1. (10 points) Consider the following dataset with 2 feature attributes $x_1$ and $x_2$ and a target attribute Class:

| ID # | $x_1$ | $x_2$ | Class |
|---|---|---|---|
| 1 | 5 | 1 | No |
| 2 | 6 | 2 | No |
| 3 | 6 | 4 | No |
| 4 | 3 | 4 | No |
| 5 | 8 | 2 | Yes |
| 6 | 8 | 4 | Yes |
| 7 | 5 | 6 | Yes |
| 8 | 6 | 7 | Yes |

(a) State explicitly the quadratic programming problem that needs to be solved to find the hyperplane with maximum margin of separation in the original feature space.

$$\min \frac{1}{2}\left(w_1^2 + w_2^2\right)$$

subject to

$$\begin{aligned}
5w_1 + 1w_2 + b &\leq -1 \\
6w_1 + 2w_2 + b &\leq -1 \\
6w_1 + 4w_2 + b &\leq -1 \\
3w_1 + 4w_2 + b &\leq -1 \\
8w_1 + 2w_2 + b &\geq 1 \\
8w_1 + 4w_2 + b &\geq 1 \\
5w_1 + 6w_2 + b &\geq 1 \\
6w_1 + 7w_2 + b &\geq 1
\end{aligned}$$

(b) Plot the data points. Which data points are likely to be support vectors? Samples with ID: 3,5,7

(c) Check if the points that you identify in part (b) are indeed support vectors by (i) finding their corresponding Lagrange multipliers, and (ii) by showing that the KT optimality conditions are satisfied.

$$\begin{aligned}
6w_1 + 4w_2 + b &= -1 \\
8w_1 + 2w_2 + b &= 1 \\
5w_1 + 6w_2 + b &= 1
\end{aligned}$$

$$\left(\begin{array}{cccc} 6 & 4 & 1 & || & -1 \\ 8 & 2 & 1 & || & 1 \\ 5 & 6 & 1 & || & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc} 6 & 4 & 1 & || & -1 \\ 2 & -2 & 0 & || & 2 \\ -1 & 2 & 0 & || & 2 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cccc} 0 & 16 & 1 & || & 11 \\ 0 & 2 & 0 & || & 6 \\ 1 & -2 & 0 & || & -2 \end{array}\right) \Rightarrow \left(\begin{array}{cccc} 0 & 0 & 1 & || & -37 \\ 0 & 1 & 0 & || & 3 \\ 1 & 0 & 0 & || & 4 \end{array}\right)$$

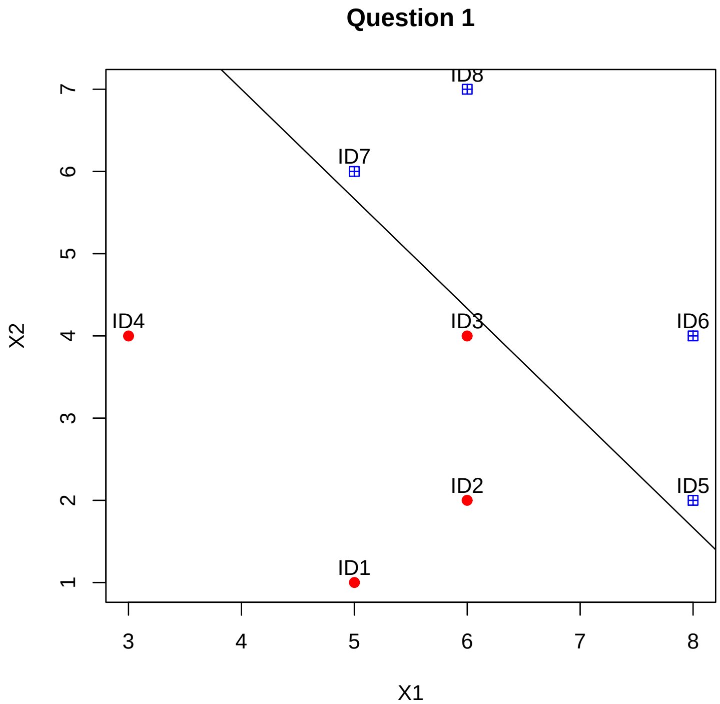


Figure 1: SVM data

Solution: $w_1 = 4, w_2 = 3, b = -37$, separating hyperplane is $4x_1 + 3x_2 - 37 = 0$.
Let $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = 0$ as the corresponding data points are not support vectors (corresponding constraints not binding).

$$\begin{aligned} \begin{pmatrix} 4 \\ 3 \end{pmatrix} &= -\alpha_3 \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \alpha_5 \begin{pmatrix} 8 \\ 2 \end{pmatrix} + \alpha_7 \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ 0 &= -\alpha_3 + \alpha_5 + \alpha_7 \end{aligned}$$

$$\begin{pmatrix} -6 & 8 & 5 & || & 4 \\ -4 & 2 & 6 & || & 3 \\ -1 & 1 & 1 & || & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 3 & 0 & || & 4 \\ 2 & -4 & 0 & || & 3 \\ -1 & 1 & 1 & || & 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & -3 & 0 & || & -4 \\ 0 & 2 & 0 & || & 11 \\ 0 & -2 & 1 & || & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & || & \frac{25}{2} \\ 0 & 1 & 0 & || & \frac{11}{2} \\ 0 & 0 & 1 & || & 7 \end{pmatrix}$$

Solution: $\alpha_3 = \frac{25}{2}, \alpha_5 = \frac{11}{2}, \alpha_7 = 7$. All the KT conditions are satisfied:

- Primal feasibility
- Stationary conditions
- Complementary conditions
- Dual multipliers $\geq 0$.

