Optimization and decision making

Optimization and Decision Making

Objectives:

- To understand the role of optimization in decision making

- To know the optimality conditions for linear and quadratic programming

Outline:

1. The Kuhn-Tucker optimality conditions
2. Optimality conditions for quadratic programming problems
3. Optimality condition for linear programming problems
4. Dual of a linear programming problem
5. Standard form linear program
6. Economic interpretation of the dual problem
7. Dual of nonlinear programming problem
8. Application of optimization in decision making

1. The Kuhn-Tucker optimality conditions

The problem NLP (Non Linear Programming):

$$\text{minimize} \quad f(x_1,x_2,x_3,\ldots\ldots, x_n)$$

$$\begin{aligned}\text{subject to:} \quad & g_1(x_1,x_2,x_3,\ldots\ldots, x_n) \le b_1 \\ & g_2(x_1,x_2,x_3,\ldots\ldots, x_n) \le b_2 \\ & \vdots \\ & g_m(x_1,x_2,x_3,\ldots\ldots, x_n) \le b_m\end{aligned}$$

Note that

- The number of variables is n, the number of constraints is m
- $\mathbf{max\ f(x_1,x_2,x_3,\ldots\ldots, x_n)}$ is equivalent to $\mathbf{-min\ -f(x_1,x_2,x_3,\ldots\ldots, x_n)}$
- The constraint $\mathbf{g_k(x_1,x_2,x_3,\ldots\ldots, x_n) \ge b_k}$ is equivalent to

$$\mathbf{-\ g_k(x_1,x_2,x_3,\ldots\ldots, x_n) \le -b_k}$$

The Kuhn-Tucker optimality conditions

The Kuhn-Tucker (KT) necessary conditions:

Suppose that the point $\boldsymbol{x} = (x_1, x_2, x_3, \ldots\ldots, x_n)$ is an optimal solution,

then $\boldsymbol{x} = (x_1, x_2, x_3, \ldots\ldots, x_n)$ must satisfy all the constraints, that is,

$$g_1(x_1, x_2, x_3, \ldots\ldots, x_n) \leq b_1$$

$$g_2(x_1, x_2, x_3, \ldots\ldots, x_n) \leq b_2$$

$$\vdots$$

$$g_m(x_1, x_2, x_3, \ldots\ldots, x_n) \leq b_m$$

and there must exist multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$ such that

$$\partial f(\boldsymbol{x}) / \partial x_j + \sum_{i=1}^{m} \lambda_i \, \partial g_i(\boldsymbol{x}) / \partial x_j = 0 \qquad (j = 1,2, \ldots, n)$$

$$\lambda_i \, [b_i - g_i(\boldsymbol{x})] = 0 \qquad (i = 1,2, \ldots, m)$$ **complementarity conditions**

$$\lambda_i \geq 0 \qquad (i = 1,2, \ldots, m)$$

The Kuhn-Tucker optimality conditions

The complementarity conditions:

$$\lambda_i \; [b_i - g_i(\boldsymbol{x})] = 0$$

imply

- If $\lambda_i > 0$, $b_i = g_i(\boldsymbol{x})$, the i^{th} constraint is said to be <u>binding</u>
- If $g_i(\boldsymbol{x}) < b_i$, then $\lambda_i = 0$. In this case, the constraint is said to be <u>non-binding</u>.

