Paraméteres integrál

Egyismeretlenes

Pl: $f(a) = \int_0^{\frac{\pi}{2}} ln\frac{1+a\cos x}{1-a\cos x} * \frac{f}{\cos x}dx = ?; |a| < 1 \Rightarrow -1 < a < 1$
$f'(a) = \int_0^{\frac{\pi}{2}} (ln\frac{1+a\cos x}{1-a\cos x})'_a \frac{dx}{\cos x} = \cdots = \frac{\pi}{\sqrt{1-a^2}}$
$f(a) = \int f'(a)da = \int \frac{\pi}{\sqrt{1-a^2}} = \pi \arcsin a + C$ C-t még meg kell határozni
$f(0) = 0 \Rightarrow f(0) = \pi \arcsin 0 + C = - \Rightarrow C = 0 \Rightarrow f(a) = \pi \arcsin a$

Kétismeretlenes

Pl:

$f(a,b) = \int_0^{\infty} \frac{e^{-a^2x^2} - e^{-b^2x^2}}{x} dx = ?$

$\frac{\partial f}{\partial a} = \int_0^{\infty} (e^{-a^2x^2} - e^{b^2x^2})'_a \frac{dx}{x} = \cdots = -\frac{1}{a}$
$f(a,b) = \int \frac{\partial f}{\partial a} da = -\ln a + \underline{\mathbf{C(b)}} \leftarrow$ még ki kell számolni

$f(b,b) = 0 \Rightarrow f(b,b) = -\ln b + C(b) = 0 \Rightarrow C(b) = \ln b \Rightarrow f(a,b) = -\ln a + \ln b (*) = ln\frac{b}{a}$ *: C(b)

Inproprius integrál

$\lim_{b\to\infty} \int_a^b f(x)dx = L,$ ha L véges $\Rightarrow$ konvergens Jel: $L = \int_a^{\infty} f(x)dx$
$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t f(x)dx \leftarrow$ így kell kiszámolni
pl: $\int_1^{\infty} \frac{x}{(x^2+2)^2}dx = \lim_{t\to\infty} \int_1^t \frac{x}{(x^2+2)^2}dx = \lim_{t\to\infty} \frac{-1}{2(x^2+2)}\Big|_1^t = \frac{1}{6}$

Konvergencia kritérium: ha az integrál kiszámolása nélkül kell a konvergens/divergens
Ha $\lim_{x\to\infty} x^L * f(x) = l \quad , l \in \mathbb{R}^*$
$L > 1 \Rightarrow \int_1^{\infty} f(x)dx$ konvergens
$L \leq 1 \Rightarrow \int_1^{\infty} f(x)dx$ divergens

pl:

$f(x) = \frac{x}{(x^2+2)^2}$
$\lim_{x\to\infty} x^3 f(x) = \lim_{x\to\infty} \frac{x^4}{x^4+4x^2+4} = 1$
$L = 3 > 1 \Rightarrow f(x)$ konvergens

Dupla integrál

Pl:

$\iint\limits_D 4xy\, dx\, dy = ?$

$D : \begin{cases} y = x \\ y = \frac{1}{x} \\ x = 3 \end{cases}$

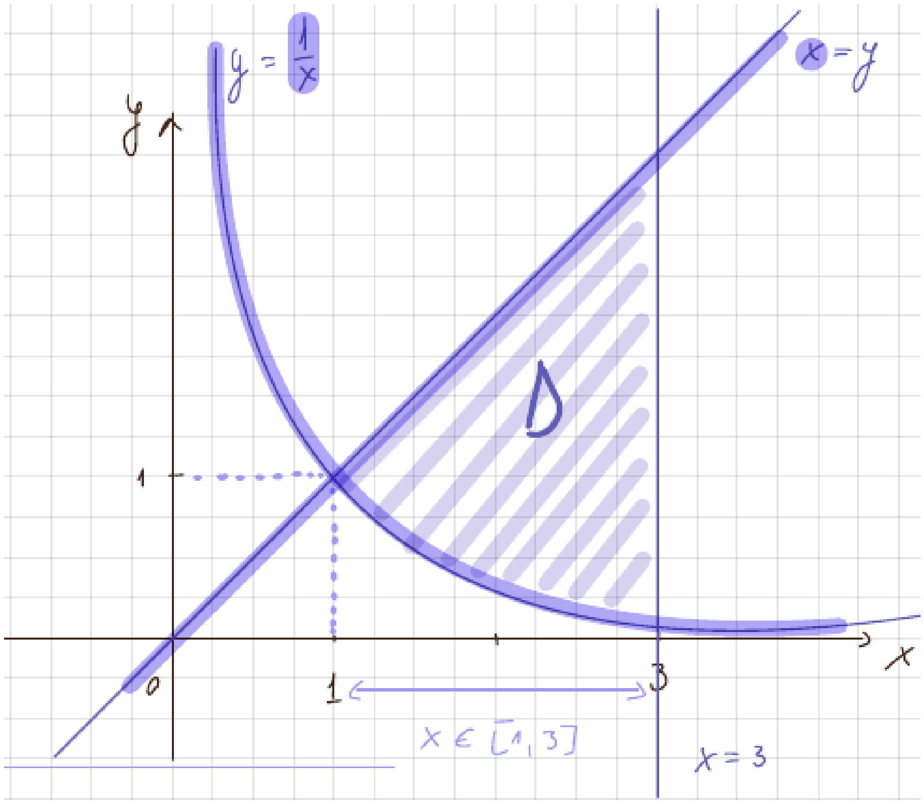


Figure 1: Dupla_integral.png

$\int_1^3 (\int_{\frac{1}{x}}^x 4xy \, dy) dx$

Pl:

$\iint\limits_D x^2 \sqrt{(x^2 + y^2)^3} \, dx \, dy$

$D : \begin{cases} x^2 + y^2 \leq 4 \\ y \geq 0 \end{cases}$

Polár koordináta

$\begin{cases} x = r \cos \Theta \\ y = r \sin \Theta \end{cases} \qquad , dx \, dy = \underline{r}(*) dr \, d\Theta \qquad , r \in [0, R]$

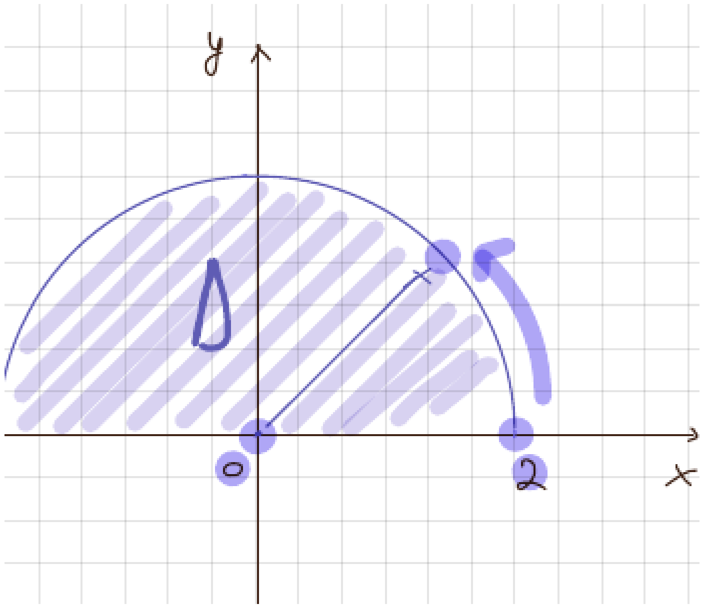


Figure 2: Dupla_integral2.png

*: det I

$$\iint\limits_{d} \underline{(r\cos\Theta)^2}(*)\sqrt{[\underline{(r\cos\Theta)^2}(*)+\underline{(r\sin\Theta)^2}(**)]^3}*\underline{r\,dr\,d\Theta}(***) = \int_0^2 (\int_0^2 r^6\cos^2\Theta\,d\Theta)dr$$

