Constant Proportion Portfolio Insurance

PETER CARR

Global Head of Market Modeling, Morgan Stanley, New York
Executive Director of the Masters Program in Math Finance, Courant Institute, NYU

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The Asset Allocation Problem

- Consider a simple economy with just a riskless asset called a bond and a risky asset called a stock.
- We assume zero interest rates and dividends for simplicity.
- The investor has continuous trading opportunities over his horizon $[t_0, T]$.
- The investor has a known positive initial wealth W_0 and faces known initial asset prices $B_{t_0} = 1$ and $S_{t_0} = S_0$.
- The asset allocation problem is to determine how much wealth to have invested in the stock, with the remainder invested in the bond.

Economics 101

- In economics, the standard solution to this problem begins with a specification of the investor's utility function defined over terminal wealth.
- Given exact knowledge of both the utility function and the stock price distribution, the investor maximizes the expected value of terminal utility, subject to the budget constraint that total initial investment equals initial wealth.
- In the absence of borrowing constraints, this approach yields an interior optimal solution which maximizes the tradeoff between risk and expected return.

An Alternative Approach

- A big problem with the standard economics approach is that the desired asset allocations are highly sensitive to the stock's expected return, whose magnitude is unknown in reality. Furthermore, econometrics is of little help in identifying the stock's expected return over investor lifetimes, even if it is constant.
- A second problem is that most investors don't know their utility function (What's yours?).
- An alternative approach is to maximize expected return subject to the constraint that wealth never fall below a floor.
- CPPI is a (family of) dynamic trading strategies which can be described either as a solution to a particular expected utility maximization strategy or as a solution to the problem of maximizing expected return subject to a floor constraint.

Geometric Brownian Motion

- Suppose that the stock price follows geometric Brownian motion with known initial level $S_0 > 0$, known constant volatility $\sigma > 0$, but with an unknown bounded random drift process μ:
$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dB_t, \qquad t \in [t_0, T],$$
where B is standard Brownian motion.
- We again assume no dividends and that the stock always has a higher expected return than the bond, i.e. $\mu_t > 0$. The standard assumption that investors are risk-averse would imply that the risky stock has a higher expected return than the riskless bond.
- We will generalize to random volatility and jumps later in this talk.

Stock Dynamics

- Recall our present assumption that the stock price S follows geometric Brownian motion with known initial level $S_0 > 0$, known constant volatility $\sigma > 0$, and unknown drift process $\mu_t > 0$:
$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dB_t \qquad t \in [t_0, T].$$
- The unique solution to this stochastic differential equation (SDE) is:
$$S_t = S_0 e^{\int_0^t \mu_s ds} G_t, \qquad t \in [t_0, T],$$
where $G_t \equiv e^{\sigma B_t - \frac{\sigma^2}{2} t}, t \in [t_0, T]$ is a mean one positive martingale.
- In words, the stock price at any future time t is the product of
  1. its starting point $S_0 > 0$
  2. its random excess growth $e^{\int_0^t \mu_s ds} > 1$
  3. a random unbiased shock $G_t > 0$.
- Notice that the stock price is always positive.

Safety First

- Consider first the naive idea of putting all of one's wealth in the stock.
- Since the stock price can get arbitrarily small over any noninfinitessimal time horizon, there is positive probability that the floor F on future wealth is violated.
- Now consider the strategy of first putting $F < W_0$ dollars in the bond and the rest $W_0 - F > 0$ (called the initial cushion) in the stock. We have:

$$W_T = F + \frac{W_0 - F}{S_0} S_T.$$

- Since the terminal stock price S_T is always positive, the floor constraint will definitely be met.
- While this safety first strategy meets the floor constraint, some other strategy might also meet this constraint and have higher mean return.

Other People's Money

- Suppose that we also assume that the investor has no borrowing constraints. In contrast, short selling constraints are allowed as they will never bind.
- Consider the following *leveraged* safety-first strategy. The investor first puts $F < W_0$ dollars in the riskless asset, borrows the initial cushion $\$W_0 - F$, and invests twice this initial cushion $\$2(W_0 - F)$ in the stock.
- If these positions are held static, then:

$$W_T = F + (W_0 - F)\left[2\frac{S_T}{S_0} - 1\right].$$

- If the stock price falls by more than half its initial level, this position violates the floor.

