Enhancing the Black-Litterman and Related Approaches:
Views and Stress-test on Risk Factors[1]

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Abstract

The Black-Litterman and related approaches modify the return distribution of a normally distributed market according to views or stress-test scenarios. We discuss how to broaden the range of applications of these approaches significantly by letting them act on the risk factors underlying the market, instead of the returns of the securities. Code implementing the models discussed here can be found at MATLAB Central File Exchange.

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1 Introduction

The pathbreaking approach pioneered by Black and Litterman (1990), BL in the sequel, allows the users to blend their subjective views on the market into a normally distributed reference model centered around the CAPM equilibrium.

The equilibrium-based assumption might at first appear to restrict the potential applications of this approach to the tactical management of a global diversified fund. However, as pointed out by the authors of BL, the posterior formulas to include views in the allocation process can be applied to any normal market, not necessarily the CAPM equilibrium.

On the other hand, the assumption that the markets be normally distributed seems to prevent the application of BL to such markets as short-maturity derivatives with highly skewed, and thus fully non-Gaussian, p&l profile. Here we show how this is not the case: as long as the risk factors underlying the randomness in the market is normal, BL can be effectively used to process views on these risk factors. The methodology discussed here also applies to other related models that assume normality: among others, regression-based scenario analysis as in Mina and Xiao (2001), and correlation stress-testing as in Qian and Gorman (2001).

In Section 2 we review the original BL approach. In Section 3 we discuss how to enhance BL and related models to accommodate long-short option trading, views on macro factors and any other non-tradable securities, non-mean-variance optimization, as well as a variety of other trading and portfolio management problems. In Section 4 we present a detailed case study: trading call options in a mean-value-at-risk framework.

Code implementing the models discussed here can be found at MATLAB Central File Exchange.

2 The original model

Here we summarize from Meucci (2008), to which the reader is referred for all the proofs and further references.

The market model

We consider a market of N securities or asset classes, whose returns are normally distributed:

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{1}$$

The covariance $\boldsymbol{\Sigma}$ is estimated by exponential smoothing of the past return realizations. To set $\boldsymbol{\mu}$, BL invoke an equilibrium argument. Assume that all investors maximize a mean-variance trade-off and that the optimization is unconstrained:

$$\mathbf{w}_{\lambda} \equiv \underset{\mathbf{w}}{\operatorname{argmax}} \left\{\mathbf{w}'\boldsymbol{\mu} - \lambda \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right\}. \tag{2}$$

By setting to zero the derivative of the the term in curly brackets, we obtain the relationship between the equilibrium portfolio $\widetilde{\mathbf{w}}$ which stems from an average risk-aversion level $\overline{\lambda}$ and the reference expected returns:

$$\boldsymbol{\mu} \equiv 2\overline{\lambda}\boldsymbol{\Sigma}\widetilde{\mathbf{w}}. \tag{3}$$

Therefore, $\boldsymbol{\mu}$ can be set in terms of $\widetilde{\mathbf{w}}$. BL set exogenously $\overline{\lambda} \approx 1.2$. However, this is not a sensitive parameter.

The views

A view is a statement on the market that can potentially clash with the reference market model (1). BL consider views on expectations. In the normal market (1), this corresponds to statements on the parameter $\boldsymbol{\mu}$. Furthermore, BL focus on linear views: K views are represented by a $K \times N$ "pick" matrix $\mathbf{P}$, whose generic k-th row determines the relative weight of each expected return in the respective view. In order to associate uncertainty with the views, BL use a normal model:

$$\mathbf{P}\boldsymbol{\mu} \sim \mathrm{N}\left(\mathbf{v}, \boldsymbol{\Omega}\right), \tag{4}$$

where the meta-parameters $\mathbf{v}$ and $\boldsymbol{\Omega}$ quantify views and uncertainty thereof respectively.

If the user has only qualitative views, it is convenient to set the entries of $\mathbf{v}$ in terms of the volatility induced by the market:

$$v_k \equiv (\mathbf{P}\boldsymbol{\mu})_k + \eta_k \sqrt{\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)_{k,k}}, \quad k = 1, \ldots, K, \tag{5}$$

where $\eta_k \in \{-\beta, -\alpha, +\alpha, +\beta\}$ defines "very bearish", "bearish", "bullish" and "very bullish" views respectively. Typical choices for these parameters are $\alpha \equiv 1$ and $\beta \equiv 2$. Also, it is convenient to set

$$\boldsymbol{\Omega} \equiv \frac{1}{c}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}', \tag{6}$$

where the scatter structure of uncertainty is inherited from the market volatilities and correlations and c represents an overall level of confidence in the views.

