Review of Dynamic Allocation Strategies
Utility Maximization, Option Replication, Insurance, Drawdown Control, Convex/Concave Management

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Abstract

We review the main approaches to dynamically reallocate capital between a risky portfolio and a risk-free account: expected utility maximization; option-based portfolio insurance (OBPI); and drawdown control, closely related to constant proportion portfolio insurance (CPPI). We present a refresher of the theory under general assumptions. We discuss the connections among the different approaches, as well as their relationship with convex and concave strategies. We provide explicit, practicable solutions with all the computations as well as numerical examples. Fully documented code for all the strategies is also provided.

JEL Classification: C1, G11

Keywords: CPPI, OBPI, drawdown control, option replication, dynamic programming, Hamilton-Jacobi-Bellman equation, Pontryagin principle, geometric Brownian motion, power utility, constant exposure portfolio, buy-and-hold

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This article provides a quick overview of the main strategies for dynamically allocating capital between a risky portfolio and a risk-free account. Other material and pointers to relevant references can be found in Browne and Kosowski (2010).

The article is organized as follows. In Section 1 we introduce the dynamics of the two investment vehicles and in Section 2 we discuss the rules to build strategies in full generality.

In Section 3 we discuss strategies that maximize the expected utility of the strategy at the investment horizon. The solution of this type of problem in full generality requires advanced techniques, but a special notable case can be solved explicitly with little effort. In Section 4 we discuss strategies that achieve a desired, arbitrary payoff that is designed as a function of the random outcome of the risky asset at the horizon. These strategies are also known under the misnomer of option-based-portfolio-insurance. In Section 5 we cover drawdown control and constant proportion portfolio insurance. In Section 6 we discuss the connections among the above approaches and how they relate to convex and concave strategies.

1 The market

We denote by D_t the value of one unit of the risk-free asset at the generic time t, which evolves deterministically. The reference model for the evolution of the deterministic asset D_t is the exponential deterministic growth with constant risk-free rate r

$$\frac{dD_t}{D_t} = r dt. \tag{1}$$

We can think of this asset as the money market.

We denote by P_t the value of one unit of the risky asset at the generic time t, which evolves stochastically. We assume that P_t follows a general diffusive process

$$dP_t = \mu(t, P_t)\, dt + \sigma(t, P_t)\, dB_t. \tag{2}$$

In this expression the deterministic drift $\mu(t,p)$ and the volatility $\sigma(t,p)$ are smooth functions of their arguments; B_t is a Brownian motion, i.e. a stochastic process such that all non-overlapping increments are independently and normally distributed with null expectation and variance equal to the time step $B_{t+s} - B_t \sim \mathrm{N}(0,s)$; and dB_t denotes the infinitesimal increments of the Brownian motion $dB_t \equiv B_{t+dt} - B_t \sim \mathrm{N}(0,dt)$.

In reality, there exists no truly risk-free investment that evolves as (1). At best, one can identify a risk-free security for a specific horizon, i.e. a zero-coupon government bond that matures at that horizon. However, before the horizon such a security is risky. Similarly, the process (2) does not represent the most generic dynamics for a risky asset. For instance, jumps or hidden factors are not included in the model. Therefore (1)-(2) are to be considered acceptable, not too restrictive approximations.

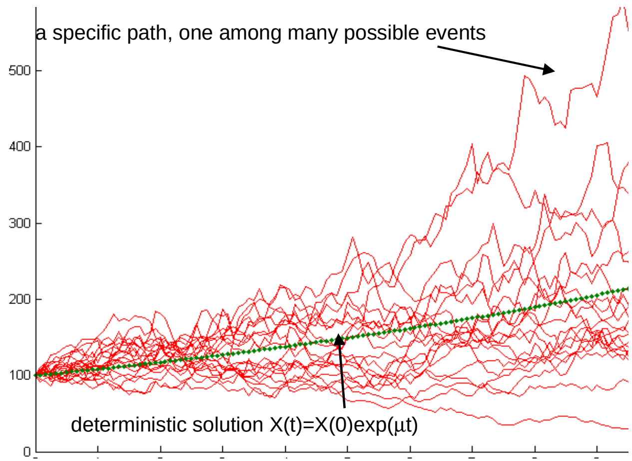

Figure 1: Geometric Brownian motion

The reference model for the evolution of the stochastic asset P_t is the geometric Brownian motion with constant drift and constant volatility adopted among others by Merton (1969) and Black and Scholes (1973)

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t, \tag{3}$$

which is the special case of the diffusion (2) with

$$\mu(t,p) \equiv \mu p, \quad \sigma(t,p) \equiv \sigma p. \tag{4}$$

As we show in Appendix A.2, this process integrates to a lognormal random variable at any horizon

$$P_t = P_0 e^{Y_t}, \tag{5}$$

where Y_t is normal

$$Y_t \sim \mathrm{N}\left((\mu - \frac{\sigma^2}{2})t, \sigma^2 t\right). \tag{6}$$

In Figure 1 we plot some paths of this process.

2 Strategies

We denote by α_t the number of units of the risky asset at time t and by β_t the number of units of the risk-free asset. We denote the current time by $t \equiv 0$ and the investment horizon by $t \equiv \tau$.

A strategy is a sequence of allocations that rebalances between the two assets at the generic time t throughout the investment period based on the information available at time t

$$\{\alpha_t, \beta_t\}_{t\in[0,\tau]} \,. \tag{7}$$

The combined value S_t of a the strategy at the generic time t is

$$S_t = \alpha_t P_t + \beta_t D_t. \tag{8}$$

The strategy must be self-financing, i.e. whenever a rebalancing occurs $(\alpha_t, \beta_t) \mapsto (\alpha_{t+\delta t}, \beta_{t+\delta t})$ the following must hold true

$$S_{t+\delta t} \equiv \alpha_t P_{t+\delta t} + \beta_t D_{t+\delta t} \equiv \alpha_{t+\delta t} P_{t+\delta t} + \beta_{t+\delta t} D_{t+\delta t}. \tag{9}$$

The strategy (7) can be represented equivalently in terms of the relative weights of the risky and risk-free investments respectively

$$w_t \equiv \frac{\alpha_t P_t}{S_t}, \quad u_t \equiv \frac{\beta_t D_t}{S_t}. \tag{10}$$

The self-financing constraint (9) is equivalent to the weight of the risk-free asset being equal to $u_t \equiv 1 - w_t$ at all times.

