Review Session 2

August 12, 2014

Projection and Pricing

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- p64 5.6 Derivatives

Statistical arbitrage

- p28 3.2.4 Cointegration

SYMMYS
Advanced Risk and Portfolio Management®

5.5.2 Student t invariants

Consider a fixed-income market, where the changes in yield-to-maturity, or rate changes, are the market invariants.

Assume that the weekly changes in yield for all maturities are fully codependent, i.e. co-monotonic. In other words, assume that the copula of any pairs of weekly yield changes is (2.106) in Meucci (2005). Assume that the marginal distributions of the weekly changes in yield for all maturities are:

$$\Delta_{\tilde{\tau}} Y^{(\upsilon)} \sim \mathrm{St}\left(\nu, \mu, \sigma_{\upsilon}^{2}\right). \tag{326}$$

In this expression υ denotes the time to maturity (in years) and

$$\nu \equiv 8, \quad \mu \equiv 0, \quad \sqrt{\sigma_{\upsilon}^{2}} \equiv \left(20 + \frac{5}{4}\upsilon\right) \times 10^{-4}. \tag{327}$$

Consider bonds with current times to maturity 4, 5, 10, 52 and 520 weeks, and assume that the current yield curve, as defined in (3.30) in Meucci (2005) is flat at 4% (measuring time in years).

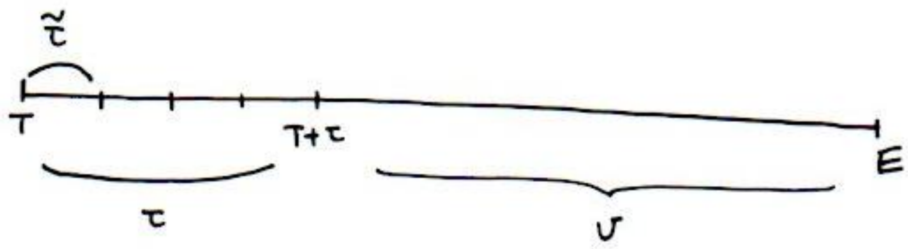
Use the function `ProjectionT` that takes as inputs the estimation parameters of the t-distributed invariants and the horizon-to-estimation ratio $\tau/\tilde{\tau}$ to compute the cdf of the invariants at the investment horizon τ. You do not need to know how this function works. Make sure you properly compute the necessary inputs (see hints below).

Use the cdf obtained above to generate a joint simulation of the bond prices at the investment horizon τ of four weeks.

Plot the histogram of the *linear* returns $L_{T+\tau,\tau}$ of each bond over the investment horizon, where the linear return is defined consistently with (3.10) in Meucci (2005) as follows:

$$L_{t,\tau} \equiv \frac{Z_t^{(E)}}{Z_{t-\tau}^{(E)}} - 1. \tag{328}$$

Notice that the long-maturity (long duration) bonds are much more volatile than the short maturity (short duration) bonds.



T: TODAY

$\tilde{\tau}$: ESTIMATION INTERVAL (1 week)

τ : INVESTMENT HORIZON (4 weeks)

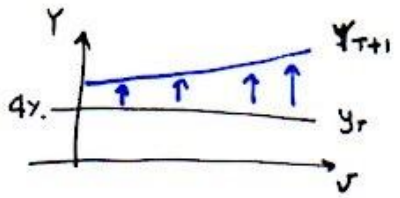
E : BOND EXPIRY = T+τ+υ 5 BONDS ($E_1, \dots, E_5$)

↓

- TIME TO MAT TODAY ($\tau+\upsilon$) $\begin{cases} 4 \\ 5 \\ 10 \\ 52 \\ 520 \end{cases}$ weeks
- TIME TO MAT AT THE HORIZON (υ) $\begin{cases} 0 \\ 1 \\ 6 \\ 48 \\ 516 \end{cases}$ weeks

- $\underset{\text{COMONOTONIC}}{\Delta_{\tilde{\tau}} Y^{(v)}} = Y_t^{(v)} - Y_{t-\tilde{\tau}}^{(v)} \sim St(\nu, \mu, \sigma_v^2)$, with $\nu = 8$, $\mu = 0$, $\sigma_v = \left(20 + \frac{5}{4} v\right) \cdot 10^{-4}$ (v in years)

- $y_T^{(v)} \equiv 4\%$ $(\forall v)$



(?) ① JOINT SIMULATION OF $Z_{T+\tau}^{(E_i)}$ $\quad E_i = E_1, \dots, E_5$ (5 BONDS)