Partial derivative example:

$$f(\boldsymbol{x}) = f(x_1,x_2,x_3) = -x_1(30-x_1) - x_2(50-2x_2) + 3x_1 + 5x_2 + 10x_3$$

$$\partial f(\boldsymbol{x})/\partial x_1 = -30 + 2x_1 + 3$$

$$\partial f(\boldsymbol{x})/\partial x_2 = -50 + 4x_2 + 5$$

$$\partial f(\boldsymbol{x})/\partial x_3 = 10$$

The Kuhn-Tucker optimality conditions

KT **sufficient** conditions:

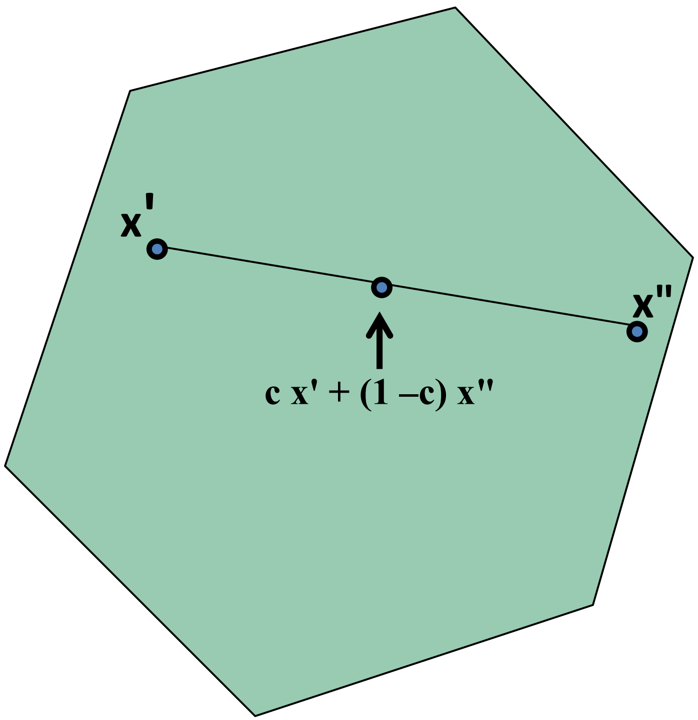
If $f(x_1,x_2,x_3,\ldots, x_n)$ is a convex function, and

$g_1(x_1, x_2, x_3,\ldots, x_n)$, $g_2(x_1, x_2, x_3,\ldots, x_n)$, …. , $g_m(x_1, x_2, x_3,\ldots, x_n)$ are **convex** functions,

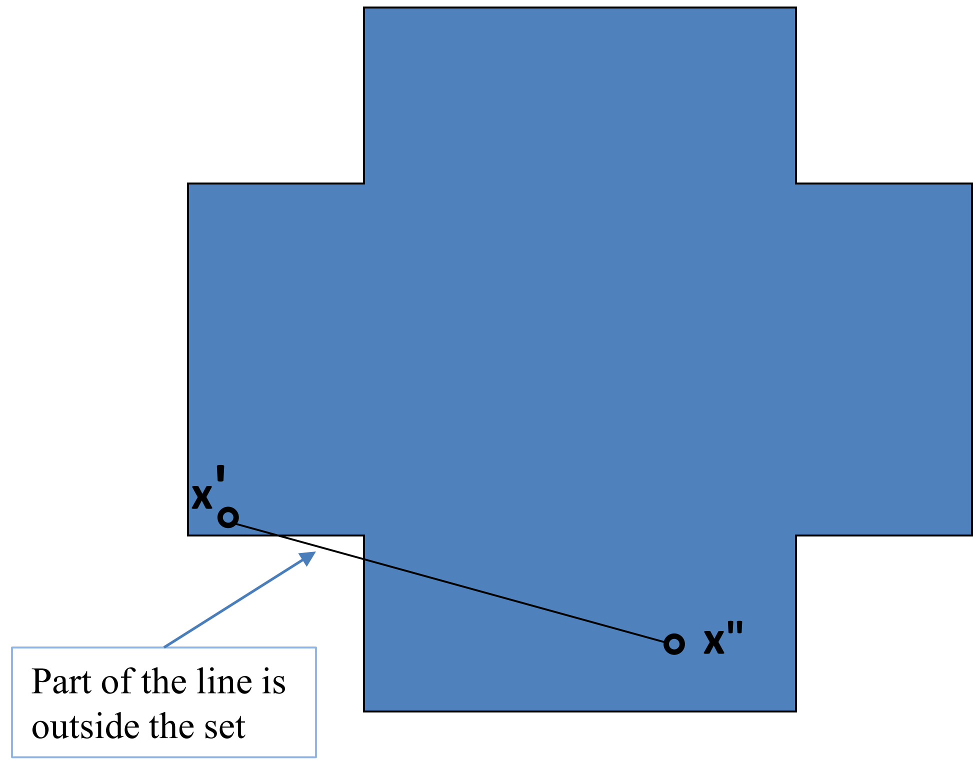
then any points $\boldsymbol{x} = (x_1, x_2, x_3,\ldots, x_n)$ that satisfies the KT necessary conditions is an optimal solution to the NLP.