Therefore the strategy (7) can be represented equivalently by the free evolution of the weight of the risky asset $\{w_t\}_{t\in[0,\tau]}$. When w_t is close to 1 the strategy evolves as the risky asset P_t. Conversely, if w_t is close to zero, the strategy evolves as the risk-free asset. More in general, as we show in Appendix A.1, the strategy evolves as

$$\frac{dS_t}{S_t} = (1 - w_t)\, r dt + w_t \frac{dP_t}{P_t}. \tag{11}$$

The dynamic rebalancing path for the weight of the risky asset $\{w_t\}_{t\in[0,\tau]}$ and the initial budget constraint

$$S_0 \text{ given.} \tag{12}$$

determine the distribution of the final payoff of the strategy

$$S_0, \{w_t\}_{t\in[0,\tau]} \mapsto S_\tau. \tag{13}$$

As we show in Appendix A.2, when the risky investment follows a geometric Brownian motion (3), the final value S_τ is lognormally distributed

$$S_\tau = S_0 e^{Y_{w(\cdot)}}, \tag{14}$$

where Y is normal

$$Y_{w(\cdot)} \sim \mathrm{N}\left(m_{w(\cdot)}, s^2_{w(\cdot)}\right) \tag{15}$$

with expected value

$$m_{w(\cdot)} \equiv r\tau + \int_0^\tau \left((\mu - r)\, w_t - \frac{\sigma^2}{2} w_t^2 \right) dt \tag{16}$$

and variance

$$s_{w_{(\cdot)}}^2 = \int_0^\tau \sigma^2 w_t^2 dt. \tag{17}$$

In the above the notation $w_{(\cdot)}$ is short for the dynamic rebalancing path $\{w_t\}_{t\in[0,\tau]}$. As expected, when the strategy fully invests in the risky asset at all times, i.e. $w_t \equiv 1$, then the strategy value (14) and the risky asset value (5) are the same.

We emphasize that the strategy framework (8) can be hierarchical. Indeed, the risky investment P_t can be as a single security, but also another strategy, managed exogenously.

For instance, the risky investment (3) that evolves as a geometric Brownian motion can be obtained as a portfolio of N securities where the weights $\mathbf{w}_t \equiv (w_{t,1}, \ldots, w_{t,N})'$ remain constant $\mathbf{w}_t \equiv \mathbf{w}$, and the prices $\mathbf{P}_t \equiv (P_{t,1}, \ldots, P_{t,N})'$ evolve according to a multivariate geometric Brownian motion

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\mathbf{B}_t, \tag{18}$$

where the division is meant entry-by-entry, $\boldsymbol{\mu}$ is a N-dimensional vector, $\boldsymbol{\Sigma}$ is a full-rank $N \times N$ matrix, and $\mathbf{B}_t$ are N independent Brownian motions. Indeed, as we show in Appendix A.2, in this case the portfolio follows the geometric Brownian motion (3), where

$$\mu \equiv \mathbf{w}'\boldsymbol{\mu}, \quad \sigma^2 \equiv \mathbf{w}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{w}. \tag{19}$$

3 Expected utility maximization

The first class of strategies that we consider is arguably the most intuitive: the investor maximizes the expected utility of the payoff at a given investment horizon, an approach, pioneered by Merton (1969)

First of all, we introduce a utility function u that allows us to measure the satisfaction ensuing from the horizon payoff S_τ of a strategy in terms of the expected utility $\mathrm{E}\{u(S_\tau)\}$. Assuming that "rich is better than poor", the function u must be increasing

$$u' > 0. \tag{20}$$

Also, assuming that "richer is better when poor, than when rich", it is common practice to assume that u is concave

$$u'' < 0. \tag{21}$$

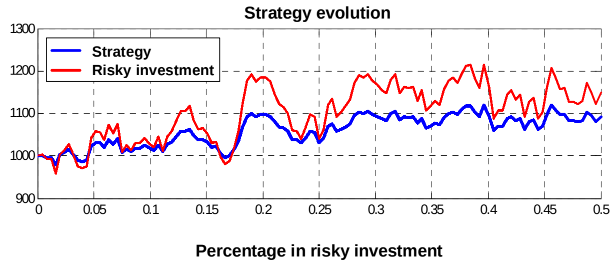

Figure 2: Constant weight dynamic strategy: one path

The most standard utility function is the power function

$$u(s) = \frac{s^{\gamma}}{\gamma}, \tag{22}$$

where $\gamma < 1$. This function satisfies (20) and (21). Indeed

$$u'(s) = s^{\gamma-1} > 0 \tag{23}$$

$$u''(s) = (\gamma - 1) s^{\gamma-2} < 0. \tag{24}$$

As in (13), the final payoff S_τ of a strategy depends on the budget constraint (12) on the initial value S_0 and the dynamic rebalancing path $w_{(\cdot)}$. The investor selects the optimal dynamic allocation path $w^*_{(\cdot)}$ in such a way to maximize the expected utility of the strategy at the horizon under the budget constraint and a potential set of additional constraints $\mathcal{C}$

$$w^*_{(\cdot)} \equiv \underset{S_0, w_{(\cdot)} \in \mathcal{C}}{\operatorname{argmax}} \left(\mathrm{E}\left\{ u\left(S_\tau\right) \right\} \right). \tag{25}$$

There exist two approaches to solve this problem in general: dynamic programming, as in Merton (1969) and Merton (1992), and martingale methods explored by Pliska (1986), Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987).

