MAT-MEK4270 - Assignment 1

Edvin Jarve

October 2023

Problem 1.2.3

First, we are given that the imaginary solution of the wave equation is given as

$$u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)}. \tag{1}$$

We will now check that if we put this into the wave equation, which is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{2}$$

that 1 indeed satisfied the wave equation. First, we clearly see that:

$$\frac{\partial^2 u}{\partial t^2} u(t,x,y) = -\omega^2 u(t,x,y), \tag{3}$$

and

$$\nabla^2 u(t,x,y) = -(k_x^2 + k_y^2) u(t,x,y). \tag{4}$$

Putting all of this into the wave equation 2 yields

$$\omega^2 u(t,x,y) = c^2 (k_x^2 + k_y^2) u(t,x,y) \tag{5}$$

which is indeed satisfied when $\omega = \pm c\sqrt{k_x^2 + k_y^2}$, which is the famous dispersion relation.

Problem 1.2.4

Assuming that $m_x = m_y$ and $k_x = k_y = k$ a discrete version of 1 will read

$$u_{ij}^n = e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \tag{6}$$

where $\tilde{\omega}$ is a numerical approximation of ω. We will now calculate what happens when we put equation 6 in

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right), \tag{7}$$

which is the discretized central difference in both time and space. First, we calculate the right hand side.

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} = \frac{1}{\Delta t^2}\left(e^{\imath(kh(i+j)-\tilde{\omega}(n+1)\Delta t)} - 2e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)} + e^{\imath(kh(i+j)-\tilde{\omega}(n-1)\Delta t)}\right) \tag{8}$$

which simplifies to

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} = \frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}}{\Delta t^2}\left(e^{\imath\tilde{\omega}\Delta t} - 2 + e^{-\imath\tilde{\omega}\Delta t}\right). \tag{9}$$

Calculating the first term in the parenthesis on the left side yields

$$\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{h^2} = \frac{1}{h^2}\left(e^{\imath(kh(i+1+j)-\tilde{\omega}n\Delta t)} - 2e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)} + e^{\imath(kh(i+j-1)-\tilde{\omega}n\Delta t)}\right) \tag{10}$$

which simplifies to

$$\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{h^2} = \frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}\left(e^{\imath kh} - 2 + e^{-\imath kh}\right). \tag{11}$$

By inspection, we see that the second terms reduced to the exact same expression

$$\frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{h^2} = \frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}\left(e^{\imath kh} - 2 + e^{-\imath kh}\right). \tag{12}$$

Putting equations 9, 11 and 12 in equation 7 gives:

$$\frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}}{\Delta t^2}\left(e^{\imath\tilde{\omega}\Delta t} - 2 + e^{-\imath\tilde{\omega}\Delta t}\right) = \frac{2c^2}{h^2}\left(e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}\right)\left(e^{\imath kh} - 2 + e^{-\imath kh}\right) \tag{13}$$

which reduces to

$$e^{\imath\tilde{\omega}\Delta t} - 2 + e^{-\imath\tilde{\omega}\Delta t} = \frac{2\Delta t^2 c^2}{h^2}\left(e^{\imath kh} - 2 + e^{-\imath kh}\right) \tag{14}$$

Setting the CFL number $C = \frac{\Delta t c}{h} = \frac{1}{\sqrt{2}}$ reduces to

$$e^{\imath\tilde{\omega}\Delta t} + e^{-\imath\tilde{\omega}\Delta t} = e^{\imath kh} + e^{-\imath kh}. \tag{15}$$

Clearly,

$$\tilde{\omega}\Delta t = kh \tag{16}$$

using that

$$\omega = c\sqrt{k_x^2 + k_y^2} = c\sqrt{2k^2} \Leftrightarrow k = \frac{\omega}{\sqrt{2}c} \tag{17}$$

gives

$$\tilde{\omega} = \frac{\omega h}{\sqrt{2}c\Delta t} = \omega \tag{18}$$

which indeed shows that $\tilde{\omega} = \omega$ when the CFL number $C = \frac{\Delta t c}{h} = \frac{1}{\sqrt{2}}$.