# Review Session 4

July 16, 2015

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### 6.6.2 No-Greek hedging

This exercise is discussed in greater depth and placed into a broader context in Meucci (2010c), see also Meucci (2010b), both freely available online at ssrn.com.

Consider the market of call options on the S&P 500 described in Exercise 5.6, namely call options on the S&P 500, with current time to maturity of 100, 150, 200, 250, and 300 days and strikes equal 850, 880, 910, 940, and 970 respectively.

Consider the time series of the underlying and the implied volatility surface provided in DB_ImplVol. Fit a joint normal distribution to the weekly invariants, namely the log-changes in the underlying and ~~the residuals from a vector autoregression of order one in~~ the log-changes in the implied volatilities surface $\boldsymbol{\sigma}_t$.

$$\begin{pmatrix} \ln S_{t+\tau} - \ln S_t \\ \ln \boldsymbol{\sigma}_{t+\tau} - \ln \boldsymbol{\sigma}_t \end{pmatrix} \sim \mathrm{N}\left(\tau\boldsymbol{\mu}, \tau\boldsymbol{\Sigma}\right) \tag{429}$$

Assume that the investment horizon is 8 weeks. We want to represent the linear returns on the options $\mathbf{R}_C$ in terms of the linear returns $R$ of the underlying S&P 500 by means of a linear model

$$\mathbf{R}_C \equiv \mathbf{a} + \mathbf{b}R + \mathbf{U}. \tag{430}$$

Notice that the specification (430) is the interpretation side of a "factors on demand" model.

Generate joint simulations for $\mathbf{R}_C$ and $R$ as in Exercise 5.6 and scatter-plot the results. Then compute $\mathbf{a}$ and $\mathbf{b}$ by OLS.

Compute the cash and underlying amounts necessary to hedge $\mathbf{R}_C$ based on the delta of the Black-Scholes formula and compare with $\mathbf{a}$ and $\mathbf{b}$.

Repeat the above exercise when the investment horizon shifts further or closer in the future.

PORTFOLIO STOCK (UNDERLYING) + CALL

$\downarrow$ holdings $h_s$ $\qquad$ $\downarrow$ holdings $h_c$

$$V_t = h_s S_t + h_c C_t$$

$h_c = 1 \quad \rightarrow$ 1 CALL

DELTA HEDGING : $\dfrac{\partial V_t}{\partial S_t} = 0$

$$\frac{\partial V_t}{\partial S_t} = h_s + \frac{\partial C_t}{\partial S_t} \overset{!}{=} 0 \quad \rightarrow \quad h_s = -\overbrace{\frac{\partial C_t}{\partial S_t}}^{\Delta}$$

$\rightarrow$ PORTFOLIO : $V_t = C_t - \Delta S_t$

$$\begin{cases} dV_t = dC_t - \Delta dS_t & \rightarrow \text{SELF FINANCING} \\ dV_t = r V_t dt & \rightarrow \text{HEDGED PTF IS LOCALLY RISK FREE} \end{cases}$$

$\downarrow$

$$dC_t - \Delta dS_t = r \cdot (C_t - \Delta S_t) dt$$

$$\underbrace{\frac{\delta C_t}{C_t}}_{R^c} \approx \underbrace{\frac{r(C_t - \Delta S_t)\,\delta t}{C_t}}_{a} + \underbrace{\frac{\Delta}{C_t} S_t}_{b} \underbrace{\frac{\delta S_t}{S_t}}_{R^s}$$

$R^c \approx a + b R^s \quad \rightarrow$ compare with $\hat{a}, \hat{b}$ OLS

### 7.4.1 VaR in elliptical markets

Consider an $N$-dimensional market that as in (2.144) in Meucci (2005) is uniformly distributed on an ellipsoid (surface and internal points):

$$\mathbf{M} \sim \mathrm{U}\left(\mathcal{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}\right). \tag{456}$$

Write the quantile index $\mathrm{Q}_c(\boldsymbol{\alpha})$ of the objective (5.10) as defined in (5.159) in Meucci (2005) as a function of the allocation.