② HIST OF THE LIN RETURNS FOR EACH BOND

$$L_{T \to T+\tau}^{(i)} = \frac{Z_{T+\tau}^{(E_i)}}{Z_T^{(E_i)}} - 1$$

① PRICE OF A BOND WITH EXPIRY E AT THE HORIZON

$$Z_{T+\tau}^{(E)} = \exp\left\{-v Y_{T+\tau}^{(v)}\right\}$$

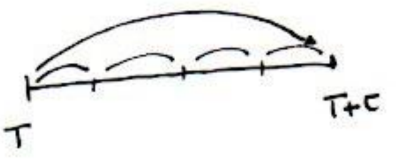
$$= \exp\left\{-v \left[y_T^{(v)} + \Delta_{\tilde{\tau}} Y_{T+\tilde{\tau}}^{(v)} + \dots + \Delta_{\tilde{\tau}} Y_{T+4\tilde{\tau}}^{(v)} \right]\right\}$$

$$= \exp\left\{-v\, y_T^{(v)}\right\} \cdot \exp\left\{-v \underbrace{\sum_{k=1}^{4} \underbrace{\Delta_{\tilde{\tau}} Y_{T+k\tilde{\tau}}^{(v)}}_{\sim St(\nu, \mu, \sigma_v^2)}}_{S^{(v)}}\right\}$$

$S^{(v)} \to$ THE CDF OF $S^{(v)}$ IS OBTAINED VIA FFT USING THE FUNCTION

$[x, f, F] = \text{ProjectionT}(\nu, \mu, \sigma_v, \frac{\tau}{\tilde{\tau}})$

$\Delta_{\tilde{\tau}} Y_t^{(v)} = Y_t^{(v)} - Y_{t-\tilde{\tau}}^{(v)} \sim St$

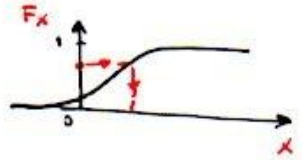


$\tau = 4\tilde{\tau}$

- SIMULATION

recall that $U \sim \mathrm{Unif}[0,1] \rightarrow F_X^{-1}(U) \sim X$

QUANTILE FUNCTION OF X

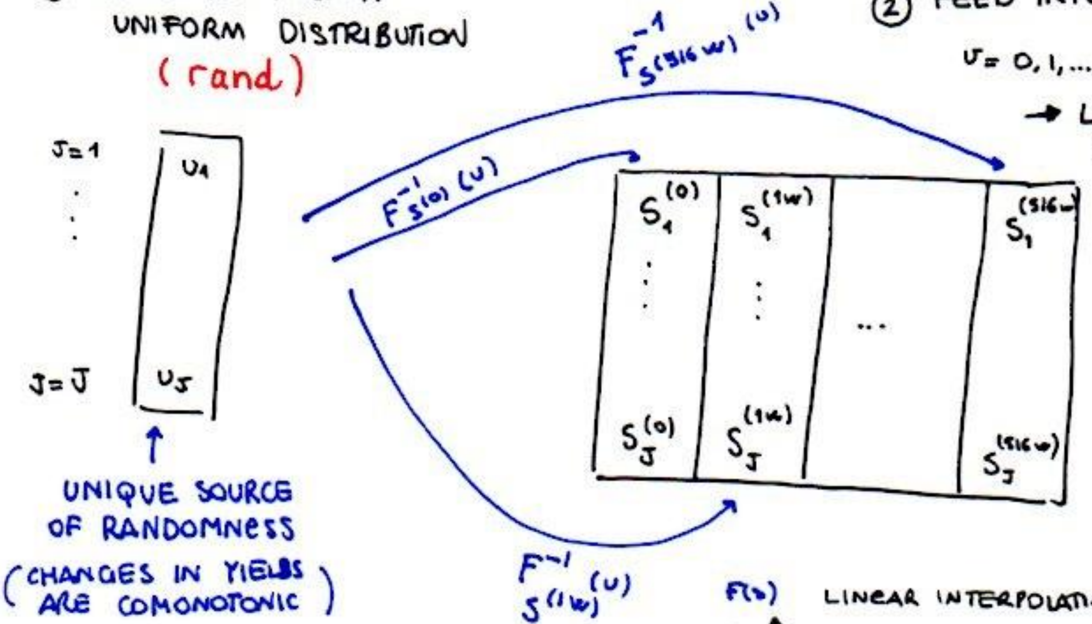


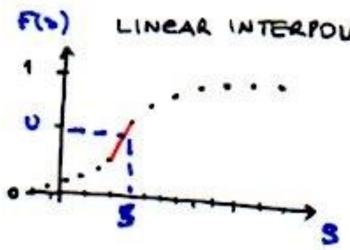
① SIMULATE FROM A UNIFORM DISTRIBUTION (rand)

