Lab 2: Mine Crafting

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I. Introduction

To calculate how long it takes for a 1 kg test mass to fall to the bottom of a 4 km vertical mine shaft, we must apply some basic physics to understand how gravity and drag impact fall time. Gravity is a force that accelerates objects downwards, and as a result gravitational acceleration decreases with height, which increases fall time. Drag opposes an object's motion and increases as the object speeds up, causing the object to fall slower with drag than without it, which increases fall time. Using Python, I calculated the ideal case of free fall, which assumes constant gravity and no drag. After this I accounted for drag and variable gravity due to height. Also, I considered the Earth's rotation by including the Coriolis force, which impacts the test mass's trajectory by deflecting it eastward because the force is proportional to the velocity of the mass and the rotational velocity of the Earth. Finally, crossing times for a homogenous and non-homogenous Earth and Moon model with varying density profiles were calculated. By computing numerical calculations and visualizations, I investigated how gravity and drag impact fall time to present these findings, which will offer valuable information for the mining company.

II. Calculation of Fall Times

To determine how long it takes for a 1 kg test mass to fall about 4 km into a vertical shaft, I modeled the motion using constant gravity with no drag force. Using the equation $d = \frac{1}{2}gt^2$, I calculated an analytical fall time of 28.6 s. To improve accuracy the accuracy of the calculation, I reduced the second order differential equation $\frac{d^2y}{dt^2} = -g + \boldsymbol{\alpha}\left|\frac{dy}{dt}\right|^{\gamma}$ into a system of coupled first order differential equations using $v = \frac{dy}{dt}$. I used Python's solve_ivp which integrates the equations using initial conditions, and the time span over to integrate. Also I used an event function with solve_ivp which stops the integration once the mass reaches the bottom of the shaft. This confirmed our earlier time and allowed me to plot position and velocity vs time, using the twinx() to display both on one graph (Figure 1). Gravity varies with height from the Earth's center, using the equation $g(r) = g_0 * \frac{r}{Re}$, confirming the fall time of 28.6 s. Finally, I added air resistance, calibrating the drag coefficient to 0.004 based on a terminal velocity of 50 m/s. Including drag raised the fall time to 83.5 s, essentially tripling the amount of time it takes for the mass to hit the bottom of the shaft .

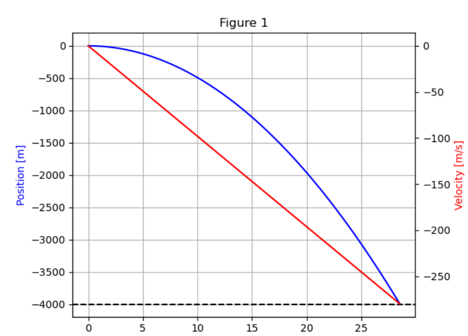

Figure 1: Plot showing position and velocity vs time for kinematics free fall equation.

III. Feasibility of Depth Measurement Approach

The motion was modeled to include the Coriolis force due to the Earth's rotation. The Coriolis force is $F_c = -2m(\vec{\Omega} x \vec{v})$, where $\vec{\Omega}$ is the Earth's angular velocity and $\vec{v}$ is the mass's velocity. The coordinate system has the x-axis along East, y down into the shaft, and z along North, giving the force components $F_{c_x} =+ 2m\Omega v_y$, $F_{c_y} =- 2m\Omega v_x$, and $F_{c_z} = 0$. I included this in the equations of motion as an acceleration term, which affected the transverse position and lateral movement. The motion is described by a system of coupled first-order differential equations with position and velocity vectors. I used solve_ivp to integrate the equations accounting for gravity and the Coriolis effect on the test mass's movement. The Coriolis force causes the test mass hitting the wall at a depth of 1847.9 m after 40.5 s as seen in Figure 3. This confirms the Coriolis force affects the motion, causing the mass to hit the wall before reaching the bottom. I do not recommend proceeding with this measuring technique as the test mass never reaches the bottom with or without drag due to the Coriolis force. I visualized the trajectory by plotting transverse position against the depth (Figure 3). The plot confirms that the mass hits the wall at 1847.8 m after 40.5 s before reaching the bottom.

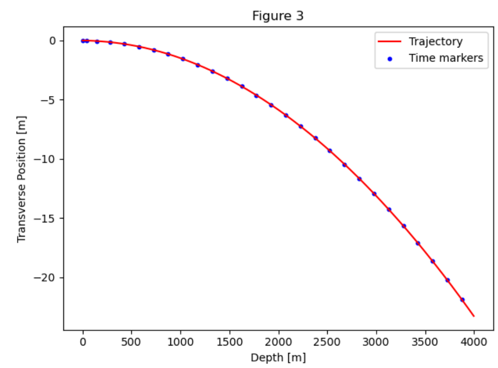

Figure 3: Plot of transverse position as a function of depth, assuming no drag, with time markers to see how the mass moves.