Use the above results to factor $\mathrm{Q}_c(\boldsymbol{\alpha})$ in terms of its marginal contributions.
**Hint**. Compare with (5.189) in Meucci (2005).

Consider the case $N \equiv 3$. Generate randomly the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Generate a sample of $J \equiv 1{,}000$ simulations of the market (456).

Generate a random allocation vector $\boldsymbol{\alpha}$. Set $c \equiv 0.95$ and compute $\mathrm{Q}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.159) in Meucci (2005).

Compute the marginal contributions to $\mathrm{Q}_c(\boldsymbol{\alpha})$ from each security in terms of the empirical derivative of $\mathrm{Q}_c(\boldsymbol{\alpha})$:

$$\frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\alpha_n} \approx \frac{\mathrm{Q}_c\left(\boldsymbol{\alpha}+\epsilon\boldsymbol{\delta}^{(n)}\right)-\mathrm{Q}_c(\boldsymbol{\alpha})}{\epsilon}, \tag{467}$$

where $\mathrm{Q}_c(\mathbf{x})$ is calculated as in the previous point; $\boldsymbol{\delta}^{(n)}$ is the Kronecker delta ($A$.15) in Meucci (2005); and $\epsilon$ is a small number, as compared with the average size of the entries of $\boldsymbol{\alpha}$.

Display the result using the built-in plotting function bar.

Use the result above to compute $\mathrm{Q}_c(\boldsymbol{\alpha})$ in a different way, i.e. semi-analytically.

**Hint**. You will have to compute the quantile of the standardized univariate generator, use the simulations generated above.

Use the previous results to compute the marginal contributions to $\mathrm{Q}_c(\boldsymbol{\alpha})$ from each security. Display the result using the built-in plotting function bar.

N dim. market $\quad M \sim U(\mathcal{E}_{\mu,\Sigma})$

objective $\quad \Psi_\alpha = \alpha' M$

Quantile based index of satisfaction : $Q_c(\alpha) = Q_{\Psi_\alpha}(1-c)$

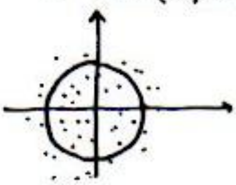
$(= -\mathrm{Var}_c(\alpha))$

<u>Recall</u> $\quad M \sim El(\mu, \Sigma, g)$

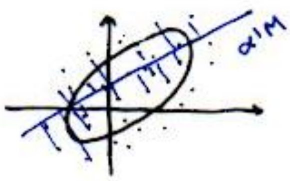
$\alpha' M \sim El(\alpha'\mu, \alpha'\Sigma\alpha, \tilde{g})$

e.g. 2 Dimensions

$Y \sim El(0, I, g_N)$



$\longrightarrow \quad M = \mu + AY$ where $AA' = \Sigma \quad \sim El(\mu, \Sigma, g_N)$




$\alpha' M \sim El(\alpha'\mu, \alpha'\Sigma\alpha, \tilde{g}_1)$

In this exercise

$$M = \mu + AY \quad \text{where } Y \sim \text{Unif (unit ball)}$$

- Quantile index of satisf. for elliptical R.V.

$$\Psi_\alpha = \alpha' M \sim El\left(\alpha'\mu, \alpha'\Sigma\alpha, \tilde{g}\right)$$

$$Q_c(\alpha) = Q_{\Psi_\alpha}(1-c) = Q_{\alpha' M}(1-c) = Q_{\alpha'\mu + \sqrt{\alpha'\Sigma\alpha}\, Y}(1-c)$$

$$= \alpha'\mu + \sqrt{\alpha'\Sigma\alpha} \cdot \underbrace{Q_Y(1-c)}_{\text{indep of } \alpha \text{ (evaluated numerically)}}$$

