Review Session 5

July 17, 2015

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7.5.2 Expected shortfall and linear factor models

Assume a linear factor model for the market

$$\mathbf{M} \equiv \mathbf{B}\mathbf{F} + \mathbf{U}, \tag{494}$$

where $\mathbf{B}$ is a $N \times K$ matrix with entries of the order of the unit; $\mathbf{F}$ is a K-dimensional vector; $\mathbf{U}$ is a N-dimensional vector; and

$$\begin{pmatrix} \ln \mathbf{F} \\ \ln(\mathbf{U} + \mathbf{a}) \end{pmatrix} \sim \mathrm{St}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{495}$$

where $\boldsymbol{\mu} \equiv \mathbf{0}$, a is such that $\mathrm{E}\{\mathbf{U}\} \equiv \mathbf{0}$ and

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} \epsilon \boldsymbol{\Sigma}^f & \mathbf{0} \\ \mathbf{0} & \epsilon^2 \boldsymbol{\Sigma}^u \end{pmatrix}, \tag{496}$$

with $\boldsymbol{\Sigma}^f$ a correlation matrix and $\boldsymbol{\Sigma}^u$ a Toepliz correlation matrix

$$\Sigma^u_{n,m} \equiv e^{-\gamma|n-m|}, \tag{497}$$

with $\epsilon \ll 1$ and γ arbitrary.

Assume $N \equiv 30$, $K \equiv 10$ and $\nu \equiv 10$. Generate randomly the parameters in $\boldsymbol{\Sigma}$ and the allocation $\boldsymbol{\alpha}$. Then generate $J \equiv 10,000$ Monte Carlo scenarios from the market distribution (494).

Set $c \equiv 0.95$ and compute $\mathrm{ES}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.208) in Meucci (2005).

Compute the K marginal contributions to $\mathrm{ES}_c(\boldsymbol{\alpha})$ from each factor and the one aggregate contribution from all the residuals, as the sample counterpart of (5.238) in Meucci (2005) adapted to the factors. Display the result in a subplot using the built-in plotting function bar.

Hint. Represent the objective as a linear function

$$\Psi \equiv \boldsymbol{\beta}\mathbf{F} + u. \tag{498}$$

$$\underset{n\times 1}{M} = \underset{n\times k}{B}\,\underset{k\times 1}{F} + \underset{n\times 1}{U}$$

$n = 30$
$K = 10$

$$\begin{pmatrix} \ln F \\ \ln(U+a) \end{pmatrix} \sim St(\nu, \mu, \Sigma)$$

$\ln(U+a)$: such that $E[U] = 0$

$\nu = 10$, $\mu = 0$, $\Sigma = \begin{pmatrix} \epsilon \Sigma^f & 0 \\ 0 & \epsilon^2 \Sigma^U \end{pmatrix}$

$\epsilon = 0.1$

Σ^f: correl. matrix

Σ^U: TOEPLITZ corr matrix

$\downarrow$

$\Sigma^U_{n,m} = e^{-\gamma|n-m|}$ $\quad \gamma > 0$

- Σ and α generated randomly

① MC scenarios for the market

② compute $ES_c(\alpha)$ $\quad c = 0.95$
as its sample counterpart

③ Marginal contributions of the factors and of the residuals

<u>Recall</u>

TOEPLITZ MATRIX

$$\begin{bmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \\ g & f & e & a \end{bmatrix}$$

TOEPLITZ <u>CORRELATION</u> MATRIX

$\downarrow$

(semi) DEF. POS, SYMMETRIC, 1 on the principal diagonal

Here: $\Sigma^U = \begin{bmatrix} 1 & e^{-\gamma} & e^{-2\gamma} \cdots & e^{-29\gamma} \\ & 1 & e^{-\gamma} & \cdots \\ & & 1 & \cdots \\ & & & \ddots \end{bmatrix}$

