Quant Nugget 5

Return Calculations for Leveraged Securities and Portfolios[1]

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this version: November 12 2010
last version available at http://ssrn.com/abstract=1675067

Abstract

How can we report returns for a swap that has zero value? How can we perform return optimization for a zero-value long-short portfolio? By introducing a suitable "basis", it is possible to extend the definition of returns to leveraged products in such a way that performance attribution and portfolio optimization are feasible. Risk-adjusted performance attribution and connections of performance attribution with probability theory are also discussed.

JEL Classification: C1, G11

Keywords: Leverage, portfolio weights, swaps, futures, risk-adjusted performance attribution, hierarchical portfolios, conditional probability

[1] This article appears as *"The Quant Classroom by Attilio Meucci - Return Calculations for Leveraged Securities and Portfolios"*, GARP Risk Professional, October 2010, p. 40-43

[2] The author is grateful to Eva Chan and Luca Spampinato

1 The "usual" returns: pros and cons

The return is the ultimate measure of performance of an investment. In Meucci (2010b) we analyzed the two different notions of return widely used in the quantitative finance industry: compounded and linear return. For a generic tradable asset that has value V_t at time t, which can be a single security or a portfolio, the linear return is defined as

$$R \equiv \frac{V_{t+1}}{V_t} - 1, \tag{1}$$

and the compounded return is defined as $C \equiv \ln(V_{t+1}/V_t)$.

In markets such as equities and foreign exchanges, the compounded return is a better candidate for time series estimation, see Meucci (2010a). On the other hand, the linear return is the only option for portfolio management across all asset classes, due to its remarkable aggregation properties at a given fixed time.

More precisely, consider a market of N assets and a portfolio that contains u_n units of the generic n-th asset, with value $V_{n,t}$ at time t. For instance, in the equity world u_n is the number of shares. The relative weight of each asset in the portfolio is defined as

$$w_n \equiv \frac{u_n V_{n,t}}{\sum_{m=1}^{N} u_m V_{m,t}}, \quad n = 1, \ldots, N. \tag{2}$$

Then, the aggregation property of linear returns prescribes that the linear return of the portfolio R_P is the weighted average of the linear returns of the securities

$$\begin{aligned} R_P &\equiv \frac{\sum_{n=1}^{N} u_n V_{n,t+1} - \sum_{n=1}^{N} u_n V_{n,t}}{\sum_{m=1}^{N} u_m V_{m,t}} \\ &= \sum_{n=1}^{N} \frac{u_n V_{n,t}}{\sum_{m=1}^{N} u_m V_{m,t}} \frac{V_{n,t+1} - V_{n,t}}{V_{n,t}} \equiv \sum_{n=1}^{N} w_n R_n \end{aligned} \tag{3}$$

The aggregation property (3) is key for portfolio optimization. To illustrate, the mean-variance efficient frontier can be computed as the set of portfolios that maximizes the trade-off between the expected value of the portfolio return and its variance. Using the aggregation property (3), we obtain

$$\mathbb{E}\{R_P\} = \mathbf{w}'\mathbb{E}\{\mathbf{R}\}, \quad \mathbb{V}ar\{R_P\} = \mathbf{w}'\mathbb{C}ov\{\mathbf{R}\}\mathbf{w}, \tag{4}$$

where $\mathbf{w} \equiv (w_1, \ldots, w_N)'$, $\mathbf{R} \equiv (R_1, \ldots, R_N)'$, $\mathbb{E}\{\mathbf{R}\}$ is the $N \times 1$ vector of the expected values of $\mathbf{R}$ and $\mathbb{C}ov\{\mathbf{R}\}$ is the $N \times N$ matrix of the covariances of $\mathbf{R}$. Therefore the efficient frontier solves

$$\mathbf{w}^{\lambda} \equiv \underset{\mathbf{w} \text{ sat } \mathcal{C}}{\operatorname{argmax}} \{\mathbf{w}'\mathbb{E}\{\mathbf{R}\} - \lambda \mathbf{w}'\mathbb{C}ov\{\mathbf{R}\}\mathbf{w}\}, \tag{5}$$

where "$\mathbf{w}$ sat $\mathcal{C}$" denotes that the weights $\mathbf{w}$ must satisfy a set of investment constraints $\mathcal{C}$, such as the budget constraint, long-only or sector boundaries, maximum number of securities, etc. We emphasize that (5) only makes sense if we use linear returns, but not compounded returns, because (4) does not hold true for compounded returns, see Meucci (2010b).

