DIMENSION REDUCTION: Principal Component Analysis of Swap Market

Risk and Asset Allocation – Springer – `symmys.com`

Attilio Meucci

www.symmys.com

Formulas and figures in this presentation refer to the book **Risk and Asset Allocation**, Springer.

The notation, say, (5.24) refers to Formula 24 in Chapter 5 of the book

The notation, say, (T4.12) refers to Formula 12 in the Technical Appendices for Chapter 4, which can be downloaded from www.symmys.com

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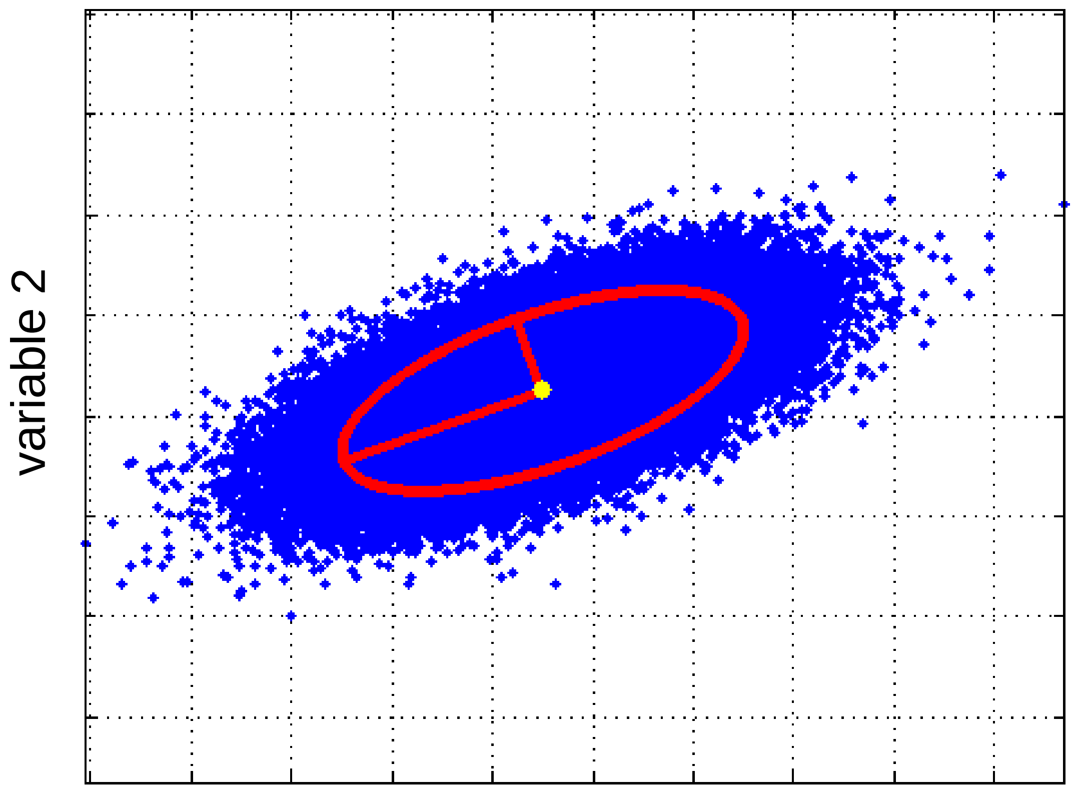

location dispersion ellipsoid

$$(\boldsymbol{r}-\boldsymbol{m})'\boldsymbol{S}^{-1}(\boldsymbol{r}-\boldsymbol{m}) \equiv \text{constant}$$

$$\boldsymbol{m} \equiv \mathrm{E}\{\boldsymbol{X}\}$$

$$\boldsymbol{S} \equiv \mathrm{Cov}\{\boldsymbol{X}\} \equiv \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}'$$

algebra		statistics		geometry
orthogonal eigenvectors	$\Leftrightarrow$	uncorrelated directions	$\Leftrightarrow$	direction of principal axes
square root of eigenvalues	$\Leftrightarrow$	volatility in uncorr. dir.	$\Leftrightarrow$	length of principal axes

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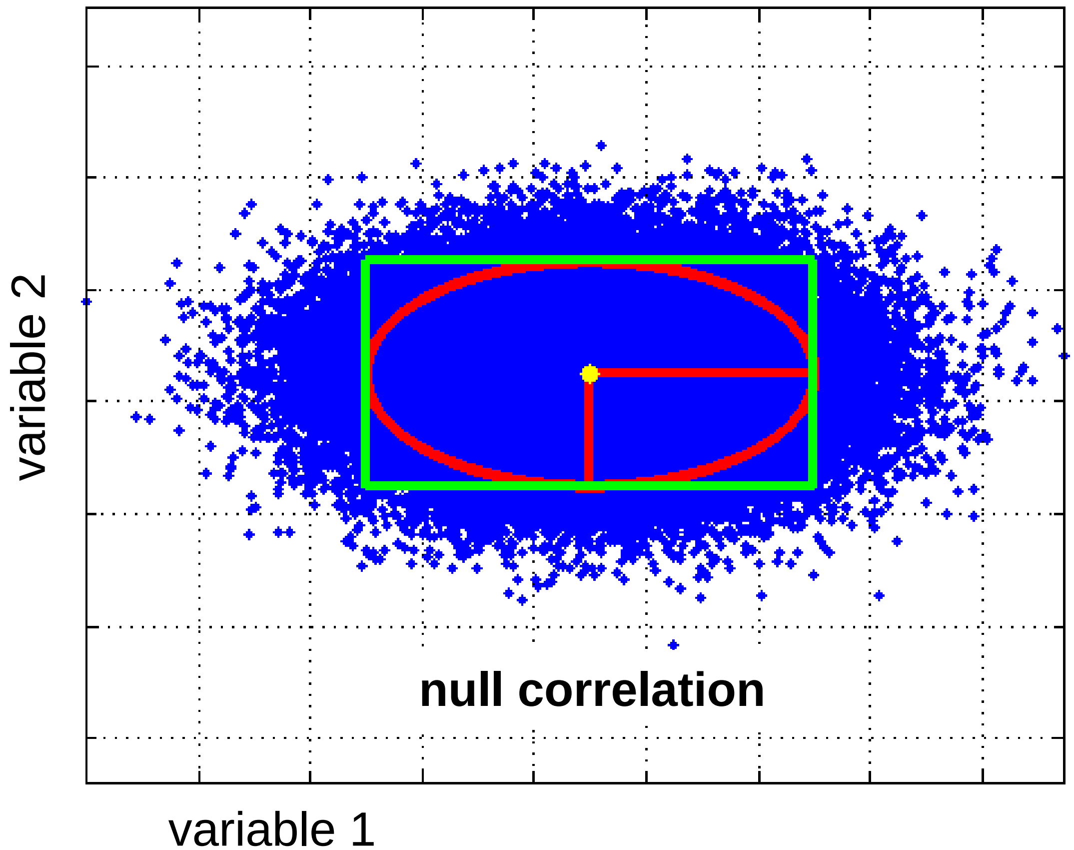