② FEED INTO $F^{-1}_{S^{(v)}}$

$v = 0, 1, \ldots, 516$ weeks

→ LINEAR INTERPOLATION





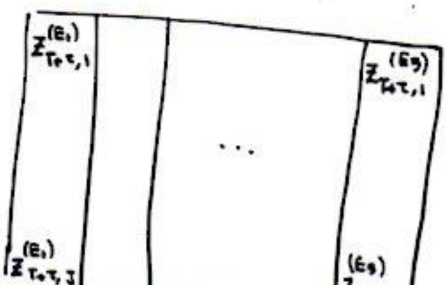
③ PRICING

$$Z^{(E_i)}_{T+\tau,j} = \exp\{-v_i \cdot 0.04\}$$

$$* \exp\{-v_i \cdot S_J^{(v_i)}\}$$

$$v_i = \begin{cases} 0 \\ 1/52 & (i=1) \\ \vdots \\ 516/52 & (i=9) \end{cases}$$

↑ years



PANEL OF BOND PRICES AT THE HORIZON

④ LINEAR RETURNS FOR EACH BOND

$$L^{(i)}_{T\to T+\tau,J} = \frac{Z^{(E_i)}_{T+\tau,J}}{Z^{(E_i)}_{T}} - 1$$

→ SIMULATIONS FROM THE PANEL OF PRICES

→ KNOWN AT TIME T

$$Z^{(E_i)}_{T} = \exp\left\{-(v+\tau)\, y^{(v+\tau)}_{T}\right\}$$

↑ 4%

$$\begin{array}{cccc} \left[\begin{array}{c|c|c|c} L^{(1)}_1 & L^{(2)}_1 & & L^{(S)}_1 \\ \vdots & \vdots & \cdots & \vdots \\ L^{(1)}_J & L^{(2)}_J & \cdot & L^{(S)}_J \end{array}\right] \\ \downarrow \quad\quad \downarrow \quad\quad\quad \downarrow \\ \text{HIST} \quad \text{HIST} \quad\quad \text{HIST} \end{array}$$

5.6 Derivatives

Consider a market of call options on the S&P 500, with current time to maturity of 100, 150, 200, 250, and 300 days and strikes equal 850, 880, 910, 940, and 970 respectively. Assume that the investment horizon is 8 weeks.

Consider the time series of the underlying and the implied volatility surface provided in `DB_ImplVol`. Fit a joint normal distribution to the weekly invariants, namely the log-changes in the underlying and ~~the residuals from a vector autoregression of order one in~~ the log-changes in the implied volatilities surface $\boldsymbol{\sigma}_t$.

$$\begin{pmatrix} \ln S_{t+\tau} - \ln S_t \\ \ln \boldsymbol{\sigma}_{t+\tau} - \ln \boldsymbol{\sigma}_t \end{pmatrix} \sim \mathrm{N}\left(\tau\boldsymbol{\mu}, \tau\boldsymbol{\Sigma}\right) \tag{339}$$

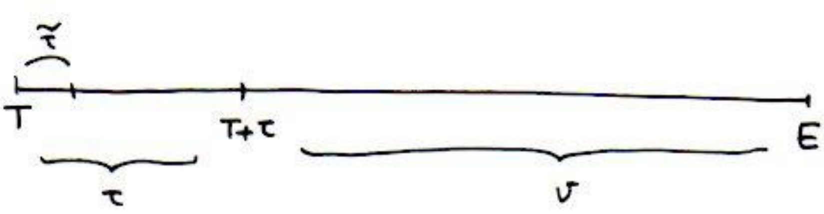
Generate simulations for the invariants and jointly project underlying and implied volatility surface to the investment horizon.

Price the above simulations through the full Black-Scholes formula at the investment horizon, assuming a constant risk-free rate at 4%.

