Solution for Coupled First Order Differential Equations

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1 Vector Method

Having the following system of differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = cx_1 + dx_2$$

It can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

The 2x2 matrix would be the characteristic matrix known as **M**. The roots of the equation can be found with:

$$\left|\mathbf{M} - r\mathbf{I}\right| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = 0 \tag{2}$$

$$(a-r)(d-r) - bc = 0$$
$$r^2 - (a+d)r + ad - bc = 0$$

$$r_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \tag{3}$$

1.1 General Solutions

As long as the square root is greater or equal to zero then the general solution is:

$$\mathbf{x} = \mathbf{u}_1 e^{r_1 t} + \mathbf{u}_2 e^{r_2 t} \tag{4}$$

However if the root is complex $\alpha + \beta i$:

$$\mathbf{x} = \mathbf{u}_1 e^{\alpha_1 t}(\cos\beta_1 t + i\sin\beta_1 t) + \mathbf{u}_2 e^{\alpha_2 t}(\cos\beta_2 t + i\sin\beta_2 t) \tag{5}$$

The root will only be complex if the square root is negative. Therefore, let's verify when this happens:

$$\begin{aligned}(a+d)^2 - 4(ad-bc) &< 0\\ (a-d)^2 + 4bc &< 0\\ bc &> \frac{(a-d)^2}{4}\end{aligned}$$

If b or c is negative and it fulfills the former equation then the roots will be complex.

1.2 Roots of Equation

Once then roots (eigevalues) are known, we need to find the eigenvectors U_i. Substituting the roots to equation 1:

$$\begin{pmatrix} a-r_1 & b \\ c & d-r_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} a-r_2 & b \\ c & d-r_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

You can verify if the matrix is correct if it looks like this:

$$\begin{pmatrix} p & q \\ p & q \end{pmatrix}$$

in such a way that:

$$(a-r_1)u_1 + bu_2 = cu_1 + (d-r_1)u_2$$

Now to solve for the eigenvectors using arbitrary number (s_i):

$$
\begin{aligned}
(a - r_1)u_1 + bu_2 &= 0 \\
u_2 &= s_1 \\
u_1 &= -s_1 \frac{b}{a - r_1} \\
\vec{u}_1 &= s_1 \begin{pmatrix} -\frac{b}{a-r_1} \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(a - r_2)u_1 + bu_2 &= 0 \\
u_1 &= s_2 \\
u_2 &= -s_2 \frac{a - r_2}{b} \\
\vec{u}_2 &= s_2 \begin{pmatrix} 1 \\ -\frac{a-r_2}{b} \end{pmatrix}
\end{aligned}
$$

The General Solution wil be:

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s_1 \begin{pmatrix} -\frac{b}{a-r_1} \\ 1 \end{pmatrix} e^{r_1 t} + s_2 \begin{pmatrix} 1 \\ -\frac{a-r_2}{b} \end{pmatrix} e^{r_2 t} \tag{6}
$$

2 Substitution method

Using the same system of differential equations We use the first equation to substitute x_2 of the second equation:

$$
x_2 = \frac{1}{b} \left(\frac{\mathrm{d}x_1}{\mathrm{d}t} - ax_1 \right) \tag{7}
$$

$$
\begin{aligned}
\frac{1}{b} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x_1}{\mathrm{d}t} - ax_1 \right) &= cx_1 + \frac{d}{b} \left(\frac{\mathrm{d}x_1}{\mathrm{d}t} - ax_1 \right) \\
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} - (a + d) \frac{\mathrm{d}x_1}{\mathrm{d}t} + (ad - bc)x_1 &= 0 \\
r^2 - (a + d)r + (ad - bc) &= 0
\end{aligned}
$$

Notice the quadratic formula of this equation is identical to equation 1. wrong reference

$$\begin{aligned} x_1 &= s_1 e^{r_1 t} + s_2 e^{r_2 t} \\ \frac{\mathrm{d}x_1}{\mathrm{d}t} &= s_1 r_1 e^{r_1 t} + s_2 r_2 e^{r_2 t} \end{aligned}$$

Substituting in equation 2 the general solution of x_2 can be derivated: wrong reference

$$\begin{aligned} x_2 &= \frac{1}{b}\left[s_1 r_1 e^{r_1 t} + s_2 r_2 e^{r_2 t} - s_1 e^{r_1 t} - s_2 e^{r_2 t}\right] \\ &= s_1 \frac{r_1 - 1}{b} e^{r_1 t} + s_2 \frac{r_2 - 1}{b} e^{r_2 t} \end{aligned}$$

So the general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ \frac{r_1 - 1}{b} \end{pmatrix} e^{r_1 t} + s_2 \begin{pmatrix} 1 \\ \frac{r_2 - 1}{b} \end{pmatrix} e^{r_2 t} \tag{8}$$

3 Example

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \tag{10}$$

4 Conclusion

So this two should give the same result. However, I suggest the Vector Method instead of Substitution as it's simpler and less prone to mistakes.

References

[1] University of Bristol. *Continuous dynamical systemst. coupled first order differential equations*. type. University of Bristol.

[2] Nagle Saf Snider. *Fundamentals of Differential Equations*. Vol. 8. Reading, Massachusetts: Pearson, 1993.