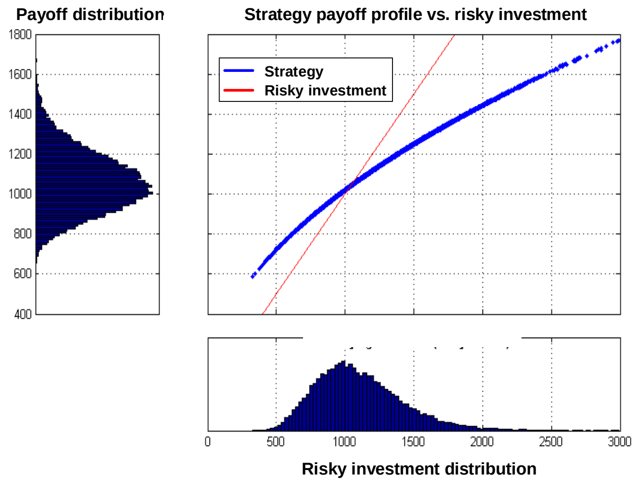

Figure 3: Constant weight dynamic strategy: final payoff in many paths

In our simplified market dynamics (11) with power utility (22) and under no other constraint than the budget, we can solve for the optimal strategy (25) by direct computation, as in Omberg (2001).

From the lognormal expression for the strategy payoff (14) and the expression for the expectation a lognormal variable $X \sim \text{LogN}\left(\mu, \sigma^2\right)$, which reads $\mathrm{E}\left\{X\right\} = e^{\mu+\sigma^2/2}$, we obtain

$$\mathrm{E}\left\{u\left(S_\tau\right)\right\} = \mathrm{E}\left\{\frac{S_\tau^\gamma}{\gamma}\right\} = \frac{S_0^\gamma}{\gamma}\mathrm{E}\{e^{\gamma Y_{w_{(\cdot)}}}\} = \frac{S_0^\gamma}{\gamma}e^{\gamma\left(m_{w_{(\cdot)}}+\frac{\gamma}{2}s^2_{w_{(\cdot)}}\right)}. \tag{26}$$

Therefore the optimal dynamic rebalancing path $w^*_{(\cdot)}$ maximizes $m_{w_{(\cdot)}}+\gamma s^2_{w_{(\cdot)}}/2$. From the definition of $m_{w_{(\cdot)}}$ and $s^2_{w_{(\cdot)}}$ in (16)-(17) this means

$$w^*_{(\cdot)} \equiv \underset{w_{(\cdot)}}{\operatorname{argmax}}\left\{\int_0^\tau \left(w_t\left(\mu - r\right) - w_t^2\frac{\sigma^2}{2}\left(1-\gamma\right)\right)dt\right\}. \tag{27}$$

The solution to this problem is the value that maximizes the integrand at each time. Therefore, the solution is the constant

$$w^*_{(\cdot)} \equiv w \equiv \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}. \tag{28}$$

As in the standard mean-variance approach, the more promising the risky investment, i.e. large μ or small σ, the more the investor will allocated in the risky asset. Also, γ can be interpreted as a risk-propensity parameter: the closer the parameter γ to the upper boundary 1, the more the investor borrows cash to buy the risky asset.

In Figure 2 we plot the evolution of the constant weight strategy in a specific path. In Figure 3 we plot the payoff profile of this strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

4 Dynamic payoff replication

Here we discuss a second class of dynamic strategies, namely option-based portfolio insurance (OBPI), proposed by Rubinstein and Leland (1981). OBPI is quite different from the approach discussed in Section 3, where the investor optimizes the expected utility of the random payoff S_τ of a dynamic strategy. Instead, OBPI realizes that such a payoff is a *deterministic* function of the random outcome P_τ of the risky asset at the horizon. Therefore, in OBPI we design exogenously a desired, *arbitrary* payoff function

$$S_\tau \equiv s\left(P_\tau\right). \tag{29}$$

For instance, we can design and achieve a payoff such as $s(p) \equiv \sin(p)$. As another, more interesting, example, we can choose a call option payoff with strike K

$$s(p;K) \equiv \max\left(0, p-K\right). \tag{30}$$

As it turns out, this profile is very special, because any arbitrary profile $s(p)$ can be obtained by means of a buy-and-hold portfolio of a few strategies, each of which attains a call option profile $s(p;K)$ with a different strike K, see the proof in Appendix A.4.

Then we follow a strategy that attains the desired payoff (29). More precisely we follow a four-step process, see all the proofs in Appendix A.3.

First, we estimate the volatility of the risky investment as a function of time and the risky investment $\sigma(t,p)$ in (2). Second, we compute the solution $G(t,p)$ of the following partial differential equation

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial p} r + \frac{1}{2}\frac{\partial^2 G}{\partial p^2}\left(\sigma\left(t,p\right)\right)^2 - Gr = 0, \tag{31}$$

with the strategy payoff as a boundary condition

$$G(\tau,p) \equiv s(p), \quad p > 0. \tag{32}$$

Third, we set aside the required budget to achieve the desired payoff profile, which is

$$S_0 \equiv G\left(0, P_0\right). \tag{33}$$

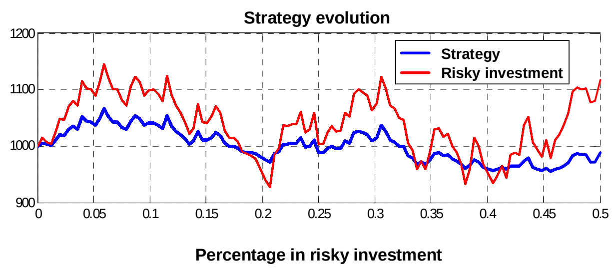

Figure 4: Dynamic replication of call option profile: one path

Fourth, we rebalance dynamically the total budget S_t by investing at each time t the following portion in the risky asset

$$w_t \equiv \frac{P_t}{S_t} \frac{\partial G\left(t, P_t\right)}{\partial P_t}. \tag{34}$$

This dynamic strategy provides the desired payoff (29).

