A Fully Integrated Liquidity and Market Risk Model

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Latest version and code at `http://www.symmys.com/node/350`

Abstract

Going beyond the simple bid–ask spread overlay for a particular value at risk, we introduce a framework that integrates liquidity risk, funding risk, and market risk.

We overlay a whole distribution of liquidity uncertainty on future market risk scenarios and we allow the liquidity uncertainty to vary from one scenario to another, depending on the liquidation or funding policy implemented. The result is one easy-to-interpret, easy-to-implement formula for the total liquidity-plus-market-risk profit and loss distribution.

Using this formula we can stress-test different market risk P&L distributions and different scenario-dependent liquidation policies and funding policies; compute total risk and decompose it into a novel liquidity-plus-market risk formula; and define a liquidity score as a monetary measure of portfolio liquidity.

Our approach relies on three pillars: first, the literature on optimal execution, to model liquidity risk as a function of the actual trading involved; second, an analytical conditional convolution, to blend market risk and liquidity/funding risk; third the Fully Flexible Probabilities framework, to model and stress-test market risk even in highly non-normal portfolios with complex derivatives.

Our approach can be implemented efficiently with portfolios of thousand of securities. The code for the case study is available at http://www.symmys.com/node/350.

JEL Classification: C1, G11

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1 Introduction

Market risk management and liquidity/funding risk management are among the top challenges in buy-side quantitative finance. Loosely speaking, market risk is the uncertainty of the profit and loss (P&L), at a given investment horizon in the future; and liquidity risk is the potential loss with respect to a reference mark-to-market value, due to the action of trading.

The literature on market risk management, market risk estimation and estimation error is enormous, refer e.g. to Meucci (2005) for a review. Liquidity risk has been addressed in a variety of contexts and possibly under different names in the financial literature, see e.g. Hibbert, Kirchner, Kretzschmar, Li, McNeil, and Stark (2009) for a review. To mention only a few: measures of liquidity, see e.g. Amihud (2002); axiomatic definition of liquidity impact, see e.g. Cetin, R., and Protter (2004) and Acerbi and Scandolo (2007); optimal execution, see e.g. Bertsimas and Lo (1998), Almgren and Chriss (2000), Obizhaeva and Wang (2005), Gatheral, Schied, and Slynko (2011), Bouchard, Dang, and Lehalle (2011); transaction cost-aware portfolio optimization, see e.g. Lobo, Fazel, and Boyd (2007), Lo, Petrov, and Wierzbicki (2003), Engle and Ferstenberg (2007).

In this article we propose a new methodology to integrate market risk, liquidity risk, and funding risk for all asset classes. In our approach, we overlay liquidity/funding uncertainty to each market-risk scenario at the future investment horizon. The distribution of the liquidity uncertainty depends on the amount liquidated at the future horizon, which in turn depends on the specific market scenario.

The main result is the liquidity-plus-market-risk P&L distribution formula (10), which is easy to interpret and easy to implement.

Other approaches to jointly model market and liquidity risk have been explored, see e.g. Bangia, Diebold, Schuermann, and Stroughair (2002), Jorion (2007) and references therein. Our methodology improves on the existing literature in seven directions.

First, our liquidity model goes beyond a deterministic bid-ask spread overlay to a pure market risk component. Indeed, we model the full impact of any actual liquidation schedule, including impact uncertainty and impact correlations, as well as the differential impact between trading quickly and trading slowly.

Second, our liquidity model is state-dependent: in those scenarios where the market is down and volatile, then the adverse impact of any liquidation schedule is worse, and hence so is the liquidity of our portfolio.

Third, we model both exogenous liquidity risk, due to the above market conditions, and funding risk, or endogenous liquidity risk: in our framework, we can model more aggressive liquidation schedules on capital intensive securities specifically in those market scenarios that give rise to very negative P&L, all while no liquidation takes place in positive P&L scenarios.

Fourth, we model all the features of the market risk component beyond mean and variance. In particular, we model the P&L of non-symmetrical tail events and of non-linear securities such as complex derivatives.

Fifth, we explicitly address the issue of estimation error, allowing for fast distributional stress-testing via the Fully Flexible Probabilities methodology, which we discuss below.

Sixth, our methodology allows for a novel decomposition of risk into a market risk component and a liquidity risk component.

Seventh, our approach allows for a natural definition of a the liquidity score for of the our portfolio in monetary units.

This article is organized as follows. In Section 2 we introduce in full generality our framework. In Section 3 we present a case study for a book of equities. In the appendix we discuss the technical details. Fully commented code for our methodology and for the case study is available for download at http://www.symmys.com/node/350.

2 The general framework

To build our market-plus-liquidity risk model we follow four steps, which we discuss in four separate sub-sections, also refer to Figure 1 to support intuition.

First, we model the pure market risk component of the P&L from now to the investment horizon via exact scenario repricing and the Fully Flexible Probabilities methodology.

Second, we overlay to the pure market risk the impact of the liquidation schedules at the horizon, which can be different in different scenarios. This way, we can model both exogenous liquidity risk and endogenous funding risk.

Third, we aggregate all the market scenarios and their liquidity adjustments into one total liquidity- and funding-adjusted P&L distribution.

Fourth, from the total liquidity-adjusted P&L distribution we compute all summary statistics, including standard deviation, VaR, CVaR; we compute the decomposition of such statistics into the market risk contribution and the liquidity risk contribution via a novel explicit formula; and we compute a novel liquidity score in monetary terms.

2.1 Market risk: Fully Flexible Probabilities

Consider a general market of N securities. Let us denote by $\overline{\Pi}_n$ the mark-to-market P&L delivered by one unit of the generic n-th security between the current time and the future investment horizon of the portfolio manager. The security unit is one share for stocks, one contract for futures, a given reference notional for swaps, etc. The investment horizon varies widely across portfolio managers, although typically it is of the order of one day.