The posterior

From the above inputs one can compute the posterior market distribution

$$\mathbf{X}|\mathbf{v};\boldsymbol{\Omega} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}\right), \tag{7}$$

where

$$\boldsymbol{\mu}_{BL} \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\left(\mathbf{v} - \mathbf{P}\boldsymbol{\mu}\right) \tag{8}$$

$$\boldsymbol{\Sigma}_{BL} \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{9}$$

see details in Meucci (2008). The normal posterior distribution (7) represents the modification of the reference model that incorporates the views (4): indeed, when the confidence in the views is null, i.e. $\boldsymbol{\Omega} \to \infty$, then the BL posterior equals the reference model (1).

Scenario analysis

Consider the conditional distribution of the market (1) given a user-defined set of deterministic scenarios $\mathbf{v}$ for the combinations $\mathbf{PX}$. As proved in Meucci (2005) this distribution is normal

$$\mathbf{X}|\mathbf{v} \sim \mathrm{N}\left(\boldsymbol{\mu}|\mathbf{v}, \boldsymbol{\Sigma}|\mathbf{v}\right), \tag{10}$$

where

$$\boldsymbol{\mu}|\mathbf{v} \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\left(\mathbf{v} - \mathbf{P}\boldsymbol{\mu}\right) \tag{11}$$

$$\boldsymbol{\Sigma}|\mathbf{v} \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{12}$$

The conditional distribution (10) is the core of scenario analysis: the user selects a set of scenarios $\mathbf{v} \equiv (v_1, \ldots, v_K)'$ for the combinations of factor $\mathbf{PX}$ and analyzes their effect on the reference model. Indeed, (11)-(12) generalize the classical regression-like result utilized e.g. in Mina and Xiao (2001).

When the confidence in the views is full, i.e. $\boldsymbol{\Omega} \to \mathbf{0}$, the BL posterior (7) equals the conditional distribution (10). Therefore, BL can be interpreted as a generalization of scenario analysis that allows for uncertainty in the views.

The allocation

With the posterior distribution it is now possible to set and solve a mean-variance optimization such as (2), possibly under a set of linear constraints, such as boundaries on securities/asset classes, or a budget constraint. This problem, is an instance of quadratic programming and can be easily solved numerically. The ensuing efficient frontier represents a gentle twist to equilibrium that reflects the views without extreme corner solutions.

3 The enhanced model

As pointed out by the authors of BL soon after the model was created, the posterior formulas to include linear views in the allocation process (8)-(9) can be applied to any normal distribution, not necessarily the CAPM equilibrium (3).

Expanding on their rationale, we notice further steps can be undertaken to broaden the range of applications of the BL model, as well as any other model that assumes normally distributed markets.

First and foremost, instead of returns, the variable $\mathbf{X}$ in the reference model (1), which we report here

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right), \tag{13}$$

can represent generic risk factors linearly or non-linearly responsible for the p&l of a given market of securities: for instance, underlying and implied volatility for options, or the value at different monitoring times of a key rate for path-dependent, prepayment-driven mortgage-backed securities.

Second, any management style can be handled, including long-short positions, as well as any optimization framework other than mean-variance. In general, the user allocates a budget by solving

$$\mathbf{w}^* \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{\mathcal{S}\left(\mathbf{w}; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\right\}, \tag{14}$$

where $\mathcal{C}$ is a set of constraints, such as value- or -market-neutral constraints, budget and VaR limits, diversification constraints; and $\mathcal{S}$ is an index of satisfaction, such as expected utility or a (negative) measure of risk such as a spectral measure, see Meucci (2005). Satisfaction depends on the p&l distribution, which depends on the allocation decision $\mathbf{w}$ as well as on the market distribution and thus on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

Finally, $\mathbf{X}$ can include factors that indirectly determine the p&l, but on which traders want to express views, such as macro-economic drivers. An instance of this generalization is discussed in Cheung (2007).