$\Theta \in [0,\pi]$

: x^2
: y^2
: dxdy

Tripla integrál

Gömbi koordináta

$$\begin{cases} x = r * \cos\varphi\sin\Theta \\ y = r * \sin\varphi\sin\Theta \qquad ,\det I = r^2 * \sin\Theta \quad ,r\in[0,R] \\ z = r*\cos\Theta \end{cases}$$

gömb: $\qquad x^2+y^2+z^2=R^2$
$R\in[0,r]$

Pl: $I = \iiint\limits_{C}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \iiint\sqrt{\underline{(r\cos\varphi\sin\Theta)^2}(*)+\underline{(r\sin\varphi\sin\Theta)^2}(**)+\underline{(r\cos\Theta)^2}(***)}\underline{r^2\sin\Theta}$
...
$[0,r]x[0,\frac{\pi}{2}]x[0,\frac{\pi}{2}]$

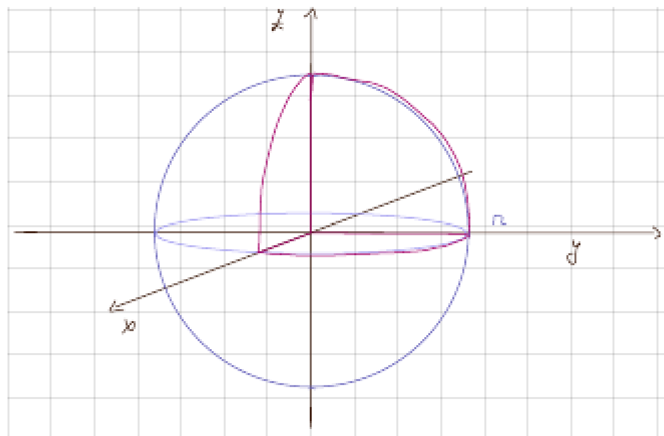


Figure 3: Tripla_Integral.png

: x^2
: y^2
: z^2
****: det I

Görbe ívhossza (görbe grafikonjának a hossza)

Ha $r: \qquad x = x_{(t)}, y = y_{(t)}, z = z_{(t)} \quad ; t \in [a, b]$
$L(r) = \int_a^b \sqrt{(x'_{(t)})^2 + (y'_{(t)})^2 + (z'_{(t)})^2}\, dt$

Ha

$f : [a, b] \to \mathbb{R} \quad f(x) = \dots$
$L(f) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Elsőfajú görbementi integrál

r - a görbe jele

$y = f(x) \qquad , x \in [a, b]$
$L(f) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

$\int_r f(x, y, z)\, \underline{ds} \leftarrow$ ívhossz elem
$ds = \sqrt{(x'_{(t)})^2 + (y'_{(t)})^2 + (z'_{(t)})^2}\, dt$

Pl:
$\int_r (x^2 + y^2) \ln z\, ds = ?$

$$r : \begin{cases} x = e^t \cos t \\ y = e^t \sin t \\ z = e^t \end{cases} \qquad t \in [0, 1]$$

$$\begin{aligned} x'_{(t)} &= e^t(\cos t - \sin t) \\ y'_{(t)} &= e^t(\sin t + \cos t) \Rightarrow ds = e^t\sqrt{3}dt \\ z'_{(t)} &= e^t \end{aligned}$$

$\int_0^1 [\underline{(e^t \cos t)^2}(*) + \underline{(e^t \sin t)^2}(**)]\underline{\ln e^t}(***) * \underline{e^t\sqrt{3}}(****)\, dt$