Across all asset classes and in full generality, the stochastic behavior of the projected P&L $\overline{\Pi}_n$ is fully determined by the evolution of D risk drivers $\mathbf{X} \equiv (X_1, \ldots, X_D)'$. Thus, the P&L of the generic n-th security is a deterministic function π_n of the risk drivers.

$$\overline{\Pi}_n = \pi_n(\mathbf{X}), \quad n = 1, \ldots, N, \tag{1}$$

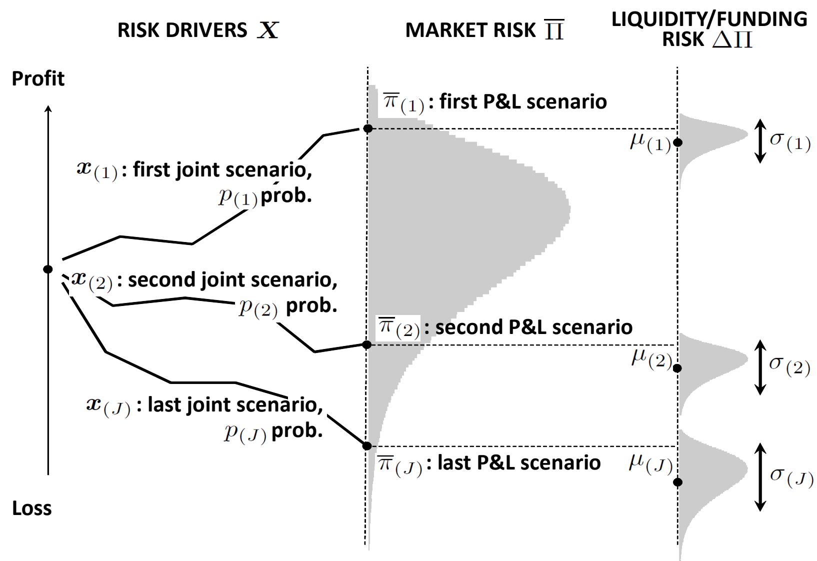

Figure 1: Liquidity/funding risk execution price uncertainty at the investment horizon, overlayed to market risk

This framework is completely general for all asset classes, see Meucci (2011c). For instance, for a stock, the risk driver is its log-price at the horizon $X_n = \ln S_{n,t}$ and the mark-to-market P&L is $\overline{\Pi}_n = e^{X_n} - s_{n,0}$. For an option, $\boldsymbol{X}$ represents the underlying and all the entries of the implied volatility surface at the investment horizon, and the pricing function π_n can be specified in terms of the exact Black-Scholes formula, or it can be approximated by a Taylor expansion whose coefficients are the well-known "deltas", "vegas", "gammas", "vannas", "volgas", etc.

Consider a portfolio of N holdings $\boldsymbol{h} \equiv (h_1, \ldots, h_N)'$. By holdings we mean the number of units of the given securities, i.e. the number of shares for stocks, the number of contracts for futures, etc. By holdings we do not mean portfolio weights, which might not be properly defined for long-short positions or leveraged instruments such as swaps or futures, see Meucci (2010b).

. For the portfolio $\boldsymbol{h}$ the mark-to-market P&L is the sum of the contributions from each position, i.e. $\overline{\Pi} = \sum_n h_n \pi_n (\boldsymbol{X})$.

A flexible approach to model market risk, i.e. the distribution of the risk drivers $\boldsymbol{X}$, is via the Fully Flexible Probabilities framework in Meucci (2010a), refer to Figure 1 for the intuition. In the Fully Flexible Probabilities approach, the distribution of $\boldsymbol{X}$ is modeled by two sets of variables: a set of joint scenarios $j = 1, \ldots, J$ for the risk drivers $\boldsymbol{x}_{(j)} \equiv \left(x_{1,(j)}, \ldots, x_{D,(j)}\right)'$, which can be generated historically or via Monte Carlo; and their respective relative probability

$p_{(j)}$, one probability for each joint scenario

$$\boldsymbol{X} \sim \left\{ \boldsymbol{x}_{(j)}, p_{(j)} \right\}_{j=1,\ldots,J}. \tag{2}$$

The Fully Flexible Probabilities framework (2) is computationally efficient, because the distribution of the portfolio P&L follows immediately by computing the portfolio P&L in each scenario, all while the relative probabilities remain unchanged

$$\overline{\Pi} \sim \left\{ \overline{\pi}_{(j)}, p_{(j)} \right\}_{j=1,\ldots,J}, \tag{3}$$

where each P&L scenario reads

$$\overline{\pi}_{(j)} = \textstyle\sum_n h_n \pi_n(\boldsymbol{x}_{(j)}). \tag{4}$$

One special case of the Fully Flexible Probabilities framework (2) is when all the probabilities are equal $p_{(j)} = 1/J$, which is typical in the so-called "historical simulations" approach.

One significant benefit of the Fully Flexible Probabilities framework is that it allows us to model all sorts of non-normal market distributions, as well as non-linear instruments, such as close-to-expiry options.

Furthermore, in the Fully Flexible Probabilities framework we can stress-test the probabilities $p_{(j)}$ to emphasize different time periods or different market conditions, using all sorts of advanced techniques: exponential smoothing, kernels, Entropy Pooling, etc.: this way we can perform all sorts of stress-tests on the P&L distribution, as we show in the case study in Section 3.

Finally, with the Fully Flexible Probabilities approach we can separate the computationally heavy part of the process, namely the computation of the P&L's of each security $\pi_n(\boldsymbol{x}_{(j)})$, from the computationally light part, namely the sum that yields the portfolio P&L scenarios $\overline{\pi}_{(j)}$ in (4) and the stress-testing of the market distribution via $p_{(j)}$. The heavy part can be run overnight via a batch process, whereas the light part can be performed on the fly for portfolios of thousands of securities.

We refer the reader to Meucci (2010a) for more details on the Fully Flexible Probabilities framework.

2.2 Liquidity risk: market impact over liquidation horizon

In standard risk and portfolio management applications, the P&L function (1) yields the distribution of the P&L. However, for a specific realization $\boldsymbol{x}$ of the risk drivers $\boldsymbol{X}$ at the investment horizon, there exists an uncertainty $\Delta\Pi_n$ in the P&L generated by the generic n-th position, due to liquidity-related issues, see Figure 1. This liquidity adjustment $\Delta\Pi_n$ is determined by three factors.

The first factor affecting $\Delta\Pi_n$ is the state of the market, which can be included among the risk drivers $\boldsymbol{X}$. For instance, when the VIX spikes, liquidity can suddenly decrease.

The second factor affecting the liquidity adjustment $\Delta\Pi_n$ is the amount liquidated at the future investment horizon. Let us denote by $\Delta\boldsymbol{h} \equiv (\Delta h_1, \ldots, \Delta h_N)'$

this action, or liquidation schedule. For mark-to-market purposes we do not liquidate any position, i.e. $\Delta \boldsymbol{h} = \mathbf{0}$, and thus there is no impact on the P&L, $\Delta \Pi_n = 0$. On the other extreme, in a full-liquidation of a large portfolio, i.e. $\Delta \boldsymbol{h} = -\boldsymbol{h}$, we generate an impact on the prices and thus on the P&L. The price impact has two components: a permanent component, which must be linear and thus cannot contribute to the cost of a round-trip trade, otherwise there would be arbitrage opportunities, see Gatheral (2010); and a temporary component, which we expect to adversely affect the portfolio, $E\{\Delta \Pi_n\} < 0$, refer to Figure 1.

The third factor affecting the liquidity adjustment $\Delta \Pi_n$ are the execution horizons $\boldsymbol{\tau} \equiv (\tau_1, \ldots, \tau_N)'$ for the liquidations $\Delta \boldsymbol{h} \equiv (h_1, \ldots, h_N)'$: as in Almgren and Chriss (2000), longer execution horizons generate lesser impact on the prices, but more uncertainty; shorter execution horizons generate more impact with less uncertainty.