2. (10 points) The Ministry of Education wants to evaluate the efficiency of four elementary schools in the following towns: Clementi, Ang Mo Kio, Yishun and Bedok. The three outputs of the schools are defined to be:

- Output 1 = average reading score
- Output 2 = average writing score
- Output 3 = average science score

The three inputs to the schools are defined to be:

- Input 1 = average educational level of mothers (defined by highest grade completed; 12 = high school graduate, 16 = university graduate, and so on).
- Input 2 = number of parent visits to the school per child.
- Input 3 = teacher to student ratio.

The relevant information for the four schools is given in the table below.

| | Inputs | | | Outputs | | |
|---|---|---|---|---|---|---|
| School | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 (Clementi) | 13 | 4 | 0.05 | 90 | 70 | 60 |
| 2 (Ang Mo Kio) | 14 | 5 | 0.05 | 95 | 80 | 75 |
| 3 (Yishun) | 11 | 6 | 0.06 | 90 | 76 | 80 |
| 4 (Bedok) | 15 | 8 | 0.08 | 90 | 90 | 90 |

The Data Envelopment Analysis method is used to identity school(s) that could be inefficient. The Linear Programming outputs from SAS for each of the schools are as follows.

```
School 1:
                    The LP Procedure
                    Constraint Summary
        Constraint           S/S                      Dual
Row     Name          Type     Col  Rhs  Activity     Activity
1       z             OBJECTVE   .    0         1            .
2       School1       GE         7    0         0           -1
3       School2       GE         8    0         0            0
4       School3       GE         9    0  0.26562             0
5       School4       GE        10    0  0.6859933           0
6       Input1        EQ         .    1         1            1


School 2:
                    The LP Procedure
                    Constraint Summary
        Constraint           S/S                      Dual
Row     Name          Type     Col  Rhs  Activity     Activity
1       z             OBJECTVE   .    0         1            .
```

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | School1 | GE | 7 | 0 | 0.0541158 | 0 |
| 3 | School2 | GE | 8 | 0 | 0 | -1 |
| 4 | School3 | GE | 9 | 0 | 0.2511358 | 0 |
| 5 | School4 | GE | 10 | 0 | 0.6485758 | 0 |
| 6 | Input2 | EQ | . | 1 | 1 | 1 |

School 3:

The LP Procedure
Constraint Summary

| Row | Constraint Name | Type | S/S Col | Rhs | Activity | Dual Activity |
|---|---|---|---|---|---|---|
| 1 | z | OBJECTVE | . | 0 | 1 | . |
| 2 | School1 | GE | 7 | 0 | 0.0113119 | 0 |
| 3 | School2 | GE | 8 | 0 | 0 | 0 |
| 4 | School3 | GE | 9 | 0 | 0 | -1 |
| 5 | School4 | GE | 10 | 0 | 0.3461923 | 0 |
| 6 | Input3 | EQ | . | 1 | 1 | 1 |

School 4:

The LP Procedure
Constraint Summary

| Row | Constraint Name | Type | S/S Col | Rhs | Activity | Dual Activity |
|---|---|---|---|---|---|---|
| 1 | z | OBJECTVE | . | 0 | 0.8761191 | . |
| 2 | School1 | GE | 7 | 0 | 0.0637822 | 0 |
| 3 | School2 | GE | 8 | 0 | 0 | -0.060565 |
| 4 | School3 | GE | 9 | 0 | 0 | -1.120458 |
| 5 | School4 | GE | 10 | 0 | 0.1238809 | 0 |
| 6 | Input4 | EQ | . | 1 | 1 | 0.8781965 |

(a) Determine which (if any) schools are inefficient.

From the objective function, School 4 is the only one that has efficiency less than 1.

(b) For each school that is inefficient, explain in terms of the inputs and outputs, why it is not efficient.