MATLAB

$\Sigma^U =$ toeplitz $\big(\underbrace{\text{exp}(-\gamma\,[0:29])}_{\text{FIRST ROW}}\big)$

$$\Psi_\alpha = \alpha' M = \alpha'(BF + U) = \underbrace{\alpha' B}_{\beta} F + \underbrace{\alpha' U}_{\tilde{u}}$$

RANDOM

$$= \beta F + \tilde{u}$$

$$ES_c(\alpha) = \frac{1}{1-c}\int_0^{1-c} Q_{\psi_\alpha}(p)\,dp = \frac{1}{1-c}\int_0^{1-c} Q_{\beta F+\tilde{u}}(p)\,dp =$$

$$= E\left[\beta F + \tilde{u} \mid \psi_\alpha \le Q_{\psi_\alpha}(1-c)\right]$$

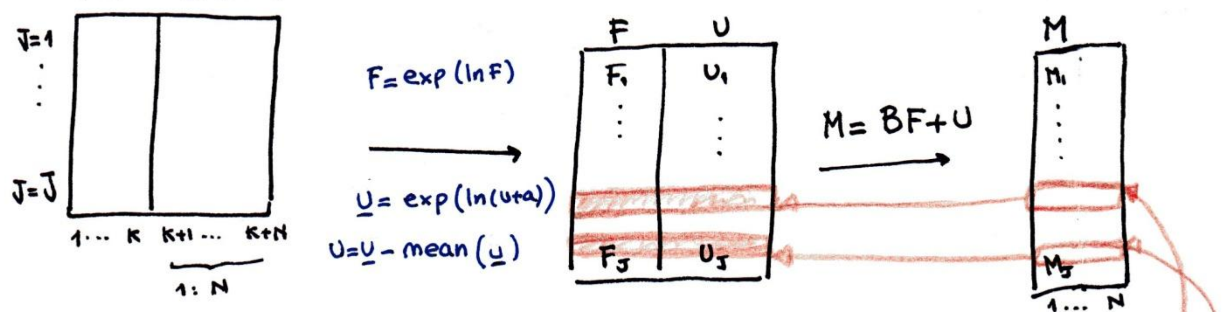
$$= \beta E\left[F \mid \psi_\alpha \le Q_{\psi_\alpha}(1-c)\right] + E\left[\tilde{U} \mid \psi_\alpha \le Q_{\psi_\alpha}(1-c)\right]$$

$$= \sum_{k=1}^{10} \underbrace{\beta_k E\left[F_k \mid \psi_\alpha \le Q\right]}_{\text{marginal contributions from the factors}} + \underbrace{E\left[\tilde{U} \mid \psi_\alpha \le Q\right]}_{\text{marg. contrib from the residuals}}$$

SIMULATION PROCEDURE

Simulate $\begin{pmatrix} \ln F \\ \ln(U+a) \end{pmatrix}$ from a mv. Student t dist

$X = \text{mvtrnd}(\text{corr}, \nu, J)$



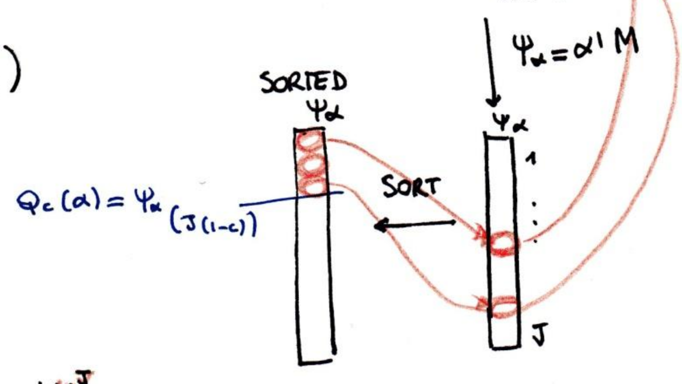
$ES_c(\alpha) = \text{mean}\left(\psi_{\alpha(1)}, \dots, \psi_{\alpha(J(1-c))}\right)$

contributions

$C_K^F = \beta_K \cdot \text{mean}\left(F_K^J \mid \psi_\alpha^J \le Q\right)$

$(\beta = \alpha' B)$

$C^U = \text{mean}\left(\tilde{U}^J \mid \psi_\alpha^J \le Q\right) \quad \tilde{U}^J = \alpha' U^J$