For now, let us model the liquidity adjustment for the P&L of each position as a normal distribution, where both mean and volatility depend on the above three factors

$$\Delta \Pi_n \sim \mathrm{N}\left(\mu_n, \sigma_n^2\right). \tag{5}$$

For the reader concerned that the normal assumption for the liquidity adjustments (5) might not be realistic, in Appendix A.1 we show how to generalize the normal assumption to more realistic skewed and thicker-tailed distributions.

In Appendix A.2 we derive explicit expressions for μ_n, which is negative, and σ_n in (5) across asset classes, for a broad class of liquidation strategies $\Delta \boldsymbol{h}$ and arbitrary execution time frames $\boldsymbol{\tau}$. Such expressions represent one of the innovations in this article. They draw on four sets of previous results: the universal square-root impact form, justified empirically and theoretically in Grinold and Kahn (1999), Almgren, Thum, Hauptmann, and Li (2005), Toth, Lemperiere, Deremble, De Lataillade, Kockelkoren, and Bouchaud (2011); the moments of a general impact function in Almgren (2003); the optimal execution paradigm in Almgren and Chriss (2000); and the equivalence between volatility and market activity in Ane and Geman (2000).

Here we report only the simplest of such expressions, namely those that follow from volume-weighed-average-price (VWAP) execution.

The VWAP derived mean of the liquidity adjustment (5) reads

$$\mu_n = -\alpha_n(\boldsymbol{x})\, e_n \left|\Delta h_n\right| - \beta_n(\boldsymbol{x})\, e_n \bar{\sigma}_n \frac{\left|\Delta h_n\right|^{\frac{3}{2}}}{\sqrt{v_n}}. \tag{6}$$

In this expression, α_n is an estimate of the commissions plus half the bid-ask spread as a percentage of the exposure e_n of one unit of the n-th security (e.g., in stocks, e_n is the price of one share), which can vary with the market conditions $\boldsymbol{x}$; β_n is a coefficient, approximately constant across securities within the same asset class, but which can vary with the liquidity and market conditions $\boldsymbol{x}$; $\bar{\sigma}_n$ is an estimate of the average annualized P&L volatility of one unit of the n-th security, as percentage of the exposure e_n; and v_n is an estimate of the total number of units of the n-th security traded by the market over the execution period τ_n.

The VWAP derived uncertainty of the liquidity adjustment (5) reads

$$\sigma_n = \delta_n(\boldsymbol{x})\sqrt{v_n} e_n |\Delta h_n|, \tag{7}$$

where δ_n is a coefficient that varies with the the market conditions $\boldsymbol{x}$ and has dimension inverse to $\sqrt{v_n}$, and the remaining quantities are defined in (6).

Notice that when the execution period τ_n of a given liquidation Δh_n increases, so does the cumulative trading activity v_n over the period. Thus, as intuition suggests, the expected impact (6) with a longer execution period τ_n decreases, whereas the impact uncertainty (7) increases. Also, notice how (6) and (7) make sense from a dimensional perspective. For instance, if a 1:2 stock split occurs, α_n, β_n and $\bar{\sigma}_n$ in (6) do not change, e_n becomes half, $|\Delta h_n|$ doubles, v_n doubles, and thus the expected liquidity adjustment in money terms μ_n does not change.

As for the correlations $\rho_{n,m}$ among the liquidity adjustments (5) for two positions, these are similar to the respective market risk correlations $\hat{\rho}_{n,m}$, but empirically are likely closer to the maximum value 1, because liquidity risk is less diversifiable than market risk. This phenomenon can be modeled via a common shrinkage parameter γ close to 1, as follows

$$\rho_{n,m} = \gamma + (1-\gamma)\hat{\rho}_{n,m}. \tag{8}$$

The parameters of the model α_n, β_n, δ_n and γ must be calibrated in each traded market, say stocks, foreign exchanges, etc. We provide an illustration in the case study in the next section.

Once we have computed the liquidity adjustments (5) for all the positions and the correlations (8), we can aggregate all the adjustments $\Delta\Pi = \sum_n \Delta\Pi_n$ and compute the distribution of the liquidity adjustment $\Delta\Pi$ for the whole portfolio

$$\Delta\Pi \sim \mathrm{N}\left(\mu, \sigma^2\right), \tag{9}$$

where $\mu = \sum_n \mu_n$ and $\sigma^2 = \sum_{n,m} \sigma_n \sigma_m \rho_{n,m}$.

2.3 Total and funding risk

The total portfolio P&L $\Pi = \overline{\Pi} + \Delta\Pi$ is the sum of the pure mark-to-market component $\overline{\Pi}$ obtained in (3) and the liquidity adjustment $\Delta\Pi$ obtained in (9). As we prove in Appendix A.1, the probability density for any generic value y of the portfolio P&L is the following simple formula

$$f_\Pi(y) = \sum_j \frac{p_{(j)}}{\sigma_{(j)}} \varphi\left(\frac{y - \overline{\pi}_{(j)} - \mu_{(j)}}{\sigma_{(j)}}\right). \tag{10}$$

In (10), φ denotes the standard normal distribution density (this assumption can be easily generalized to more realistic skewed and thicker-tailed distributions,.see Appendix A.1); $\overline{\pi}_{(j)}$ is the pure market risk P&L in the generic j-th scenario (4);

$p_{(j)}$ is the respective probability according to the Fully Flexible Probabilities framework (2); and $(\mu_{(j)}, \sigma_{(j)})$ are the portfolio liquidity adjustment parameters (μ, σ) that appear in (9), evaluated in the generic j-th scenario. In particular, in the limit of no liquidity adjustment, i.e. $\mu_{(j)} \approx 0$ and $\sigma_{(j)} \approx 0$, and when all the probabilities are equal, i.e. $p_{(j)} = 1/J$, the distribution (10) becomes the standard empirical distribution of the mark-to-market P&L scenarios.

To the best of our knowledge, the simple, yet general and flexible total market-plus-liquidity portfolio P&L distribution (10) is new and represents one of the main contributions in this article.

As one can verify in the code available at http://www.symmys.com/node/350, we can compute the total P&L distribution (10) with thousands of positions N and thousands of scenarios J in fractions of a second.

Notice how, with no additional computational cost, we can make the liquidation schedule of each position depend on the market scenario $\Delta h_n \mapsto \Delta h_{n,(j)}$. The same holds for the execution periods of the liquidation schedule $\tau_n \mapsto \tau_{n,(j)}$. Hence, in our framework we can easily model stochastic liquidations, see also Brigo and Nordio (2010).