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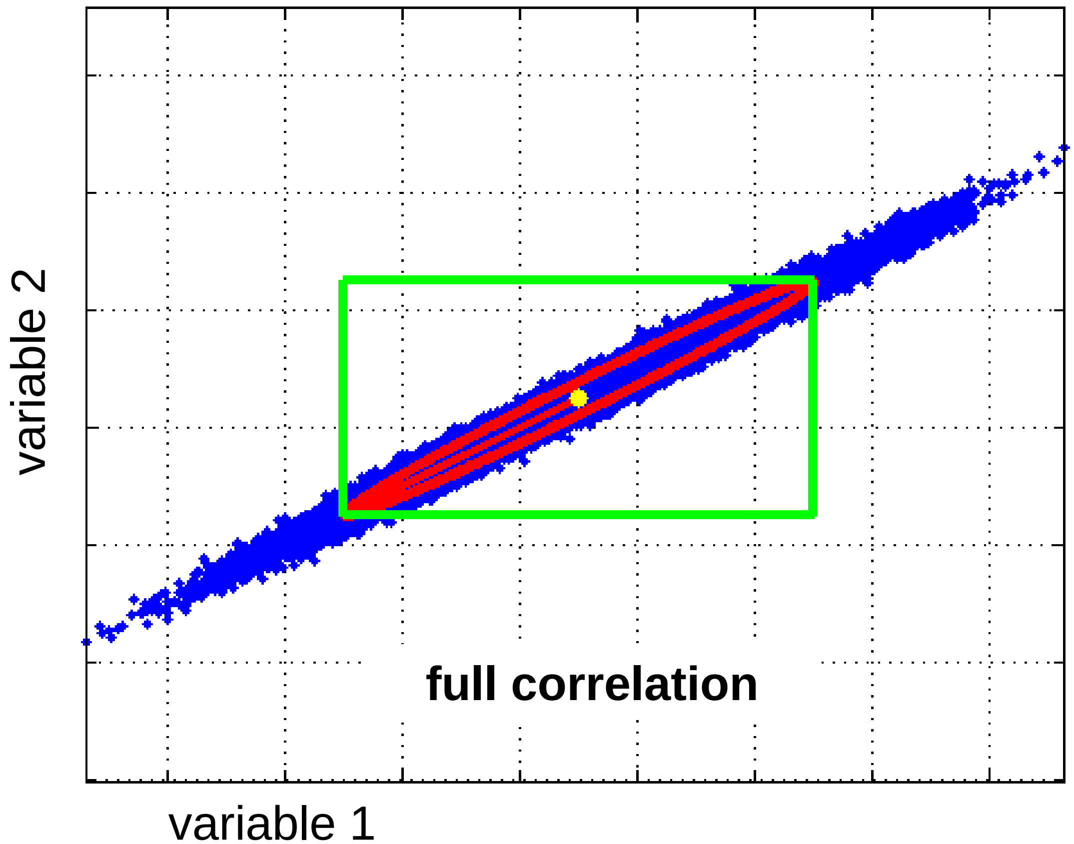

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$$\mathbf{X} \equiv \mathbf{Y}_t - \mathbf{Y}_{t-\tilde{\tau}}, \quad (3.232)$$

$$\boxed{\left(\boldsymbol{x} - \mathrm{E}\{\boldsymbol{X}\}\right)' \left(\mathrm{Cov}\{\boldsymbol{X}\}\right)^{-1} \left(\boldsymbol{x} - \mathrm{E}\{\boldsymbol{X}\}\right) \equiv 1}$$

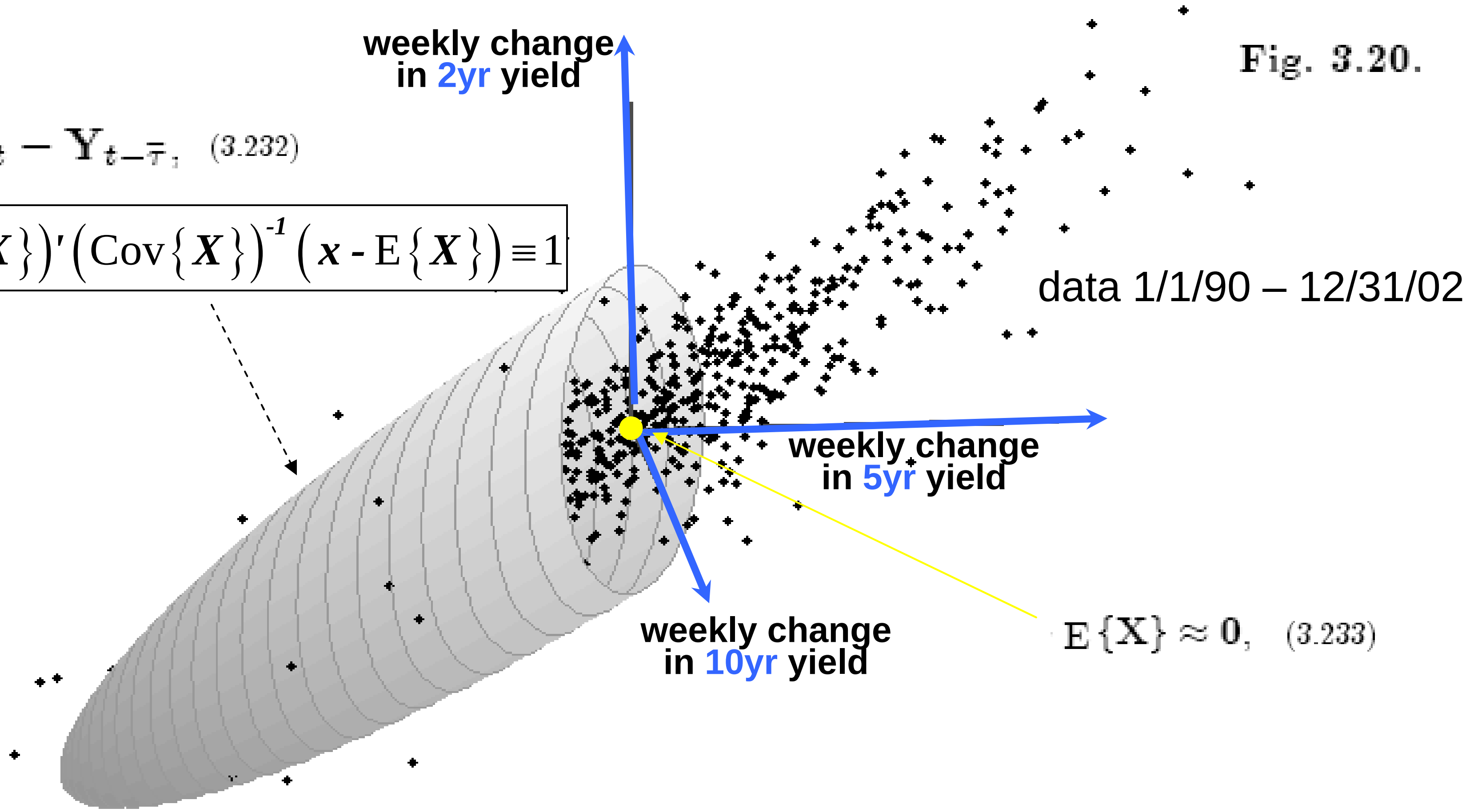

Fig. 3.20.

