**Quant Nugget 4**

# Annualization and General Projection of Skewness, Kurtosis and All Summary Statistics[1]

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If the distribution of a financial variable is highly non-normal, as is the case for the monthly return of some hedge funds or options, how do we compute the projected annualized skewness and kurtosis? We address this question in greater generality, projecting all the summary statistics of the financial variables, in addition to skewness and kurtosis, to arbitrary horizons, in addition to one year. Fully documented MATLAB code is also provided.

## 1 The annualization/projection problem

Consider a financial variable $Y$ that is the sum of other financial variables

$$Y \equiv X_1 + \cdots + X_\tau, \tag{1}$$

For instance, $Y$ can be the annual profit-and-loss and $\{X_t\}_{t=1,\ldots,\tau}$ are the monthly profit-and-loss with $\tau = 12$ months; or $Y$ can be the annual compounded return and $\{X_t\}_{t=1,\ldots,\tau}$ the monthly compounded returns.

The problem is: given the distribution of $X_t$, and its statistics such as standard deviation and skewness, how do we compute the statistics of $Y$? This problem is well-known as the *annualization*, when the horizon $\tau$ is one year. More in genera, we call this problem the *projection* to the horizon $\tau$.

To determine $Y$ and its statistics we need the distribution of $X_1, X_2, \ldots$. In this Quant Nugget we provide the solution in a special case, which is nonetheless the most relevant one: $X_t$ is an invariant, i.e. $\{X_t\}_{t=1,\ldots,\tau}$ are independent and display the same distribution, see the "quest for invariance" in Meucci (2005) and Meucci (2009b).

Assume that we have estimated the distribution of $X_t$, which does not depend on the time $t$ because $X_t$ is an invariant. Then we can compute the

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well-know summary statistics: the expectation

$$\mu_X \equiv \mathrm{E}\{X_t\}, \tag{2}$$

the standard deviation

$$\sigma_X \equiv \sqrt{\mathrm{E}\{(X_t - \mu_X)^2\}}, \tag{3}$$

the skewness

$$sk_X \equiv \mathrm{E}\{(X_t - \mu_X)^3\}/\sigma_X^3, \tag{4}$$

the kurtosis

$$ku_X \equiv \mathrm{E}\{(X_t - \mu_X)^4\}/\sigma_X^4, \tag{5}$$

and more in general the standardized summary statistics

$$\gamma_X^{(n)} \equiv \mathrm{E}\{(X_t - \mu_X)^n\}/\sigma_X^n, \quad n \geq 3. \tag{6}$$

Once we know the above statistics (2)-(6) of the invariant $X_t$, how do we compute the same statistics for $Y$, the annualization/projection of $X_t$? It is well known that the expectation of $Y$ is proportional to the projection horizon

$$\mu_Y = \tau \mu_X \tag{7}$$

and that the standard deviation of $Y$ grows as the square root of the projection horizon

$$\sigma_Y = \sqrt{\tau}\sigma_X, \tag{8}$$

see e.g. Meucci (2010b). Then, we are left with the computation of the skewness, kurtosis and all the other higher order statistics for $Y$ from the statistics for $X_t$. In Section 2 we consider the special case where $X_t$ is normally distributed. In Section 3 we discuss the special case where $X_t$ is non-normal but the projection horizon $\tau$ is very large. In Section 4 we provide an algorithm for the projection of arbitrary, non-normal $X_t$ to arbitrary horizons $\tau$ in full generality. This becomes important for instance when annualizing highly skewed monthly returns. In Section 5 we conclude, pointing out some caveats.

## 2 The normal case

Consider a normal invariant

$$X_t \sim \mathrm{N}\left(\mu_X, \sigma_X^2\right). \tag{9}$$

The sum of independent normal variables is normal. Therefore the annualization/projection $Y \equiv X_1 + \cdots + X_\tau$ is normal

$$Y \sim \mathrm{N}\left(\mu_Y, \sigma_Y^2\right), \tag{10}$$

where the expectation and the standard deviation follow from the the general rules (7)-(8), which we report here

$$\mu_Y \equiv \tau \mu_X, \quad \sigma_Y \equiv \sqrt{\tau}\sigma_X. \tag{11}$$

On the other hand, skewness, kurtosis and all the other standardized summary statistics are independent of the projection horizon $\tau$, because they do not depend on the expectation and the standard deviation. An explicit calculation yields

$$\gamma_Y^{(n)} = \begin{cases} 0 & \text{if } n \geq 3 \text{ is odd} \\ 1 \times 3 \times \cdots \times (n-1) & \text{if } n \geq 3 \text{ is even,} \end{cases} \tag{12}$$

see Meucci (2009a). In particular, the skewness $sk_Y = \gamma_Y^{(3)}$ is zero, because the normal distribution is symmetrical, and the kurtosis $ku_Y = \gamma_Y^{(4)}$ is 3.

