Quant Nugget 3

Common Misconceptions about "Beta"

Hedging, Estimation and Horizon Effects[1]

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this version: September 27 2010
last version available at: http://ssrn.com/abstract=1619923

The intuitive meaning of "beta" is well known to all risk and portfolio managers: the beta is the sensitivity of the return on a given asset to a given risk factor. The applications of the "beta" are manifold, from risk computation and analysis to hedging. However, the precise definition and computation of the beta is far from trivial.

1 Definition

Let us consider a broad market index whose value at time t is M_t and a stock that trades at the price S_t. Let us consider the return of the index from the current time $t \equiv 0$ to a given horizon $t \equiv \tau$ in the future $R_M \equiv (M_\tau - M_0)/M_0$; and the return of the stock over the same horizon $R_S \equiv (S_\tau - S_0)/S_0$. Assume that we have estimated the joint distribution of R_M and R_S. Based on the estimated distribution, we draw Monte Carlo scenarios $(R_M^{(j)}, R_S^{(j)})$ for $j = 1, \ldots, J$, where J is a large number. For instance, in Figure 1 we scatter-plot the scenarios for the one-month return of the S&P 500 versus a utility stock, refer to Meucci (2009) for a detailed description of all the steps of this example and to download the code that generates the results.

The most standard definition of β reads

$$\beta \equiv \frac{\rho_{S,M}\sigma_S}{\sigma_M}, \tag{1}$$

where $\rho_{S,M}$ is the correlation between market return scenarios and stock return scenarios and σ_M (σ_S) is the standard deviation of the market (stock) return scenarios.

A common misconception is that (1) follows from the Capital Asset Pricing Theorem. In reality, we can recover (1) without any connection with the CAPM.

[1] This article appears as Meucci, A. (2010) "Common misconceptions about 'beta' - hedging, estimation and horizon effects", *GARP Risk Professional - "The Quant Classroom"*, **2**, June, p. 42-45

[2] The author is grateful to Eva Chan and Jiang Wenli

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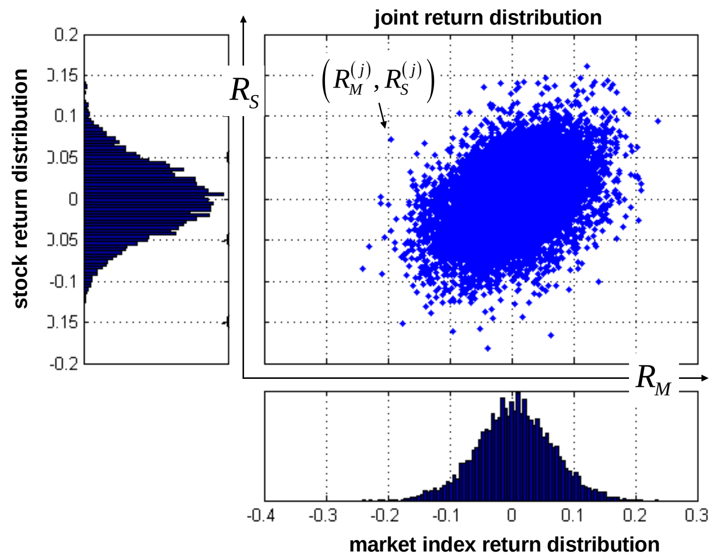

Figure 1: Estimated distribution of one-month stock and market index returns

Consider a simple linear function that transfers the randomness of the market return scenarios $R_M^{(j)}$ into the stock return scenarios $R_S^{(j)}$

$$R_S^{(j)} = \alpha + \beta R_M^{(j)} + \epsilon^{(j)}, \quad j = 1, \ldots, J, \tag{2}$$

where $\epsilon^{(j)}$ are residuals of randomness which are not due to the market.