The aggregation property of the linear return (3) is key also for performance attribution, see the discussion in Section 3.

Unfortunately, the linear return (1) is not defined when the current asset value V_t is zero, as can be the case in swaps and futures. Similarly, the weight (2) is not defined when the total portfolio value is zero, as in a simple equity-pair long-short portfolio.

2 The basis and leverage-adjusted returns

Here we extend the notion of linear returns and weights to leveraged securities and portfolios, in such a way that the aggregation property (3) still holds and thus performance attribution and portfolio optimization can be performed.

A closer inspection of the linear return definition (1) reveals that the return can be re-written as $R \equiv (V_{t+1} - V_t)/V_t$, i.e. the ratio of the "profit and loss" $\Pi \equiv V_{t+1} - V_t$ over a positive quantity that is known at time t, namely the current value V_t.

To extend the definition of the linear return to leveraged instruments, first we associate with every tradable asset, which can be a single security or a portfolio, a "basis" b, namely a generalized normalizing quantity that, similarly to the current value V_t, satisfies the following properties:

- Property 1: the basis b for a position long one unit of a tradable asset is positive
- Property 2: the basis b is measured in the same money units as the p&l.
- Property 3: the basis b is homogenous, i.e. the basis of two units of the same asset is twice the basis of one unit of that asset.
- Property 4: the basis b is known at the beginning of the return period

Then we define the return as the ratio of the p&l and the asset-specific basis

$$R \equiv \frac{V_{t+1} - V_t}{V_t} \quad \mapsto \quad R \equiv \frac{V_{t+1} - V_t}{b} \equiv \frac{\Pi}{b}. \tag{6}$$

For non-leveraged assets the most natural basis is the initial market value $b \equiv V_t$. For leveraged assets, the basis depends on the nature of the specific asset: for instance, for swaps the basis can be set as the notional, or as the historical maximum loss for the given notional; for call/put options the basis can be set as the strike, or the underlying value, or the dollar-delta; for equity-pairs portfolios, the basis can be set as the total margin. In practice, the basis is set in such a way that the magnitude of the ensuing return has an intuitive meaning to the portfolio manager.

Property 1 above guarantees that a positive p&l corresponds to a positive return for a long position in one unit of a security. Property 2 guarantees that the return is a dimensionless measure of performance. Property 3 guarantees that the return is independent of the size of a position. Property 4 guarantees that the return represents all and only the performance of the p&l.

In order to preserve the aggregation property (3) we also need to generalize the weight definition (2) as the ratio of the security basis over the portfolio basis

$$w_n \equiv \frac{u_n V_{n,t}}{\sum_{m=1}^{N} u_m V_{m,t}} \quad \mapsto \quad w_n \equiv \frac{u_n b_n}{b_P}. \tag{7}$$

Notice that the numerator $u_n b_n$ is the total basis for the n-th position, due to the homogeneity of the basis, see Property 3 above.

With the generalized definition of returns (6) and weights (7), now the portfolio return satisfies the aggregation rule (3) with arbitrary leveraged positions. Indeed, applying the definition of return (6) to the portfolio and using the fact that the portfolio p&l is the aggregation of the single-asset p&l $\Pi_P = \sum_{n=1}^{N} u_n \Pi_n$, we obtain

$$R_P \equiv \frac{\Pi_P}{b_P} = \sum_{n=1}^{N} \frac{b_n u_n}{b_P} \frac{\Pi_n}{b_n} = \sum_{n=1}^{N} w_n R_n. \tag{8}$$

Notice that the basis b_n which appears in the weight w_n cancels the term b_n that appears in the return R_n and therefore the choice of b_n does not affect the contribution from the n-th asset $w_n R_n$ to the portfolio return R_P. Nevertheless, it is important that each basis is chosen in a suitable manner, in such a way that both w_n and R_n appear intuitive to the portfolio manager.