- TRANSLATION INVARIANCE
- POSITIVE HOMOGENEITY

- MARGINAL CONTRIBUTIONS:

$$Q_c(\alpha) = \alpha'\mu + \sqrt{\alpha'\Sigma\alpha}\, Q_Y(1-c)$$

$$= \alpha' \underbrace{\left(\mu + \frac{\Sigma\alpha}{\sqrt{\alpha'\Sigma\alpha}} Q_Y(1-c)\right)}_{\frac{\partial Q_c(\alpha)}{\partial \alpha}}$$

$\frac{\partial Q_c(\alpha)}{\partial \alpha}$ : MARGINAL CONTRIBUTIONS

$$= \sum_{n=1}^{N} \alpha_n \cdot \frac{\partial Q_c(\alpha)}{\partial \alpha_n} = \sum_{n=1}^{N} c_n$$

- consider $N=3$; generate randomly $\mu, \Sigma$

① Generate scenarios for the market

(*) → SIMULATE DATA UNIF. DISTRIB in the unit Ball (?)

→ TRANSFORM THE SAMPLE AS $M^{(j)} = \mu + AY^{(j)}$ where $AA' = \Sigma$

② AGGREGATE THE SAMPLE by an arbitrary allocation $\alpha$

$$\psi_\alpha^{(j)} = \alpha' \cdot M^{(j)}$$

③ compute $Q_c(\alpha) = Q_{\psi_\alpha}(1-c)$ as sample quantile

④ Repeat ②-③ for perturbed allocations

$$\alpha_\varepsilon = \alpha + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \varepsilon \\ 0 \\ 0 \end{pmatrix} \xrightarrow{②,③} Q_{\psi_{\alpha_\varepsilon}}(1-c)$$

⑤ calculate numerically the marg. contributions as

$$\frac{\partial Q}{\partial \alpha} \approx \frac{1}{\varepsilon}\left(Q_{\psi_{\alpha_\varepsilon}} - Q_{\psi_\alpha}\right)$$

⑥ Compare with the analytical marg. contrib. relationship

(*) HOW TO SIMULATE a sample from a r.v. uniformly distributed inside the unit ball?

e.g. $N=2$ (it holds for any $N$)

$$(X_1, X_2) = (R\theta_1, R\theta_2)$$

① Simulate $(Y_1, Y_2) \sim N(0, I)$

$\rightarrow (Y_1^{(J)}, Y_2^{(J)}) \quad J = 1 \dots J$

② $(\theta_1, \theta_2) = \left( \frac{Y_1}{\|Y\|}, \frac{Y_2}{\|Y\|} \right)$ $\rightarrow$ $(\theta_1, \theta_2)$ UNIF ON THE BORDER OF THE CIRCLE

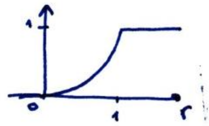
$\rightarrow \left(\theta_1^{(J)}, \theta_2^{(J)}\right) = \left( \frac{Y_1^{(J)}}{\|Y^{(J)}\|}, \frac{Y_2^{(J)}}{\|Y^{(J)}\|} \right) \quad J = 1 \dots J$

$\|Y^{(J)}\| = \sqrt{(Y_1^{(J)})^2 + (Y_2^{(J)})^2}$

③ Generate $R$ with cdf, given by $F_R(r) = \begin{cases} 0 & r < 0 \\ r^2 & 0 \leq r \leq 1 \\ 1 & r > 1 \end{cases}$

($R$ is such that the density is prop. to the radius $(N=2)$)

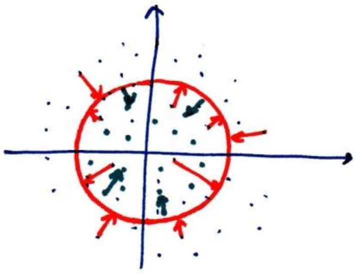
$\rightarrow Q_R(p) = \sqrt{p}$




$\downarrow$

$(R\theta_1, R\theta_2) \sim \text{Unif}(\mathcal{E}_{0,I})$

UNIF DIST. INSIDE THE CIRCLE



### 7.4.2 Cornish-Fisher approximation of VaR

Assume that the investor's objective is lognormally distributed:

$$\Psi_\alpha \sim \text{LogN}\left(\mu_\alpha, \sigma_\alpha^2\right), \tag{478}$$

where $\mu_\alpha \equiv 0.05$ and $\sigma_\alpha \equiv 0.05$.