$$\mathrm{E}\{\mathbf{X}\} \approx \mathbf{0}, \quad (3.233)$$

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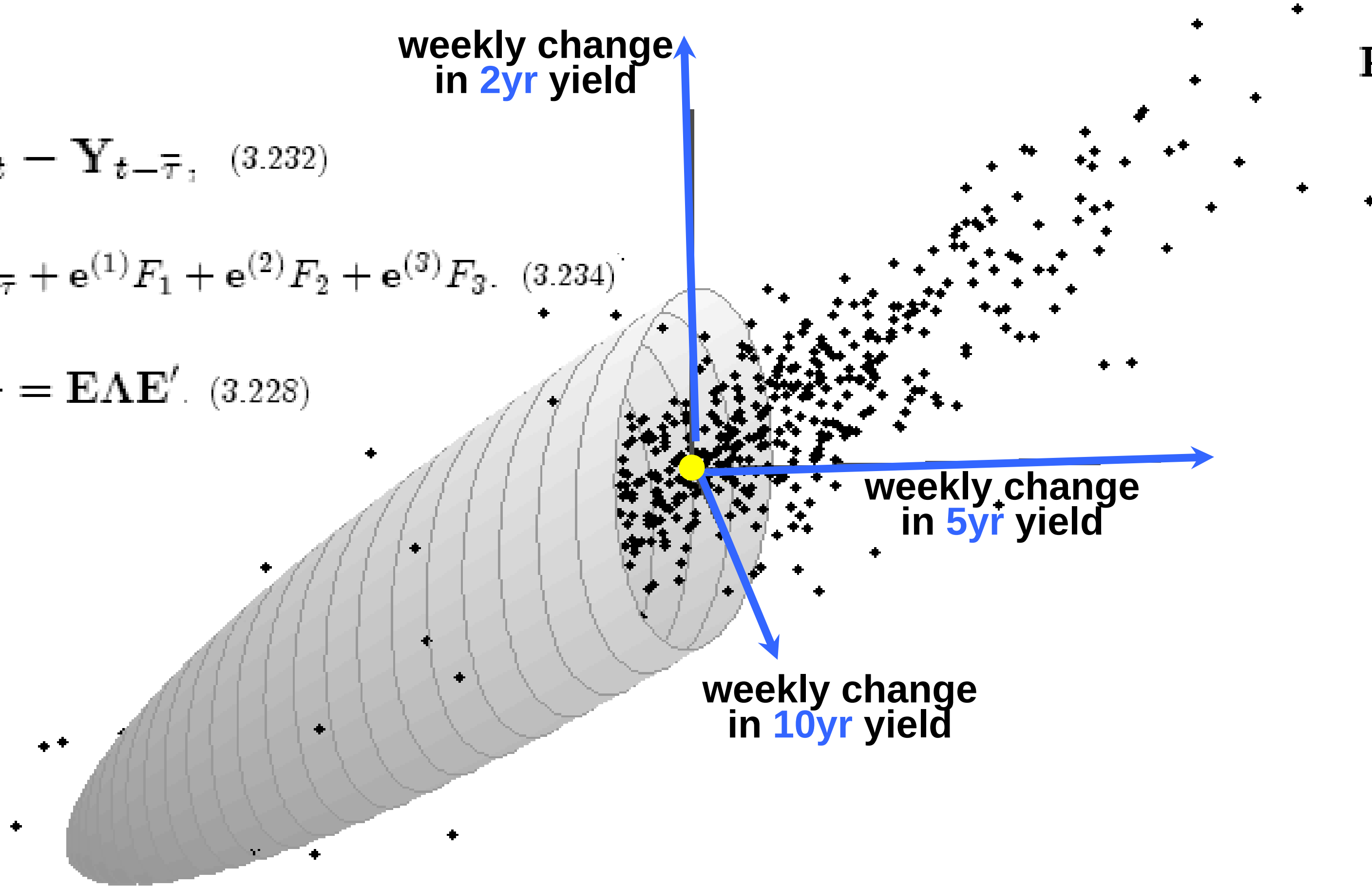
$$\mathbf{X} \equiv \mathbf{Y}_t - \mathbf{Y}_{t-\tilde{\tau}}, \quad (3.232)$$

$$\widetilde{\mathbf{Y}}_t \equiv \mathbf{Y}_{t-\tilde{\tau}} + \mathbf{e}^{(1)} F_1 + \mathbf{e}^{(2)} F_2 + \mathbf{e}^{(3)} F_3. \quad (3.234)$$

$$\mathrm{Cov}\{\mathbf{X}\} = \mathbf{E}\Lambda\mathbf{E}'. \quad (3.228)$$



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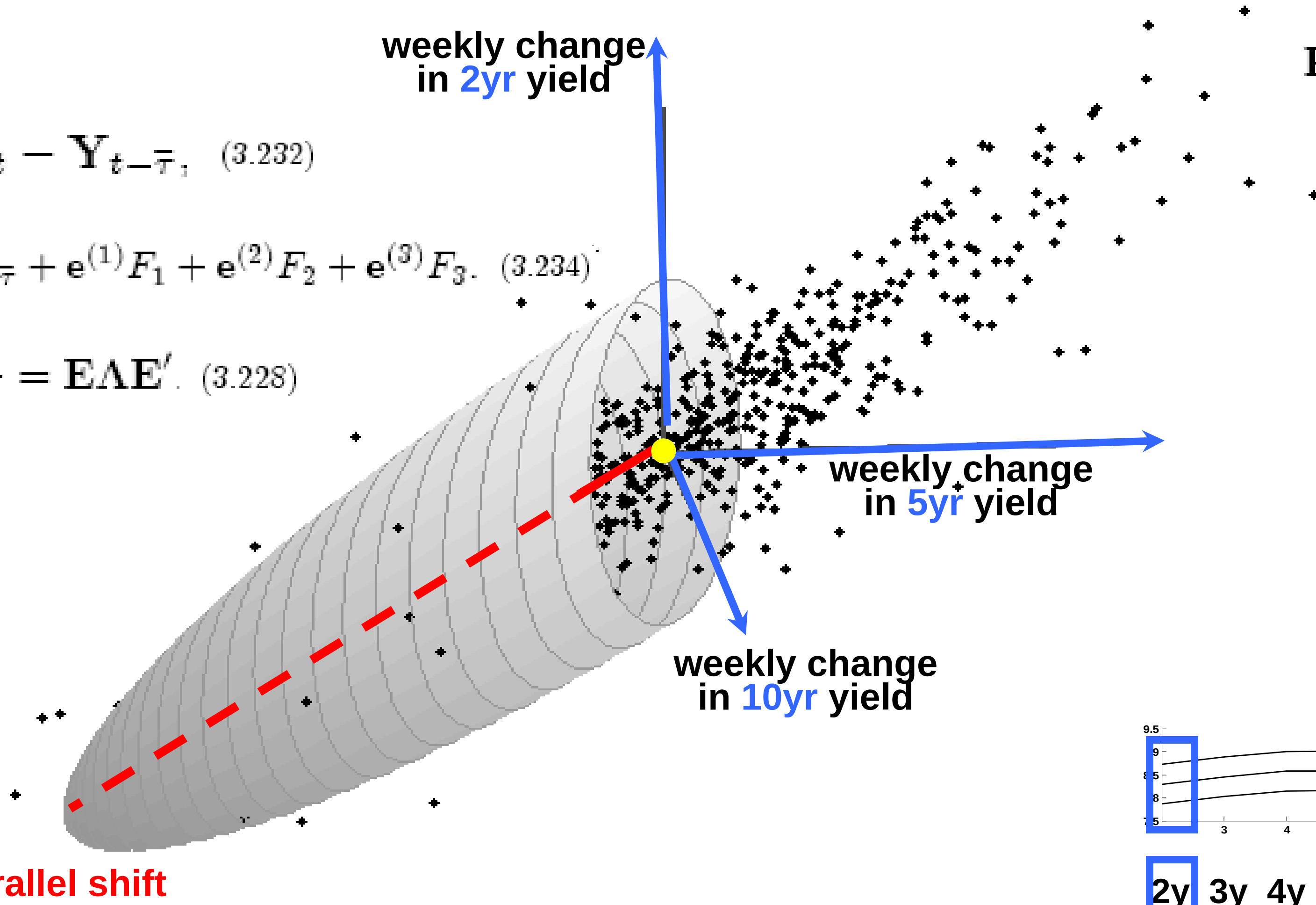
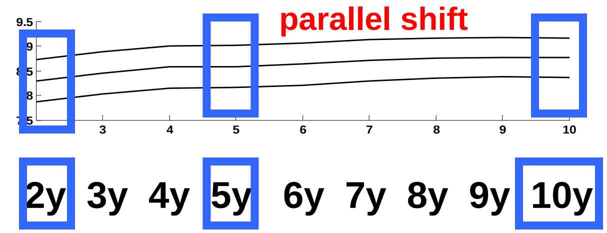