# 3 The case of long horizons

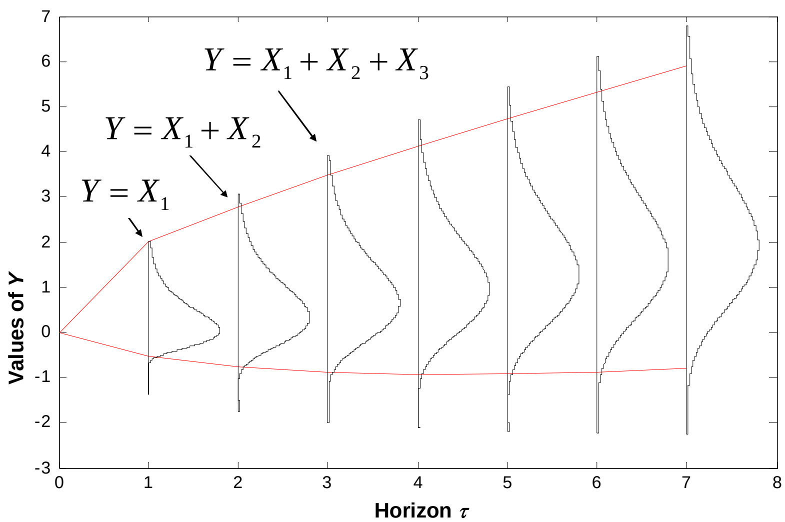


Figure 1: Central limit theorem and projection of non-normal invariants

Consider an invariant $X_t$ with a fully general distribution with *finite* variance. The central limit theorem states that, when the horizon $\tau$ is large enough, the distribution of the annualization/projection $Y \equiv X_1 + \cdots + X_\tau$ is approximately normal

$$Y \approx \mathrm{N}\left(\mu_Y, \sigma_Y^2\right), \tag{13}$$

where $\mu_Y \equiv \tau\mu_X$ and $\sigma_Y \equiv \sqrt{\tau}\sigma_X$ follow from the general linear and square-root rules (7)-(8).

In Figure 1 we illustrate this phenomenon in the case of a shifted-lognormal invariant

$$X_t + a \sim \mathrm{LogN}\left(m, s^2\right), \tag{14}$$

where $a \equiv 1$, $m \equiv 0.2$, and $s \equiv 0.4$. This distribution, which corresponds to the horizon $\tau \equiv 1$ in the figure, displays heavily non-normal skewness. As the horizon $\tau$ grows, the distribution of $Y$ becomes much more symmetrical.

As a result of the central limit theorem, when the projection horizon $\tau$ is large the summary statistics of $Y$ must approach the normal statistics (12), regardless of the distribution of $X_t$. Unfortunately, if the invariant $X_t$ is highly non-normal, the horizon $\tau$ where the central limit approximation is correct can be extremely large, possibly of the order of several years. Therefore, in order to handle the annualization/projection for a generic horizon $\tau$ we must use the algorithm in Section 4.

# 4 Annualization/projection in the general case

Consider an invariant $X_t$ with a fully general, non-normal distribution and well-defined summary statistics (2)-(6). Consider the annualization/projection $Y \equiv X_1 + \cdots + X_\tau$ with an arbitrary, not necessarily large, horizon $\tau$. For instance, $X_t$ can be the highly skewed profit-and-loss for a hedge fund over the next quarter and $Y$ the annualized profit-and-loss, which corresponds to $\tau = 4$ quarters.

To compute the standardized summary statistics of $Y$ we need to introduce three sets of players, defined as follows for a generic random variable $X$: the central moments

$$\mu_X^{(1)} \equiv \mu_X; \qquad \mu_X^{(n)} \equiv \mathrm{E}\{(X - \mu_X)^n\}, \quad n = 2, 3, \ldots; \tag{15}$$

the non-central moments

$$\widetilde{\mu}_X^{(n)} \equiv \mathrm{E}\{X^n\}, \quad n = 1, 2, \ldots; \tag{16}$$

and the cumulants

$$\kappa_X^{(n)} \equiv \left. \frac{d^n \ln \left( \mathrm{E}\left\{ e^{zX} \right\} \right)}{dz^n} \right|_{z=0}, \quad n = 1, 2, \ldots. \tag{17}$$

Then we proceed as follows, refer to Figure 2 and refer to Meucci (2009a) for fully commented MATLAB code.