Dynamic Asset Allocation

- We just examined a strategy in which the investor first puts $\$F < W_0$ in bonds and then is able to put twice the initial cushion i.e. $\$ \, 2(W_0 - F)$ in the stock through borrowing. The investor then kept the number of shares held static and as a result, failed the floor constraint.
- Suppose that the investor still puts twice the initial cushion in the stock, but now dynamically adjusts the dollars invested in equity (denoted E_t for exposure), so that it is always twice the running cushion $C_t \equiv W_t - F$, for $C_t > 0$.
- When the cushion is positive, then exposure is positive, and when the cushion first hits zero, then exposure first hits zero.
- Suppose that when the cushion first vanishes, the investor stops the dynamic trading strategy and just keeps the F dollars in bonds.
- This dynamic trading strategy meets the floor constraint and its expected return is higher than without borrowing. We refer to it as CPPI with floor F and multiplier 2.

Ever a Borrower Nor a Lender Be

- Recall that CPPI with floor F and multiplier 2 is a strategy in which the investor first puts $F < W_0$ dollars in bonds and then through borrowing keeps twice the running cushion invested in stocks, so long as this cushion is positive. When the cushion first vanishes, the investor closes the leveraged share position and abandons dynamic trading.
- More generally, an investor could first put \$$F < W_0$ in the riskless asset, initially borrow \$$(m-1)(W_0 - F)$ for $m > 1$, and keep \$$m(W_t - F)$ in the stock so long as $W_t \geq F$. When W_t first reaches F, the investor pulls the plug on the dynamic trading strategy.
- Actually, so long as the multiplier $m > 0$, wealth will be floored.
- We refer to this trading strategy as CPPI with floor F and multiplier m.
- As we increase m upwards from 0, the expected return on the strategy increases without bound.

Infinite Riches With No Risk!

- Assuming unlimited borrowing opportunities and that an investor can continuously trade a continuous stock with positive drift, we have seen that an investor can send expected return to infinity, while preserving any floor $F \in (0, W_0)$ on future wealth.
- This embarassing conclusion is isomorphic to Modigliani and Miller's classic observation in 1963 that a firm with a tax advantage to debt and no bankruptcy costs will borrow an unlimited amount.
- The conclusion that firms or investors will leverage an unlimited amount is clearly at odds with reality.
- To escape this condundrum, MM63 assumed that firms can go bankrupt and face positive bankrupty costs if they do.
- In our setting, the positive floor on wealth rules out bankruptcy, so what are we to do?

What to Do?

- There at least 4 ways to rule out unlimited borrowing:
  1. **Strengthen Risk Aversion**: At present, all we have assumed is that investors place a positive floor on terminal wealth. If we alternatively assume that utility is concave everywhere, then an interior optimum results.
  2. **Add a Borrowing Constraint**: At present, we have no borrowing constraint. If we just add a borrowing constraint, then a corner solution will result.
  3. **Add Transactions Costs**: At present, we have no transaction costs. If we just add transaction costs for borrowing, then infinite leverage need not be optimal.
  4. **Add Down Jumps**: At present, the stock price is a continuous process with constant volatility. We will show that adding stochastic volatility and/or the possibility of up jumps in the stock price process does not rule out unlimited borrowing. However, adding the possibility of down jumps to the stock price process does alter the infinite borrowing conclusion.
- Although all 4 routes have merit, we will just pursue the 4th route in this talk.

Stochastic Volatility

- Suppose now that the stock has stochastic volatility, but still can't jump.
- Hence, the stock price S solves the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V}_t dB_t \qquad t \in [t_0, T],$$

  where:
  - $\mu_t > 0$ is the stochastic instantaneous drift at time t
  - $V_t > 0$ is the stochastic instantaneous variance at time t,
  - B is a standard Brownian motion.
- The unique solution to this SDE is:

$$S_t = S_0 e^{\int_0^t \mu_s ds} G_t, \qquad t \in [t_0, T],$$

  where $G_t \equiv e^{\int_0^t \sqrt{V_s} dB_s - \frac{1}{2}\int_0^t V_s ds}, t \in [t_0, T]$ is still a mean one positive martingale.
- Since S_0 and G_t are positive, so is S_t for any t.
- One can also show that $E_0 S_t > S_0$ for any t.