8.1.1 Mean-variance pitfalls: two-step approach

Assume a market of $N \equiv 4$ stocks and all possible zero-coupon bonds. The weekly compounded returns of the stocks are market invariants with the following distribution:

$$\mathbf{C}_{t,\widetilde{\tau}} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{499}$$

Estimate the matrix $\boldsymbol{\Sigma}$ and the vector $\boldsymbol{\mu}$ from the time series of weekly prices in the attached database `StockSeries`. To do this, shrink the sample mean as in (4.138) in Meucci (2005), where the target is the null vector and the shrinkage factor is set as $\alpha \equiv 0.1$. Similarly, shrink as in (4.160) in Meucci (2005) the sample covariance to a suitable multiple of the identity by a factor $\alpha \equiv 0.1$.

Assume that the weekly changes in yield to maturity for the bond market are fully codependent, i.e. co-monotonic. In other words, assume that the copula of any pairs of weekly yield changes is (2.106) in Meucci (2005). Also, assume that they have the following marginal distribution:

$$\Delta_{\widetilde{\tau}} Y^{(\upsilon)} \sim \mathrm{N}\left(0, \left(\frac{20 + 1.25\upsilon}{10,000}\right)^2\right), \tag{500}$$

where υ denotes the generic time to maturity (measuring time in years).

Assume that the bonds and the stock market are independent. Assume that the current stock prices are the last set of prices in the time series. Restrict your attention to bonds with times to maturity 4, 5, 10, 52 and 520 weeks, and assume that the current yield curve, as defined in (3.30) in Meucci (2005) is flat at 4%.

Produce joint simulations of the four stock and five bond prices at the investment horizon τ of four weeks.

Assume that the investor considers as his market one single bond with time to maturity $\upsilon \equiv$ five weeks and all the stocks.

Determine numerically the mean-variance inputs, namely expected prices and covariance of prices (*not* returns).

Determine analytically the mean-variance inputs, namely expected prices and covariance of prices (*not* returns) and compare with their numerical counterpart.

Assume that the investor's objective is final wealth. Suppose that his budget is $w \equiv 100$. Assume that the investor cannot short-sell his securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier as represented by a grid of 40 portfolios whose expected values are equally spaced between the expected value of the minimum variance portfolio and the largest expected value among the portfolios composed of only one security.

Assume that the investor's satisfaction is the certainty equivalent associated with an exponential function

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{501}$$

where $\zeta \equiv 10$.

Compute the optimal allocation according to the two-step mean-variance framework.

Hints

- Do not use portfolio weights and returns. Instead, use number of securities and prices.
- Given the no-short-sale constraint, the minimum variance portfolio cannot be computed analytically, as in (6.99) in Meucci (2005): use `quadprog` to compute it numerically.
- Given the no-short-sale constraint, the frontier cannot be computed analytically, as in (6.97)-(6.100) in Meucci (2005): use `quadprog` to compute it numerically.

Summary

0) Market 4 stocks, all ZCB

P1) QUEST FOR INVARIANCE

a) RISK DRIVERS $\ln P$ (stocks)

$Y^{(v)}$ (bonds)

b) INVARIANTS $C_{\tilde{\tau}} = \ln\left(\frac{P_t}{P_{t-\tilde{\tau}}}\right)$

$\Delta_{\tilde{\tau}} Y^{(v)} = Y_t^{(v)} - Y_{t-\tilde{\tau}}^{(v)}$

$\tilde{\tau} = 1$ week

P2) ESTIMATION

$C_{\tilde{\tau}} \sim N(\mu, \Sigma)$

INDEPENDENT TIME TO MAT (years)

$\Delta_{\tilde{\tau}} Y^{(v)} \sim N\left(0, \left(\frac{20+1.25 v}{10000}\right)^2\right)$ COMONOTONIC