The Kuhn-Tucker optimality conditions

A set S is <u>convex</u> if $x' \in S$ and $x'' \in S$ imply that all points on the line segment joining x' and x'' are members of S, that is $c\,x' + (1-c)\,x''$ is also in S for any $0 \le c \le 1$.



Convex set



Non-convex set

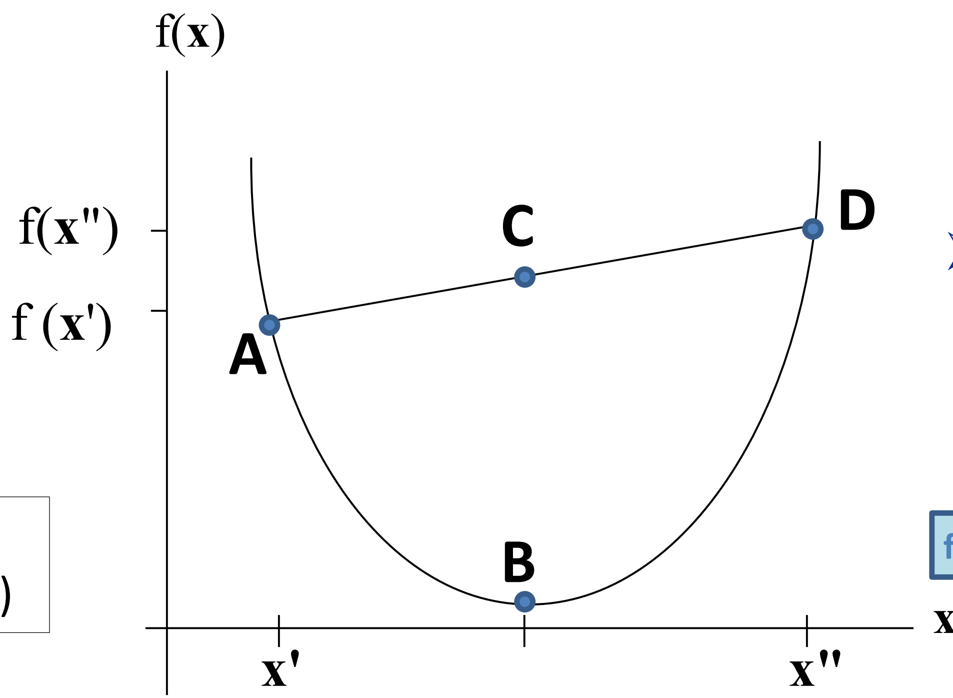
The Kuhn-Tucker optimality conditions

The function $f(x_1,x_2,x_3,\ldots., x_n)$ is a **convex function** on a convex set S if for any

$\mathbf{x}' \in S$ and $\mathbf{x}'' \in S$

$$f(c\,\mathbf{x}' + (1-c)\,\mathbf{x}'') \le c\,f(\mathbf{x}') + (1-c)\,f(\mathbf{x}'')$$

holds for $0 \le c \le 1$



A: (x',f(x'))
D: (x",f(x"))

- X-coordinate:

  B = $(c\,\mathbf{x}' + (1-c)\,\mathbf{x}'')$

  C = $(c\,\mathbf{x}' + (1-c)\,\mathbf{x}'')$

- Y-coordinate:

  B = $f(c\,\mathbf{x}' + (1-c)\,\mathbf{x}'')$

  C = $c\,f(\mathbf{x}') + (1-c)\,f(\mathbf{x}'')$

$$\boxed{f(c\,x' + (1-c)\,x'') \le c\,f(x') + (1-c)\,f(x'')}$$

The Kuhn-Tucker optimality conditions

Example 1. Describe the optimal solution to

$$\max f(x)$$

$$\text{subject to} \quad a \le x \le b$$

Answer:

$$-\min -f(x)$$

$$\text{subject to} \quad -x \le -a$$

$$x \le b$$

KT necessary conditions:

$$\begin{aligned} -f'(x) - \lambda_1 + \lambda_2 &= 0 \\ \lambda_1(a - x) &= 0 \\ \lambda_2(x - b) &= 0 \\ \lambda_1, \lambda_2 &\ge 0 \end{aligned}$$

Four possibilities:

1) $\lambda_1 = \lambda_2 = 0$, then $f'(x) = 0$
2) $\lambda_1 = 0$, $\lambda_2 > 0$, then $x = b$, $f'(b) = \lambda_2 > 0$
3) $\lambda_1 > 0$, $\lambda_2 = 0$, then $x = a$, $f'(a) = -\lambda_1 < 0$
4) $\lambda_1 > 0$, $\lambda_2 > 0$, then $x = a = b$,

   a contradiction, hence this case cannot occur

If $-f(x)$ is convex, a point that satisfies these 4 conditions is optimal since the constraints are linear

The Kuhn-Tucker optimality conditions

Example 1 (continued). Describe the optimal solution to

$$\max -x^2$$

subject to $2 \le x \le 5$

Answer:

$$-\min x^2$$

subject to $-x \le -2$

$$x \le 5$$

KT necessary conditions:

$$2x - \lambda_1 + \lambda_2 = 0$$

$$\lambda_1(2 - x) = 0$$

$$\lambda_2(x - 5) = 0$$

$$\lambda_1, \lambda_2 \ge 0$$

Here $f(x) = -x^2$ and $f'(x) = -2x$. Four possibilities:

1) $\lambda_1 = \lambda_2 = 0$, then $f'(x) = -2x = 0$

Not possible!

2. $\lambda_1 = 0$, $\lambda_2 > 0$, then $x = b = 5$, $2x - \lambda_1 + \lambda_2 > 0$.

Not good!

3) $\lambda_1 > 0$, $\lambda_2 = 0$, then $x = a = 2$ and $2x - \lambda_1 + \lambda_2 = 0$ if we let $\lambda_1 = 4$

Solution: $x = 2$ with maximum value of $-x^2 = -4$

The Kuhn-Tucker optimality conditions

Example 2.

$$\min\ -x_1(30-x_1) - x_2(50-2x_2) + 3x_1 + 5x_2 + 10x_3$$

$$\text{subject to}\quad x_1 + x_2 - x_3 \leq 0$$

$$x_3 \leq 17.25$$

KT necessary conditions:

$$-30 + 2x_1 + 3 + \lambda_1 = 0$$

$$-50 + 4x_2 + 5 + \lambda_1 = 0$$

$$10 - \lambda_1 + \lambda_2 = 0$$

$$\lambda_1(x_1 + x_2 - x_3) = 0$$

$$\lambda_2(x_3 - 17.25) = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

Four possibilities:

1) $\lambda_1 = \lambda_2 = 0$, impossible $(10 = 0)$
2) $\lambda_1 = 0$, $\lambda_2 > 0$, impossible $(\lambda_2 = -10)$
3) $\lambda_1 > 0$, $\lambda_2 = 0$, then $\lambda_1 = 10$, $x_1 = (30 - 3-10)/2 = 8.5$, $x_2 = (50 - 10 -5)/4 = 8.75$, $x_3 = x_1 + x_2 = 17.25$
4) $\lambda_1 > 0$, $\lambda_2 > 0$, need not be considered as optimal solution is already found above.