The readers familiar with the literature on option pricing initiated by Black and Scholes (1973) will recognize the partial differential equation (31). Those readers will also recognize that the partial derivative $\partial G\left(t, p\right) / \partial p$ in (34) is the "delta". For our concerns, the above is but a four-step practical recipe to achieve the desired goal.

Assume that the risky investment follows a geometric Brownian motion (3). Then $\sigma\left(t, p\right) = \sigma p$ and the partial differential equation (31) becomes

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial p} r + \frac{1}{2} \frac{\partial^2 G}{\partial p^2} \sigma^2 p^2 - G r = 0. \tag{35}$$

Suppose that we want to obtain the call option payoff profile (30). This means that the boundary condition (32) becomes

$$G\left(\tau, p\right) \equiv \max\left(0, p - K\right), \quad p > 0. \tag{36}$$

The solution of the partial differential equation (35) with boundary condition (36) can be computed explicitly and reads

$$G\left(t, p\right) = p \Phi\left(d_1\right) - e^{-r(\tau - t)} K \Phi\left(d_2\right), \tag{37}$$

where Φ is the cumulative distribution for the standard normal distribution and

$$d_1(t,p) \equiv \frac{1}{\sigma\sqrt{\tau - t}}\left(\ln\left(\frac{p}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(\tau - t)\right) \tag{38}$$

$$d_2(t,p) \equiv d_1(t,p) - \sigma\sqrt{\tau - t}. \tag{39}$$

This is the celebrated result in Black and Scholes (1973), the proof of which can be easily found on the internet.

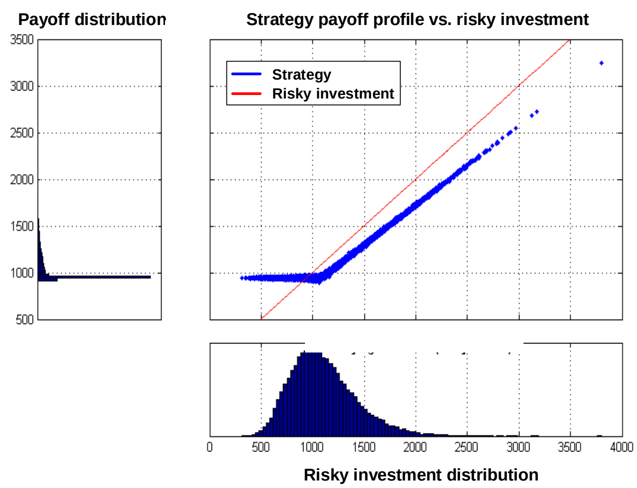

Figure 5: Dynamic replication of call option profile: final payoff in many paths

From the explicit analytical expression (37) for $G(t,p)$ we can compute the "delta" $\partial G(t,p)/\partial p = \Phi(d_1(t,p))$, again find the proof on the internet. Then the weight of the risky asset (34) becomes

$$w_t = \frac{P_t}{S_t}\Phi(d_1(t,P_t)). \tag{40}$$

In Figure 4 we plot the evolution of this strategy along a specific path. In Figure 5 we plot the payoff profile of the strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

5 Drawdown control

In Section 3 we discussed expected utility maximization, which attempts to optimize the strategy payoff. In Section 4 we covered dynamic payoff replication, which aims at shaping the payoff as a function of the underlying risk. Here we discuss heuristics that constrain the final payoff, in an attempt to offer protection against undesired scenarios.

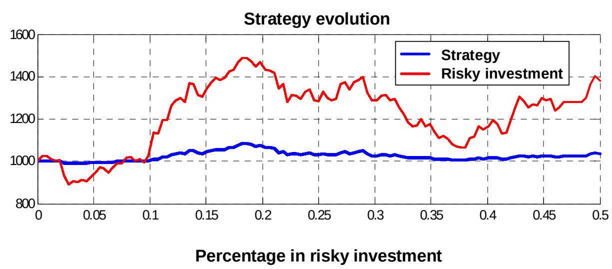

Figure 6: CPPI: one path

The most popular heuristic in this direction is the constant proportion portfolio insurance (CPPI) by Black and Perold (1992). According to this strategy, the investor first specifies a minimal final wealth level H at the horizon, which should under no circumstance be violated. Then a floor F_t is computed, which grows to the value H at the horizon and which can be replicated with a full investment in the risk-free account

$$F_t \equiv He^{-r(\tau-t)}, \quad t \in [0,\tau]. \tag{41}$$

This floor determines the minimum budget necessary to set up the strategy

$$S_0 > F_0. \tag{42}$$

At all times t, for any level of the strategy S_t there is an excess cushion

$$C_t \equiv \max(0, S_t - F_t). \tag{43}$$

Then, according to the CPPI strategy, a constant multiple m of the cushion is invested in the risky asset,. Therefore, the CPPI weight reads

$$w_t \equiv \frac{mC_t}{S_t}. \tag{44}$$

As the strategy gains value, the investor puts a relatively larger amount of money in the risky asset. As wealth shrinks toward the floor (41) the investor turns more and more conservative. In any case, the value of the strategy remains above the floor at all times $S_t > F_t$ and in particular at the horizon, at which point the value H is recovered with certainty: $S_\tau > F_\tau \equiv H$.

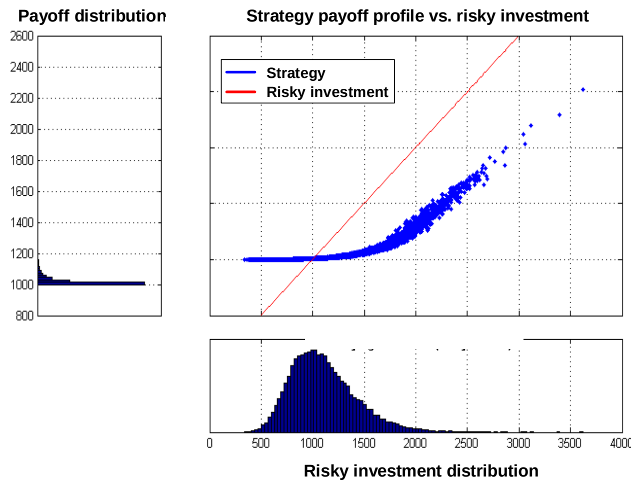

Figure 7: CPPI: final payoff in many paths

To illustrate, we can perform the CPPI under the geometric Brownian motion assumption (3) for the evolution of the risky asset. Bouye (2009) provides the analytical expression for the cushion. In Figure 6 we plot the evolution of this strategy in a specific path. In Figure 7 we plot the payoff profile of this strategy as a function of the risky asset in many paths. Please refer to Meucci (2009) for more details and to download the fully commented code that generated the figures.