Fig. 3.21

swap curve



$$\mathrm{Sd}\{F_1\} = \sqrt{\lambda_1}. \quad (3.235)$$

$$\boxed{\mathbf{Y}_t \mapsto \mathbf{Y}_{t+\tilde{\tau}} \equiv \mathbf{Y}_t \pm \sqrt{\lambda_1}\,\mathbf{e}^{(1)}} \quad (3.236)$$

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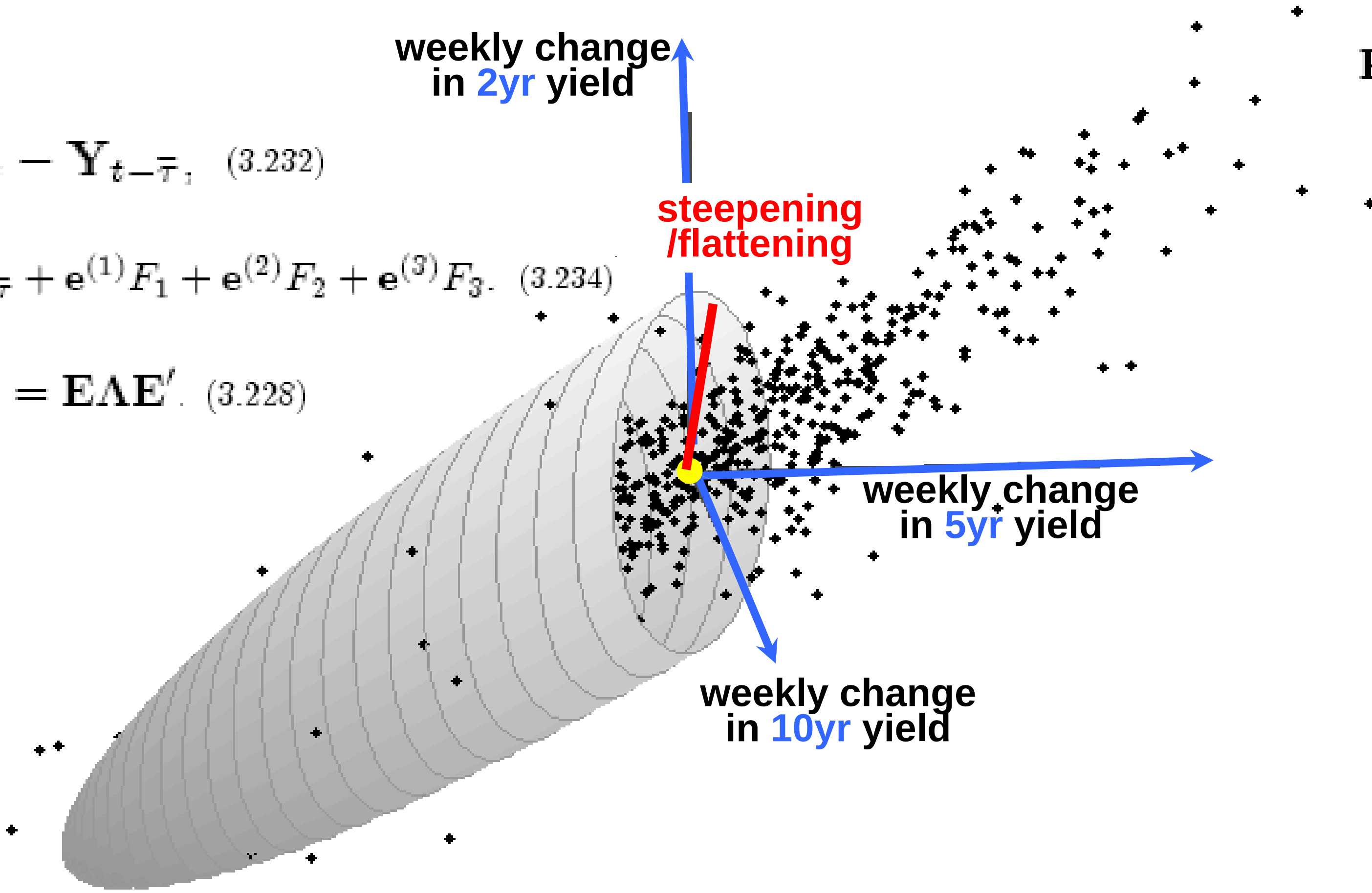
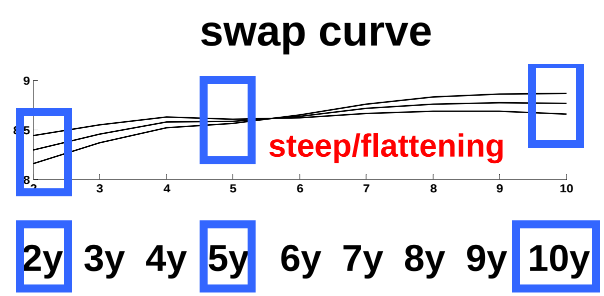


Fig. 3.20.

Fig. 3.21



$$\boxed{\mathbf{Y}_t \mapsto \mathbf{Y}_{t+\widetilde{\tau}} \equiv \mathbf{Y}_t \pm \sqrt{\lambda_2}\,\mathbf{e}^{(2)}} \quad (3.238)$$