In order to determine the reference parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (13) we can rely on historical estimation. In this case, a plethora of estimation techniques can be used: non-parametric, maximum likelihood, shrinkage, robust, Bayesian, etc., see a detailed discussion in Meucci (2005). Furthermore, a subset of the entries in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ can be calibrated to current market observables in a portfolio-specific manner. First, we choose a reference portfolio $\widetilde{\mathbf{w}}$. This portfolio, which could be the current portfolio, or a benchmark, or any other set of potentially long-short positions is in general neither optimal, nor the equilibrium: it simply represents a portfolio that the trader would not rebalance unless he had specific views. Then we modify some entries in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in such a way that $\widetilde{\mathbf{w}}$ is the solution of (14). The details of this approach are further discussed in Appendix 5. Notice that the standard BL prior (3) represents the special instance of this approach where the market $\mathbf{X}$ are returns, satisfaction $\mathcal{S}$ is the unconstrained mean-variance trade-off (2), the covariance $\boldsymbol{\Sigma}$ is fully determined by historical analysis, and the expected returns $\boldsymbol{\mu}$ are all and the only variables left to imply.

To input and process the views, one proceeds as in Section 2: the views are specified by $\mathbf{v}$ and $\boldsymbol{\Omega}$ and the BL parameters (8)-(9) then follow:

$$\boldsymbol{\mu}_{BL} \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} (\mathbf{v} - \mathbf{P}\boldsymbol{\mu}) \tag{15}$$

$$\boldsymbol{\Sigma}_{BL} \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} \mathbf{P}\boldsymbol{\Sigma}. \tag{16}$$

The p&l is now a (potentially highly non-linear) function of the allocation decision $\mathbf{w}$ and the BL parameters $\boldsymbol{\mu}_{BL}$ and $\boldsymbol{\Sigma}_{BL}$, and so is satisfaction. Therefore, we can compute the optimal portfolio

$$\mathbf{w}^* \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{\mathcal{S}\left(\mathbf{w}; \boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}\right)\right\}. \tag{17}$$

This allocation smoothly incorporates the view.

Notice that if the confidence in the views is full, this approach incorporates scenario analysis. Also notice that any other model that relies on the normal assumption, such as correlation stress-testing in Qian and Gorman (2001), can also be extended as discussed here.

4 Case study: option trading

To illustrate, here we consider a trader of European call options. The current time is t and the investment horizon is $t + \tau$. The price $P_{t+\tau}$ of the generic call option at the horizon can be written in the format $P_{t+\tau} \equiv C(\mathbf{X}, \mathcal{I}_t)$ as a deterministic non-linear function of the risk factors $\mathbf{X}$ and current information $\mathcal{I}_t$. More precisely, the price of a call option with strike κ and time to expiration T depends on two factors X_y and X_σ as follows:

$$P_{t+\tau} = C_{BS}\left(y_t e^{X_y}, h\left(y_t e^{X_y}, \sigma_t + X_\sigma, \kappa, T-\tau\right); \kappa, T-\tau, r\right). \tag{18}$$

In this expression τ is the investment horizon; y_t is the current value and $X_y \equiv \ln(y_{t+\tau}/y_t)$ is the log-change of the underlying; σ_t is the current value and $X_\sigma \equiv \sigma_{t+\tau} - \sigma_t$ is the change in ATM implied volatility; C_{BS} is the Black-Scholes formula

$$C_{BS}(y, \sigma; \kappa, T, r) \equiv y\Phi(d_1) - \kappa e^{-rT}\Phi(d_2), \tag{19}$$

where Φ is the standard normal cdf and $d_1 \equiv \left(\ln(y/\kappa) + \left(r + \sigma^2/2\right)T\right)/\sigma\sqrt{T}$, $d_2 \equiv d_1 - \sigma\sqrt{T}$; and h is a skew/smile map

$$h(y, \sigma; \kappa, T) \equiv \sigma + a\frac{\ln(y/\kappa)}{\sqrt{T}} + b\left(\frac{\ln(y/\kappa)}{\sqrt{T}}\right)^2, \tag{20}$$

for coefficients a and b which depend on the underlying and are fitted empirically, similarly to Malz (1997). Notice that, if the investment horizon τ is short, such as one day, a delta-gamma-vega approximation of (18) would suffice. For longer horizons, the second-order Taylor approximation based on the Greeks is no longer sufficient and the full pricing formula must be used instead, see Figure 1.

Consider a long-short portfolio $\mathbf{w}$ of I call options, where the generic i-th entry is the number of contracts in the respective call option, whose currently traded price we denote by $c_{i,t}$. The p&l then reads

$$\Pi_{\mathbf{w}} \equiv \sum_{i=1}^{I} w_i\left(C_i(\mathbf{X}, \mathcal{I}_t) - c_{i,t}\right), \tag{21}$$

where $C_i(\mathbf{X}, \mathcal{I}_t)$ is the pricing function (18) for the i-th call.