↑
Az integrál a és b-je a t a és b-je
: x^2
: y^2
: $\ln z$
****: ds

Másodfajú görbementi integrál

Ha függ az úttól

Pl:
$I = \int_r \underline{(y^2 - z^2)}(*)dx + \underline{2yz}(**)\, dy - \underline{x^2}(***)dz = ?$

: P
: Q
: R

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ Ha igazak függ az úttól, ha nem, akkor nem függ

$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

$$r : \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases} \quad , t \in [0, 1] \qquad \begin{aligned} dx &= x'_{(t)}dt = dt \\ dy &= y'_{(t)}dt = 2t\,dt \\ dz &= z'_{(t)}\,dt = 3t^2dt \end{aligned}$$

$I = \int_0^1 \underline{(t^4 - t^6)}(*)\underline{dt}(**) + \underline{2t^2 * t^3}(***)\underline{*2t\,dt}(****)\underline{-t^2}(*****)\underline{3t^2\,dt}(******) =$
$\dots$

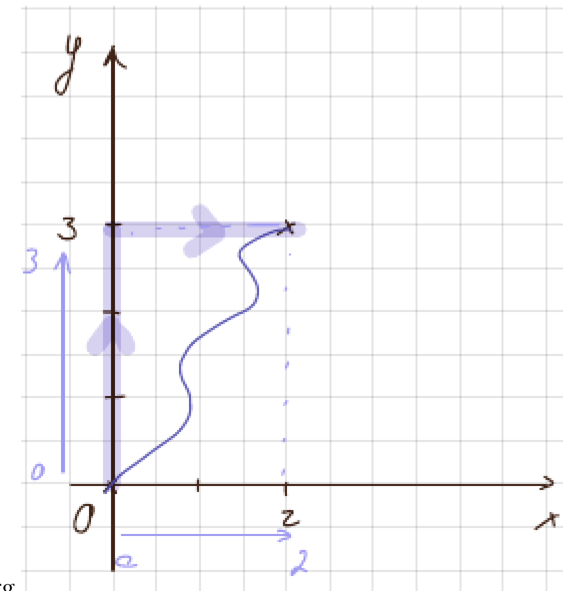
: $y^2 - z^2$
: dx
: 2yz
****: dy

*****: $-x^2$
******: dz

Ha nem függ az úttól

a. egyik módszer:

pl: $I = \int_{(0,0)}^{(2,3)} \underline{(x+y)}(*)dx + \underline{(x-y)}(**)dy = ?$
*: P
**: Q



$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial t} \Rightarrow 1 = 1$ igaz $\Rightarrow$ nem függ

$I = \int_0^3 \underline{0-y}(*)\, dy + \int_0^2 \underline{x+3}(**)\, dx$

*: függőlegesen 0-tól 3-ig, itt x = 0, mert 0-an van a függőleges
**: vízszintesen 0-tól 2-ig, itt y = 3, mert 3-ban van vízszintesen

b. másik módszer (ez könnyebb, főleg ha x,y,z van)

pl: $I = \int_{(1,1,1)}^{(2,3,-4)} \underline{x\, dx}(*) + \underline{y^2\, dy}(**) - \underline{z^3\, dz} = ?$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad 0 = 0$$
$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad 0 = 0 \Rightarrow \text{ nem függ}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad 0 = 0$$

$H(x,y,z) = \int_{x_0}^{x} P(i,y,z)\, du + \int_{y_0}^{y} Q(x_0,v,z)\, dv + \int_{z_0}^{z} R(x_0,y_0,s)\, ds = \frac{x^2}{2} - \frac{x_0^2}{2} + \frac{y^3}{3} - \frac{y_0^3}{3} - \frac{z^4}{4} + \frac{z_0^4}{4} = \frac{x^2}{2} + \frac{y^3}{3} - \frac{z^4}{4} + C$

$I = H(2,3,-4) - H(1,1,1) = \dots$

Green képlet (zárt görbék esetén)