The CPPI is closely related to drawdown control strategies, see Browne and Kosowski (2010) and references therein. The drawdown is defined in terms of the peak of the strategy $M_t \equiv \max \{S_u\}_{u\in[0,t]}$ as the difference between the peak and the current value of the strategy

$$D_t \equiv M_t - S_t. \tag{45}$$

The drawdown, which is always positive, should ideally be as small as possible. The CPPI limits the drawdown, because from (41) we obtain

$$D_t < M_t - He^{-r(\tau-t)}. \tag{46}$$

However, the constraint (46) might prove unacceptably lax if the strategy has recorded substantial profits. To address this issue, Grossman and Zhou (1993) devised a "rolling" version of the CPPI, where the fixed floor (41) is replaced with a given portion of the current maximum

$$F_t \equiv \gamma M_t, \tag{47}$$

for $\gamma \in (0,1)$. Then the investable cushion is defined as in (43) and the dynamically rebalanced weights are defined as in (44). Notice that in this framework the notion of investment horizon is lost.

6 Concave versus convex management

As we can appreciate in Figure 3, the constant-weight strategy ensuing from utility maximization is concave: concave strategies perform better when the risky asset moves sideways, but worse when the risky asset trends downward.

On the other hand, the call option dynamic replication in Figure 5 and the CPPI in Figure 7 are convex strategies: convex strategies protect the investor when the risky asset underperforms, although they fail to capture all the upside when the risky asset rallies. Therefore convex strategies are suitable to provide portfolio insurance.

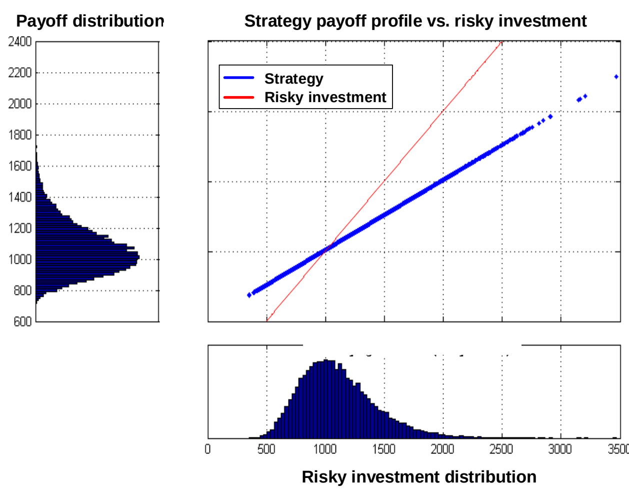

Figure 8: Buy & hold strategy: final payoff in many paths

Finally, the buy-and-hold strategy is linear, see Figure 8: only a percentage of the risk in the risky asset is reflected by the strategy, so that both upside and downside are limited. Notice that the buy-and-hold strategy is not the same as the constant weight strategy, see Figure 9: indeed, in the buy-and-hold strategy as the risky asset rallies, the respective weight in the portfolio increases.

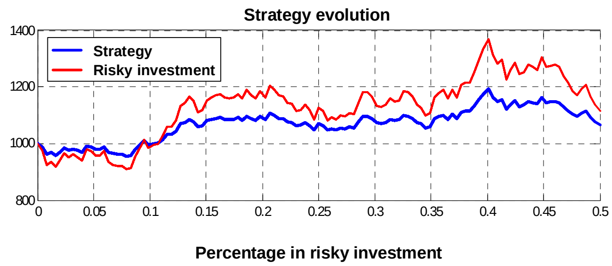

Figure 9: Buy & hold strategy: one path

In general, all the different dynamic strategies discussed here are interconnected and can display convex or concave profiles, see also Bertrand and Prigent (2001), Bouye (2009) and references therein.

Indeed, the dynamic payoff replication discussed in Section 4 can be linear, convex, or concave by definition, as this depends on the subjective choice of the shape of the final payoff (29). Therefore, the term "option based portfolio insurance" given to this approach is a misnomer, because portfolio insurance refers to convex strategies only.

The expected utility maximization discussed in Section 3 can also be both convex and concave. This follows from a result in Brennan and Solanki (1981), that connects expected utility maximization with dynamic payoff replication. Indeed, given an arbitrary utility function u, the strategy that maximizes expected utility is the same as the dynamic replication of the following payoff

$$s(p) \equiv u'^{-1}\left(\lambda \frac{\pi(p)}{f(p)}\right). \tag{48}$$

In this expression the first derivative of the utility function u' can be inverted because of (21); π is the state-price density, i.e. the discounted risk-neutral probability density function of the risky investment at the horizon P_τ; f is the real-measure probability density function of P_τ;and λ is a constant, set in terms of the quantile function Q of the real measure probability in such a way that the budget constraint is satisfied:

$$\int_0^1 \frac{\pi(Q(t))}{f(Q(t))} u'^{-1}\left(\lambda \frac{\pi(Q(t))}{f(Q(t))}\right) dt \equiv S_0. \tag{49}$$

In Appendix A.5 we provide the proof in full detail.

Finally, dynamic heuristics include dynamic payoff replication and expected utility maximization as special cases, therefore dynamic heuristics include convex profiles as the CPPI, as well as concave profiles such as the constant-weight strategy.

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A Appendix

In this appendix we discuss technical results that can be skipped at first reading.

