \frac{\boldsymbol{\Sigma}_0^{-1}}{\nu_0}\right) \tag{7.21}$$

$$\mathrm{E}\left\{\boldsymbol{\Sigma}^{-1}\right\} = \boldsymbol{\Sigma}_0^{-1}. \tag{7.25}$$

$$\mathrm{Cov}\left\{\mathrm{vec}\left[\boldsymbol{\Sigma}^{-1}\right]\right\} = \frac{1}{\nu_0}\left(\mathbf{I}_{N^2} + \mathbf{K}_{NN}\right)\left(\boldsymbol{\Sigma}_0^{-1} \otimes \boldsymbol{\Sigma}_0^{-1}\right) \tag{7.26}$$

$$e_C \equiv \{\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0; T_0, \nu_0\} \tag{7.27}$$

BAYESIAN ESTIMATION – NORMAL INVERSE WISHART

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$$f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right) = \boxed{f\left(\mathrm{x}_1|\boldsymbol{\theta}\right)} \cdots f\left(\mathrm{x}_T|\boldsymbol{\theta}\right) \qquad (7.13)$$

$$f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right) = f_{I_T|\boldsymbol{\theta}}\left(i_T|\boldsymbol{\theta}\right)\boxed{f_{\mathrm{pr}}\left(\boldsymbol{\theta}\right)} \qquad (7.15)$$

$$f_{\mathrm{po}}\left(\boldsymbol{\theta}; i_T, e_C\right) \equiv \boxed{f\left(\boldsymbol{\theta}|i_T\right)} = \frac{f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right)}{\int_{\boldsymbol{\Theta}} f_{I_T,\boldsymbol{\theta}}\left(i_T,\boldsymbol{\theta}\right) d\boldsymbol{\theta}}. \qquad (7.14)$$

$$\mathbf{X}_t|\boldsymbol{\mu},\boldsymbol{\Sigma} \sim \mathrm{N}\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right) \qquad (7.16)$$

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BAYESIAN ESTIMATION – NORMAL INVERSE WISHART

$$T_1 [i_T, e_C] \equiv T_0 + T \tag{7.28}$$

$$\boldsymbol{\mu}_1 [i_T, e_C] \equiv \frac{1}{T_1} [T_0 \boldsymbol{\mu}_0 + T \widehat{\boldsymbol{\mu}}] \tag{7.29}$$

$$\nu_1 [i_T, e_C] \equiv \nu_0 + T \tag{7.30}$$

$$\boldsymbol{\Sigma}_1 [i_T, e_C] \equiv \frac{1}{\nu_1} \left[\nu_0 \boldsymbol{\Sigma}_0 + T \widehat{\boldsymbol{\Sigma}} + \frac{(\boldsymbol{\mu}_0 - \widehat{\boldsymbol{\mu}})(\boldsymbol{\mu}_0 - \widehat{\boldsymbol{\mu}})'}{\frac{1}{T} + \frac{1}{T_0}} \right] \tag{7.31}$$

$$\mathbf{X}_t | \boldsymbol{\mu}, \boldsymbol{\Sigma} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{7.16}$$

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BAYESIAN ESTIMATION – NORMAL INVERSE WISHART

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$$\boxed{\begin{aligned} \widehat{\boldsymbol{\mu}}_{\mathrm{ce}}\left[i_T, e_C\right] &= \frac{T_0\boldsymbol{\mu}_0 + T\widehat{\boldsymbol{\mu}}}{T_0 + T}. \quad (7.35) \\ \mathrm{S}_{\boldsymbol{\mu}}\left[i_T, e_C\right] &= \frac{1}{T_1}\frac{\nu_1}{\nu_1 - 2}\boldsymbol{\Sigma}_1 \quad (7.36) \end{aligned}}$$

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$$\widehat{\boldsymbol{\Sigma}}_{\mathrm{ce}}[i_T, e_C] = \frac{1}{\nu_0 + T + N + 1}\left[\nu_0\boldsymbol{\Sigma}_0 + T\widehat{\boldsymbol{\Sigma}} + \frac{(\boldsymbol{\mu}_0 - \widehat{\boldsymbol{\mu}})(\boldsymbol{\mu}_0 - \widehat{\boldsymbol{\mu}})'}{\frac{1}{T} + \frac{1}{T_0}}\right] \tag{7.38}$$

$$\mathrm{S}_{\boldsymbol{\Sigma}}[i_T, e_C] = \frac{2\nu_1^2}{(\nu_1 + N + 1)^3}\left(\mathbf{D}_N'\left(\boldsymbol{\Sigma}_1^{-1} \otimes \boldsymbol{\Sigma}_1^{-1}\right)\mathbf{D}_N\right)^{-1} \tag{7.39}$$

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