Loan portfolio value

Using a conditional independence framework, Oldrich Vasicek derives a useful limiting form for the portfolio loss distribution with a single systematic factor. He then derives a risk-neutral distribution suitable for traded portfolios, and shows how credit migration and granularity can be incorporated into this model too

The amount of capital needed to support a portfolio of debt securities depends on the probability distribution of the portfolio loss. Consider a portfolio of loans, each of which is subject to default resulting in a loss to the lender. Suppose the portfolio is financed partly by equity capital and partly by borrowed funds. The credit quality of the lender's notes will depend on the probability that the loss on the portfolio exceeds the equity capital. To achieve a certain credit rating of its notes (say Aa on a rating agency scale), the lender needs to keep the probability of default on the notes at the level corresponding to that rating (about 0.001 for the Aa quality). It means that the equity capital allocated to the portfolio must be equal to the percentile of the distribution of the portfolio loss that corresponds to the desired probability.

In addition to determining the capital needed to support a loan portfolio, the probability distribution of portfolio losses has a number of other applications. It can be used in regulatory reporting, measuring portfolio risk, calculation of value-at-risk, portfolio optimisation, and structuring and pricing debt portfolio derivatives such as collateralised debt obligations (CDOs).

In this article, we derive the distribution of the portfolio loss under certain assumptions. It is shown that this distribution converges with increasing portfolio size to a limiting type, whose analytical form is given here. The results of the first two sections of this paper are contained in the author's technical notes, Vasicek (1987) and (1991). For a review of recent literature on the subject, see, for instance, Pykhtin & Dev (2002).

The limiting distribution of portfolio losses

Assume that a loan defaults if the value of the borrower's assets at the loan maturity T falls below the contractual value B of its obligations payable. Let A_i be the value of the i-th borrower's assets, described by the process:

$$dA_i = \mu_i A_i dt + \sigma_i A_i dx_i$$

The asset value at T can be represented as:

$$\log A_i(T) = \log A_i + \mu_i T - \tfrac{1}{2}\sigma_i^2 T + \sigma_i \sqrt{T} X_i \qquad (1)$$

where X_i is a standard normal variable. The probability of default of the i-th loan is then:

$$p_i = P\left[A_i(T) < B_i\right] = P\left[X_i < c_i\right] = N(c_i)$$

where:

$$c_i = \frac{\log B_i - \log A_i - \mu_i T + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}$$

and N is the cumulative normal distribution function.

Consider a portfolio consisting of n loans in equal dollar amounts. Let the probability of default on any one loan be p, and assume that the asset values of the borrowing companies are correlated with a coefficient ρ for any two companies. We will further assume that all loans have the same term T.

Let L_i be the gross loss (before recoveries) on the i-th loan, so that $L_i = 1$ if the i-th borrower defaults and $L_i = 0$ otherwise. Let L be the portfolio percentage gross loss:

$$L = \frac{1}{n}\sum_{i=1}^{n} L_i$$

If the events of default on the loans in the portfolio were independent of each other, the portfolio loss distribution would converge, by the central limit theorem, to a normal distribution as the portfolio size increases. Because the defaults are not independent, however, the conditions of the central limit theorem are not satisfied and L is not asymptotically normal. It turns out, however, that the distribution of the portfolio loss does converge to a limiting form, which we will now proceed to derive.

The variables X_i in Equation (1) are jointly standard normal with equal pair-wise correlations ρ, and can therefore be represented as:

$$X_i = Y\sqrt{\rho} + Z_i\sqrt{1-\rho} \qquad (2)$$

where $Y, Z_1, Z_2, \ldots, Z_n$ are mutually independent standard normal variables. (This is not an assumption, but a property of the equicorrelated normal distribution.) The variable Y can be interpreted as a portfolio common factor, such as an economic index, over the interval $(0, T)$. Then the term $Y\sqrt{\rho}$ is the company's exposure to the common factor and the term $Z_i\sqrt{(1-\rho)}$ represents the company's specific risk.

We will evaluate the probability of the portfolio loss as the expectation over the common factor Y of the conditional probability given Y. This can be interpreted as assuming various scenarios for the economy, determining the probability of a given portfolio loss under each scenario and then weighting each scenario by its likelihood.

When the common factor is fixed, the conditional probability of loss on any one loan is:

$$p(Y) = P\left[L_i = 1 \middle| Y\right] = N\left(\frac{N^{-1}(p) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right) \qquad (3)$$

The quantity $p(Y)$ provides the loan default probability under the given scenario. The unconditional default probability p is the average of the conditional probabilities over the scenarios.