- Step 0. We collect the first $n$ statistics (2)-(6) of the invariant $X_t$

$$\mu_X, \sigma_X, sk_X, ku_X, \gamma_X^{(5)}, \ldots, \gamma_X^{(n)}. \tag{18}$$

- Step 1. From (18) we compute the central moments $\mu_X^{(1)}, \ldots, \mu_X^{(n)}$ of $X_t$. To do so, notice from the definition of central moments (15) and from (3) that $\mu_X^{(2)} \equiv \sigma_X^2$ and that from (6) we obtain

$$\mu_X^{(n)} = \gamma_X^{(n)} \sigma_X^n, \quad n \geq 3. \tag{19}$$

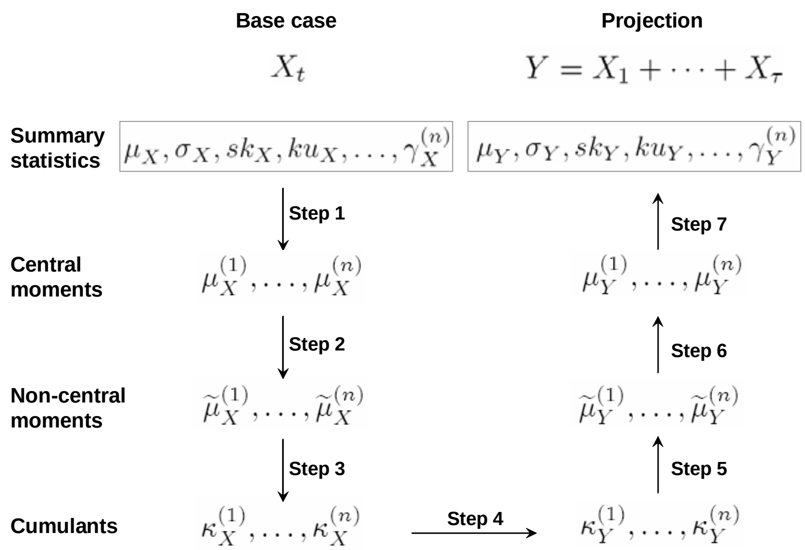


Figure 2: Projection of summary statistics to an aribtrary horizon: implementation steps

- Step 2. From the central moments $\mu_X^{(1)}, \ldots, \mu_X^{(n)}$ of $X_t$ we compute the non-central moments $\widetilde{\mu}_X^{(1)}, \ldots, \widetilde{\mu}_X^{(n)}$. To do so, we start from $\widetilde{\mu}_X^{(1)} = \mu_X^{(1)} = \mu_X$, which follows from (2), (15) and (16). Then we apply recursively the identity

$$\widetilde{\mu}_X^{(n)} = (-1)^{n+1}\,\mu_X^n + \sum_{k=1}^{n-1} \binom{n}{k} (-1)^{n-k+1}\,\mu_X^{n-k}\widetilde{\mu}_X^{(k)} + \mu_X^{(n)}, \tag{20}$$

which follows from the binomial expansion of the power in (15).
- Step 3. From the non-central moments $\widetilde{\mu}_X^{(1)}, \ldots, \widetilde{\mu}_X^{(n)}$ of $X_t$ we compute the cumulants $\kappa_X^{(1)}, \ldots, \kappa_X^{(n)}$. To do so, we start from $\kappa_X^{(1)} = \widetilde{\mu}_X^{(1)}$: this follows from the Taylor approximations $\mathrm{E}\left\{e^{zX}\right\} \approx \mathrm{E}\left\{1 + zX\right\} = 1 + z\widetilde{\mu}_X^{(1)}$ for any small $z$ and $\ln(1+x) \approx x$ for any small $x$, and from the definition of the first cumulant in (17). Then we apply recursively the identity

$$\kappa_X^{(n)} = \widetilde{\mu}_X^{(n)} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_X^{(k)} \widetilde{\mu}_X^{(n-k)}, \tag{21}$$

see Kendall and Stuart (1969).
- Step 4. From the cumulants $\kappa_X^{(1)}, \ldots, \kappa_X^{(n)}$ of $X_t$ we compute the cumulants $\kappa_Y^{(1)}, \ldots, \kappa_Y^{(n)}$ of the annualization/projection $Y \equiv X_1 + \cdots + X_\tau$. To do so, we notice that for any independent variables $X_1, \ldots, X_\tau$ we have $\mathrm{E}\left\{e^{z(X_1+\cdots+X_\tau)}\right\} =$