Plot the true quantile-based index of satisfaction $Q_c(\alpha)$ against the Cornish-Fisher approximation (5.179) in Meucci (2005) as a function of the confidence level $c \in (0, 1)$.

CORNISH - FISHER APPROXIMATION

$$Q_X(p) \cong E(X) + Sd(X)\left[\underbrace{z(p)}_{\text{standard normal quantile}} + \frac{1}{6}\left(z^2(p)-1\right) Sk(X)\right]$$

• $\Psi_\alpha \sim \log N(\mu_\alpha, \sigma_\alpha^2)$ $\quad\quad \mu_\alpha = 0.05$, $\sigma_\alpha = 0.05$

→ CF approx of $Q_c(\alpha) = Q_{\Psi_\alpha}(1-c) \cong E[\Psi_\alpha] + Sd[\Psi_\alpha]\left(z(p) + \frac{1}{6}(z^2(p)-1)\, Sk(\Psi_\alpha)\right)$

$$E[\Psi_\alpha] = \exp\left(\mu_\alpha + \frac{\sigma_\alpha^2}{2}\right)$$

$$Sd[\Psi_\alpha] = \exp\left(\mu_\alpha + \frac{\sigma_\alpha^2}{2}\right)\sqrt{\exp(\sigma_\alpha^2)-1}$$

$$Sk[\Psi_\alpha] = \sqrt{\exp(\sigma_\alpha^2)-1}\,\left(\exp(\sigma_\alpha^2)+2\right)$$

→ TRUE : $F^{-1}_{\log N(\mu_\alpha, \sigma_\alpha^2)}(1-c)$ $\quad\quad$ MATLAB: q = logninv(1-c, $\mu_\alpha$, $\sigma_\alpha$)

### 7.4.3 Extreme value theory approximation of VaR

Assume that the objective is $t$ distributed:

$$\Psi_\alpha \sim \mathrm{St}\left(\nu, \mu_\alpha, \sigma_\alpha^2\right), \tag{479}$$

where $\nu \equiv 7$, $\mu_\alpha \equiv 1$, $\sigma_\alpha^2 \equiv 4$.

Plot the true quantile-based index of satisfaction $\mathrm{Q}_c(\alpha)$ for $c \in [0.950, 0.999]$.

**Hint.** Use the built-in function `tinv`.

Generate Monte Carlo simulations from (479) and superimpose the plot of the sample counterpart of $\mathrm{Q}_c(\alpha)$ for $c \in [0.950, 0.999]$.

Consider the threshold:

$$\widetilde{\psi} \equiv \mathrm{Q}_{0.95}(\alpha). \tag{480}$$

Superimpose the plot of the EVT fit (5.186) in Meucci (2005) for $c \in [0.950, 0.999]$.

**Hint.** Estimate the parameters $\xi$ and $v$ using the built-in function `xi_v = gpfit(Excess)`, where Excess are the realizations of the random variable

$$Z \equiv \widetilde{\psi} - \Psi_\alpha | \Psi_\alpha \leq \widetilde{\psi}. \tag{481}$$

$\Psi_\alpha \sim St(\nu, \mu_\alpha, \sigma^2_\alpha)$ $\quad \nu = 7, \mu_\alpha = 1, \sigma^2_\alpha = 4$