Keeping Track of the Cushion

- Suppose that an investor first puts $F \in (0, W_0)$ dollars in the riskfree asset and then initiates a dynamic self-financing trading strategy with the initial cushion $W_0 - F > 0$.
- Denoting the dollar amount in equity by E_t, the cushion C at time t is given by:

$$C_t = W_0 - F + \int_0^t E_s \frac{dS_s}{S_s}, \qquad t \in [t_0, T].$$

- The CPPI strategy with floor $F \in (0, W_0)$ and multiplier $m > 0$ is defined by:

$$E_t = mC_t, \qquad t \in [t_0, T].$$

- Hence, the cushion C solves the SDE:

$$dC_t = mC_t \frac{dS_t}{S_t}, \qquad t \in [t_0, T].$$

- Using stochastic exponentials, the explicit solution for the cushion is given by:

$$C_t = (W_0 - F) \left(\frac{S_t}{S_0} \right)^m e^{-\frac{m(m-1)}{2} \int_0^t V_s ds}, \qquad t \in [t_0, T].$$

Loverage

- Recall that for CPPI, the cushion is given by:

$$C_t = (W_0 - F)\left(\frac{S_t}{S_0}\right)^m e^{-\frac{m(m-1)}{2}\int_0^t V_s ds}, \qquad t \in [t_0, T].$$

- Clearly, the cushion is positive for any horizon. As a result, wealth is floored for any positive multiplier.
- Now recall the SDE:

$$\frac{dC_t}{C_t} = m\frac{dS_t}{S_t}, \qquad t \in [t_0, T].$$

- Taking conditional expectations (using real world probability measure $\mathbb{P}$), we have:

$$E_t^{\mathbb{P}}\frac{dC_t}{C_t} = mE_t^{\mathbb{P}}\frac{dS_t}{S_t} = m\mu_t dt > 0, \qquad t \in [t_0, T].$$

- Hence, the larger is the multiplier m, the bigger is the expected growth in the cushion.
- Since $W_T = F + C_T$, the larger is the multiplier m, the bigger is the expected terminal wealth.

Jumps

- Now suppose that the stock price S solves the following stochastic differential equation:

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sqrt{V}_t dW_t + \int_{-\infty}^{\infty} (e^x - 1)[\gamma_t(dx, dt) - \nu_t(dx, dt)], \qquad t \geq 0,$$

where:

  - S_{t-} is the pre-jump stock price at time t.
  - $\mu_t > 0$ is the stochastic instantaneous drift at time t
  - $V_t > 0$ is the stochastic instantaneous variance at time t
  - W is a $\mathbb{Q}$ standard Brownian motion,
  - γ_t is the counting measure, and
  - ν_t is its predictable compensator.

- Since $S_0 > 0$, the stock price S that solves the above SDE is always positive.

Whoopee Cushion

- When the stock price S can jump, the cushion C still solves the SDE:

$$C_t = W_0 - F + \int_0^t mC_{s-} \frac{dS_s}{S_{s-}}, \qquad t \geq 0,$$

- Using stochastic exponentials, the explicit solution for the cushion is given by:

$$C_t = (W_0 - F)\left(\frac{S_t}{S_0}\right)^m e^{-\frac{m(m-1)}{2}\int_0^t V_s ds - m \sum\limits_{0\leq s\leq t}\left[\ln\left(1+\frac{\Delta S_s}{S_{s-}}\right) - \frac{\Delta S_s}{S_{s-}}\right]} \prod_{0\leq s\leq t}\left[1 + m\frac{\triangle S_s}{S_{s-}}\right] e^{-m\frac{\triangle S_s}{S_{s-}}}, t$$

- Keeping the cushion nonnegative requires choosing m so that $1 + m\frac{\triangle S_s}{S_{s-}} \geq 0$.
- If the largest possible down move is 100%, then the only way to keep the cushion nonnegative is to keep $m \in [0,1]$.
- However, for many assets, eg S&P500, the largest daily percentage drop has never exceeded 25%.

Nonnegative Cushion

- Recall that under stochastic volatility and jumps, the cushion is given by:

$$C_t = (W_0 - F)\left(\frac{S_t}{S_0}\right)^m e^{-\frac{m(m-1)}{2}\int_0^t V_s ds - m\sum_{0\le s\le t}\left[\ln\left(1+\frac{\Delta S_s}{S_{s-}}\right) - \frac{\Delta S_s}{S_{s-}}\right]} \prod_{0\le s\le t}\left[1 + m\frac{\triangle S_s}{S_{s-}}\right] e^{-m\frac{\triangle S_s}{S_{s-}}}, t$$

- Suppose that the largest possible percentage drop in the stock price is $\ell \in (0,1)$:

$$-\infty < -\ell \le \frac{\triangle S_s}{S_{s-}}.$$

- Then the cushion is nonnegative so long as $m \in [0, \frac{1}{\ell}]$.
- In fact, it is common practice to set $m = \frac{1}{\ell}$.
- If the stock's risk premium is positive, this approach will maximize expected return of the portfolio among all CPPI strategies which respect the floor.