As a result, we can measure and stress-test funding risk: in a scenario j where the mark-to-market P&L $\overline{\pi}_{(j)}$ is very negative, the firm will liquidate faster, via short execution schedules $\tau_{n,(j)}$, and in larger amounts, via the transaction $\Delta h_{n,(j)}$, those securities n with the smallest marginal cost of liquidation per unit of capital, i.e. the most liquid. For example, during the recent financial crisis many hedge funds faced with funding issues chose to liquidate quant-equity strategies.

We provide a detailed case study for all the above applications of the total P&L distribution (10) in Section 3.

2.4 Summary statistics and liquidity score

With the distribution (10) of the total market-plus-liquidity portfolio P&L $\Pi = \overline{\Pi} + \Delta\Pi$ we can compute all sorts of risk statistics. Standard statistics include expected return, standard deviation, Sharpe ratio, VaR and CVaR for any tail risk level, etc. More advanced statistics include expected utility and certainty equivalent for any kind of utility function, spectral measures of satisfaction for arbitrary risk-aversion spectra, etc., see e.g. Meucci (2005) for a review.

Since the total P&L $\Pi = \overline{\Pi} + \Delta\Pi$ is simply the sum of the mark-to-market component and the liquidity component, it is possible in principle to decompose all the standard risk measures such as standard deviation, VaR and CVaR into so-called marginal contributions, as in Hallerbach (2003), Gourieroux, Laurent, and Scaillet (2000), Tasche (2002). For instance, for the CVaR we obtain the following decomposition

$$CVaR\{\Pi\} = \underbrace{\partial_{\overline{\Pi}} CVaR\{\Pi\}}_{\text{market risk}} + \underbrace{\partial_{\Delta\Pi} CVaR\{\Pi\}}_{\text{liquidity risk}}. \tag{11}$$

The total CVaR on the left-hand side follows from the numerical computation of the total P&L pdf (10). The liquidity-risk contribution is provided by the following formula, which to the best of our knowledge is a new result

$$\partial_{\Delta\Pi} CVaR\{\Pi\} = \sum_j p_{(j)} \frac{\mu_{(j)} - \sigma_{(j)}\varphi\left(z_{(j)}\right)}{\Phi(z_{(j)})}, \tag{12}$$

where $z_{(j)} \equiv (F_{\Pi}^{-1}(\alpha) - \bar{\pi}_{(j)} - \mu_{(j)})/\sigma_{(j)}$; α is the CVaR confidence; and Φ is the standard normal cdf. Then, the market-risk contribution to $\partial_{\overline{\Pi}} CVaR\{\Pi\}$ follows as the difference of the CVaR and its liquidity contribution. We refer to Appendix A.3 for more details.

Also, we can measure the liquidity score for the portfolio. A portfolio is liquid when the liquidity adjustment $\Delta\Pi$ plays a minor role on the total portfolio P&L Π. Since the liquidity adjustment always hits the P&L downward, it is natural to define a liquidity score as the percentage deterioration in the left tail, as measured by a standard measure such as the conditional value at risk with 90% confidence. Accordingly, we define the liquidity score LS of a portfolio as the difference between the pure market-risk $CVaR\{\overline{\Pi}\}$ and the total, liquidity-adjusted risk $CVaR\{\Pi\}$, normalized as a return

$$\text{LS} = \frac{CVaR\{\overline{\Pi}\}}{CVaR\{\Pi\}}. \tag{13}$$

The liquidity score is always larger than zero and less than one. When the impact of liquidity is negligible, i.e. $\Pi = \overline{\Pi}$, the liquidity score approaches the upper boundary of one. When the impact of liquidity is substantial, i.e. the left tail of Π is much more negative than the left tail of $\overline{\Pi}$, the liquidity score approaches the lower boundary of zero.

3 Case study: liquidity management for equity portfolios

We consider a case study with portfolios of stocks in the S&P 500. For all the details we refer the reader to the commented code at http://www.symmys.com/node/350.

The setup

In our chosen market there are $N = 500$ securities. We choose an investment horizon t = one day. The risk drivers are the horizon log-prices of each stock $X_n \equiv \ln S_n$, and thus the P&L pricing function (1) reads

$$\overline{\Pi}_n = e^{X_n} - s_n, \tag{14}$$

where upper case letters denote future random variables, and lower case letters denote currently known numbers.

We also include an additional risk driver X_0 that summarizes the overall level of liquidity in the market. In particular, we use the Morgan Stanley Liquidity Factor, which is the cumulative P&L of a dynamic portfolio rebalanced to stay long liquid stocks and short illiquid stocks. Thus, in our case study, we have $D =$

$1 + N = 501$ risk drivers. We collect daily data over a period of approximately four years for the risk drivers and other variables such as traded volumes.

We model the distribution of the risk drivers $\boldsymbol{X} \equiv (X_0, X_1, \ldots, X_N)'$ via the historical distribution of their day-to-day changes. With our database of about four years of daily observations, we obtain $J \approx 1,000$ scenarios. To start with, we assign those historical scenarios equal probability weights $p_{(j)} = 1/J$. This concludes the specification of the Fully Flexible Probabilities framework that defines market risk (2).

Next, for each stock, we calibrate the parameters α_n, β_n, δ_n for the liquidity adjustment (6)-(7). We calibrate these parameters as functions $\alpha_n(x_0)$, $\beta_n(x_0)$, $\delta_n(x_0)$ of the liquidity index x_0, because different liquidity regimes in the market correspond to different market impact parameters. Also, we calibrate the correlation shrinkage parameter for the liquidity correlations in (8) as $\gamma = 0.90$.

Figure 2: Liquidity risk has adverse impact on the portfolio

Now we are ready to start our analysis.

Example 1: base-case equally-weighed portfolio

As a first example, we consider an equally-weighted portfolio of 50 stocks $h_1 s_1 = \cdots = h_{50} s_{50}$ whose total notional $h_1 s_1 + \cdots + h_{50} s_{50}$ amounts to 30% the average daily volume of the S&P 500. The ensuing pure market risk distribution (3) for the P&L $\overline{\Pi}$ in our portfolio is represented by the histograms in Figure 2. Then we stress-test a proportional liquidation of 20% of all the assets, i.e. $\Delta h_n = -0.2 \cdot h_n$. We also consider the same execution period $\tau_n =$ one day for all the assets, i.e. all trades are executed between the investment horizon, which is tomorrow's close, and the close one day thereafter. The distribution line in the left portion of Figure 2 represents the distribution (10) for the market-plus-liquidity P&L $\Pi = \overline{\Pi} + \Delta\Pi$, in the case of our portfolio. Notice how this liquidation can be absorbed by the market with relatively little impact. This is reflected in the liquidity score (13), which in this case is LS $\approx 96\%$.

Example 2: aggressive liquidation

As a second example, we consider the same above setting, but with an aggressive full liquidation, i.e. $\Delta h_n = -h_n$. The distribution of the total P&L

$\Pi = \overline{\Pi} + \Delta\Pi$ now becomes the line in the right portion of Figure 2. Notice how the liquidity impact, now more invasive, shifts and twists the P&L distribution toward the left tail. Indeed, the liquidity score now is LS $\approx$ 71%.