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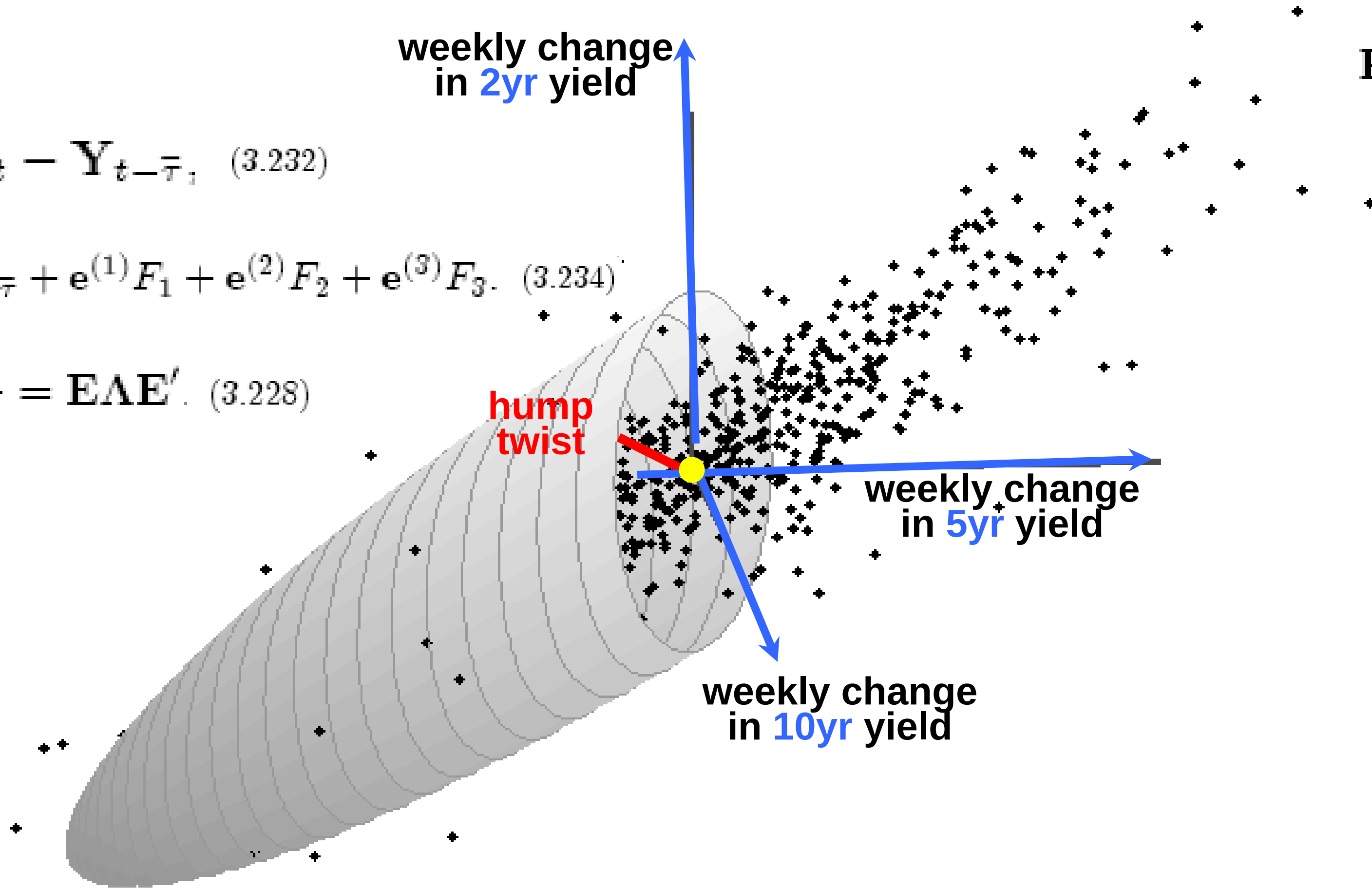

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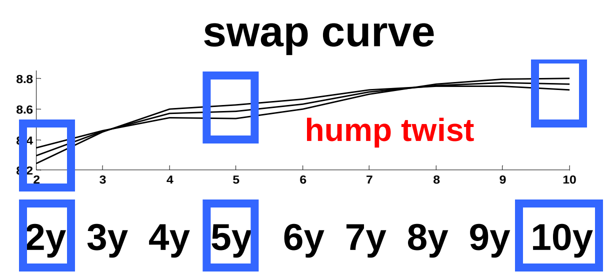

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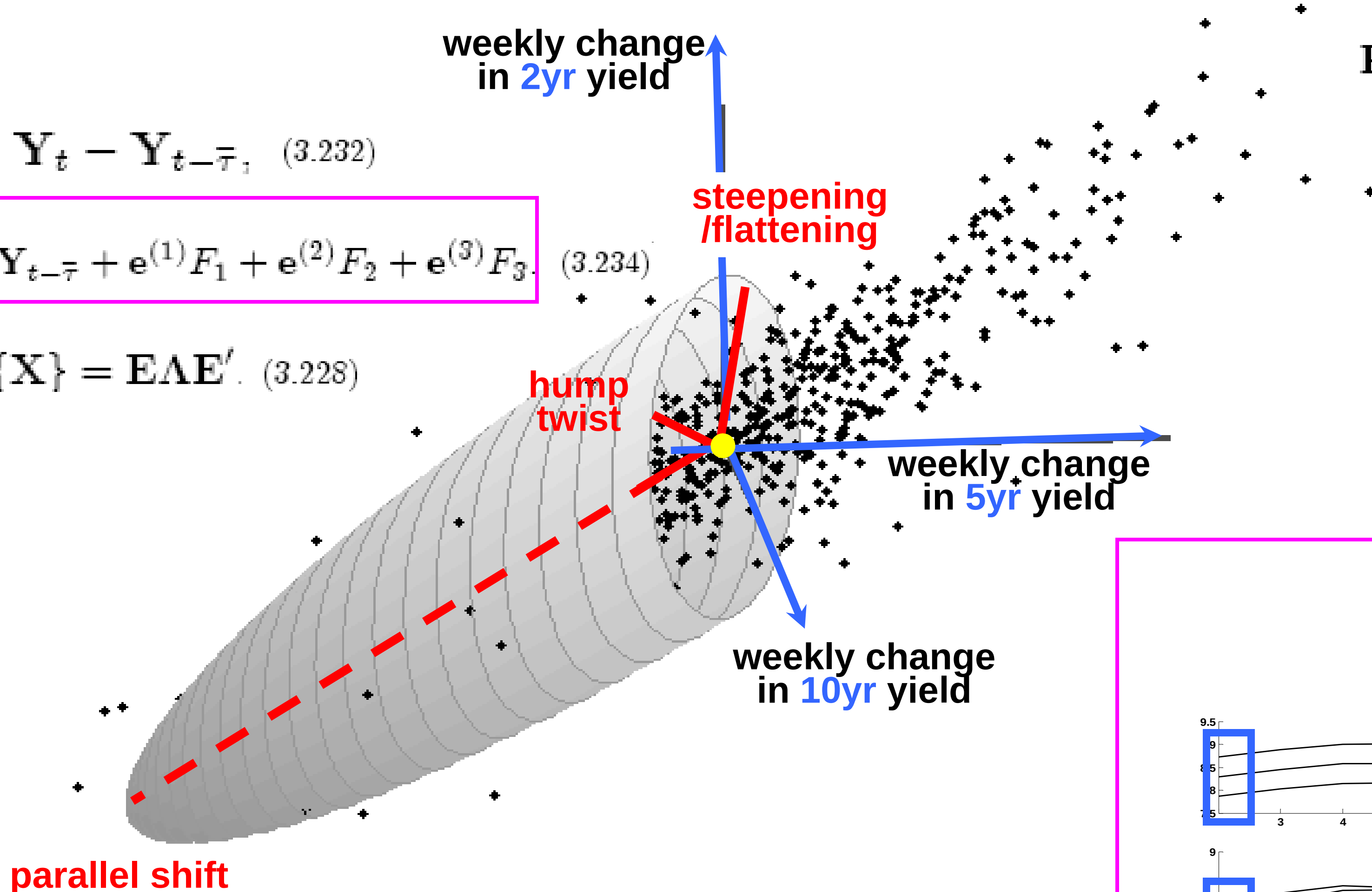