With the generalized aggregation rule (8) we can perform portfolio optimization as in (5), or decompose performance for attribution purposes. We emphasize the flexibility of this framework. First, the basis b_n of each asset is fully arbitrary. Second, the portfolio basis can be, but need not be, the aggregate portfolio value $b_P \neq \sum_{n=1}^{N} u_n V_{n,t}$. Third, the portfolio basis can be, but need not be, the sum of the single-asset bases $b_P \neq \sum_{n=1}^{N} u_n b_n$. This third phenomenon occurs for instance in a portfolio of swaps, where the basis of each security b_n can be set as the notional, but the basis for the portfolio b_P can be set as the total portfolio value. As a result of this third phenomenon $b_P \neq \sum_{n=1}^{N} u_n b_n$, in general the weights do not sum to one $\sum_{n=1}^{N} w_n \neq 1$.

3 Hierarchical portfolios

Large portfolios are often managed according to a hierarchical structure. For instance, in bond management capital is allocated according to industry sectors or rating categories first, and bond-picking at a later stage. Here we show first how our generalized framework accounts for a simple hierarchy, and later how it accounts for a multiple-layer hierarchy.

Consider a division of the whole market into S mutually exclusive buckets of assets indexed by $s = 1, \ldots, S$, such as industry sectors. First, we specify exogenously all the bucket-specific bases $\widetilde{b}_s$ (here and below the sign $\widetilde{\ }$ denotes bucket-level quantities). For instance, for a bucket of equity pairs in a given industry sector, the basis can be the total margin required for that bucket. Next, we define the bucket return $\widetilde{R}_s$ consistently with (6) in such a way that it accounts for the p&l generated by the bucket

$$\widetilde{R}_s \equiv \frac{\sum_{n\in s} u_n \Pi_n}{\widetilde{b}_s}, \tag{9}$$

where, with minor abuse of notation, $n \in s$ denotes that the summation spans all the securities in the s-th bucket. Then we define the bucket weights $\widetilde{w}_s$ consistently with (7) as the ratio of the bucket basis over the portfolio basis

$$\widetilde{w}_s \equiv \frac{\widetilde{b}_s}{b_P}. \tag{10}$$

From (9) and (10) we obtain that the aggregation property for the securities (8) also holds true for the buckets

$$\begin{aligned} R_P &\equiv \frac{\Pi_P}{b_P} = \frac{\sum_{s=1}^{S} \sum_{n\in s} u_n \Pi_n}{b_P} \\ &= \sum_{s=1}^{S} \frac{\widetilde{b}_s}{b_P} \frac{\sum_{n\in s} u_n \Pi_n}{\widetilde{b}_s} \equiv \sum_{s=1}^{S} \widetilde{w}_s \widetilde{R}_s. \end{aligned} \tag{11}$$

In particular, in the special case where the basis of each asset/bucket is its own market value, the generalized return (9) reduces to the standard definition (1) for the return of the s-th bucket

$$\widetilde{R}_s \equiv \frac{\sum_{n\in s} u_n V_{t+1,n}}{\sum_{n\in s} u_n V_{t,n}} - 1, \tag{12}$$

and the generalized weight (10) is consistent with the standard notion of weight (2)

$$\widetilde{w}_s \equiv \frac{\sum_{n\in s} u_n V_{t,n}}{\sum_{n=1}^{N} u_n V_{t,n}}. \tag{13}$$

The definitions (12)-(13) are typically assumed for performance attribution purposes, see e.g. Brinson and Fachler (1985).

Now we show how the definitions of bucket-level returns (9) and weights (10) are consistent with a multiple-layer hierarchical structure of the allocation process. Consider a two-way partition of the market. The first partition divides the market into S buckets indexed by $s = 1, \ldots, S$ and the second divides the market into K buckets indexed by $k = 1, \ldots, K$. For instance, allocation decisions can be taken according to country selection, and industry sector selection, in which case s denotes the sector and k denotes the country.

The "sector" partition $s = 1, \ldots, S$ gives rise to the return-weight pairs for the respective buckets $\{\widetilde{R}_s, \widetilde{w}_s\}_{s=1,\ldots,S}$. Similarly, and with minor abuse of notation, the "country" partition $k = 1, \ldots, K$ gives rise to the return-weight pairs for the respective buckets $\{\widetilde{R}_k, \widetilde{w}_k\}_{k=1,\ldots,K}$.