Compute the joint distribution of the linear returns of the call options, as represented by the simulations: the current prices of the options can be obtained similarly to the prices at the horizon by assuming that the current values of underlying and implied volatilities are the last observations in the database.

For each call option, plot the histogram of its distribution at the horizon and the scatter-plot of its distribution against the underlying.

Verify what happens as the investment horizon shifts further in the future.



T: TODAY

$\tilde{\tau}$: ESTIMATION INTERVAL (1 week)

τ : INVESTMENT HORIZON (8 weeks)

E : CALL OPTIONS EXPIRIES $= T + \tau + \upsilon = \begin{cases} 100 \\ 150 \\ 200 \\ 250 \\ 300 \end{cases}$ days

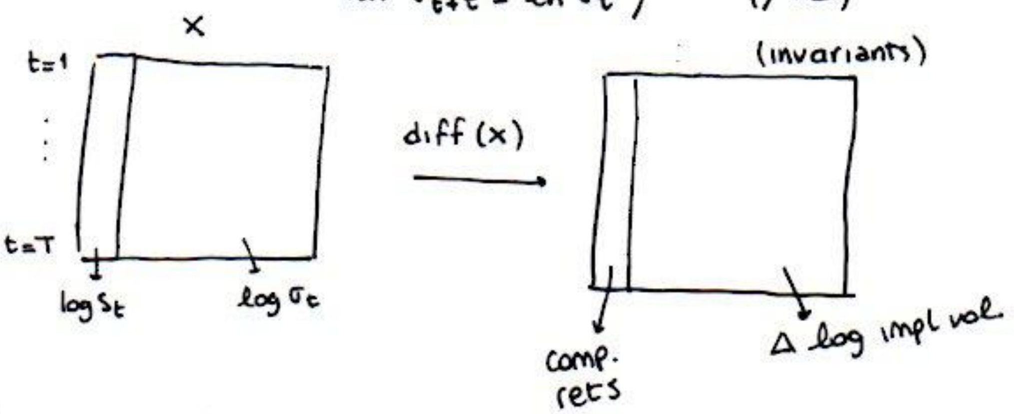
	STRIKES (k)
	850
	880
	910
	940
	970

① RISK DRIVERS $X_t = [\log S_t \,\vdots\, \log \sigma_t]$

log- underlying

log-impl.vol surface (reshaped as a vector $1 \times [I \times J]$

INVARIANTS $\begin{pmatrix} \ln S_{t+\tilde{\tau}} - \ln S_t \\ \ln \sigma_{t+\tilde{\tau}} - \ln \sigma_t \end{pmatrix} \sim N(\mu, \Sigma)$



② ESTIMATION

$\hat{\mu}$: sample mean , $\hat{\Sigma}$: sample covariance

③ PROJECTION

$$\Delta_\tau X \sim \underbrace{N\left(\hat{\mu} \cdot \frac{\tau}{\tilde{\tau}}, \ \hat{\Sigma} \frac{\tau}{\tilde{\tau}}\right)}_{N(8\hat{\mu},\ 8\hat{\Sigma})}$$

8 8

→ projected distrib of the invariants

PROJECT THE RISK DRIVERS:

Simulation for $\Delta_\tau X$: mvnrnd ($8\hat{\mu}$, $8\hat{\Sigma}$, Nsimul)

$\Delta_\tau X\,(:,1)$

$$S^{(j)}_{T+\tau} = S_T \cdot \exp\left(C^{(j)}_\tau\right)$$

$$\sigma^{(j)}_{T+\tau} = \sigma_T \cdot \exp\left(\Delta_\tau \ln \sigma^{(j)}\right)$$

$\Delta_\tau X\,(:,2:\text{end})$

$$\rightarrow \begin{array}{c|c|c|} j=1 & & \\ \vdots & & \\ j=J & & \\ \hline & \downarrow & \downarrow \\ & S_{T+\tau} & \sigma_{T+\tau} \end{array}$$

④ PRICING

$$\text{CALL}^{(j)}_{T+\tau} = BS\left(\frac{S^{(j)}_{T+\tau}}{K},\ v,\ 4\%,\ \sigma^{(j)}_{T+\tau}\underbrace{\left(\frac{S_{T+\tau}}{K}, v\right)}\right)$$