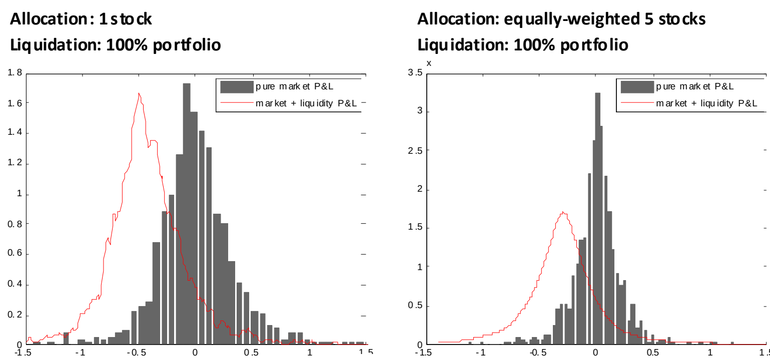

Figure 3: Market risk is more diversifiable than liquidity risk

Example 3: liquidity diversification

As a third example, we look into diversification issues, by performing the above full liquidation on a heavily concentrated portfolio of one stock, and on a mildly concentrated equally weighted portfolio of five stocks, both with the same initial capital as in the above setting. We display the results respectively in the left and right portion of Figure 3, compare also with Figure 2. The liquidity scores are LS $\approx$ 66% and LS $\approx$ 69%.respectively. Notice how market risk, i.e. the width of the histogram, shrinks as we progress from one, to 5, to 50 stocks. Liquidity, on the other hand, is not easily diversifiable.

Example 4: funding risk, with different trades in different market scenarios

As a fourth example, we look into funding risk. For this purpose, we consider our original 50-stock portfolio, but for a larger total value of assets under management, equal to the average daily volume of the S&P 500. The one-day full liquidation schedule tested above is unrealistic. Instead, we intend to free up cash as needed, in adverse market scenarios only. More precisely, suppose that, in a given scenario j, the market P&L $\overline{\pi}_{(j)}$ falls below a critical negative threshold $\overline{\pi}$, which in our case we set as a fraction of the market P&L volatility. In that scenario we will need to free up the amount of cash $\overline{\pi} - \overline{\pi}_{(j)}$. Let us denote by m_n the dollar margin to invest, long or short, one share of the n-th stock. Then the liquidation of a fraction θ of the current exposure $\Delta h_n = -\theta h_n$ makes available the amount of cash $\theta \left| h_n \right| m_n$. Therefore, to cover for our cash needs, we need to liquidate a scenario-dependent portion $\theta_{(j)}$ of our portfolio, which is zero if the P&L $\overline{\pi}_{(j)}$ is above the threshold, and otherwise it is determined by the relationship $\sum_n \theta_{(j)} \left| h_n \right| m_n = \overline{\pi} - \overline{\pi}_{(j)}$. We proceed to do so, and we display the result in the left portion of Figure 4. Notice how funding risk hits increasingly the far left tail of the P&L. Therefore, despite the overall relatively small liquidity perturbation, the liquidity score is LS $\approx 84\%$.

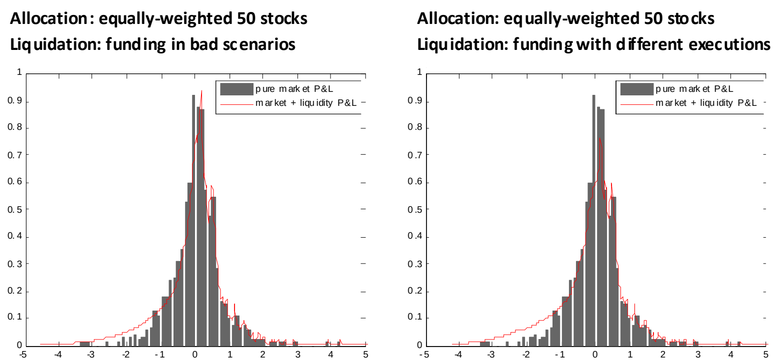

Figure 4: Funding risk: state-dependent liquidation schedules

Example 5: funding risk, with different trades and execution periods in different market scenarios

As a fifth example, we delve deeper into funding risk: not only do we need scenario-dependent liquidation policies, but also we need scenario-dependent execution schedules: in very negative market P&L scenarios $\overline{\pi}_{(j)}$, far below the critical threshold $\overline{\pi}$, the need for cash is more immediate than in milder scenarios, and thus all the execution schedules $\tau_{n,(j)}$ occur faster, creating a vicious circle of heightened liquidity risk. In our case, the liquidity score becomes LS $\approx 78\%$. In the right portion of Figure 4 we display the P&L distribution. Notice how funding risk is exacerbated further with respect to the left side of Figure 4, where all the execution times equal one day, i.e. $\tau_{n,(j)} = 1$, for all stocks n and all scenarios j.

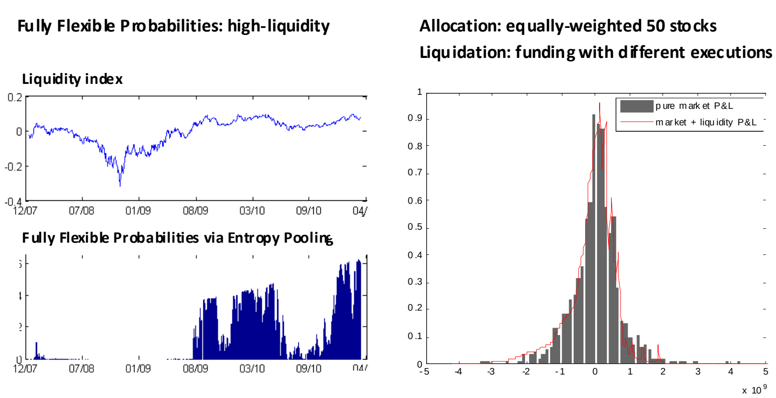

Figure 5: Fully Flexible Probabilities to stress-test market risk

Example 6: liquidity- and market-risk stress-testing

As a sixth example, we leverage the Fully Flexible Probabilities framework to address the issue of estimation risk: any market distribution estimated from past historical observations is never equal to the true, unknown, future market distribution. Hence we need to stress-test different estimates. In particular, so far, all the historical scenarios $j = 1, \ldots, J$ in the total portfolio distribution (10) have been given equal weight, by setting all the probabilities of the respective scenarios equal, i.e. $p_{(j)} = 1/J$. With the Fully Flexible Probabilities framework we can modify the relative weights $p_{(j)}$ of the scenarios to better reflect the current state of the market. More precisely, we observe in the time series of the Morgan Stanley Liquidity Index displayed in the top left portion of Figure 5 that recently the market has been relatively liquid. Accordingly, we intend to give more probability weight to recent scenarios, as well as to past scenarios where the liquidity index was high. The technique we used to perform this blending in the Fully Flexible Probabilities framework is called Entropy Pooling, refer to Meucci (2011a) for more details. The outcome are the non-equal probabilities $p_{(j)}$ displayed in the bottom left portion of Figure 5.