It is possible to create a "composite school" that requires less inputs and produces more outputs than School 4. This composite school consists of 0.060565 School 2 and 1.120458 School 3. The inputs would be

$$0.060565 \begin{pmatrix} 14 \\ 5 \\ 0.05 \end{pmatrix} + 1.120458 \begin{pmatrix} 11 \\ 6 \\ 0.06 \end{pmatrix} = \begin{pmatrix} 13.173 \\ 7.026 \\ 0.070 \end{pmatrix} < \begin{pmatrix} 15 \\ 8 \\ 0.08 \end{pmatrix}$$

and the outputs:

$$0.060565 \begin{pmatrix} 95 \\ 80 \\ 75 \end{pmatrix} + 1.120458 \begin{pmatrix} 90 \\ 76 \\ 80 \end{pmatrix} = \begin{pmatrix} 106.595 \\ 90 \\ 94.179 \end{pmatrix} > \begin{pmatrix} 90 \\ 90 \\ 90 \end{pmatrix}$$

3. (10 points) The Clementi Garment Company manufactures men shirts and women blouses. A shirt requires an average of 4 hours and a blouse an average 6 hours in the cutting and sewing center owned by the company. The workers at this center work a normal shift of 8 hours a day, 6 days a week. There are a total of 30 workers employed by the company. The unit profits for shirts and blouses are \$5 and \$10, respectively. The demands require that at least 200 shirts and 300 blouses be produced each week. For each unit of either shirt or blouse by which production falls short of demand, a penalty of \$2.50 is assessed. A \$10 penalty is incurred for each hour of overtime labor and a \$2 penalty is incurred for each hour of available labor that is unused.

   (a) Formulate a linear program that can be used to minimize the total penalty incurred by the company. Clearly define all your decision variables.

   Let $x_A$ and $x_B$ be the number of shirts and blouses to be produced, respectively.

   - Total profit: $5x_A + 10x_B$.
   - Constraint, demand for shirts $x_A \geq 200$
   - Constraint, demand for blouses $x_B \geq 300$
   - Constraint, production capacity $4x_A + 6x_B \leq 8 \times 6 \times 30 = 1440$

   Linear program:

$$
\begin{aligned}
\min\ 2.5s_2^- + 2.5s_3^- + 2s_4^- + 10s_4^+ & & \\
s.t.\ x_A + s_2^- - s_2^+ &= 200 \\
x_B + s_3^- - s_3^+ &= 300 \\
4x_A + 6x_B + s_4^- - s_4^+ &= 1440 \\
x_A, x_B, s_2^-, s_2^+, s_3^-, s_3^+, s_4^-, s_4^+ &\geq 0
\end{aligned}
$$

   (b) The company wishes to achieve the following goals, which are listed in order of importance (highest to lowest):

   - Achieve the profit goal of at least \$3000 per week.
   - Meet demand for blouses.
   - Meet demand for shirts.
   - Avoid having any underutilization of production capacity.
   - Do not use any overtime.

   Formulate and solve a preemptive goal programming model for this problem.

   Preemptive goal linear program:

$$
\begin{aligned}
\min\ P_1s_1^- + P_2s_2^- + P_3s_3^- + P_4s_4^- + P_5s_4^+ & & \\
s.t.\ 5x_A + 10x_B + s_1^- - s_1^+ &= 3000 \\
x_A + s_2^- - s_2^+ &= 200 \\
x_B + s_3^- - s_3^+ &= 300 \\
4x_A + 6x_B + s_4^- - s_4^+ &= 1440 \\
x_A, x_B, s_1^-, s_1^+, s_2^-, s_2^+, s_3^-, s_3^+, s_4^-, s_4^+ &\geq 0
\end{aligned}
$$

   Solution: 200 shirts and 300 blouses, overtime = $s_4^+ = 4(200) + 6(300) - 1440 = 1160$.

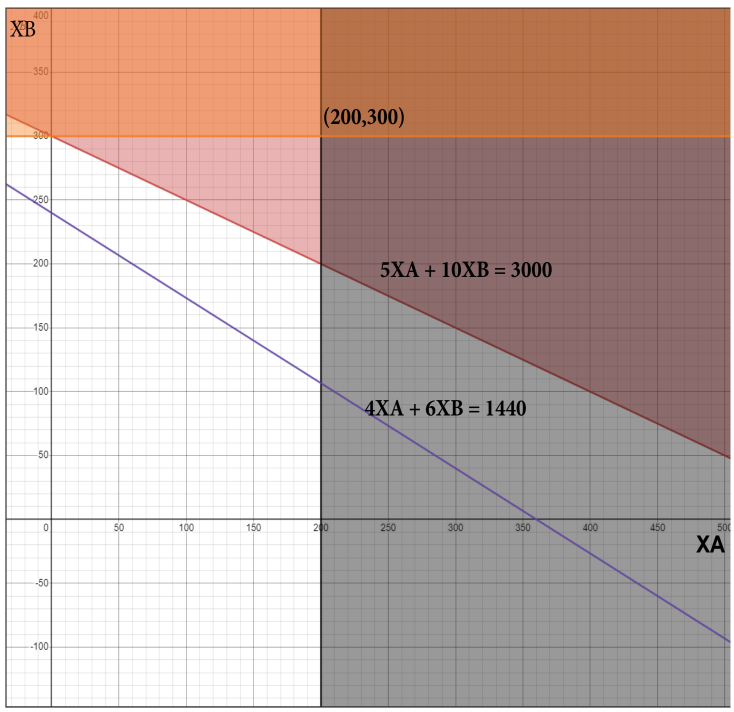


Figure 2: The Clementi Garment Company