MONEYNESS AT THE HORIZON

TIME TO MAT AT THE HORIZON

OBTAINED BY n-linear INTERPOLATION (Function: interpne)

$$\begin{bmatrix} \text{CALL}^{(1)}_{T+\tau,1} & & \text{CALL}^{(5)}_{T+\tau,1} \\ \vdots & \cdots & \vdots \\ \text{CALL}^{(1)}_{T+\tau,J} & & \text{CALL}^{(5)}_{T+\tau,J} \end{bmatrix}$$

$\downarrow$ LINEAR RETURNS$_{T\to T+\tau,j}$ $= \dfrac{\text{CALL}^{i}_{T+\tau,j}}{\text{CALL}^{i}_{T}} - 1$

KNOWN AT TIME T

$= BS\left(\frac{S_T}{K}, v+\tau, 4\%, \sigma_T\left(\frac{S_T}{K}, v+\tau\right)\right)$

$$\begin{bmatrix} L^{(1)}_{T\to T+\tau,1} & & L^{(5)}_{T\to T+\tau,1} \\ & \cdots & \\ L^{(1)}_{T\to T+\tau,J} & & L^{(5)}_{T\to T+\tau,J} \end{bmatrix}$$

$\downarrow$ HIST $\quad\downarrow$ HIST $\quad\downarrow$ HIST

SCATTER PLOTS

plot ($S_{T+\tau}$, $L^{(i)}$, '.')

marker

PROJECTION IN AR(1) MODEL

$$X_t = a + b X_{t-1} + \varepsilon_t$$

X_{t+3} in terms of X_t ?

$$X_{t+3} = a + b X_{t+2} + \varepsilon_{t+3}$$

$$= a + b\left(a + b X_{t+1} + \varepsilon_{t+2}\right) + \varepsilon_{t+3}$$

$$= a + ab + b^2 X_{t+1} + b\,\varepsilon_{t+2} + \varepsilon_{t+3}$$

$$= a + ab + b^2\left(a + b X_t + \varepsilon_{t+1}\right) + b\,\varepsilon_{t+2} + \varepsilon_{t+3}$$

$$= a + ab + ab^2 + b^3 X_t + b^2 \varepsilon_{t+1} + b\,\varepsilon_{t+2} + \varepsilon_{t+3}$$

$$= a \sum_{i=1}^{3} b^{i-1} + b^3 . X_t + \sum_{i=1}^{3} b^{3-i} \cdot \varepsilon_{t+i}$$

IN GENERAL

$$X_{\underset{\substack{\uparrow \\ \text{INTEGER}}}{t+\tau}} = a \sum_{i=1}^{\tau} b^{i-1} + b^{\tau} X_t + \sum_{i=1}^{\tau} b^{\tau-i} \varepsilon_{t+i}$$

3.2.4 Cointegration

Upload the database `DB_SwapParRates` of the daily series of a set of par swap rates.

Determine the (in-sample) decreasingly most cointegrated combination of the above par swap rates using principal component analysis.

Fit an AR(1) process to these combinations and compute the unconditional (long term, equilibrium) expectation and standard deviation. Plot the 1-z-score bands around the long-term mean to generate signals to enter or exit a trade.

COINTEGRATION: find a non-explosive linear combination of the risk drivers

X_t typically RW $\sim I(1)$

we look for $c' X_t \sim I(0)$ $\quad$ ($\|c\| = 1$ for convenience)

→ Best candidate for cointegration:

$$c^* = \operatorname{argmin} \operatorname{Var}\{c' X\} = \operatorname{argmin} \left[c' \operatorname{Cov}(X)\, c\right]$$

↓ solution obtained via PCA factorization

sample cov: proxy for the long term unconditional cov of the risk drivers [$\operatorname{Cov}(X_\infty)$]

$$\hat{\Sigma} = \operatorname{cov}(X) = E \Lambda E'$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$

EIGENVALUES IN DECREASING ORDER

$$E = \left[\underline{e}_1 \,\vdots\, \underline{e}_2 \,\vdots\, \dots \,\vdots\, \boxed{\underline{e}_N}\right]$$ → corresponding EIGENVECTORS

$c^* = \underline{e}_N$ → eigenvector corresponding to the smallest eigenvalue

- TO TEST FOR THE STATIONARITY OF $Y = e_N' \cdot X$

→ FIT Y TO AN AR(1)/OU PROCESS AND VERIFY THAT IT IS MEAN REVERTING

Recall: OU process $\quad dX_t = -(\theta X_t - \mu)\,dt + \sigma\, dW_t$

$\theta > 0 \rightarrow$ Mean reversion (non explosive behaviour)