Then we use these probabilities to recompute the total market-plus-liquidity portfolio P&L distribution (10) in the same portfolio with funding-driven liquidation schedules considered in the previous example. Notice that the Fully Flexible Probabilities stress-test comes at no additional cost: in order to obtain any new estimates of the P&L distribution we simply have to plug the new probabilities $p_{(j)}$ in the total distribution formula (10). In Figure 5 we display the results. As expected, the liquidity score has risen to LS $\approx 93\%$.

4 Conclusions

We have presented a framework to model, measure and take action on market, liquidity and funding risk, across all financial instruments and trading styles and we have illustrated our framework in practice in a case study.

Our approach goes beyond the simple bid-ask spread adjustment to a VaR number because we model the full random impact on the prices, and thus on the portfolio P&L, stemming from any liquidation schedule. Furthermore, in our framework, liquidity depends on the state of the market at the future investment horizon.

Our approach culminates in the simple, new formula (10) for the market-plus-liquidity distribution of our portfolio P&L, which can be computed in fractions of a second for portfolios with thousands of securities.

The market-plus-liquidity P&L formula (10) allows us to dissect, stress-test and eventually intervene on all the components of risk.

By changing the probabilities $p_{(j)}$ according to the Fully Flexible Probabilities framework, we can explore on the fly the effects of different market environments on the liquidity-adjusted P&L distribution, such as low/high liquidity, low/high volatility, etc.

By modifying the liquidation stress-test Δh_n of each security $n = 1, \ldots, N$ and the respective execution periods τ_n, we obtain different liquidity adjustment parameters $\mu_{(j)}$ and $\sigma_{(j)}$ in each future market scenario j.

By making also the liquidation actions $\Delta h_{n,(j)}$ and their respective execution periods $\tau_{n,(j)}$ depend on the market scenarios j, we can stress-test more flexible and realistic situations, such as funding-driven massive and fast liquidations only when the portfolio incurs large losses.

Once we have computed the market-plus-liquidity P&L distribution (10), we can extract all sorts of risk indicators, such as standard deviation, VaR, CVaR, and the newly introduced liquidity score (13) which is the tail return of the liquidity-adjusted portfolio over the tail return for the same portfolio, if there were no liquidity issues.

Finally, we can compute explicitly the contributions to standard deviation, VaR, and CVaR from liquidity-risk and market- risk as in (11), using another original contribution of the present work.

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A Appendix

A.1 The conditional convolution

Let us start with the conditional distribution of the liquidity adjustment (9), i.e.

$$\Delta\Pi|\boldsymbol{x} \sim \mathrm{N}\left(\mu\left(\boldsymbol{x}\right), \sigma^2\left(\boldsymbol{x}\right)\right), \tag{15}$$

where, with more generality, we let the liquidation policy depend on the market scenario, i.e. $\mu\left(\boldsymbol{x}\right) = \mu\left(\boldsymbol{x}, \Delta\boldsymbol{h}\left(\boldsymbol{x}\right), \boldsymbol{\tau}\left(\boldsymbol{x}\right)\right)$ and $\sigma^2\left(\boldsymbol{x}\right) = \sigma^2\left(\boldsymbol{x}, \Delta\boldsymbol{h}\left(\boldsymbol{x}\right), \boldsymbol{\tau}\left(\boldsymbol{x}\right)\right)$. We can write $\Delta\Pi|\boldsymbol{x} \overset{d}{=} \mu\left(\boldsymbol{x}\right) + \sigma\left(\boldsymbol{x}\right) Z$, where $Z \sim \mathrm{N}\left(0, 1\right)$ is standard normal and has the familiar pdf $\varphi\left(z\right) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. Then the conditional pdf of (15) for an arbitrary value y reads

$$f_{\Delta\Pi|\boldsymbol{x}}\left(y\right) = \frac{1}{\sigma\left(\boldsymbol{x}\right)} \varphi\left(\frac{y - \mu\left(\boldsymbol{x}\right)}{\sigma\left(\boldsymbol{x}\right)}\right), \tag{16}$$

Using the conditional-marginal representation of a generic pdf, we obtain

$$\begin{aligned} f_{\Pi}\left(y\right) &= \int f_{\Pi|\boldsymbol{x}}\left(y\right) f_{\boldsymbol{X}}\left(\boldsymbol{x}\right) d\boldsymbol{x} \\ &= \int f_{\overline{\Pi}+\Delta\Pi|\boldsymbol{x}}\left(y\right) f_{\boldsymbol{X}}\left(\boldsymbol{x}\right) d\boldsymbol{x} \\ &= \int f_{\Delta\Pi|\boldsymbol{x}}\left(y - \textstyle\sum_n h_n \pi_n\left(\boldsymbol{x}\right)\right) f_{\boldsymbol{X}}\left(\boldsymbol{x}\right) d\boldsymbol{x}. \end{aligned} \tag{17}$$

The pdf of the Fully Flexible Probabilities risk drivers distribution (3) reads

$$f_{\boldsymbol{X}}\left(\boldsymbol{x}\right) = \textstyle\sum_j p_{(j)} \delta^{\boldsymbol{x}_{(j)}}\left(\boldsymbol{x}\right), \tag{18}$$

where $\delta^{\boldsymbol{y}}\left(\boldsymbol{x}\right)$ denotes the Dirac delta centered in $\boldsymbol{y}$, see Meucci (2010a).Then, substituting the risk driver pdf (18) in the expression for the portfolio P&L pdf (17), we obtain

$$\begin{aligned} f_{\Pi}\left(z\right) &= \int f_{\Delta\Pi|\boldsymbol{x}}\left(y - \textstyle\sum_n h_n \pi_n\left(\boldsymbol{x}\right)\right) \textstyle\sum_j p_{(j)} \delta^{\boldsymbol{x}_{(j)}}\left(\boldsymbol{x}\right) d\boldsymbol{x} \\ &= \textstyle\sum_j p_{(j)} f_{\Delta\Pi|\boldsymbol{x}_{(j)}}\left(y - \overline{\pi}_{(j)}\right). \end{aligned} \tag{19}$$

Substituting in (19) the portfolio conditional pdf (16) we obtain the result in the main text (10). All the above operations can be applied if instead of the normal distribution in (5) we use the t distribution or the skew-t distribution by Azzalini and Capitanio (2003) to model the liquidity adjustment.