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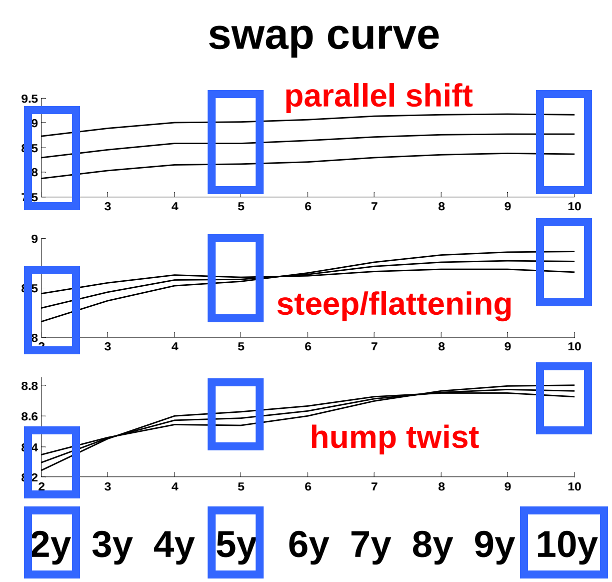

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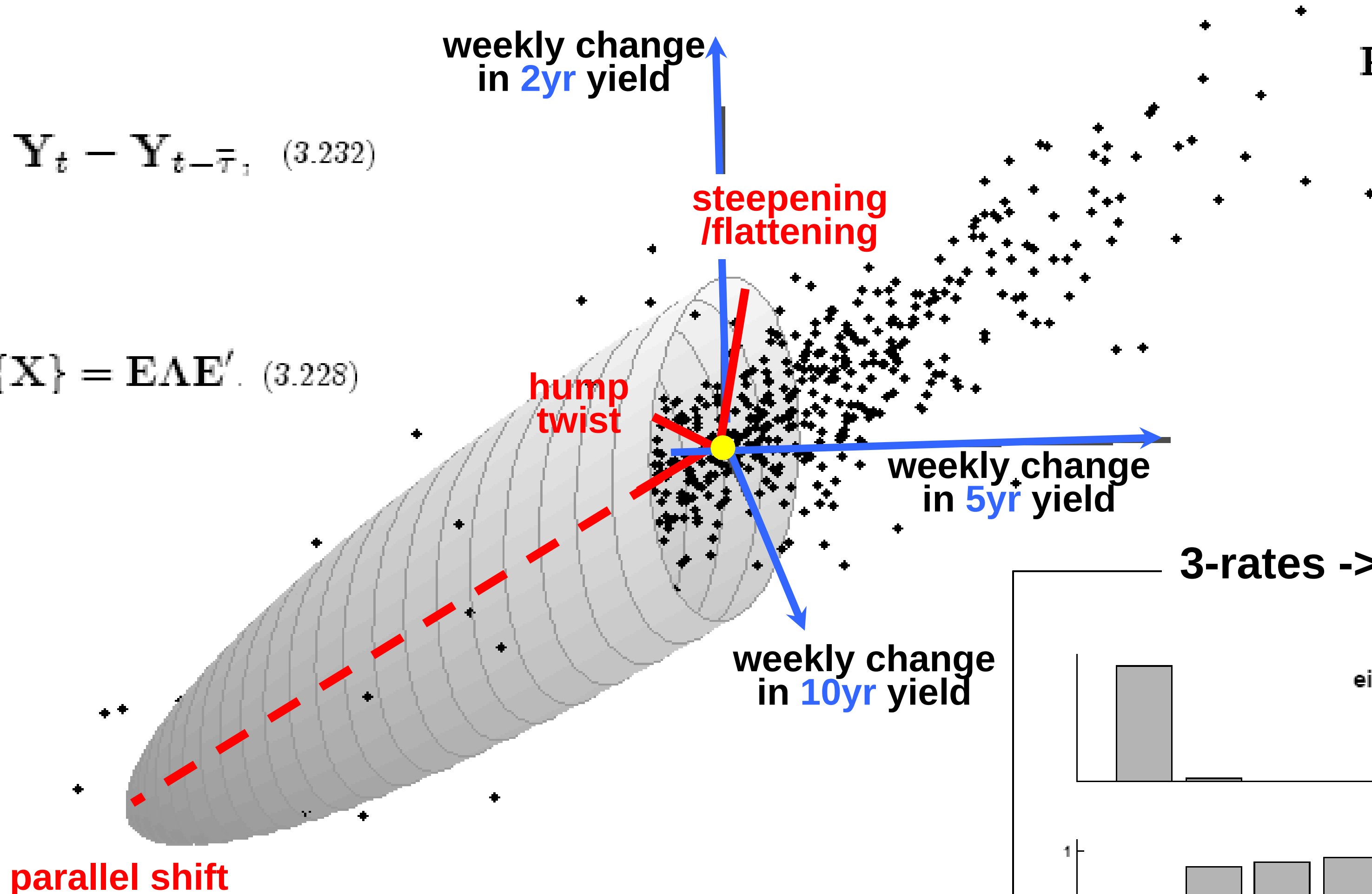

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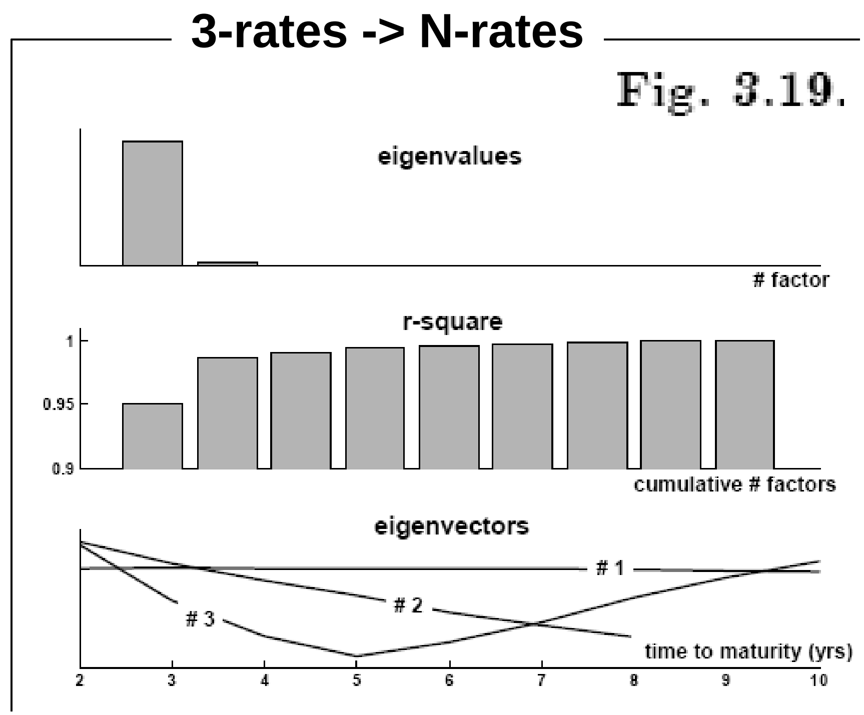

Fig. 3.19.

$$\mathrm{Cov}\{\mathbf{X}\}\,\mathbf{e}^{(n)} = \lambda_n \mathbf{e}^{(n)} \quad (3.215)$$

$$\Updownarrow$$

$$\mathrm{Cov}\{\mathbf{X}\} = \mathbf{E}\Lambda\mathbf{E}'. \quad (3.228)$$

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$$\int_{\mathbb{R}} \mathrm{Cov}\left\{X^{(v)}, X^{(p)}\right\} e^{(\omega)}(p)\, dp = \lambda_{\omega} e^{(\omega)}(v) \qquad (3.216)$$

$$\mathrm{Cov}\{\mathbf{X}\}\, \mathbf{e}^{(n)} = \lambda_n \mathbf{e}^{(n)} \qquad (3.215)$$