For any value of the transfer coefficients (α, β) we obtain a different set of residuals $\epsilon^{(j)} \equiv R_S^{(j)} - \alpha - \beta R_M^{(j)}$. What choice of (α, β) gives the residuals $\epsilon^{(j)}$ the most desirable features? If we aim at minimizing the standard deviation of the residuals

$$\beta^* \equiv \underset{\beta}{\operatorname{argmin}} \{\sigma_\epsilon\}, \tag{3}$$

then the solution β^* is exactly the specification (1), see Meucci (2005).

The constructive definition of the beta (3) also applies directly to hedging problems. Indeed, the residuals in (2) are the returns of a portfolio long the stock and short cash plus the market index. By minimizing the volatility as in (3) we are actually computing the minimum-risk hedged position. More in general, we can think of hedging any security, not necessarily a stock, with any product, not necessarily a market index. For instance, we can hedge a call option with cash and the underlying

$$\epsilon^{(j)} \equiv R_C^{(j)} - \alpha - \beta R_U^{(j)}. \tag{4}$$

This generalized beta, which is one of the pillars of the Factors on Demand (FoD) approach in Meucci (2010a), is closely related to the "delta" of the Black-

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Scholes-Merton formula but it is more flexible. For instance, instead of minimizing the volatility of the hedged portfolio we can minimize only its downside, as represented by the conditional value at risk (CVaR); also, unlike the delta, the FoD beta properly accounts for horizon effects.

	100 days	150 days	200 days	250 days	300 days
FOD	5.8	5.3	5.0	4.9	4.8
BS	5.7	5.4	5.2	5.1	5.0

Figure 2: Number of stocks for Factors on Demand beta hedging versus Black-Scholes delta hedging

In Figure 2 we report the number of stocks necessary to hedge a call option with different expiries according to the FoD beta and to the Black-Scholes delta, please refer to Meucci (2009) for a detailed description of all the steps of this example and to download the code that generates the results.

2 Computation

Another common misconception is that in order to compute the beta we should run regressions on the realized time series of the returns of the securities.

This is incorrect: the beta is the sensitivity of the yet-to-be-realized return of a stock to the yet-to-be-realized return of the market, as represented by the joint Monte Carlo scenarios. The history of the past returns is a representation of the true forward distribution only if the returns are invariants, i.e. if they are identically and independently (i.i.d.) distributed across time. In general, this is not the case. For instance, volatilities and correlations can change through time: then models such as regime switches or GARCH become more adequate than the i.i.d. assumption to describe the behavior of the returns.

Furthermore, even if returns were invariants, the horizon of the yet-to-be realized return whose beta we want to compute is typically inconsistent with the time step of the regression. For instance, if we want to estimate the beta of a one-month ahead return, we would waste precious information if we only relied on the necessarily very few non-overlapping monthly observations available.

Finally, as highlighted in Section 1, the concept of beta applies to any asset, not only stocks. For such securities as options, returns are definitely not invariants and running a regression would not make sense.

Therefore, to compute the beta we must first estimate the invariants; then generate scenarios from the invariants distribution; next project the invariants from the estimation step to the return horizon; and finally compute the scenarios for the returns. At this point, the beta is computed as in (1). Notice that

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these are the same steps necessary in a more general framework to build a risk platform, see Meucci (2010b).

For instance, to generate Figure 1 we fit a multivariate GARCH to the daily series of the compounded returns C of market and stock as in Ledoit, Santa-Clara, and Wolf (2003); then we draw Monte Carlo scenarios iteratively to obtain the distribution of the respective monthly compounded return; finally we map the compounded returns C into the linear returns R using the pricing equation $R = e^C - 1$ to obtain the distribution of the monthly returns. For a detailed description of all the steps of this example and to download the code that generates the results see Meucci (2009).

3 Horizon-dependence

A third common misconception is related to the dependence of the beta on the horizon. Naturally, the beta depends on the horizon when returns are not invariants. However, the beta changes with the horizon even when returns are invariants.