Numerical Example

- To illustrate CPPI in action, suppose that initial wealth is \$100 and the floor is \$90.
- To create the floor, \$90 is invested in a riskless asset and held static.
- If the stock price can drop by at most one third, then the multiplier is set at 3.
- The initial cushion is \$100 - \$90 = \$10 and the multiplier is 3, so the initial exposure should be \$30.
- The investor has \$10 so he/she initially borrows \$20 and invests \$30 in the stock.
- If the worst possible drop occurs immediately, then exposure falls from \$30 to \$20 and the stock is sold.
- Otherwise, exposure is set at thrice the new cushion and we roll the dice again.

CPPI as Expected Utility Maximization

- Suppose that an investor craves no intermediate consumption and that the utility of terminal wealth is in the Hyperbolic Absolute Risk Aversion (HARA) class:

$$U(W) = \frac{(W-\phi)^{1-R}}{1-R}, \qquad W > \phi,$$

where $\phi \in [0, W_0)$ and $R > 0$ are the 2 given parameters defining preferences.

- Defining $\gamma \equiv W - \phi$, let $u(\gamma) \equiv U(W-\phi)$. Then one can check that:

$$-\frac{\gamma u''(\gamma)}{u'(\gamma)} = R.$$

- Suppose that the continuously compounded riskfree rate is constant at r, so that one dollar invested in the bond at t_0 grows to $e^{r(T-t_0)}$ at T.

CPPI as Expected Utility Maximization (Con'd)

- Further suppose that the stock price S follows geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \qquad t \in [t_0, T].$$

Hence, S_T is lognormally distributed with $E\frac{S_T}{S_0} = e^{\mu T}$ and $\mathrm{Var} \ln(S_T/S_0) = \sigma^2 T$.

- Then one can show that the optimal terminal wealth is:

$$W_T = \phi + \frac{W_0 - \phi}{V_0} \left(\frac{S_T}{S_0}\right)^{\frac{\mu - r}{R\sigma^2}},$$

where V_0 is the initial cost of replicating the payoff $\left(\frac{S_T}{S_0}\right)^{\frac{\mu - r}{R\sigma^2}}$.

- Thus, when interest rates are constant and the stock price follows geometric Brownian motion, then the HARA investor should choose $F = \phi$ and $m = \frac{\mu - r}{R\sigma^2}$.
- Black and Perrold[1] consider a generalization with a borrowing constraint.

Expected Utility Maximization vs Satisficing

- Consider a second investor who has the same initial wealth as the HARA investor, but doesn't know his utility function aside from requiring a floor of ϕ.
- The second investor also doesn't know anything about the stock price dynamics, except that the largest percentage drop in the stock price is $\ell \in [0,1)$.
- Suppose that the second investor follows a CPPI strategy with floor ϕ and multiplier $m = \frac{1}{\ell}$.
- If $\ell = \frac{R\sigma^2}{\mu - r}$, then the two investors follow the exact same strategy.
- It is arguable whether the multiplier observed in practice is due to risk aversion or due to crash concerns.
- However, I suspect that the popularity of CPPI is more due to its simplicity and to its robustness, rather than to its resonance with EU maximization.

Summary

- Stock prices are nonnegative. Given some bound on the largest possible percentage move down, a leveraged stock portfolio can also be kept nonnegative by keeping a fixed proportion of the wealth of the strategy invested in this stock.
- This class of trading strategies is known as CPPI.
- To maximize the expected return of the portfolio while ensuring that wealth weakly exceeds a floor, one chooses the proportion in the stock so that a realization of the worst possible move zeroes out the cushion.
- If the stock price follows geometric Brownian motion with drift μ and volatility σ, then HARA investors with utility function $U(W) = \frac{(W-\phi)^{1-R}}{1-R}, W > \phi$, will find that CPPI maximizes their expected utility if they choose the floor $F = \phi$ and set the multiplier $m = \frac{\mu - r}{R\sigma^2}$.

Future Research

- Suppose that we distinguish between a non-recurring default event and a possibly recurring downward percentage move, which is smaller in absolute value.
- It is tempting to use the smaller move to gain a larger multiplier and then use credit default swaps to insure CPPI wealth against default.
- Dynamic trading in delta and vega hedged variance swaps are also tuned to large moves and hence can be used to insure CPPI.
- Finally, a market has emerged for options on CPPI wealth.
- In principle, dynamic trading in options and their underlying stock can be used to robustly replicate the payoff to options on CPPI wealth.

References

[1] Black, F., and A. Perrold, 1992, “Theory of Constant Proportion Portfolio Insurance”, *Journal of Economic Dynamics and Control*, **16**, 403–426.