A.2 The moments of the liquidity adjustment

Let us consider the cost of trading a lot Δh between inception of trading, which we set at $t = 0$ and the execution horizon $t = \tau$, according to a given liquidation trajectory h_t, where

$$h_0 = \Delta h, \qquad h_\tau = 0. \tag{20}$$

As in Almgren and Chriss (2000), let us assume that the value of the security evolves as a Brownian motion, and let us assume that the price actually received/paid $\tilde{S}_t$ is modified by a temporary impact of the liquidation trajectory $\mathrm{t}\left(\dot{h}_t\right)$, as follows

$$dS_t = \sigma_t dW_t \tag{21}$$

$$\tilde{S}_t = S_{t^-} - \mathrm{t}\left(|\dot{h}_t|\right). \tag{22}$$

Then the liquidity adjustment p&l $\Delta\Pi$ has a distribution whose first two moments were computed in Almgren (2003) and read

$$\mu \equiv \mathrm{E}\{\Delta\Pi\} = -\int_0^\tau |\dot{h}_t|\,\mathrm{t}\left(|\dot{h}_t|\right) dt \tag{23}$$

$$\sigma^2 \equiv \mathrm{V}\{\Delta\Pi\} = \int_0^\tau \sigma_t^2 h_t^2 dt. \tag{24}$$

Let us use square-root temporary impact, as in Grinold and Kahn (1999), Almgren, Thum, Hauptmann, and Li (2005), Toth, Lemperiere, Deremble, De Lataillade, Kockelkoren, and Bouchaud (2011)

$$\mathrm{t}\left(|\dot{h}_t|\right) = \alpha p_0 + \beta\bar{\sigma}p_0 |\frac{\dot{h}_t}{\dot{v}_t}|^{\frac{1}{2}}, \tag{25}$$

where α is the bid-ask spread plus commissions, as a percentage of the exposure p_0 of one contract, β is a coefficient common to all securities in a given asset class, $\bar{\sigma}$ is the average volatility of the security, as a percentage of the reference exposure p_0 of one contract, and v_t is the total number of contracts traded in the market since our trading started at inception $t = 0$ up to time t. Notice that we can interpret v_t as a time change

$$v_t \equiv \int_0^t dv_z. \tag{26}$$

Then the expectation of the liquidity adjustment $\Delta\Pi$ in (23) reads

$$\begin{aligned}\mu &= -\alpha \int_0^\tau |\dot{h}_t| dt + \beta\bar{\sigma} \int_0^\tau |\frac{dh_t}{dv_t}|^{\frac{1}{2}} \frac{|dh_t|}{dv_t} dv_t \\ &= -\alpha\,|\Delta h| - \beta\bar{\sigma} \int_0^{v_\tau} |\frac{dh}{dv}|^{\frac{3}{2}} dv,\end{aligned} \tag{27}$$

where we made the reasonable assumption that dh never changes sign.

Also, we can write the price (21) as a time-changed Brownian motion

$$S_t \stackrel{d}{=} \bar{W}_{u_t}, \tag{28}$$

where the time change reads

$$u_t \equiv \int_0^t \sigma_z^2 dz. \tag{29}$$

This follows from

$$\bar{W}_{u_t} \sim \mathrm{N}\left(0, \int_0^t \sigma_z^2 dz\right) \sim S_t. \tag{30}$$

From Ane and Geman (2000) we know that the time change that makes the price process a Brownian motion is proxied by the total volume (26), therefore

$$\sigma_t^2 dt \approx \gamma p_0^2 dv_t, \tag{31}$$

where p_0 is the reference contract exposure of one contract at the beginning of trading and γ is a suitable coefficient. Thus the variance of the implementation p&l $\Delta\Pi$ in (24) reads

$$\sigma^2 = \gamma p_0^2 \int_0^\tau h_t^2 dv_t = \gamma p_0^2 \int_0^{v_\tau} h_v^2 dv. \tag{32}$$

Now let us assume that the execution strategy is as follows

$$h_v \equiv \frac{\Delta h}{1 - e^{-\lambda v_\tau}} \left(1 - e^{-\lambda v}\right), \tag{33}$$

where $\lambda > 0$. The execution strategy (33) is not exactly implementable because the time change $t \to v_t$ defined in (26) is not deterministic. However, the strategy can be approximated by the best possible real-time estimate of the time change $t \to v_t$.

The execution strategy (33) completes the trade in the allotted time, i.e. it satisfies

$$h_0 = 0 \tag{34}$$

$$h_{v_\tau} = \Delta h. \tag{35}$$

Also, the execution strategy (33) is an approximation of the optimal execution strategy in Almgren and Chriss (2000). In particular, when $\lambda \approx 0$ the strategy is VWAP $h_v \approx \Delta h \cdot v / v_\tau$. We can readily compute the trading rate of the execution strategy (33), which reads

$$\frac{dh_v}{dv} = \frac{\lambda \Delta h}{1 - e^{-\lambda v_\tau}} e^{-\lambda v}. \tag{36}$$

Thus we can compute

$$\begin{aligned}
\int_0^{v_\tau} \left|\frac{dh}{dv}\right|^{\frac{3}{2}} dv &= \left|\frac{\lambda\,|\Delta h|}{1 - e^{-\lambda v_\tau}}\right|^{\frac{3}{2}} \int_0^{v_\tau} e^{-\frac{3\lambda}{2} v} dv \\
&= -\frac{2}{3\lambda} \left|\frac{\lambda\,|\Delta h|}{1 - e^{-\lambda v_\tau}}\right|^{\frac{3}{2}} \int_0^{v_\tau} \frac{d}{dv} e^{-\frac{3\lambda}{2} v} dv \\
&= -\frac{2}{3\lambda} \left|\frac{\lambda\,|\Delta h|}{1 - e^{-\lambda v_\tau}}\right|^{\frac{3}{2}} e^{-\frac{3\lambda}{2} v} \Big|_{v_0}^{v_\tau} \\
&= \frac{2}{3\lambda} \left|\frac{\lambda\,|\Delta h|}{1 - e^{-\lambda v_\tau}}\right|^{\frac{3}{2}} \left(1 - e^{-\frac{3\lambda}{2} v_\tau}\right) \\
&= \frac{2}{3} \sqrt{\lambda}\,|\Delta h|^{\frac{3}{2}} \frac{1 - e^{-\lambda v_\tau \frac{3}{2}}}{\left(1 - e^{-\lambda v_\tau}\right)^{\frac{3}{2}}}.
\end{aligned} \tag{37}$$

Also, we can compute

$$\int_0^{v_\tau} h_v^2 dv = \frac{(\Delta h)^2}{\left(1 - e^{-\lambda v_\tau}\right)^2} \int_0^{v_\tau} \left(1 - 2e^{-\lambda v} + e^{-2\lambda v}\right) dv. \tag{38}$$