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$$\int_{\mathbb{R}} \operatorname{Cov}\left\{X^{(\upsilon)}, X^{(p)}\right\} e^{(\omega)}(p)\, dp = \lambda_{\omega} e^{(\omega)}(\upsilon) \quad (3.216)$$

$$C(\upsilon, p) \equiv \operatorname{Cov}\left\{X^{(\upsilon)}, X^{(\upsilon+p)}\right\} \approx h(p) \quad (3.213)$$

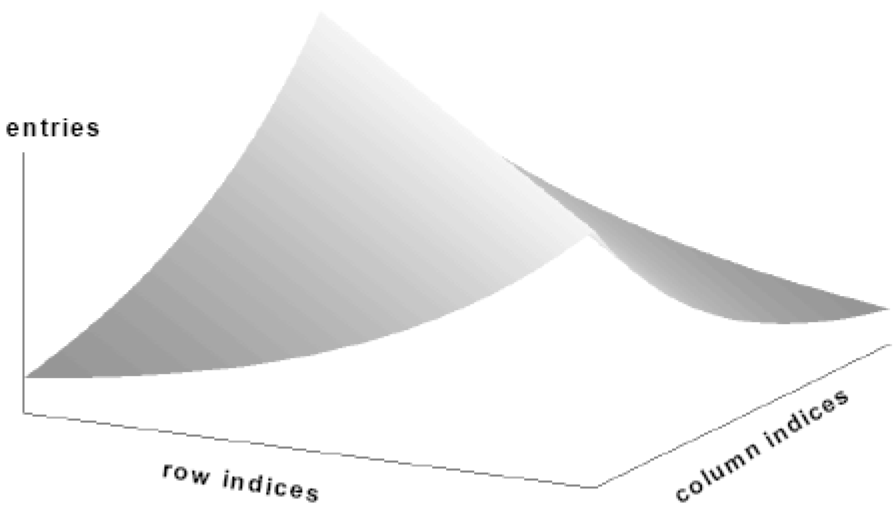

Fig. 3.16. Toeplitz matrix

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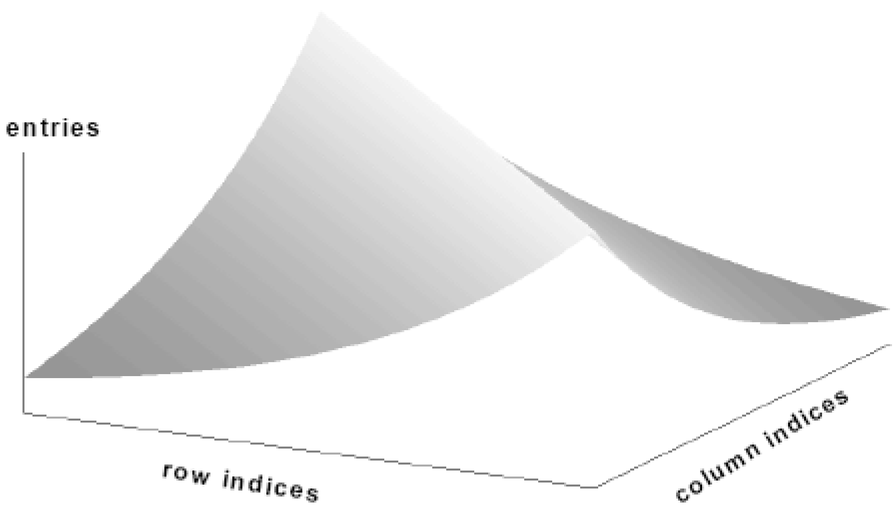

Fig. 3.16. Toeplitz matrix

$$C(\upsilon, p) \equiv \mathrm{Cov}\left\{X^{(\upsilon)}, X^{(\upsilon+p)}\right\} \approx h(p) \quad (3.213)$$

$$\begin{cases} e^{(\omega)}(\upsilon) \equiv e^{i\omega\upsilon} & (3.217) \\ \lambda_\omega = \mathcal{F}[h](\omega) & (3.218) \\ R^2\left\{X, \widetilde{X}\right\} \equiv \dfrac{\int_\Omega \lambda_\omega d\omega}{\int_0 \lambda_\omega d\omega}. & (3.220) \end{cases}$$

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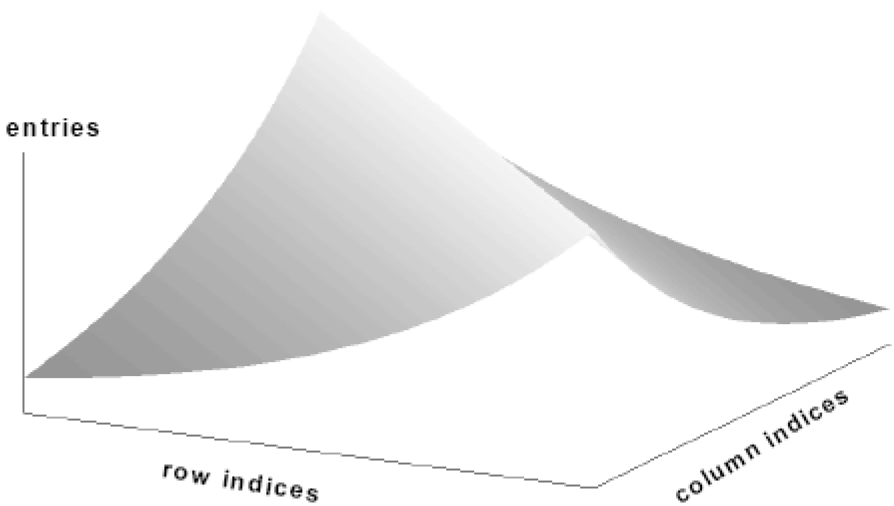

Fig. 3.16. Toeplitz matrix

$$h(p) = \sigma^2 \exp(-\gamma |p|) \quad (3.222)$$

$$\begin{cases} e^{(\omega)}(\upsilon) \equiv e^{i\omega\upsilon} & (3.217) \\ \lambda_\omega = \dfrac{2\sigma^2}{\sqrt{\gamma^2}}\left(1 + \dfrac{\omega^2}{\gamma^2}\right)^{-1} & (3.223) \\ R^2\left\{X, \widetilde{X}\right\} = \dfrac{2}{\pi} \arctan\left(\dfrac{\overline{\omega}}{\gamma}\right) & (3.226) \end{cases}$$