$\mathrm{E}\left\{e^{zX_1}\right\}\cdots\mathrm{E}\left\{e^{zX_\tau}\right\}$. Substituting this in the definition of the cumulants (17) we obtain

$$\kappa^{(n)}_{X_1+\cdots+X_\tau} = \kappa^{(n)}_{X_1} + \cdots + \kappa^{(n)}_{X_\tau}. \tag{22}$$

In particular, since $X_t$ is an invariant, all the $X_t$'s are identically distributed. Therefore the projected cumulants read

$$\kappa^{(n)}_Y = \tau\kappa^{(n)}_X, \tag{23}$$

see also Duc and Schorderet (2008).
Step 5. From the cumulants $\kappa^{(1)}_Y, \ldots, \kappa^{(n)}_Y$ of $Y$ we compute the non-central moments $\widetilde{\mu}^{(1)}_Y, \ldots, \widetilde{\mu}^{(n)}_Y$. To do so, we use recursively the identity

$$\widetilde{\mu}^{(n)}_Y = \kappa^{(n)}_Y + \sum_{k=1}^{n-1} \tbinom{n-1}{k-1}\kappa^{(k)}_Y\widetilde{\mu}^{(n-k)}_Y, \tag{24}$$

which follows from applying (21) to $Y$ and rearranging the terms.
- Step 6. From the non-central moments $\widetilde{\mu}^{(1)}_Y, \ldots, \widetilde{\mu}^{(n)}_Y$ of $Y$ we compute the central moments $\mu^{(1)}_Y, \ldots, \mu^{(n)}_Y$. To do so, we use recursively the identity

$$\mu^{(n)}_Y = (-1)^n \mu^n_Y + \sum_{k=1}^{n-1} \tbinom{n}{k} (-1)^{n-k} \widetilde{\mu}^{(k)}_Y \mu^{n-k}_Y + \widetilde{\mu}^{(n)}_Y, \tag{25}$$

which follows from applying (20) to $Y$ and rearranging the terms.
- Step 7. From the central moments $\mu^{(1)}_Y, \ldots, \mu^{(n)}_Y$ of $Y$ we compute the standardized summary statistics

$$\mu_Y, \sigma_Y, sk_Y, ku_Y, \gamma^{(5)}_Y, \ldots, \gamma^{(n)}_Y \tag{26}$$

of the projected multi-period invariant $Y$, by applying to $Y$ the definitions (2)-(6).

Notice that in order to compute the first $n$ statistics (26) of $Y$ we only need the first $n$ statistics (18) of the invariant $X_t$.

Also notice that Steps 3-4-5 can be replaced by a direct computation of $\widetilde{\mu}^{(n)}_Y = \mathrm{E}\left\{(X_1 + \cdots + X_\tau)^n\right\}$ in terms of the non-central moments $\widetilde{\mu}^{(1)}_X, \ldots, \widetilde{\mu}^{(n)}_X$ by expanding the power on the right hand side in terms of the multinomial coefficient. However, such expansion is more complex to code than the binomial expansions (21) and (24). Also, (23) isolates the projection into one simple operation, which becomes easier to handle and to generalize.

When applied to the computation of the first two statistics $\mu_Y$ and $\sigma_Y$, Steps 1-7 yield the well-known linear and square-root projection rules (7) and (8). When applied to the computation of the statistics stemming from a normal invariant $X_t$, Steps 1-7 yield the normal statistics (11) and (12). More generally, Steps 1-7 become necessary to compute the annualization or arbitrary projection of skewness, kurtosis and other statistics for arbitrary distributions.