IV. Crossing Times of Trans-Earth/Moon

Determining the crossing time for a homogenous Earth requires neglecting Coriolis and drag, resulting in the equation of motion with variable gravity. I used solve_ivp and its event detection to end integration when the mass hit Earth's center, which happened at 1266.5 s. For the crossing times for a non-homogenous Earth and Moon, I modeled density as a function of distance from the Earth and Moon's center where $\rho = \rho_n (1 - \frac{r^2}{R_\oplus^2})^n$, where n = 0,1,2,9 and ρ_n is a normalizing constant. An n value of 0 is the constant Earth density and n = 2 is closer to the real value. Using scipy.integrate.quad which evaluates definite integrals, I solved the volume integral of the density function and calculated ρ_n. Using ρ_n, I calculated the gravitational force using the equation $F = \frac{GM(r)m}{r^2}$, where $M(r)$, is mass enclosed at r and I plotted this force as a function of radius as seen in Figure 6, which indicates at higher densities the force is stronger. I then rewrote the second-order differential equation into a system of first-order equations and solved using solve_ivp. The equations of motion now includes a radius dependent gravitational force due to varying density, making the equation $\frac{d^2y}{dt^2} = -\rho_n M(r) + \boldsymbol{\alpha}\left|\frac{dy}{dt}\right|^{\gamma}$. Event detection ended integration at the center of the planet and Moon, giving fall times for different profiles. The results show that for n = 0, the fall time is 1267.2 s and at n = 9 the fall time decreased to 943.8 s, indicating that as density increases, crossing time decreases and max velocity increases. For a trans-planetary and trans-lunar tunnel, for n = 0 on the Moon, the fall time is 1624.9 s and for n = 9 the fall time is 1210.2 s, showing again that an increase in density decreases crossing time and increases max velocity. Denser cores lead to shorter crossing times because more mass is concentrated near the center, which increases the gravitational force. On the other hand, a uniform-density results in a slower acceleration and longer crossing time because mass is spread evenly.

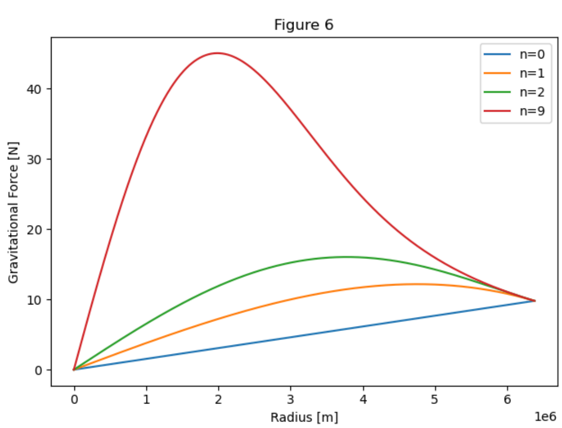

Figure 6: Gravitational Force as a function of Radius for n = 0,1,2,9.

V. Discussion and Future Work

This experiment modeled the motion of a 1 kg test mass dropped 4 km down a vertical mine shaft to estimate fall time. Initial assumptions were constant gravity, a homogenous Earth, and neglecting the Coriolis and drag force. Under these assumptions, the analytical fall time was 28.6 s. Including variable gravity confirmed this result. Adding drag force, with the drag coefficient calibrated for a terminal velocity of 50 m/s, increased the fall time to 83.5 s. The Coriolis force due to Earth's rotation was included later and had a significant effect, causing the test mass to hit a wall in the shaft at a depth of 1847.9 m after 40.5 s, ruling out this method for measuring the depth of the shaft. For Earth crossing times through a planet-wide tunnel, assuming a homogeneous Earth and variable gravity, the fall time was 1266.5 s. For a non-homogenous Earth with varying density profiles, the fall time decreased to 943.8 s for n = 9 from 1267.2 seconds for n = 0. A similar Moon tunnel resulted in fall times of 1624.9 s for n = 0 to 1210.2 s for n = 9. To enhance the model, we can consider how the Earth is not a perfect sphere which affects gravity at different locations, varying atmospheric conditions which affects the drag force, and better density profiles. The Coriolis force can be explored better by considering the shaft's orientation. Considering the shaft's structure would improve accuracy in predicting the trajectory of the test mass. These enhancements will result in more realistic models, which will refine the method for measuring the depth of mines.