Assume that, in order to account for market asymmetries and downside risk, the trader monitors the mean-VaR trade-off of his positions under a set of linear constraints. Then the optimization (14) becomes:

$$\mathbf{w}_\lambda \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{\mathrm{E}\{\Pi_{\mathbf{w}}\} + \lambda Q_{1-\gamma}\{\Pi_{\mathbf{w}}\}\right\}, \tag{22}$$

where $\mathrm{E}\{Y\}$ denotes the expectation and $Q_\gamma\{Y\}$ the quantile with tail level γ of a random variable Y; and $\mathbf{B}$, $\underline{\mathbf{b}}$, and $\overline{\mathbf{b}}$ are a matrix and vectors that represent the linear constraints.

To illustrate, we set $\gamma \equiv 95\%$. Also, we impose delta-neutrality, i.e. the long-short positions offset to give a zero-budget initial investment, and that the absolute investment in each option should not exceed a fixed threshold. We set the investment horizon τ as one week. We consider a limited market of $I \equiv 9$

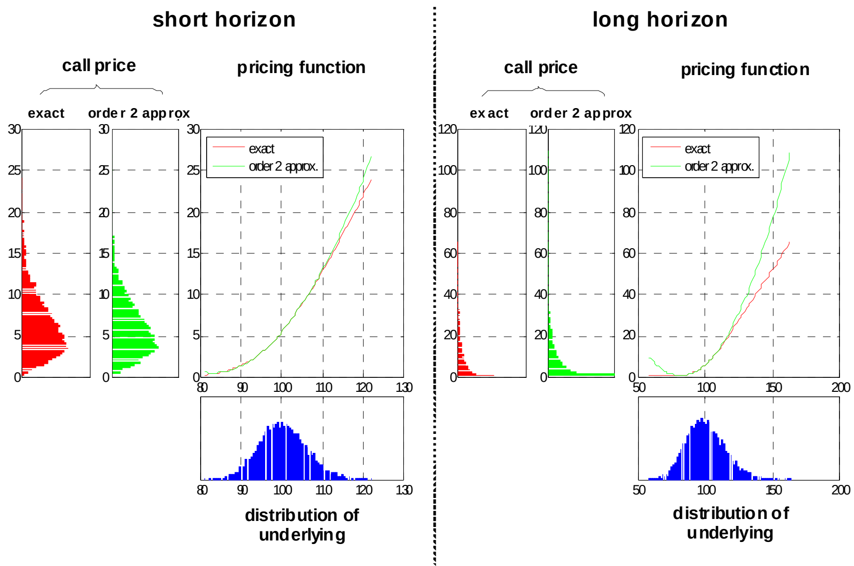

Figure 1: Goodness of delta-gamma approximation at different horizons

securities: 1-month, 2-month and 6-month calls on the three technology stocks Microsoft (M), Yahoo (Y) and Google (G).

In addition to the respective underlyings and implied volatilities, we assume that the trader has views on growth or inflation, as represented by the slope of the interest rate curve: therefore we add the changes in the two- and ten-year points of the curve, for a total of $N \equiv 14$ factors:

$$\mathbf{X} \equiv \left(X^M, X^M_{1m}, X^M_{2m}, X^M_{6m}, \ldots, , X^G_{6m}, X_{2y}, X_{10y}\right)'. \tag{23}$$

The distribution of these factors is in first approximation jointly normal and can be modeled as in (13). For illustration purposes, we estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as the sample mean and sample standard covariance respectively. For a more refined approach where a subset of the parameters is implied from the currently held book see Appendix 5.

We simulate a large number J of Monte Carlo scenarios from the reference model $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, thereby generating a $J \times N$ panel $\mathcal{X}$: the generic j-th row of $\mathcal{X}$ represents one in a very large number of joint scenarios for the N factors $\mathbf{X}$, whereas the generic n-th column of $\mathcal{X}$ represents the marginal distribution of the n-th factor X_n. To ensure precision we match exactly the first and second sample moments of this panel with their desired population counterpart $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ by solving a simple continuous-time algebraic Riccati equation as in Meucci (2009).