↓

sample mean, sample cov + shrinkage

↓

$\hat{\mu} = (1-\alpha)\,\hat{\mu}_{sample} + \alpha \cdot 0$

$\hat{\Sigma} = (1-\alpha)\,\hat{\Sigma}_{sample} + \alpha \cdot \epsilon I$

P3) PROJECTION

$\tilde{\tau} = 1$ week (estimation step)

$\tau = 4$ weeks (investment horizon) $\rightarrow \frac{\tau}{\tilde{\tau}} = 4$

$C_{\tau} \sim N(4\hat{\mu}, 4\hat{\Sigma})$

$\Delta_{\tau} Y^{(v)} \sim N(0, 4 \cdot (\dots)^2)$

P4) PRICING

$$P_{T+4} = P_T \cdot \exp(C_4) \sim \log N(\log P_T + 4\hat{\mu},\ 4\hat{\Sigma})$$

FOCUS ON THE 5 weeks ZCB

T T+4 E=T+5

$\underbrace{\quad}_{v=1}$

$$Z^{(T+5)}_{T+4} = \exp(\underset{\substack{\uparrow \\ \text{known } (4\%)}}{-y_T^{(1)}}) \cdot \exp(-\underbrace{\Delta_4 Y^{(1)}}_{\text{Normal}}) \sim \log N(\dots)$$

P5) AGGREGATION

objective is final wealth

$$\Psi_\alpha = \alpha' M$$

holdings $(\alpha^0, \underbrace{\alpha^1, \dots, \alpha^4}_{\text{stocks}})$

α^0: 5 WEEKS BOND

M: $\left(Z^{(T+5)}_{T+4}, P^{(1)}_{T+4}, \dots, P^{(4)}_{T+4}\right)$

P6) ATTRIBUTION $\rightarrow$ security level

P7) EVALUATION : index of satisfaction is the CE for exponential utility $u(\Psi) = -e^{-\frac{1}{10}\Psi}$

P8) OPTIMIZATION $\rightarrow$ MV

STEP 1 : Dimension reduction $\rightarrow$ find the efficient frontier

STEP 2 : Find the allocation which gives rise to the highest satisfaction over the efficient frontier

① $\mu_M = E[M]$

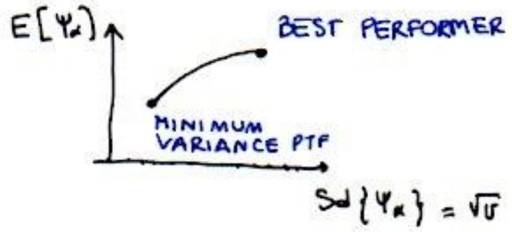
$\Sigma_M = Cov[M]$

$$\begin{cases} \alpha(v) = \underset{\alpha}{\operatorname{arg\,max}}\ \alpha'\mu_M & \leftarrow E[\Psi_\alpha] \\ \text{st} \quad \alpha'\Sigma_M\alpha = v & \rightarrow Var[\Psi_\alpha] = v \\ \quad\ \ \alpha' P_T = b & \rightarrow \text{budget} \\ \quad\ \ \alpha \geq 0 & \rightarrow \text{no short selling} \end{cases}$$

P_T = current prices (including the bond)

Frontier → curve in the N dimensional space.

Representation in the expected objective / std space :

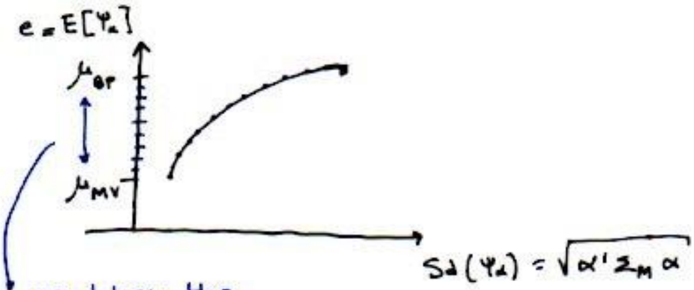


IN THE CODE : Dual representation of the pb (LINEAR CONSTRAINTS) RISK AVERSION

$$\begin{cases} \alpha(e) = \underset{\alpha}{\arg\min} \quad \alpha' \Sigma_M \alpha \\ \alpha' \mu_M = e \\ \alpha' p_T = b \\ \alpha \geq 0 \end{cases}$$