$\int_r Pdx + Qdy = \iint\limits_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$

pl:
$\int_r \underline{(x+y)}(*)dx - \underline{(x-y)}(**)dy = ?$

$r : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad T_{ellipszis} = a * b * \pi$

$$\begin{cases} x = a * r * \cos\Theta \\ y = b * r * \cos\Theta \end{cases} \qquad r \in [0,1]$$

$\iint\limits_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = -2\underline{\iint\limits_D dxdy}(*) = -2\int_0^{25?}[\int_0^1(-abr)dr]d\Theta = \dots$

Felület darabterülete

$$\text{Ha S:}\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad ,(u,v) \in D$$

$E = (\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2 + (\frac{\partial z}{\partial u})^2$

$F = \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 z}{\partial u \partial v}$

$G = (\frac{\partial x}{\partial v})^2 + (\frac{\partial y}{\partial v})^2 + (\frac{\partial z}{\partial v})^2$

$T_{(S)} = \iint\limits_D \sqrt{EG - F^2}du\,dv \leftarrow$ felület területe

$$\text{Ha S: }\begin{cases} z = z(x,y) \\ x = u \\ y = z \end{cases} \qquad (x,y) \in D$$

$T_{(S)} = \iint\limits_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}dx\,dy \leftarrow$ felület területe

Elsőfajú felületi integrál

$f : D \to \mathbb{R} \qquad$ folytonos, $S \subset D \subset \mathbb{R}^3$

S:
$x = x(u,v)$
$y = y(u,v) \qquad (u,v) \in \Omega$
$z = z(u,v)$

$$\iint\limits_S f(x,y,z)d\nabla = \iint\limits_\Omega f(x_{(u,v)}, y_{(u,v)}, z_{(u,v)}) * \sqrt{EG-F^2}\,dudv$$

pl: $I = \iint\limits_S (x+y+z)d\nabla = ?$

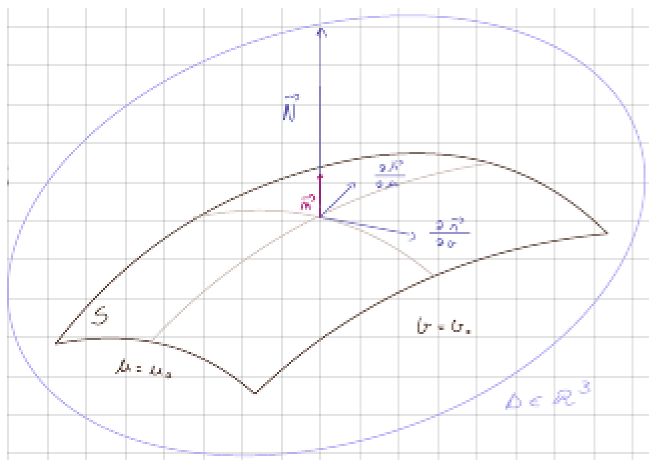
d$\gamma = \sqrt{EG-F^2}$ - felület területe

S: $x^2+y^2+z^2 = a^2, z \geq 0 \leftarrow$ gömb

$$\begin{cases} z = \sqrt{a^2-x^2-y^2} = z(x,y) & \\ x = x & x, y \in D \\ y = y & \end{cases}$$

$$I = \iint\limits_D (x+y+\sqrt{a^2-x^2-y^2}) * \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy = \dots$$

Másodfajú felületi integrál



S: $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $; (u,v) \in D \subset \mathbb{R}^3$

x(u,v);y(u,v);z(u,v) folytonos függ.