$\theta \leq 0$ explosive behavior ($\theta = 0$: random walk)

$m = \frac{\mu}{\theta} \rightarrow$ Long term expectation

OU: $\quad dX_t = \theta(m - X_t)\,dt + \sigma\, dW_t$

$\downarrow$ ⊛ PROOF

$$X_{t+\Delta t} = e^{-\theta \Delta t} x_t + (1 - e^{-\theta \Delta t}) m + \varepsilon_{t+\Delta t}$$

$$\varepsilon_{t+\Delta t} \sim N\left(0, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta \Delta t})\right)$$

→ LONG TERM UNCONDITIONAL VARIANCE ($\Delta t \to \infty$) $= \frac{\sigma^2}{2\theta}$

$\Delta t = 1 \downarrow$

$$X_{t+1} = \underbrace{e^{-\theta}}_{b} x_t + \underbrace{(1 - e^{-\theta}) m}_{a} + \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \sim N\left(0, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta})\right)$$

$\downarrow$

AR(1) : $\quad X_{t+1} = b X_t + a + \varepsilon_{t+1}$

① FIT AN AR(1) TO THE TIME SERIES OF Y by REGRESSION
→ find $\hat{a}, \hat{b}$

② $b = e^{-\theta} \rightarrow \hat{\theta} = -\log \hat{b}$ $\quad$ (mean rev if $\theta > 0$ i.e. $0 < b < 1$) $\quad$ MEAN REV

③ $(1 - e^{-\theta}) m = a \rightarrow \hat{m} = \frac{\hat{a}}{1 - e^{-\hat{\theta}}} = \frac{\hat{a}}{1 - \hat{b}}$ $\quad$ LONG TERM EXPECT.

④ compute the sample variance of the residuals and find σ^2 using $\hat{\sigma}^2_{RES} = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta})$

z-score $\quad Z_{t,\infty} = \dfrac{|Y_t - E[Y_\infty]|}{Sd\{Y_\infty\}}$ $\qquad E[Y_\infty] = m$

$Sd[Y_\infty] = \sqrt{\dfrac{\sigma^2}{2\theta}}$

(univariate case)

(+) PROOF

$$dX_t = \theta(m - X_t)dt + \sigma dW_t$$

multiply by $e^{\theta t}$

$$e^{\theta t} dX_t = m\theta e^{\theta t} dt - \theta e^{\theta t} X_t dt + \sigma e^{\theta t} dW_t$$

change side to ~, integrate

$$\int_s^{s+\Delta s} \underbrace{\left(e^{\theta t} dX_t - \theta e^{\theta t} X_t dt\right)}_{d(e^{\theta t} X_t)} = m\int_s^{s+\Delta s} \theta e^{\theta t} dt + \sigma \int_s^{s+\Delta s} e^{\theta t} dW_t$$

$$\left[e^{\theta t} X_t\right]_s^{s+\Delta s} = m\left[e^{\theta t}\right]_s^{s+\Delta s} + \sigma \int_s^{s+\Delta s} e^{\theta t} dW_t$$

$$e^{\theta(s+\Delta s)} X_{s+\Delta s} - e^{\theta s} X_s = m\left(e^{\theta(s+\Delta s)} - e^{\theta s}\right) + \sigma \int_s^{s+\Delta s} e^{\theta t} dW_t$$

divide by $e^{\theta(s+\Delta s)}$

$$X_{s+\Delta s} = e^{-\theta \Delta s} X_s + m\left(1 - e^{-\theta\Delta s}\right) + \underbrace{\sigma \int_s^{s+\Delta s} e^{\theta(t-s-\Delta s)} dW_t}$$

$$X_{s+\Delta s} = e^{-\theta\Delta s} X_s + m\left(1 - e^{-\theta\Delta s}\right) + \varepsilon_{s+\Delta s}$$

$$\varepsilon_{s+\Delta s} \sim N\left(0, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\Delta s}\right)\right)$$

QED

$$\varepsilon_{s+\Delta s} \sim N\left(0, \sigma^2 \underbrace{E\left[\left(\int_s^{s+\Delta s} e^{\theta(t-s-\Delta s)} dW_t\right)^2\right]}\right)$$

ITO ISOMETRY

$$\int_s^{s+\Delta s} e^{2\theta(t-s-\Delta s)} dt$$

$$= \frac{1}{2\theta}\left(1 - e^{-2\theta\Delta s}\right)$$