→ TRUE $\quad Q_c(\alpha) = F^{-1}_{St(..)}(1-c)$ $\quad$ MATLAB $q = \underbrace{\text{tinv}}_{F^{-1}_{St(\nu,0,1)}}(1-c, \nu)\cdot\sigma + \mu$

→ SIMULATIONS

$X = \text{trnd}(\nu, \text{Nsim}, 1)$

$\Psi_\alpha = \mu + \sigma X$

$q_{sim} = \text{prctile}(\Psi_\alpha, (1-c)\cdot 100)$ $\quad$ → order statistic

→ EVT Approximation

$$\widetilde{\Psi} = Q_{0.95}(\alpha) \left(\equiv Q_{\Psi_\alpha}(0.05)\right) \quad \to \text{THRESHOLD}$$

$$Q_c(\alpha) \cong \widetilde{\Psi} + \frac{\upsilon(\alpha)}{\xi(\alpha)}\left[1 - \left(\frac{1-c}{F_{\Psi_\alpha}(\widetilde{\Psi})}\right)^{-\xi(\alpha)}\right]$$

$\upsilon(\alpha)$ and $\xi(\alpha)$ → PARAMETERS of a GENERALIZED PARETO fitted to $Z = \widetilde{\Psi} - \Psi_\alpha \mid \Psi_\alpha \leq \widetilde{\Psi}$

gpfit(Excess) $\quad$ EXCESS

$$P(Z \leq z) = P(\widetilde{\Psi} - \Psi_\alpha \leq z \mid \Psi_\alpha \leq \widetilde{\Psi})$$

$$= 1 - \underbrace{P\left(\Psi_\alpha \leq \widetilde{\Psi} - z \mid \Psi_\alpha \leq \widetilde{\Psi}\right)}_{L_{\widetilde{\Psi}}(z) = \frac{F_{\Psi_\alpha}(\widetilde{\Psi}-z)}{F_{\Psi_\alpha}(\widetilde{\Psi})}} \approx G_{\xi,\upsilon}(z)$$

→ EXCESS FUNCTION OF $\Psi_\alpha$

### 7.5.1 Expected shortfall in elliptical markets

Assume that the market is multivariate $t$ distributed:

$$\mathbf{M} \sim \mathrm{St}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{484}$$

Write the expected shortfall $\mathrm{ES}_c(\boldsymbol{\alpha})$ defined in (5.207) in Meucci (2005) as a function of the allocation.

Use the previous results to factor the $\mathrm{ES}_c(\boldsymbol{\alpha})$ in terms of its marginal contributions.

**Hint**. Compare with (5.236) in Meucci (2005).

Assume $N \equiv 40$ and $\nu \equiv 5$. Generate randomly the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the allocation $\boldsymbol{\alpha}$. Then generate $J \equiv 10,000$ Monte Carlo scenarios from the market distribution (484).

Generate a random allocation vector $\boldsymbol{\alpha}$. Set $c \equiv 0.95$ and compute $\mathrm{ES}_c(\boldsymbol{\alpha})$ as the sample counterpart of (5.208) in Meucci (2005).

Compute the marginal contributions to $\mathrm{ES}_c(\boldsymbol{\alpha})$ from each security as the sample counterpart of (5.238) in Meucci (2005). Display the result in a subplot using the built-in plotting function `bar`.

Use the previous results to compute $\mathrm{ES}_c(\boldsymbol{\alpha})$ in a different way, i.e. semi-analytically. Never at any stage use simulations.