→ variance of Ψ_α minimized for each level of $E[\Psi_\alpha]$

Quadprog : quadratic objective
linear constraints



partition the interval into 40 points

Solve the problem using quadprog for each of the points

→ 40 portfolios ∈ Frontier : $\alpha_1, \alpha_2 \dots, \alpha_{40}$
MIN VARIANCE — BEST PERFORMER

② compute the expected utility (better: the C.E.) of the 40 allocations and pick the best one.

$$\alpha^* = \arg\max \left(S(\alpha(B) \right)$$

8.1.2 Mean-variance pitfalls: horizon effect

Consider the stock market described in the previous exercise and the investment horizon τ of one day.

Determine analytically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns and compare with their numerical counterpart.

Assume that the investor's objective is final wealth. Suppose that the budget is $b \equiv 100$. Assume that the investor cannot short-sell the securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Assume that the investor's satisfaction is the certainty equivalent associated with an exponential function

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{502}$$

where $\zeta \equiv 10$.

Compute the optimal allocation according to the two-step mean-variance framework.

Repeat the above steps an the investment horizon τ of four years

Same settings as 8.1.1.

investment horizon $\tau = 1$ day ; $\tau = 4$ years

MV inputs $\rightarrow$ weights & returns

correct way to proceed:

INVARIANTS (comp. rets for stocks) $\longrightarrow$ PROJECTION $\longrightarrow$ PRICES M $\longrightarrow$ LINEAR RETURNS R

otherwise the optimal allocation ends up to be independent from the horizon (pitfall)

OBJECTIVE $\Psi_w = w'R$

STEP 1

$\mu_R = E[R]$

$\Sigma_R = Cov[R]$

$$\begin{cases} w(v) = \operatorname{argmax} \; w'\mu_R & \\ w'\Sigma_R w = v & Var(\Psi_w) = v \\ w \geq 0 & \text{no shortselling} \\ \sum w_i = 1 & \text{budget constraint} \end{cases}$$

$\downarrow$

in the code: dual representation
QUADPROG.

STEP 2 $\quad w^* = \operatorname{argmax} \; S(\alpha(v))$

$\uparrow$

CE exponential utility

8.1.3 Benchmark driven allocation

Consider the market described in Exercise 8.1.1, namely one single bond with time to maturity $v \equiv$ five weeks and all the stocks.

Produce joint simulations of the four stock and the one bond prices at the investment horizon τ of four weeks.

Determine numerically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns.

Determine analytically the mean-variance inputs in terms of weights and returns, namely expected linear returns and covariance of linear returns and compare with their numerical counterpart.

Assume first that the investor's objective is final wealth. Suppose that the budget is $b \equiv 100$. Assume that the investor cannot short-sell the securities, i.e. the allocation vector cannot include negative entries.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Now assume that the investor's objective is wealth relative to an equal-weight benchmark, i.e. a benchmark that invests an equal amount of money in each security in the market.

Compute the mean-variance efficient frontier in terms of portfolio weights as represented by a grid of 40 portfolios whose expected linear returns are equally spaced between the expected value of the linear return on the minimum variance portfolio and the largest expected value among the linear returns of all the securities.

Project the two efficient frontiers computed above in the expected outperformance/tracking error plane.