Conditional on the value of Y, the variables L_i are independent equally distributed variables with a finite variance. The portfolio loss conditional on Y converges, by the law of large numbers, to its expectation $p(Y)$ as $n \to \infty$. Then:

$$P\left[L \le x\right] = P\left[p(Y) \le x\right] = P\left[Y \ge p^{-1}(x)\right] = N\left(-p^{-1}(x)\right)$$

and on substitution, the cumulative distribution function of loan losses on a very large portfolio is in the limit:

$$P\left[L \le x\right] = N\left(\frac{\sqrt{1-\rho}N^{-1}(x) - N^{-1}(p)}{\sqrt{\rho}}\right) \qquad (4)$$

This result is given in Vasicek (1991).

The convergence of the portfolio loss distribution to the limiting form above actually holds even for portfolios with unequal weights. Let the portfolio weights be $w_1, w_2, \ldots, w_n$ with $\Sigma w_i = 1$. The portfolio loss:

$$L = \sum_{i=1}^{n} w_i L_i$$

conditional on Y converges to its expectation $p(Y)$ whenever (and this is a necessary and sufficient condition):

$$\sum_{i=1}^{n} w_i^2 \to 0$$

In other words, if the portfolio contains a sufficiently large number of loans without it being dominated by a few loans much larger than the rest, the limiting distribution provides a good approximation for the portfolio loss.

Properties of the loss distribution

The portfolio loss distribution given by the cumulative distribution function:

$$F(x;p,\rho) = N\left(\frac{\sqrt{1-\rho}N^{-1}(x) - N^{-1}(p)}{\sqrt{\rho}}\right) \tag{5}$$

is a continuous distribution concentrated on the interval $0 \le x \le 1$. It forms a two-parameter family with the parameters $0 < p$, $\rho < 1$. When $\rho \to 0$, it converges to a one-point distribution concentrated at $L = p$. When $\rho \to 1$, it converges to a zero-one distribution with probabilities p and $1 - p$, respectively. When $p \to 0$ or $p \to 1$, the distribution becomes concentrated at $L = 0$ or $L = 1$, respectively. The distribution possesses a symmetry property:

$$F(x;p,\rho) = 1 - F(1-x;1-p,\rho)$$

The loss distribution has the density:

$$f(x;p,\rho) = \sqrt{\frac{1-\rho}{\rho}}\exp\left(-\frac{1}{2\rho}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(p)\right)^2 + \frac{1}{2}\left(N^{-1}(x)\right)^2\right)$$

which is unimodal with the mode at:

$$L_{mode} = N\left(\frac{\sqrt{1-\rho}}{1-2\rho}N^{-1}(p)\right)$$

when $\rho < ½$, monotone when $\rho = ½$ and U-shaped when $\rho > ½$. This density is shown in figure 1 for $p = 0.02$, $\rho = 0.1$. The mean of the distribution is $E(L) = p$ and the variance is:

$$s^2 = Var\,L = N_2\left(N^{-1}(p), N^{-1}(p), \rho\right) - p^2$$

where N_2 is the bivariate cumulative normal distribution function. The inverse of this distribution, that is, the α-percentile value of L, is given by:

$$L_\alpha = F(\alpha;1-p,1-\rho)$$

The portfolio loss distribution is highly skewed and leptokurtic. Table A lists the values of the α-percentile L_α expressed as the number of standard deviations from the mean, for several values of the parameters. The α-percentiles of the standard normal distribution are shown for comparison.

These values manifest the extreme non-normality of the loss distribution. Suppose a lender holds a large portfolio of loans to firms whose pairwise asset correlation is $\rho = 0.4$ and whose probability of default is $p = 0.01$. The portfolio expected loss is $E(L) = 0.01$ and the standard deviation is $s = 0.0277$. If the lender wishes to hold the probability of default on his notes at $1 - \alpha = 0.001$, he will need enough capital to cover 11.0 times the portfolio standard deviation. If the loss distribution were normal, 3.1 times the standard deviation would suffice.

The risk-neutral distribution

The portfolio loss distribution given by equation (4) is the actual probability distribution. This is the distribution from which to calculate the probability of a loss of a certain magnitude for the purposes of determining the necessary capital or of calculating VAR. This is also the distribution to be used in structuring CDOs, that is, in calculating the probability of loss and the expected loss for a given tranche. For the purposes of pricing the tranches, however, it is necessary to use the risk-neutral probability distribution. The risk-neutral distribution is calculated in the same way as above, except that the default probabilities are evaluated under the risk-neutral measure P^*:

$$p^* = P^*\left[A(T) < B\right] = N\left(\frac{\log B - \log A - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

where r is the risk-free rate. The risk-neutral probability is related to the actual probability of default by the equation:

$$p^* = N\left(N^{-1}(p) + \lambda\rho_M\sqrt{T}\right) \tag{6}$$

where ρ_M is the correlation of the firm asset value with the market and

A. Values of $(L_\alpha - p)/s$ for the portfolio loss distribution

p	ρ	α = 0.9	α = 0.99	α = 0.999	α = 0.9999
0.01	0.1	1.19	3.8	7.0	10.7
0.01	0.4	0.55	4.5	11.0	18.2
0.001	0.1	0.98	4.1	8.8	15.4
0.001	0.4	0.12	3.2	13.2	31.8
Normal		1.28	2.3	3.1	3.7