A.1 Strategy dynamics, the general case

Consider a market of N securities with prices $\mathbf{P}_t \equiv (P_{t,1}, \ldots, P_{t,N})'$. Consider strategies that invest the amount $\boldsymbol{\alpha}_t \equiv (\alpha_{t,1}, \ldots, \alpha_{t,N})'$. The self-financing condition (9) in this N-dimensional market reads

$$\sum_{n=1}^{N} \alpha_{t,n} (P_{t,n} + dP_{t,n}) \equiv \sum_{n=1}^{N} (\alpha_{t,n} + d\alpha_{t,n}) (P_{t,n} + dP_{t,n}), \tag{50}$$

which simplifies to

$$\sum_{n=1}^{N} d\alpha_{t,n} (P_{t,n} + dP_{t,n}) \equiv 0 \tag{51}$$

The evolution of the value of a strategy $S_t \equiv \sum_{n=1}^{N} \alpha_{t,n} P_{t,n}$ is described by Ito's rule

$$\begin{aligned} dS &= \sum_{n=1}^{N} d(\alpha_{t,n} P_{t,n}) = \sum_{n=1}^{N} d\alpha_{t,n} P_{t,n} + \alpha_{t,n} dP_{t,n} + d\alpha_{t,n} dP_{t,n} \\ &= \sum_{n=1}^{N} \alpha_{t,n} dP_{t,n} = \sum_{n=1}^{N} \alpha_{t,n} P_{t,n} \frac{dP_{t,n}}{P_{t,n}}, \end{aligned} \tag{52}$$

where we made use of equation (51). The last expression implies

$$\frac{dS_t}{S_t} = \sum_{n=1}^{N} \frac{\alpha_{t,n} P_{t,n}}{S_t} \frac{dP_{t,n}}{P_{t,n}} = \sum_{n=1}^{N} w_{t,n} \frac{dP_{t,n}}{P_{t,n}}, \tag{53}$$

where $\mathbf{w}_t \equiv (w_{t,1}, \ldots, w_{t,N})'$ are the portfolio weights.

A.2 Strategy dynamics, integral in gBm market

If as in (18) the prices follow a multivariate geometric Brownian motion

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\mathbf{B}_t. \tag{54}$$

Then using (53) we obtain

$$\frac{dS_t}{S_t} = \mathbf{w}_t' \frac{d\mathbf{P}_t}{\mathbf{P}_t} = \mathbf{w}_t' \boldsymbol{\mu} dt + \mathbf{w}_t' \boldsymbol{\Sigma} d\mathbf{B}_t. \tag{55}$$

Since $d\mathbf{B}_t \sim \mathrm{N}(0, \mathbf{I} dt)$ it follows

$$\mathbf{w}_t' \boldsymbol{\Sigma} d\mathbf{B}_t \sim \mathrm{N}\left(0, \mathbf{w}_t' \boldsymbol{\Sigma} \boldsymbol{\Sigma}' \mathbf{w}_t dt\right). \tag{56}$$

Therefore

$$\frac{dS_t}{S_t} \stackrel{d}{=} \mu_{\mathbf{w}_t} dt + \sigma_{\mathbf{w}_t} dB_t, \tag{57}$$

where

$$\mu_{\mathbf{w}_t} \equiv \mathbf{w}_t' \boldsymbol{\mu}, \quad \sigma^2_{\mathbf{w}_t} \equiv \mathbf{w}_t' \boldsymbol{\Sigma} \boldsymbol{\Sigma}' \mathbf{w}_t. \tag{58}$$

In particular, notice that if the weights are constant, i.e. $\mathbf{w}_t \equiv \mathbf{w}$, then the strategy dynamics (57) becomes a geometric Brownian motion.

Applying Ito's rule to (57) we obtain

$$\begin{aligned} d \ln S &= \frac{1}{S} dS - \frac{1}{2S^2} (dS)^2 \\ &= \mu_{\mathbf{w}_t} dt + \sigma_{\mathbf{w}_t} dB - \frac{1}{2S^2} \left(S \mu_{\mathbf{w}_t} dt + S \sigma_{\mathbf{w}_t} dB_t \right)^2 \\ &= \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + \sigma_{\mathbf{w}_t} dB_t, \end{aligned} \tag{59}$$

or

$$\ln S_\tau - \ln S_0 = \int_0^\tau \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + \int_0^\tau \sigma_{\mathbf{w}_t} dB_t. \tag{60}$$

Using Ito's isometry we obtain

$$\ln S_\tau - \ln S_0 \stackrel{d}{=} \int_0^\tau \left(\mu_{\mathbf{w}_t} - \frac{\sigma^2_{\mathbf{w}_t}}{2} \right) dt + Z \int_0^\tau \sigma^2_{\mathbf{w}_t} dt, \tag{61}$$

where $Z \sim \mathrm{N}(0,1)$.

In particular, the above results apply to a market of only two securities, one of which is risk-free, by specializing the notation as follows

$$\boldsymbol{\mu} \equiv \begin{pmatrix} r \\ \mu \end{pmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad \mathbf{w}_t \equiv \begin{pmatrix} 1 - w_t \\ w_t \end{pmatrix}. \tag{62}$$

Also, the above results apply to a market of one risky security, where $\mathbf{w}_t \equiv 1$. Then (61) yields the result (5) quoted in the main text.