| | $\tau$ | $\mu_Y$ | $\sigma_Y$ | $sk_Y$ | $ku_Y$ | $\gamma_Y^{(5)}$ | $\gamma_Y^{(6)}$ |
|---|---|---|---|---|---|---|---|
| Quarter P&L (shifted lognormal) | **1** | 0.32 | 0.55 | 1.32 | 6.26 | 25.53 | 145.80 |
| | **2** | 0.65 | 0.78 | 0.93 | 4.63 | 13.70 | 64.29 |
| | **3** | 0.97 | 0.95 | 0.76 | 4.09 | 10.00 | 44.28 |
| Annual P&L (unknown distribution) | **4** | 1.29 | 1.10 | 0.66 | 3.82 | 8.15 | 35.62 |
| | **5** | 1.62 | 1.23 | 0.59 | 3.65 | 7.01 | 30.85 |
| | **6** | 1.94 | 1.35 | 0.54 | 3.54 | 6.23 | 27.85 |
| | **7** | 2.26 | 1.46 | 0.50 | 3.47 | 5.66 | 25.80 |
| Long-horizon P&L (normal approx.) | **100** | 32.31 | 5.51 | 0.13 | 3.03 | 1.33 | 15.67 |

Figure 3: Projection of summary statistics for non-normal invariants to arbitrary horizons

To illustrate, we consider the shifted lognormal distribution (14) displayed in Figure 1, which models the distribution of the profit-and-loss of a hedge fund over the next quarter. In the first row of Figure 1 we report the statistics (18) for the quarterly horizon. The last row reports the statistics for the large-horizon projection. These statistics can be well approximated by the central limit theorem: the skewness $sk_Y$ and $\gamma_Y^{(5)}$ converge to zero, the kurtosis $ku_Y$ converges to 3, and $\gamma_Y^{(6)}$ converges to $3 \times 5 = 15$, which are the normal statistics (12). The darker area displays the statistics for the projection of the quarterly distribution to $2, \ldots, 7$ quarters. In order to compute these numbers, and in particular the annualized profit-and-loss statistics for $\tau = 4$, we must use Steps 1-7 above, refer to Meucci (2009a) for more details.

# 5 Conclusions and remarks

In this Quant Nugget we provide an algorithm to annualize/project to any horizon all the statistics of any financial variable $X_t$ under three hypotheses, which cover the most important cases in risk and portfolio management: $X_t$ is an invariant; the statistics of $X_t$ are well defined; and the annualized/projected variable $Y$ is the sum of $X_t$ as in (1). The algorithm generalizes to skewness and higher statistics the well-known square-root rule (8), which only applies to the standard deviation.

When $X_t$ is not an invariant the annualization/projection is still feasible, but we must account for the autocorrelations. When the standard deviation, the skewness and the other statistics are not defined, as for instance when the distribution of $X_t$ displays fat tails, the projection is still feasible, but we must explore alternative ways to summarize the features of $X_t$ by means of sensible numbers. We will discuss these issues in future Quant Nuggets.

When the projection is not a sum as in (1), the present algorithm, and in particular the square-root rule (8), do not apply. For instance, this algorithm and the square-root rule do *not* apply to the annualization/projection of the linear return $R_t \equiv P_t/P_{t-1} - 1$, because $Y \equiv R_1 + \cdots + R_\tau$ is *not* the linear return over the horizon $\tau$. To project a linear return we must first transform it into a compounded return, for which the present methodology holds, see Meucci (2010a).

Finally, we recall that in Meucci (2010b) we generalize the square-root rule (8) to the multi-variate case, providing an interpretation in terms of ever-expanding ellipsoids. The multivariate projection of skewness, kurtosis and other statistics, although feasible, becomes hard to interpret.

# References


Duc, F., and Y. Schorderet, 2008, *Risk Management for Hedge Funds* (Wiley).

Kendall, M.G., and A. Stuart, 1969, *The Advanced Theory of Statistics, Volume* (Griffin) third edn.

Meucci, A., 2005, *Risk and Asset Allocation* (Springer).

——— , 2009a, Exercises in advanced risk and portfolio management - with step-by-step solutions and fully documented code, *Free E-Book* available at http://ssrn.com/abstract=1447443.

——— , 2009b, Review of discrete and continuous processes in finance: Theory and applications, *Working paper* Available at http://ssrn.com/abstract=1373102.

——— , 2010a, Linear vs. compounded returns - common pitfalls in portfolio management, *GARP Risk Professional - "The Quant Classroom"* April, 52–54 Available as "Quant Nugget 2" at http://ssrn.com/abstract=1586656.

——— , 2010b, Square-root rule, covariances and ellipsoids - how to analyze and visualize the propagation of risk, *GARP Risk Professional - "The Quant Classroom"* February, 52–53 Available as "Quant Nugget 1" at http://ssrn.com/abstract=1548162.