$\lambda = (\mu_M - r)/\sigma_M$ is the market price of risk. The risk-neutral portfolio loss distribution is then given by:

$$P^*[L \le x] = N\left(\frac{\sqrt{1-\rho}N^{-1}(x) - N^{-1}(p^*)}{\sqrt{\rho}}\right) \tag{7}$$

Thus, a derivative security (such as a CDO tranche written against the portfolio) that pays at time T an amount $C(L)$ contingent on the portfolio loss is valued at:

$$V = e^{-rT}E^*\left[C(L)\right]$$

where the expectation is taken with respect to the distribution (7). For instance, a default protection for losses in excess of L_0 is priced at:

$$V = e^{-rT}E^*\left[(L - L_0)_+\right] = e^{-rT}\left(p^* - N_2\left(N^{-1}(p^*), N^{-1}(L_0), \sqrt{1-\rho}\right)\right)$$

The portfolio market value

So far, we have discussed the loss due to loan defaults. Now suppose that the maturity date T of the loan is past the date H for which the portfolio value is considered (the horizon date). If the credit quality of a borrower deteriorates, the value of the loan will decline, resulting in a loss (this is often referred to as the loss due to 'credit migration'). We will investigate the distribution of the loss resulting from changes in the marked-to-market portfolio value.

The value of the debt at time 0 is the expected present value of the loan payments under the risk-neutral measure:

$$D = e^{-rT}\left(1 - Gp^*\right)$$

where G is the loss-given default and p^* is the risk-neutral probability of default. At time H, the value of the loan is:

$$D(H) = e^{-r(T-H)}\left(1 - GN\left(\frac{\log B - \log A(H) - r(T-H) + \frac{1}{2}\sigma^2(T-H)}{\sigma\sqrt{T-H}}\right)\right)$$

Define the loan loss L_i at time H as the difference between the risk-free value and the market value of the loan at H:

$$L_i = e^{-r(T-H)} - D(H)$$

This definition of loss is chosen purely for convenience. If the loss is defined in a different way (for instance, as the difference between the accrued value and the market value), it will only result in a shift of the portfolio loss distribution by a location parameter.

The loss on the i-th loan can be written as:

$$L_i = aN\left(b\sqrt{\frac{T}{T-H}} - X_i\sqrt{\frac{H}{T-H}}\right)$$

where:

$$a = Ge^{-r(T-H)}, b = N^{-1}(p) + \lambda\rho_M\frac{T-H}{\sqrt{T}}$$

and the standard normal variables X_i defined over the horizon H by:

$$\log A_i(H) = \log A_i + \mu_i H - \tfrac{1}{2}\sigma_i^2 H + \sigma_i\sqrt{H}X_i$$

are subject to equation (2).

Let L be the market value loss at time H of a loan portfolio with weights w_i. The conditional mean of L_i given Y can be calculated as:

$$\mu(Y) = E(L_i|Y) = aN\left(b\sqrt{\frac{T}{T-\rho H}} - Y\sqrt{\frac{\rho H}{T-\rho H}}\right)$$

The losses conditional on the factor Y are independent, and therefore the portfolio loss L conditional on Y converges to its mean value $E(L|Y) = \mu(Y)$ as $\Sigma w_i^2 \to \infty$. The limiting distribution of L is then:

$$P[L \le x] = P[\mu(Y) \le x] = F\left(\frac{x}{a}; N(b), \frac{\rho H}{T}\right) \tag{8}$$

We see that the limiting distribution of the portfolio loss is of the same type (5) whether the loss is defined as the decline in the market value or the realised loss at maturity. In fact, the results of the section on the distribution of loss due to default are just a special case of this section for $T = H$.