Same framework

compare efficient frontier when
- OBJECTIVE = Final wealth
- OBJECTIVE = wealth relative to an equally weighted benchmark

Benchmark driven allocation

- P1 - P4 as above
  (INVARIANTS → ESTIMATION → PROJECTION → PRICING)

- P5 AGGREGATION

  Linear returns - weights setting for the MV optimization:

  OBJECTIVE = OVERPERFORMANCE = $\alpha' P_{T+\tau} - \gamma \beta' P_{T+\tau}$

  normalization constant (makes initial investment comparable)

  $$\Psi_\alpha = \alpha' M = \alpha' \underbrace{\left(I - \frac{P_T \beta'}{\beta' P_T} \right)}_{K} P_{T+\tau} \qquad \gamma = \frac{\alpha' P_T}{\beta' P_T}$$

  α: allocation

  β: benchmark portfolio

  NOTE: $\Psi_\beta \equiv 0 \rightarrow \begin{cases} EOP_\beta = 0 \\ TE_\beta = 0 \end{cases}$

  [EOP: EXPECTED OVERPERF. $E[\Psi_\alpha]$

  [TE: TRACKING ERROR $Sd[\Psi_\alpha]$

For a budget b, the return based objective is

$$\frac{\Psi_\alpha}{b} = \frac{\alpha' K P_{T+\tau}}{\underset{\alpha' P_T}{b}} = \frac{\alpha' P_{T+\tau}}{\alpha' P_T} - \frac{\cancel{(\alpha' P_T)} \beta' P_{T+\tau}}{\cancel{(\alpha' P_T)} \beta' P_T} =$$

$$= \underbrace{\left[\frac{\alpha' P_{T+\tau}}{\alpha' P_T} - 1 \right]}_{L_\alpha} - \underbrace{\left[\frac{\beta' P_{T+\tau}}{\beta' P_T} - 1 \right]}_{L_\beta}$$

LIN RETS (PORTFOLIO) LIN RETS (BENCHMARK)

P8) MV optimization

$$\begin{cases} \alpha(e) = \underset{\substack{\alpha' p_T = b \\ \alpha \geq 0 \\ E\{L_\alpha - L_\beta\} = e}}{\operatorname{argmin}} \ Var\{L_\alpha - L_\beta\} \end{cases}$$

$= Var\{L_\alpha\} + \cancel{Var\{L_\beta\}} - 2\,Cov\{L_\alpha, L_\beta\}$

indep of our decision variable α → can be drop

↓ objective function

$Var\{L_\alpha\} - 2\,Cov\{L_\alpha, L_\beta\}$

↓ weights - return space

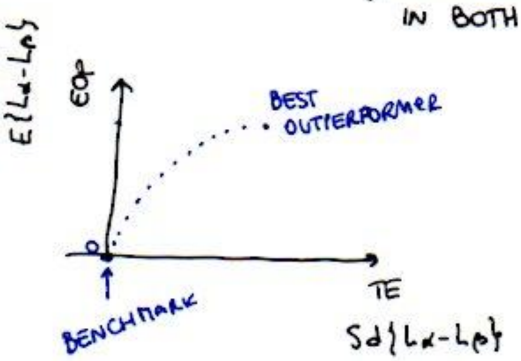
w: portfolio weights

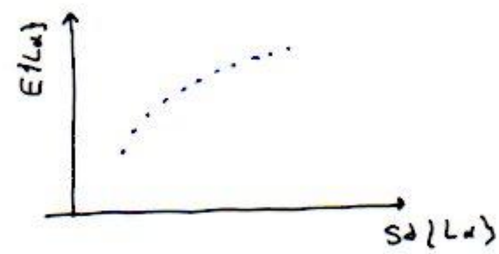
w_b: benchmark weights (known)

securities returns : L → Estimate $E\{L\}$, $Cov\{L\}$

$$\begin{cases} w(e) = \underset{\substack{w' 1 = 1 \\ w \geq 0 \\ w' E\{L\} - w_b' E[L] = e}}{\operatorname{argmin}} \ w' Cov\{L\} w - 2\, w' Cov\{L\} w_b \end{cases}$$

↓ THE EFFICIENT FRONTIER w(e) CAN BE REPRESENTED IN BOTH THE EOP/TE space and $E\{L_\alpha\}/Sd\{L_\alpha\}$ space





↑ compare with the "final wealth" efficient frontier in the two spaces (see code)

• MV STEP 2

pick the allocation maximizing the satisfaction