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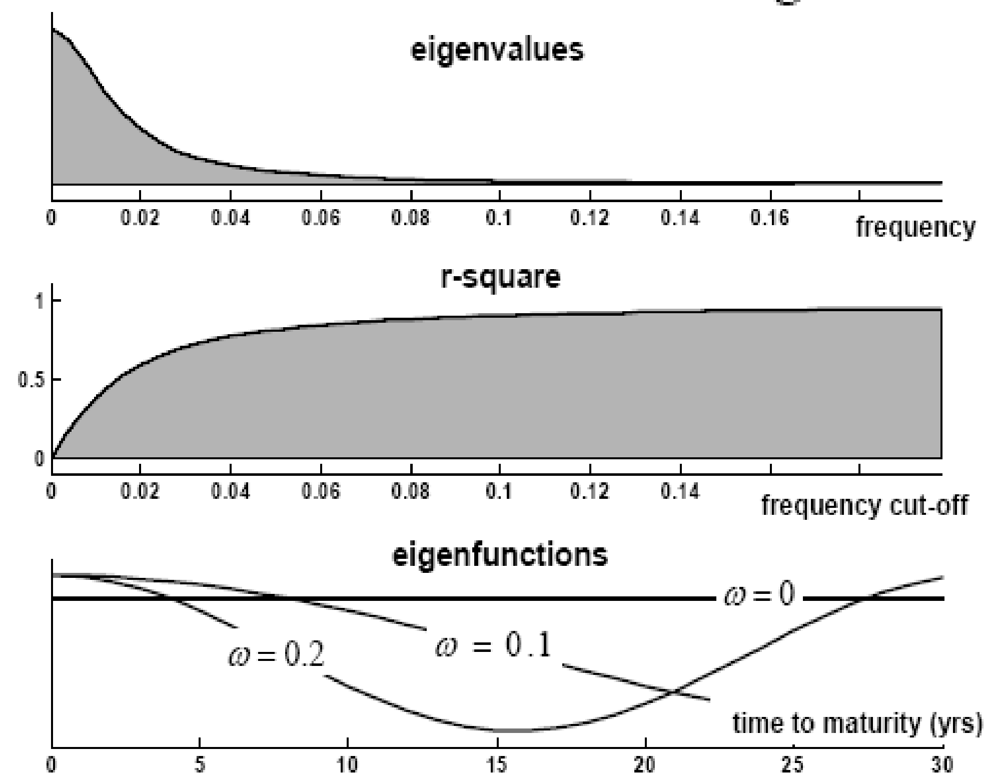
Fig. 3.18.



$$e^{(\omega)}(v) \equiv e^{i\omega v}. \qquad (3.217)$$

$$\lambda_\omega = \frac{2\sigma^2}{\sqrt{\gamma^2}}\left(1+\frac{\omega^2}{\gamma^2}\right)^{-1} \qquad (3.223)$$

$$R^2\left\{X, \widetilde{X}\right\} = \frac{2}{\pi}\arctan\left(\frac{\overline{\omega}}{\gamma}\right) \qquad (3.226)$$

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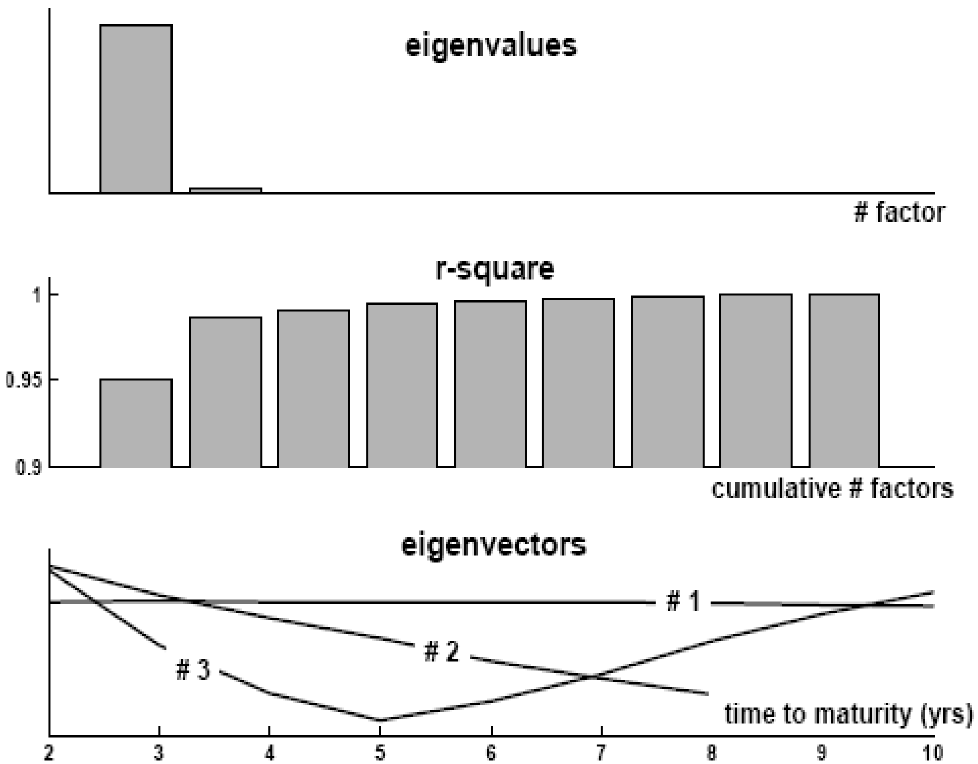

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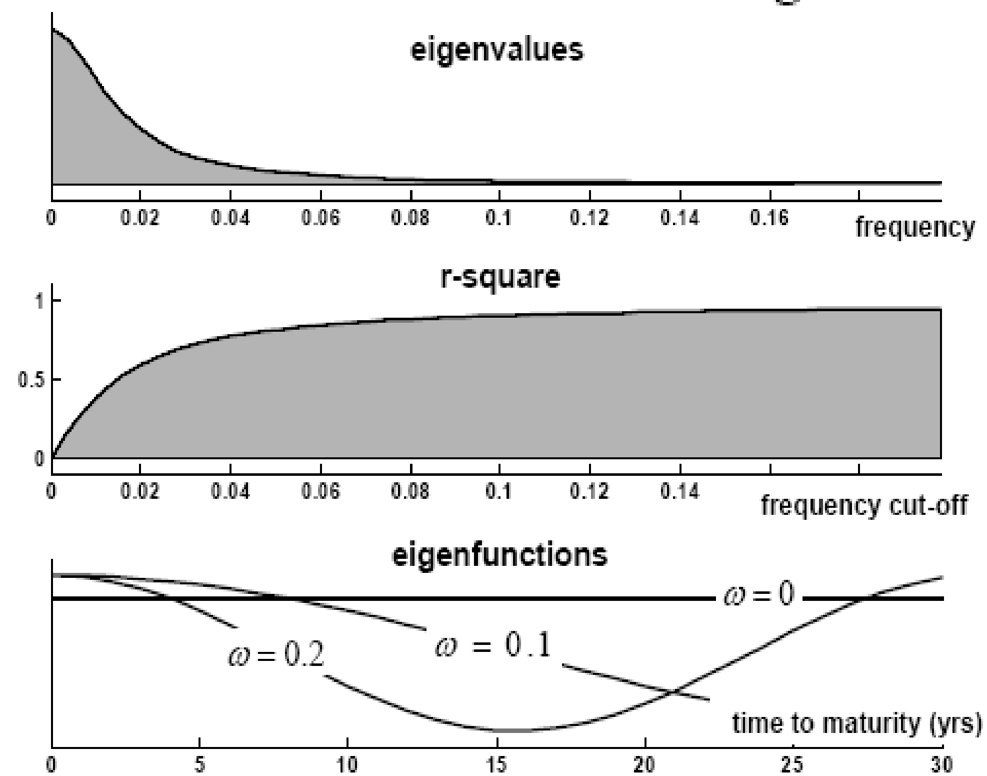

$$e^{(\omega)}(v) \equiv e^{i\omega v}. \tag{3.217}$$

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