The risk-neutral distribution for the loss due to market value change is given by:

$$P^*[L \le x] = F\left(\frac{x}{a}; p^*, \frac{\rho H}{T}\right) \tag{9}$$

Adjustment for granularity

Equation (8) relies on the convergence of the portfolio loss L given Y to its mean value $\mu(Y)$, which means that the conditional variance $Var(L|Y) \to 0$. When the portfolio is not sufficiently large for the law of large numbers to take hold, we need to take into account the non-zero value of $Var(L|Y)$. Consider a portfolio of uniform credits with weights $w_1, w_2, \ldots, w_n$ and put:

$$\delta = \sum_{i=1}^{n} w_i^2$$

The conditional variance of the portfolio loss L given Y is:

$$Var(L|Y) = \delta a^2\left(N_2\left(U, U, \frac{(1-\rho)H}{T-\rho H}\right) - N^2(U)\right)$$

where:

$$U = b\sqrt{\frac{T}{T-\rho H}} - Y\sqrt{\frac{\rho H}{T-\rho H}}$$

The unconditional mean and variance of the portfolio loss are $E(L) = aN(b)$ and:

$$\begin{aligned} Var\, L &= E[Var(L|Y)] + Var[E(L|Y)] \\ &= \delta a^2 N_2\left(b, b, \frac{H}{T}\right) + (1-\delta)a^2 N_2\left(b, b, \frac{\rho H}{T}\right) - a^2 N^2(b) \end{aligned} \tag{10}$$

Taking the first two terms in the tetrachoric expansion of the bivariate normal distribution function $N_2(x, x, \rho) = N^2(x) + \rho n^2(x)$, where n is the normal density function, we have approximately:

$$\begin{aligned} Var(L) &= \delta a^2 n^2(b)\frac{H}{T} + (1-\delta)a^2 n^2(b)\frac{\rho H}{T} \\ &= a^2 N_2\left(b, b, (\rho + \delta(1-\rho))\frac{H}{T}\right) - a^2 N^2(b) \end{aligned}$$

Approximating the loan loss distribution by the distribution (5) with the same mean and variance, we get:

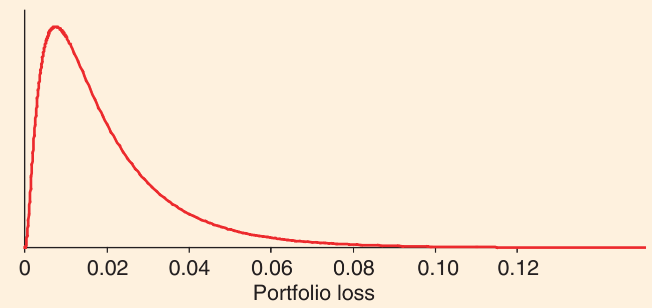
$$P[L \le x] = F\left(\frac{x}{a}; N(b), (\rho + \delta(1-\rho))\frac{H}{T}\right) \tag{11}$$

This expression is in fact exact for both extremes $n \to \infty$, $\delta = 0$ and $n = 1$, $\delta = 1$.

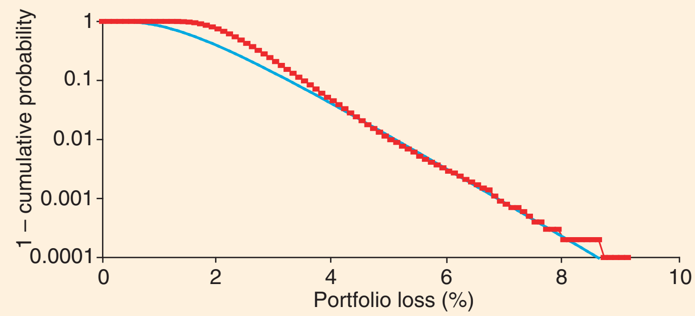
Equation (11) provides an adjustment for the 'granularity' of the portfolio. In particular, the finite portfolio adjustment to the distribution of the gross loss at the maturity date is obtained by putting $H = T$, $a = 1$ to yield:

$$P[L \le x] = F(x; p, \rho + \delta(1-\rho)) \tag{12}$$

1. Portfolio loss distribution (p = 0.02, ρ = 0.1)



2. Simulated loss distribution for an actual portfolio



Summary

We have shown that the distribution of the loan portfolio loss converges, with increasing portfolio size, to the limiting type given by equation (5). It means that this distribution can be used to represent the loan loss behaviour of large portfolios. The loan loss can be a realised loss on loans maturing before the horizon date, or a market value deficiency on loans whose term is longer than the horizon period.

The limiting probability distribution of portfolio losses has been derived under the assumption that all loans in the portfolio have the same maturity, the same probability of default and the same pair-wise correlation of the borrower assets. Curiously, however, computer simulations show that the family (5) appears to provide a reasonably good fit to the tail of the loss distribution for more general portfolios. To illustrate this point, figure 2 gives the results of Monte Carlo simulations of an actual bank portfolio. The portfolio consisted of 479 loans in amounts ranging from 0.0002% to 8.7%, with $\delta = 0.039$. The maturities ranged from six months to six years and the default probabilities from 0.0002 to 0.064. The loss-given default averaged 0.54. The asset returns were generated with 14 common factors. The plot shows the simulated cumulative distribution function of the loss in one year (dots) and the fitted limiting distribution function (solid line). ■

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