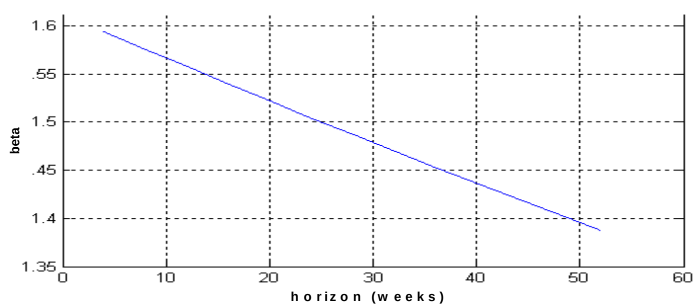

Figure 3: Dependence of beta on return horizon

To illustrate, consider the most basic assumption for the joint evolution of our stock price S_t and a market index M_t, namely the geometric Brownian motion a-la Black-Scholes-Merton

$$\begin{pmatrix} \ln S_\tau \\ \ln M_\tau \end{pmatrix} \sim \mathrm{N}\left(\tau \begin{bmatrix} \widetilde{\mu}_S \\ \widetilde{\mu}_M \end{bmatrix}, \tau \begin{bmatrix} \widetilde{\sigma}_S^2 & \widetilde{\rho}_{S,M}\widetilde{\sigma}_S\widetilde{\sigma}_M \\ \widetilde{\rho}_{S,M}\widetilde{\sigma}_S\widetilde{\sigma}_M & \widetilde{\sigma}_M^2 \end{bmatrix}\right), \tag{5}$$

where for simplicity we have normalized the initial values such that $S_0 \equiv M_0 \equiv 1$. In this framework, drift, correlation and volatilities are constant and the returns are invariants. Furthermore, the beta of the *compounded* stock return with the compounded market return is independent of the horizon τ

$$\widetilde{\beta} \equiv \frac{\widetilde{\rho}_{S,M}\widetilde{\sigma}_S}{\widetilde{\sigma}_M}. \tag{6}$$

Although we can be interested in such beta from a statistical perspective, as highlighted in Section 1, the beta must be computed on the *linear* returns, both for its relationship with the CAPM and for hedging purposes. The beta of the linear returns depends on the horizon because of the distortion introduced by the pricing function, see Meucci (2010b). Indeed, as we prove in Meucci (2009)

$$\beta = \frac{e^{\widetilde{\mu}_S t + \widetilde{\sigma}_S^2 t/2}\left(e^{\widetilde{\rho}_{S,M}\widetilde{\sigma}_S\widetilde{\sigma}_M t} - 1\right)}{e^{\widetilde{\mu}_M t + \widetilde{\sigma}_M^2 t/2}\left(e^{\widetilde{\sigma}_M^2 t} - 1\right)}. \tag{7}$$

In Figure 3 we plot the distortion effect as a function of the return horizon for the case considered in Figure 1.

4 Summary

To summarize, the CAPM prescribes a sensitivity β as defined in (1) between the *linear* return on *any* asset S, such as stocks, but also bonds, options, etc., and the *linear* return on the market portfolio M.

For hedging purposes we compute β such that the hedged *linear* return

$$\epsilon^{(j)} \equiv R_S^{(j)} - \alpha - \beta R_M^{(j)} \tag{8}$$

displays any desired features. In this context, S is *any* asset, such as stocks, but also bonds, options, etc., and M is *any* hedging instrument, not necessarily the market portfolio. In particular, if we want to minimize the variance of the hedged return than β is the same as (1).

Notice that both CAPM and hedging beta refer to linear returns, not compounded returns, see Meucci (2010b) for common pitfalls related to these two definitions of returns.

The beta of the linear returns always depends on the return horizon even when the returns are invariants, i.e. their distribution is i.i.d.

Finally, to compute the β we should refrain from running regressions on returns. Instead, we should estimate the invariants; project their distribution to the horizon; map the invariants into returns, and only then compute the beta as in (1).

References

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