Then we generate a $J \times I$ panel $\mathcal{P}$ of the p&l of the I securities in the J

scenarios. This panel is defined entry-wise as follows

$$\mathcal{P}_{j,i} \equiv C_{BS,i}\left(\mathcal{X}_{j,\cdot}, \mathcal{I}_t\right) - C_{i,t}, \tag{24}$$

where $\mathcal{X}_{j,\cdot}$ is the j-th scenario (row) of $\mathcal{X}$ and $C_{BS,i}$ and $C_{i,t}$ are as in (21). The sample counterpart of the mean-VaR frontier (22) now reads

$$\mathbf{w}_\lambda \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{ \left(\mathbf{w}'\mathcal{P}'\mathbf{1}/J\right) + \lambda \left(\mathcal{P}\mathbf{w}\right)_{(1-\gamma)J:J} \right\}, \tag{25}$$

where $\mathbf{z}_{j:J}$ denotes the j-th smallest among the J numbers in $\mathbf{z} \equiv (z_1, \ldots, z_J)'$. This problem can be solved heuristically as in Meucci (2005) by a two-step approach: first determine the mean-variance efficient frontier, then perform a uni-variate grid search for the optimal trade-off (25).

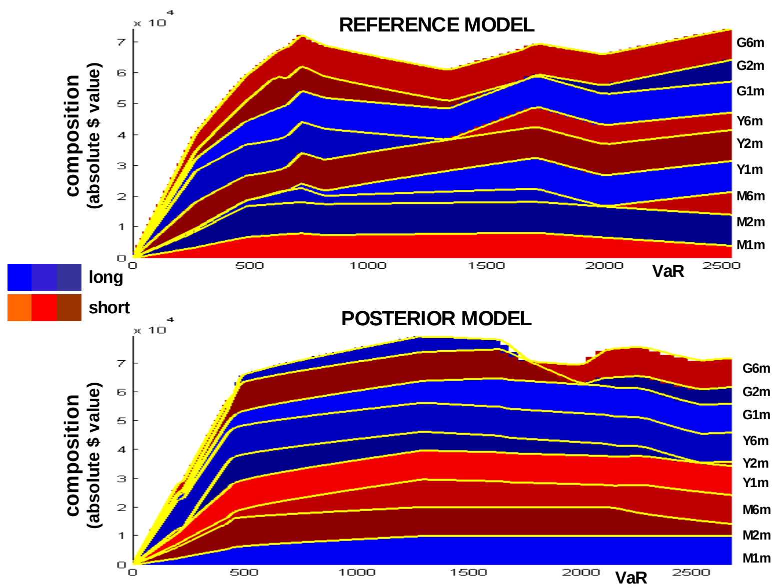

Figure 2: Mean-VaR efficient frontier

In the top portion of Figure 2 we display the mean-VaR frontier ensuing from the reference market in our example. For the extreme case of zero risk appetite, not investing at all is optimal. As the risk appetite increases, leverage increases, always respecting the constraint of a zero net initial investment, as well as delta-neutrality. When the risk appetite increases further, the remaining constraints enter the picture.

Now we are ready to input the views. To illustrate, we consider a bullish feeling on the Microsoft 1-month implied volatility and a bearish hunch on the Yahoo 1-month implied volatility. As in (5), we quantify our qualitative views in terms of the market volatility. Therefore the inputs of (15)-(16) are a $(K \equiv 2) \times (N \equiv 14)$ pick matrix $\mathbf{P}$ of zeros, except for $\mathbf{P}_{1,2} \equiv 1$ and $\mathbf{P}_{2,6} \equiv 1$; and

$$\boldsymbol{\Omega} \equiv \mathbf{P\Sigma P}' \tag{26}$$

$$\mathbf{v} \equiv \left((\mathbf{P}\boldsymbol{\mu})_1 + \sqrt{(\mathbf{P\Sigma P}')_{1,1}}, (\mathbf{P}\boldsymbol{\mu})_2 - \sqrt{(\mathbf{P\Sigma P}')_{2,2}} \right)'. \tag{27}$$

Furthermore, as in (6) we model the uncertainty on the views with the same order as the market volatility and correlations. Notice that it is trivial to input views on relative-value spreads between implied volatilities on the same stock with different maturities; views on the performance on the underlying; and flattening/steepening views for the curve.

With the above inputs we can compute the BL parameters (15)-(16). Then we proceed as above to generate new joint normal scenarios according to these parameters, where to ensure precision we match exactly the first and second sample moments of this panel with their desired population counterpart $(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL})$ by solving a simple continuous-time algebraic Riccati equation as in Meucci (2009). Next we price these scenarios and optimize the mean-VaR trade-off (25) with these new inputs, as prescribed by (17).

In the bottom portion of Figure 2 we display the mean-VaR frontier ensuing from the BL posterior: as expected, the Microsoft 1-month call, short in the reference model, is now long across all levels of risk aversion; and the Yahoo 1-month call, long in the reference model, is now shorted.

