Solution for Coupled First Order Differential Equations

Edvin Alvarado Velez

April 10, 2019

1 Vector Method

Having the following system of differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = cx_1 + dx_2$$

It can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

The 2x2 matrix would be the characteristic matrix known as $\mathbf{M}$. The roots of the equation can be found with:

$$\left|\mathbf{M} - r\mathbf{I}\right| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix} = 0 \tag{2}$$

$$(a - r)(d - r) - bc = 0$$
$$r^2 - (a + d)r + ad - bc = 0$$

$$r_{1,2} = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} \tag{3}$$

1.1 General Solutions

As long as the square root is greater or equal to zero then the general solution is:

$$\mathbf{x} = \mathbf{u}_1 e^{r_1 t} + \mathbf{u}_2 e^{r_2 t} \tag{4}$$

However if the root is complex $\alpha + \beta i$:

$$\mathbf{x} = \mathbf{u}_1 e^{\alpha_1 t}(\cos\beta_1 t + i\sin\beta_1 t) + \mathbf{u}_2 e^{\alpha_2 t}(\cos\beta_2 t + i\sin\beta_2 t) \tag{5}$$

The root will only be complex if the square root is negative. Therefore, let's verify when this happens:

$$(a+d)^2 - 4(ad-bc) < 0$$

A 3D Model can be done by joining the symmetric independent variables such as b with c and a with d. They're symmetric or conjoined in such a way that two random values can be assgined to each element to a pair and it would give the same result as if you had assigned the values in reverse. In mathematical notation:

$$f(a,b,c,d) = (a+d)^2 - 4(ad-bc) \tag{6}$$

$$f(a,x_0,x_1,d) = f(a,x_1,x_0,d) \tag{7}$$

This is also true for the pair a and d. This is thanks the variables being multiplied and/or being inside a even exponential. So the model is the following:

$$\begin{aligned} A &= ad \\ B &= bc \end{aligned}$$

$$\begin{aligned} f(a,b,c,d) &= (a+d)^2 - 4(ad-bc) \\ &= \left(\frac{ad}{d} + d\right)^2 - 4(ad-bc) \\ f(A,B,d) &= \left(\frac{A}{d} + d\right)^2 - 4(A-B) \end{aligned}$$

Although d couldn't be fully eliminated, as A includes variable d, The 3D model should still accurately portray the function of the square root.

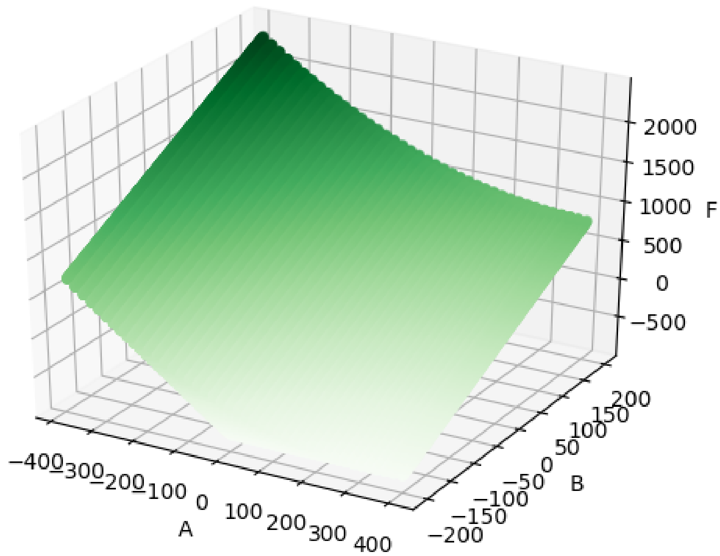

Figure 1: Python Model

1.2 Roots of Equation

Once then roots (eigevalues) are known, we need to find the eigenvectors U_i. Substituting the roots to equation 1:

$$\begin{pmatrix} a-r_1 & b \\ c & d-r_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} a-r_2 & b \\ c & d-r_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

You can verify if the matrix is correct if it looks like this:

$$\begin{pmatrix} p & q \\ p & q \end{pmatrix}$$

in such a way that:

$$(a-r_1)u_1 + bu_2 = cu_1 + (d-r_1)u_2$$

Now to solve for the eigenvectors using arbitrary number (s_i):

$$\begin{aligned}(a-r_1)u_1+bu_2&=0\\ u_2&=s_1\\ u_1&=-s_1\frac{b}{a-r_1}\\ \vec{u}_1&=s_1\begin{pmatrix}-\frac{b}{a-r_1}\\ 1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}(a-r_2)u_1+bu_2&=0\\ u_1&=s_2\\ u_2&=-s_2\frac{a-r_2}{b}\\ \vec{u}_2&=s_2\begin{pmatrix}1\\ -\frac{a-r_2}{b}\end{pmatrix}\end{aligned}$$

The General Solution wil be:

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=s_1\begin{pmatrix}-\frac{b}{a-r_1}\\ 1\end{pmatrix}e^{r_1t}+s_2\begin{pmatrix}1\\ -\frac{a-r_2}{b}\end{pmatrix}e^{r_2t} \tag{8}$$

2 Substitution method

Using the same system of differential equations We use the first equation to substitute x_2 of the second equation:

$$x_2=\frac{1}{b}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}-ax_1\right) \tag{9}$$

$$\begin{aligned}\frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}-ax_1\right)&=cx_1+\frac{d}{b}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}-ax_1\right)\\ \frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}-(a+d)\frac{\mathrm{d}x_1}{\mathrm{d}t}+(ad-bc)x_1&=0\\ r^2-(a+d)r+(ad-bc)&=0\end{aligned}$$

Notice the quadratic formula of this equation is identical to equation 1. wrong reference

$$x_1 = s_1 e^{r_1 t} + s_2 e^{r_2 t}$$
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = s_1 r_1 e^{r_1 t} + s_2 r_2 e^{r_2 t}$$

Substituting in equation 2 the general solution of x_2 can be derivated: wrong reference

$$\begin{aligned} x_2 &= \frac{1}{b}\left[s_1 r_1 e^{r_1 t} + s_2 r_2 e^{r_2 t} - s_1 e^{r_1 t} - s_2 e^{r_2 t}\right] \\ &= s_1 \frac{r_1 - 1}{b} e^{r_1 t} + s_2 \frac{r_2 - 1}{b} e^{r_2 t} \end{aligned}$$

So the general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ \frac{r_1 - 1}{b} \end{pmatrix} e^{r_1 t} + s_2 \begin{pmatrix} 1 \\ \frac{r_2 - 1}{b} \end{pmatrix} e^{r_2 t} \tag{10}$$

3 Example

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \tag{12}$$

4 Conclusion

So this two should give the same result. However, I suggest the Vector Method instead of Substitution as it's simpler and less prone to mistakes.