In turn

$$\int_0^{v_\tau} dv = v_\tau \tag{39}$$

$$\int_0^{v_\tau} 2e^{-\lambda v} dv = -\frac{2}{\lambda} e^{-\lambda v}\Big|_0^{v_\tau} = \frac{2}{\lambda}\left(1 - e^{-\lambda v_\tau}\right) \tag{40}$$

$$\int_0^{v_\tau} e^{-2\lambda v} dv = -\frac{1}{2\lambda} e^{-2\lambda v}\Big|_0^{v_\tau} = \frac{1}{2\lambda}\left(1 - e^{-2\lambda v_\tau}\right). \tag{41}$$

Thus

$$\begin{aligned}\int_0^{v_\tau} h_v^2 dv &= (\Delta h)^2 \left[\frac{v_\tau}{\left(1 - e^{-\lambda v_\tau}\right)^2}\right. \\ &\quad \left. - \frac{2/\lambda}{1 - e^{-\lambda v_\tau}} + \frac{1}{2\lambda} \frac{1 - e^{-2\lambda v_\tau}}{\left(1 - e^{-\lambda v_\tau}\right)^2}\right] \\ &= \frac{4e^{-\lambda v_\tau} - e^{-2\lambda v_\tau} + 2\lambda v_\tau - 3}{2\lambda e^{-2\lambda v_\tau} - 4\lambda e^{-\lambda v_\tau} + 2\lambda}.\end{aligned} \tag{42}$$

Substituting (37) in (27), we obtain the expression for the expectation liquidity adjustment $\Delta\Pi$ in (23)-(24).

$$\mu = -\alpha p_0 \left|\Delta h\right| - \frac{2}{3} \beta \bar{\sigma} p_0 \sqrt{\lambda} \left|\Delta h\right|^{\frac{3}{2}} \frac{1 - e^{-\lambda v_\tau \frac{3}{2}}}{\left(1 - e^{-\lambda v_\tau}\right)^{\frac{3}{2}}}. \tag{43}$$

Substituting (42) in (32), we obtain the expression for the variance of the liquidity adjustment $\Delta\Pi$ in (23)-(24).

$$\sigma^2 = \frac{\gamma p_0^2}{\lambda} (\Delta h)^2 \frac{4e^{-\lambda v_\tau} - e^{-2\lambda v_\tau} + 2\lambda v_\tau - 3}{2e^{-2\lambda v_\tau} - 4e^{-\lambda v_\tau} + 2}. \tag{44}$$

The expectation (43) and the variance (44) are general expressions. In particular, the case $\lambda = 0$ yields the VWAP. It is easy to check that, in this limit, (43) and (44) simplify as follows

$$\mu \approx -\alpha p_0 \left|\Delta h\right| - \beta p_0 \bar{\sigma} \frac{\left|\Delta h\right|^{\frac{3}{2}}}{\sqrt{v_\tau}}. \tag{45}$$

$$\sigma^2 \approx \frac{\gamma p_0^2}{3} v_\tau (\Delta h)^2. \tag{46}$$

A.3 Decomposition of CVaR into liquidity and market risk component

Let us recall

$$\Pi = \bar{\Pi} + \Delta\Pi. \tag{47}$$

Now, let us define, for a statistic σ the following operator

$$\partial_X \sigma \{X + Y\} \equiv \left. \frac{d\sigma \{uX + Y\}}{du} \right|_{u=1} \tag{48}$$

Since the CVaR is positive homogeneous the following identity, due to Euler, holds true

$$CVaR\{\Pi\} = \partial_{\bar{\Pi}} CVaR\{\Pi\} + \partial_{\Delta\Pi} CVaR\{\Pi\}. \tag{49}$$

Here we want to compute the three terms in (49).

Using the analytical expression for the pdf

$$f_\Pi(y) = \textstyle\sum_j \frac{p_{(j)}}{\sigma_{(j)}} \varphi \left(\frac{y - \overline{\pi}_{(j)} - \mu_{(j)}}{\sigma_{(j)}} \right), \tag{50}$$

we can compute the cdf

$$F_\Pi(y) = \textstyle\sum_j p_{(j)} \Phi \left(\frac{y - \overline{\pi}_{(j)} - \mu_{(j)}}{\sigma_{(j)}} \right). \tag{51}$$

Then by linear interpolation on a grid as in Meucci (2011b) we obtain the function F_Π^{-1}, and thus in particular the quantile with tail integral α

$$z \equiv F_\Pi^{-1}(\alpha). \tag{52}$$

Also, from the function F_Π^{-1} we can compute the CVaR by quadratures

$$CVaR\{\Pi\} = \frac{1}{\alpha} \int_0^\alpha F_\Pi^{-1}(u)\, du. \tag{53}$$

This is the term on the left hand side in (49).

Now, let us recall

$$\partial_X CVaR\{X + Y\} = E\left\{X | X + Y \leq F_{X+Y}^{-1}(\alpha)\right\}. \tag{54}$$

Then, the contribution to $CVaR\{\Pi\}$ stemming from the liquidity component is

$$\begin{aligned} \partial_{\Delta\Pi} CVaR\{\Pi\} &= E\left\{\Delta\Pi | \bar{\Pi} + \Delta\Pi \leq z\right\} \\ &= E\left\{\Delta\Pi | \Delta\Pi \leq z - \bar{\Pi}\right\} \\ &= \int E\{\Delta\Pi | \bar{\Pi} = y, \Delta\Pi \leq z - y\} f_{\bar{\Pi}}(y)\, dy \\ &= \textstyle\sum_j p_{(j)} E\{\Delta\Pi | \bar{\Pi} = \bar{\pi}_{(j)}, \Delta\Pi \leq z - \bar{\pi}_{(j)}\} \end{aligned} \tag{55}$$

Let us recall a general result for normal variables $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, namely

$$E\left\{X|X \leq z\right\} = \mu - \sigma \frac{\varphi\left(\left(z-\mu\right)/\sigma\right)}{\Phi((z-\mu)/\sigma)}. \tag{56}$$

Since

$$\Delta\Pi|\bar{\Pi} = \bar{\pi}_{(j)} \sim \mathrm{N}(\mu_{(j)}, \sigma^2_{(j)}), \tag{57}$$

then

$$\begin{aligned} \partial_{\Delta\Pi} CVaR\left\{\Pi\right\} &= \textstyle\sum_j p_{(j)} E\{\Delta\Pi|\bar{\Pi} = \bar{\pi}_{(j)}, \Delta\Pi \leq z - \bar{\pi}_{(j)}\} \qquad (58) \\ &= \textstyle\sum_j p_{(j)} (\mu_{(j)} - \sigma_{(j)} \dfrac{\varphi\left(\left(z - \bar{\pi}_{(j)} - \mu_{(j)}\right)/\sigma_{(j)}\right)}{\Phi((z - \bar{\pi}_{(j)} - \mu)_{(j)}/\sigma_{(j)})}). \end{aligned}$$

This is the third term in (49).

The second and final term follows as the difference of the first term and the third term.