**Hint**. Use the numerical integration function `quad` applied to the built-in quantile function `tinv`.

Compute the marginal contributions to $\mathrm{ES}_c(\boldsymbol{\alpha})$ from each security using previous results. Never at any stage use simulations. Display the result in a second subplot using the built-in plotting function `bar`.

$$M \sim St(\nu, \mu, \Sigma)$$

continuous R.V. TCE / CVaR

$$ES_c(\alpha) = \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_\alpha}(s)\, ds = E\left[\Psi_\alpha \mid \Psi_\alpha \le Q_{\Psi_\alpha}(1-c)\right]$$

$$\Psi_\alpha \sim St(\nu, \alpha'\mu, \alpha'\Sigma\alpha) \qquad \Psi_\alpha \overset{d}{=} \alpha'\mu + \sqrt{\alpha'\Sigma\alpha} \cdot X$$

$$= \alpha' M \qquad X \sim St(\nu, 0, 1)$$

$$ES_c(\alpha) = \frac{1}{1-c} \int_0^{1-c} Q_{\alpha'\mu + \sqrt{\alpha'\Sigma\alpha}\, X}(s)\, ds =$$

$$= \frac{1}{1-c} \int_0^{1-c} \alpha'\mu + \sqrt{\alpha'\Sigma\alpha}\, Q_X(s)\, ds =$$

$$= \alpha'\mu + \sqrt{\alpha'\Sigma\alpha} \cdot \boxed{\frac{1}{1-c}\int_0^{1-c} Q_X(s)\, ds}$$

$ES_c(X)$, indep of $\alpha$ evaluated numerically

MARGINAL CONTRIBUTIONS:

$$ES_c(\alpha) = \alpha'\mu + \sqrt{\alpha'\Sigma\alpha} \cdot ES_c(X)$$

$$= \alpha' \underbrace{\left(\mu + \frac{\Sigma\alpha}{\sqrt{\alpha'\Sigma\alpha}} \cdot ES_c(X)\right)}_{\frac{\partial ES_c(\alpha)}{\partial \alpha}}$$ ← analytical

$$= \sum_{n=1}^{N} \alpha_n \cdot \frac{\partial ES_c(\alpha)}{\partial \alpha_n} = \sum_{n=1}^{N} c_i$$

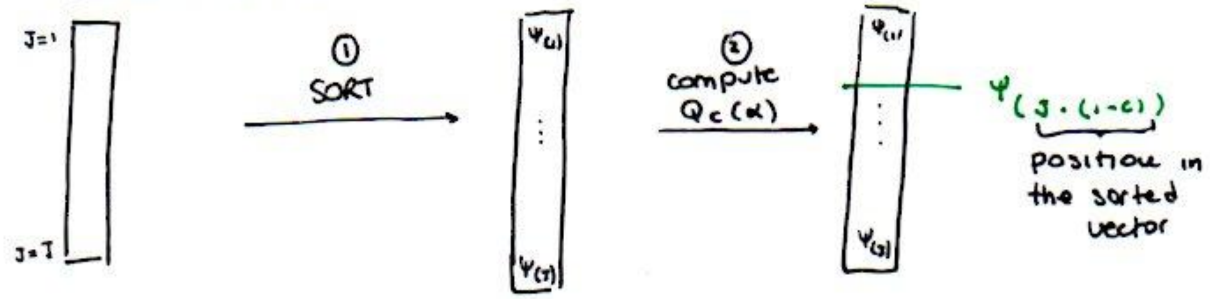
using simulations:

$$\frac{\partial ES_c(\alpha)}{\partial \alpha_n} = E\left[ M \mid \Psi_\alpha \leq Q_c(\alpha) \right]$$

↑ take the sample counterpart.

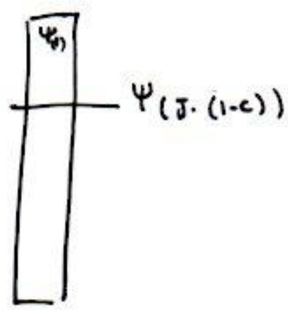
$ES_c(\alpha) = E\left[\Psi_\alpha \mid \Psi_\alpha \leq Q_c(\alpha)\right]$

scenarios for $\Psi_\alpha$




$\Psi_{(1)} \leq \dots \leq \Psi_{(J)}$

③ → take the sample mean of the $\Psi_{(i)} \leq Q_c(\alpha)$




} sample mean $= \widehat{ES}_c(\alpha)$