A.3 Payoff replication

In this section we consider the two-security market (1)-(2). Assume that we can compute the solution $G(t,p)$ of the following partial differential equation

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial p} pr + \frac{1}{2} \frac{\partial^2 G}{\partial p^2} (\sigma(t,p))^2 = Gr, \tag{63}$$

with boundary condition

$$G(\tau, p) \equiv s(p), \quad p > 0. \tag{64}$$

Now define a strategy S_t that invests the initial budget

$$S_0 \equiv G(0, P_0), \tag{65}$$

and continuously rebalances between a risk-free asset and the risky asset, with the following weight

$$w_t \equiv \frac{P_t}{S_t} \frac{\partial G(t, P_t)}{\partial P_t}. \tag{66}$$

Then we prove below that this strategy evolves in such a way that the following identity holds at all times

$$S_t \equiv G(t, P_t), \quad t \in [0, \tau]. \tag{67}$$

In particular, the identity (67) holds at the investment horizon $S_\tau = G(\tau, P_\tau)$. Since at the investment horizon the boundary condition (64) holds, we obtain

$$S_\tau = G(\tau, P_\tau) = s(P_\tau), \quad t \in [0, \tau]. \tag{68}$$

To prove our claim, we denote by $\mu_t \equiv \mu(t, P_t)$ and $\sigma_t \equiv \sigma(t, P_t)$ the functions in the diffusion process and we apply Ito's rule to $G_t \equiv G(t, P_t)$.

$$\begin{aligned} dG_t &= \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial P_t} dP_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} (dP_t)^2 \\ &= \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB_t) + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 dt \\ &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial P_t} \mu_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 \right) dt + \frac{\partial G}{\partial P_t} \sigma_t dB_t. \end{aligned} \tag{69}$$

Using the general evolution of a strategy (11) and the expression for the weight in this strategy (66) we obtain

$$\begin{aligned} dS_t &= S_t r dt + S_t w_t \left(\frac{dP_t}{P_t} - r dt \right) \\ &= S_t r dt + P_t \frac{\partial G}{\partial P_t} \left(\frac{dP_t}{P_t} - r dt \right) \\ &= S_t r dt + \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB_t - P_t r dt). \end{aligned} \tag{70}$$

Therefore

$$\begin{aligned} d(G_t - S_t) &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial P_t} \mu_t + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 \right) dt + \frac{\partial G}{\partial P_t} \sigma_t dB_t \\ &\quad - S_t r dt - \frac{\partial G}{\partial P_t} (\mu_t dt + \sigma_t dB_t - P_t r dt) \\ &= \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} \sigma_t^2 - S_t r + \frac{\partial G}{\partial P_t} P_t r \right) dt \end{aligned} \tag{71}$$

Using now the fact that G solves the partial differential equation (63) we write (71) as

$$d(G_t - S_t) = (G_t - S_t) r dt, \tag{72}$$

which integrates as

$$(G_\tau - S_\tau) = (G_0 - S_0)\, e^{r\tau}. \tag{73}$$

Since from the budget constraint (65) we have $G_0 - S_0 = 0$, it follows that $S_t = G_t$ at all times, and thus (67) holds true.

A.4 General derivative payoffs in terms of calls

Define

$$\begin{aligned} g(x) \quad &\equiv \quad f(k) + \left.(\partial_x f)\right|_k (x-k) + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u (x-u)^+ \, du \\ &\quad + \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u (u-x)^+ \, du, \end{aligned} \tag{74}$$

where $(x)^+ \equiv \max(0, x)$, f is a smooth function and k is a given arbitrary fixed point. Notice that $g(x)$ is a continuum linear combination of call option profiles (30). We want to prove that $g(x) = f(x)$. We will do this by showing that the two functions are the same in the point k: $g(k) = f(k)$.and the derivatives of any order s are the same $\left.(\partial_x^s g)\right|_x = \left.(\partial_x^s f)\right|_x$. If we prove this, then we easily obtain

$$g(x) = g(k) + \sum_{s=1}^{\infty} \left.(\partial_x^s g)\right|_k \frac{(x-k)^s}{s!} = f(k) + \sum_{s=1}^{\infty} \left.(\partial_x^s f)\right|_k \frac{(x-k)^s}{s!} = f(x) \tag{75}$$

The first point follows from:

$$\begin{aligned} g(k) &= f(k) + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u (k-u)^+ \, du + \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u (u-k)^+ \, du \\ &= f(k). \end{aligned} \tag{76}$$

As for the second point, we first recall some results. Denote by Θ the Heaviside function

$$\Theta(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases} \tag{77}$$

and by δ the Dirac delta

$$\int_{-\infty}^{+\infty} \delta(x)\, f(x)\, dx = f(0), \tag{78}$$

Then

$$\partial_x (x-u)^+ = \Theta(x-u) \tag{79}$$

$$\partial_x^2 (x-u)^+ = \delta(x-u) \tag{80}$$

$$\partial_x (u-x)^+ = -\Theta(u-x) \tag{81}$$

$$\partial_x^2 (u-x)^+ = \delta(x-u). \tag{82}$$

For a proof, it suffices to smoothen the Dirac delta with a Gaussian kernel as in Meucci (2005), derive the results for the smooth version, and then consider the limit as the bandwidth of the kernel goes to zero.

First we consider the first derivative of both sides of (74). By the above differentiation rules we obtain

$$\begin{aligned}
\left.(\partial_x g)\right|_x &= \left.(\partial_x f)\right|_k + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x (x-u)^+\right|_x du \\
&+ \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x (u-x)^+\right|_x du \\
&= \left.(\partial_x f)\right|_k + \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \Theta(x-u)\, du \qquad (83) \\
&- \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \Theta(u-x)\, du \\
&= \left.(\partial_x f)\right|_k + \Theta(x-k) \int_k^x \left.\left(\partial_u^2 f\right)\right|_u du \qquad (84) \\
&- \Theta(k-x) \int_x^k \left.\left(\partial_u^2 f\right)\right|_u du \\
&= \left.(\partial_x f)\right|_k + \left.(\partial_x f)\right|_x - \left.(\partial_x f)\right|_k \\
&= \left.(\partial_x f)\right|_x
\end{aligned}$$

Next we consider the second derivative of both sides of (74)

$$\begin{aligned}
\left.\left(\partial_x^2 g\right)\right|_x &= \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x^2 (x-u)^+\right|_x du + \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x^2 (u-x)^+\right|_x du \qquad (85) \\
&= \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \delta(x-u)\, du + \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \delta(x-u)\, du \\
&= \Theta(x-k) \int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \delta(x-u)\, du + \Theta(k-x) \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \delta(x-u)\, du \\
&= \left.\left(\partial_x^2 f\right)\right|_x
\end{aligned}$$

Finally we consider the higher order $s > 2$ derivatives of both sides of (74)

$$\begin{aligned}
\left.(\partial_x^s g)\right|_x &= \partial_x^s \left(\int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u (x-u)^+ du + \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u (u-x)^+ du \right) \\
&= \partial_x^{s-2} (\int_k^{+\infty} \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x^2 (x-u)^+\right|_x du \qquad (86) \\
&+ \int_{-\infty}^k \left.\left(\partial_u^2 f\right)\right|_u \left.\partial_x^2 (u-x)^+\right|_x du) \\
&= \partial_x^{s-2} \left.\left(\partial_x^2 f\right)\right|_x \\
&= \left.(\partial_x^s f)\right|_x
\end{aligned}$$

Therefore the result follows.