Additionally, the intersection of the two partitions generates a nested partition, as well as the respective return-weight pairs $\{\widetilde{R}_{s,k}, \widetilde{w}_{s,k}\}_{k=1,\ldots,K}^{s=1,\ldots,S}$: for instance, with S industry sectors and K countries, $\widetilde{R}_{s,k}$ is the return due to sector s within country k and $\widetilde{w}_{s,k}$ is the weight of the sector-s /country-k-th bucket in the portfolio.

Then the relationship between the k-th "country" return $\widetilde{R}_k$ and its nested "sector" returns $\widetilde{R}_{s,k}$ reads

$$\begin{aligned} \widetilde{R}_k &= \frac{\sum_{n\in k} u_n \Pi_n}{\widetilde{b}_k} = \sum_{s=1}^{S} \frac{\widetilde{b}_{s,k}}{\widetilde{b}_k} \frac{\sum_{n\in(s\cap k)} u_n \Pi_n}{\widetilde{b}_{s,k}} \\ &\equiv \sum_{s=1}^{S} \widetilde{w}_{s|k} \widetilde{R}_{s,k} \end{aligned} \tag{14}$$

In this expression $\widetilde{w}_{s|k}$ is the weight of the s-th "sector" nested within the k-th "country", defined as

$$\widetilde{w}_{s|k} \equiv \frac{\widetilde{b}_{s,k}}{\widetilde{b}_k} = \frac{\widetilde{b}_{s,k}}{b_P} \frac{b_P}{\widetilde{b}_k} = \frac{\widetilde{w}_{s,k}}{\widetilde{w}_k}. \tag{15}$$

Formulas (14)-(15) are useful when the allocation proceeds in two steps: first "country" selection and then "sector" allocation.

Again, these formulas specialize to the standard expressions used for performance attribution when the basis of each asset is its own market value, and thus returns and weights are defined as in (12)-(13).

4 Points of interest

In this section we highlight three topics that can foster further discussions.

The first topic is leverage. The introduction of the basis makes it possible to quantify and monitor a generalized notion of leverage for every security, bucket, or portfolio, defined as the ratio of the basis and the absolute value of the security, bucket, or portfolio

$$L \equiv \frac{b}{|V|}. \tag{16}$$

The second topic are risk-adjusted returns. A possible choice for the basis across all asset classes and aggregation levels could be the risk, as measured for instance by the standard deviation of the p&l, $b \equiv \mathbb{S}d\{\Pi\}$. With this choice, the generalized return (6) becomes a risk-adjusted measure of performance, see also Menchero (2007)

$$R \equiv \frac{\Pi}{\mathbb{S}d\{\Pi\}}. \tag{17}$$

Although the choice $b \equiv \mathbb{S}d\{\Pi\}$ seems quite natural, it presents two problems. First, for assets with very low risk the return (17) is distorted, in a way similar to the standard return (1) in the presence of highly leveraged instruments. To illustrate, consider the extreme case of a zero-coupon government bond that matures at the end of the return period, for which $\mathbb{S}d\{\Pi\} = 0$. Second, the standard deviation satisfies Properties 1,2 and 3 of a basis, in that $\mathbb{S}d\{\Pi\}$ is positive, it is measured in the same units as the p&l Π, and it is homogeneous; however, it does not satisfy Property 4. Indeed the risk of a position is not known at the beginning of the return period and it can only be estimated, thereby inserting a level of subjectivity that can be unsuitable for performance attribution.

The third topic are the conditional weights. Expression (15) for the weight of sector s within country k, namely $\widetilde{w}_{s|k} = \widetilde{w}_{s,k}/\widetilde{w}_k$, is reminiscent of the relationship $p_{s|k} = p_{s,k}/p_k$ among the conditional probability $p_{s|k}$ of an event s given another event k, the joint probability $p_{s,k}$ of the two events, and the marginal probability p_k of the event k. Indeed, the portfolio weight $\widetilde{w}_k$ can be interpreted as the probability of investing in the generic k-th "country" if we randomly select a "country" among $k = 1, \ldots K$; the portfolio weight $\widetilde{w}_{s,k}$ can be interpreted as the probability of investing in the generic s-th "sector" and k-th "country" if we randomly select from all the "country-sector" combinations; and the "conditional" weight $\widetilde{w}_{s|k}$ can be interpreted as the probability of investing in the generic s-th "sector", once the k-th "country" has been chosen. We will expand on this subject and its applications in a future publication.

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