$\vec{r}(u,v) = \vec{i}x(u,v) + \vec{j}y(u,v) + \vec{k}z(u,v)$ $\vec{N} = \frac{\partial \vec{r}}{\partial u}(u_0,v_0) * \frac{\partial \vec{r}}{\partial v}(u_0,v_0) =$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} = \vec{i}A + \vec{j}B + \vec{k}C$$

A $= \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$ B $= \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{bmatrix}$ C $= \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$

$$\vec{n} = \frac{\vec{N}}{||\vec{N}||} = \vec{i}\frac{A}{\sqrt{A^2+B^2+C^2}}(*) + \vec{j}\frac{B}{\sqrt{A^2+B^2+C^2}}(**) + \vec{k}\frac{C}{\sqrt{A^2+B^2+C^2}}(***)$$

$* : \cos\alpha$

$**: \cos\beta$
$***: \cos\gamma$

$\alpha = m(\vec{n}, \vec{ox})$
$\beta = m(\vec{n}, \vec{oy})$
$\gamma = m(\vec{n}, \vec{oz})$
$\vec{n} = \vec{i} * \cos\alpha + \vec{j} * \cos\beta + \vec{k} * \cos\gamma$

$\iint\limits_S (P\cos\alpha + Q\cos\beta + R\cos\gamma) d\nabla = ? \leftarrow$ Ezt kell kiszámolni
$= \iint\limits_S P\cos\alpha d\nabla + \iint\limits_S Q\cos\beta d\nabla + \iint\limits_S R\cos\gamma d\nabla$

$\cos\alpha = \frac{A}{\sqrt{A^2+B^2+C^2}} = \frac{A}{\sqrt{EG-F^2}}$

$\iint\limits_S P\cos\alpha d\nabla = \iint\limits_D P\frac{A}{\sqrt{EG-F^2}}\sqrt{EG-F^2}\,du\,dv = \iint\limits_D PA\,du\,dv$

S: $\begin{cases} x = h(y,z) \\ y = y = u \qquad\qquad y, z \in D \\ z = z = v \end{cases}$

$\iint\limits_S P\cos\alpha d\nabla = \iint\limits_D P(x_{(y,z)}, y, z)\,dy\,dz$

pl: S: $x^2 + y^2 + z^2 = a^2$ a sugarú gömb

$\iint\limits_S (x\cos\alpha + y\cos\beta + z\cos\gamma) d\nabla = ?$

$\vec{N} = \vec{i}x + \vec{j}y + \vec{k}z$

$\vec{n} = \frac{\vec{N}}{||\vec{N}||} = \vec{i}\frac{x}{\sqrt{x^2+y^2+z^2}} + \vec{j}\frac{y}{\sqrt{x^2+y^2+z^2}} + \vec{k}\frac{z}{\sqrt{x^2+y^2+z^2}}$

$I = \iint\limits_S \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} d\nabla = \iint\limits_S \sqrt{x^2+y^2+z^2} d\nabla = a\iint\limits_S d\nabla = a*Ter(s) = a*4a^2\pi = 4\pi a^3$

Gauss - Osztrogradszkij

$D \subset \mathbb{R}^3 \quad, F : D \to \mathbb{R}^3 \quad, F = (P, Q, R)$ **Képlet:**

$$\iint\limits_S P\cos\alpha + Q\cos\beta + R\cos\gamma d\nabla = \iiint\limits_T (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})\,dx\,dy\,dz$$

pl:
$\iint\limits_S x\cos\alpha + y\cos\beta + z\cos\gamma d\nabla = \iiint\limits_{Gömb} (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) dx\,dy\,dz = 3\iiint\limits_{Gömb} dx\,dy\,dz = 3 * Térf(Gömb) = 3 * \frac{4\pi a^3}{3} = 4\pi a^3$

Stokes

$F^* : D \to \mathbb{R}^3 \quad , F* = (P, Q, R), \quad S \in D$

$$\int_r Pdx + Qdy + Rdz = \iint\limits_S \begin{bmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} d\nabla$$

$$\int_r Pdx + Qdy + Rdz = \iint\limits_S (\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z})\cos\alpha + (\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x})\cos\beta + (\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}) cos\gamma \, d\nabla$$