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5 Appendix: Estimation/calibration of the reference model

In general, the prior market model $f_{\mathbf{X}}$ is determined by a combination of econometric estimation and ad-hoc market-implied parametrization. To imply the parameters, first a parametric model $f_{\mathbf{X}}^{\boldsymbol{\theta}}$ is chosen for the prior, where $\boldsymbol{\theta}$ is a vector of parameters that fully determine the distribution of $\mathbf{X}$. The dimension of $\boldsymbol{\theta}$ can be very large, as this framework also includes non-parametric approaches.

Next, by time series analysis some parameters are estimated and $\boldsymbol{\theta}$ is thereby bound to a given subspace $\boldsymbol{\Theta}$.

Then, other parameters in $\boldsymbol{\theta}$ are implied from a preferred reference book $\widetilde{\mathbf{w}}$, which can be a benchmark in standard asset management, or a general-equilibrium portfolio as in BL, or the current book in a trading environment. The parameters are implied as follows

$$\widehat{\boldsymbol{\theta}} \equiv \underset{\widehat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}}{\operatorname{argmin}}\, L\left(\widetilde{\mathbf{w}}, \boldsymbol{\theta}\right), \tag{28}$$

where L is a non-negative loss such as the norm

$$L\left(\widetilde{\mathbf{w}}, \boldsymbol{\theta}\right) \equiv \left\| \widetilde{\mathbf{w}} - \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathcal{S}\left(\mathbf{w}; f_{\mathbf{X}}^{\boldsymbol{\theta}}\right) \right\} \right\|, \tag{29}$$

and $\mathcal{S}$ and $\mathcal{C}$ are defined in (14).

Finally, in case (28) has multiple solutions then $\widehat{\boldsymbol{\theta}}$ can be picked by symmetry principles such as entropy maximization.

Once all the parameters are determined, the reference model is set as $f_{\mathbf{X}} \equiv f_{\mathbf{X}}^{\widehat{\boldsymbol{\theta}}}$.

For instance, consider the normal assumption

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{30}$$

The parameters $\boldsymbol{\theta}$ are in this case the expectation $\boldsymbol{\mu}$ and the covariance $\boldsymbol{\Sigma}$. To determine these parameters, first we specialize (14) by choosing an index of satisfaction. Here we choose the classical mean-variance trade-off. For alternative specifications such as mean-VaR, mean-CVaR, certainty-equivalent, or spectral measures, see Meucci (2005). Accordingly, the book $\mathbf{w}^*$ is optimal if it satisfies

$$\mathbf{w}^* \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{ \mathbf{w}'\mathbf{E}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) - \delta \mathbf{w}'\mathbf{S}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)\mathbf{w} \right\}. \tag{31}$$

In this expression $\mathbf{E}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ is the vector of the expectations of the p&l of each security in the book, fully determined by (30) through the securities pricing functions $P_{t+\tau} \equiv P\left(\mathbf{X}, \mathcal{I}_t\right)$; in a similar way $\mathbf{S}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ is the covariance matrix of the p&l of all securities; δ is a risk-aversion coefficient; and $\mathbf{B}$, $\underline{\mathbf{b}}$ and $\overline{\mathbf{b}}$ determine a set of linear constraints.

Then we estimate the covariance $\widehat{\mathbf{\Sigma}}$ from time series analysis of the market $\mathbf{X}$. Next, we compute the implied expected values $\widehat{\boldsymbol{\mu}}$, defined in such a way that

$$\widetilde{\mathbf{w}} \equiv \underset{\underline{\mathbf{b}} \leq \mathbf{B}\mathbf{w} \leq \overline{\mathbf{b}}}{\operatorname{argmax}} \left\{ \mathbf{w}'\mathbf{E}\left(\widehat{\boldsymbol{\mu}}, \widehat{\mathbf{\Sigma}}\right) - \delta \mathbf{w}'\mathbf{S}\left(\widehat{\boldsymbol{\mu}}, \widehat{\mathbf{\Sigma}}\right)\mathbf{w} \right\}, \tag{32}$$

where $\widetilde{\mathbf{w}}$ is a preferred such as a benchmark, or a general-equilibrium portfolio, or the current book. If the number of parameters in $\widehat{\boldsymbol{\mu}}$ exceeds those in $\widetilde{\mathbf{w}}$, as is often the case, we pin down some values in $\widehat{\boldsymbol{\mu}}$ based on historical estimates. This is (28) in the current context. More in general, we might impose structure on the covariance and we might imply terms in the covariance as well.