A.5 Utility maximization versus payoff replication

First we recall a result from control theory. Consider an objective functional to be minimized

$$J[\psi, x] \equiv \int_0^T L(x_t, \psi_t)\, dt + \Gamma(x_T), \tag{87}$$

where the control $\psi_t \in \Psi$ and where the controlled path satisfies

$$\dot{x}_t = h(x_t, \psi_t) \tag{88}$$

$$x_0 \equiv a, \quad x_T \equiv b \tag{89}$$

To determine the optimal path x_t^* and the optimal control ψ_t^* we introduce the Hamiltonian

$$H(x, \psi, \lambda) \equiv \lambda h(x, \psi) + L(x, \psi), \tag{90}$$

Then Pontryagin's principle states

$$\psi_t^* = \underset{\psi \in \Psi}{\operatorname{argmin}} \{H(x_t^*, \psi_t, \lambda_t^*)\} \tag{91}$$

$$\dot{\lambda}_t^* = -\partial_x H(\lambda_t^*, x_t^*, \psi_t^*). \tag{92}$$

Going back to our original problem of the equivalence between final payoff allocation and expected utility maximization, we want to prove that, given a utility function u we can find a strategy payoff S_τ such that
a) The payoff S_τ is a function of the risky asset at the horizon P_τ as in (29)

$$S_\tau \equiv \varphi(P_\tau). \tag{93}$$

b) The payoff S_τ maximizes the expected utility as in (25). Denoting by f the pdf of P_τ and using (93) this condition becomes

$$\varphi^* \equiv \underset{\varphi}{\operatorname{argmax}} \left\{ \int_0^\infty u(\varphi(p)) f(p)\, dp \right\}, \tag{94}$$

where
c) The payoff S_τ satisfies the budget constraint (12). Using the fundamental theorem of asset pricing, there exists a probability measure, called risk-neutral, such that

$$\frac{S_0}{D_0} = \mathrm{E}\left\{ \frac{S_\tau}{D_\tau} \right\}. \tag{95}$$

Denoting by π the risk-neutral pdf of P_τ times the discounting $D_0/D_\tau = e^{-r\tau}$, the budget constraints reads

$$S_0 \equiv \int_0^\infty \varphi(p)\, \pi(p)\, dp \tag{96}$$

Therefore, denoting by Q quantile function of the distribution f the problem (93)-(94)-(95) becomes as in Brennan and Solanki (1981)

$$\varphi^* \equiv \underset{\varphi}{\operatorname{argmax}} \left\{ \int_0^1 u\left(\varphi\left(Q\left(t\right)\right)\right) dt \right\}, \tag{97}$$

such that

$$S_0 \equiv \int_0^1 \varphi\left(Q\left(t\right)\right) \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)} dt \tag{98}$$

Defining

$$x_t \equiv \int_0^t \varphi\left(Q\left(z\right)\right) \frac{\pi\left(Q\left(z\right)\right)}{f\left(Q\left(z\right)\right)} dz \tag{99}$$

This means that the constraint reads

$$\dot{x}_t \equiv \varphi\left(Q\left(t\right)\right) \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)}, \quad x_0 \equiv 0, \quad x_1 \equiv S_0. \tag{100}$$

We can rephrase (97)-(98) as (87)-(88)-(89) with the following substitutions

$$\begin{aligned} L\left(x_t, \psi_t\right) &\equiv -u\left(\psi_t\right) && (101) \\ h\left(x_t, \psi_t\right) &\equiv \psi_t \gamma_t && (102) \\ \Gamma &\equiv 0 && (103) \\ \Psi &\equiv \mathbb{R} && (104) \\ T &\equiv 1 && (105) \\ a &\equiv 0, \quad b \equiv S_0 && (106) \end{aligned}$$

where

$$\begin{aligned} \psi_t &\equiv \varphi\left(Q\left(t\right)\right) && (107) \\ \gamma_t &\equiv \frac{\pi\left(Q\left(t\right)\right)}{f\left(Q\left(t\right)\right)}. && (108) \end{aligned}$$

Therefore (91)-(92) become

$$\begin{aligned} \psi_t^* &= \underset{\psi}{\operatorname{argmin}} \left\{\lambda_t^* \psi \gamma_t^* - u\left(\psi\right)\right\} && (109) \\ \dot{\lambda}_t^* &= -\partial_x \left\{\lambda_t^* \psi_t^* \gamma_t - u\left(\psi_t^*\right)\right\} = 0. && (110) \end{aligned}$$

From (110) we obtain that λ_t^* must be constant

$$\lambda_t^* \equiv \lambda^*. \tag{111}$$

From (109) we obtain

$$\psi_t^* = u'^{-1}\left(\lambda^* \gamma_t\right), \tag{112}$$

or

$$\varphi^* (q) = u'^{-1} \left(\lambda^* \frac{\pi (q)}{f (q)} \right), \tag{113}$$

where λ^* is set from (98) in such a way that

$$\int_0^1 \frac{\pi (Q (t))}{f (Q (t))} u'^{-1} \left(\lambda^* \frac{\pi (Q (t))}{f (Q (t))} \right) dt \equiv S_0. \tag{114}$$