2. Optimality conditions for quadratic programming problems

Any quadratic programming problem with linear constraints can be put in the following standard form:

$$\min \ \tfrac{1}{2}\, \mathbf{x}^t \mathbf{P}\, \mathbf{x} + p^t \mathbf{x}$$

$$\text{subject to } \ \mathbf{A}\,\mathbf{x} \le a$$

$$\mathbf{x} \ge 0$$

Where **P** is an n by n matrix, p is an n-dimensional vector,

A is an m by n matrix, a is an m-dimensional vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \\ x_n \end{pmatrix}$$ is an n-dimensional variable.

Optimality conditions for quadratic programming problems

QP: min $\frac{1}{2}\, \mathbf{x}^t P\, \mathbf{x} + p^t\mathbf{x}$

subject to $\mathbf{A}\, \mathbf{x} \leq \mathbf{a}$

$\mathbf{x} \geq 0$

KT necessary conditions:

If $\mathbf{x}$ is an optimal solution, then $\mathbf{x}$ must satisfy the constraints $\mathbf{A}\, \mathbf{x} \leq \mathbf{a}$, $\mathbf{x} \geq 0$ and there must exist an n-dimensional vector $\mathbf{v}$, and an m-dimensional $\mathbf{u}$, such that

- $\mathbf{P}\, \mathbf{x} + \mathbf{p} + \mathbf{A}^t \mathbf{u} - \mathbf{v} = 0$ (note: there are n equations here)
- $\mathbf{u}^t(\mathbf{Ax} - \mathbf{a}) = 0$ (note: there are m complementarity conditions here)
- $\mathbf{v}^t\, \mathbf{x} = 0$ (note: there are n complementarity conditions here)
- $\mathbf{u} \geq 0$
- $\mathbf{v} \geq 0$

Optimality conditions for quadratic programming problems

<u>A note on Complementarity condition $\mathbf{v}^t\,\mathbf{x} = 0$</u>

- Since $\mathbf{v} \geq 0$ and $\mathbf{x} \geq 0$,

  then if $v_j > 0$, x_j must be 0 for the complementarity conditions to be satisfied.

Proof: Since $\mathbf{x} \geq 0$, x_j cannot be negative. Suppose $x_j > 0$, then $v_j\, x_j > 0$, and

$$\mathbf{v}^t\mathbf{x} = v_1\, x_1 + v_2\, x_2 + \ldots.. + v_j\, x_j + \ldots\ldots\, v_n\, x_n > 0.$$ A contradiction.

- Similarly, if $x_j > 0$, then $v_j = 0$, for all $j = 1,2, \ldots. ,n$.

- If $x_j = 0$, then v_j may be 0 or positive.

- Similarly if $v_j = 0$, then x_j may be 0 or positive.

Optimality conditions for quadratic programming problems

QP: min $\frac{1}{2}\, \mathbf{x}^t \mathbf{P}\, \mathbf{x} + \mathbf{p}^t \mathbf{x}$

subject to $\mathbf{A}\mathbf{x} \le \mathbf{a}$

$\mathbf{x} \ge 0$

KT <u>sufficient</u> conditions: If $\mathbf{x}$ is feasible and if there exist an n-dimensional vector $\mathbf{v}$, and an m-dimensional $\mathbf{u}$, such that

- $\mathbf{P}\,\mathbf{x} + \mathbf{p} + \mathbf{A}^t \mathbf{u} - \mathbf{v} = 0$ (note: there are n equations here)
- $\mathbf{u}^t(\mathbf{A}\mathbf{x} - \mathbf{a}) = 0$ (note: there are m complementarity conditions here)
- $\mathbf{v}^t\, \mathbf{x} = 0$ (note: there are n complementarity conditions here)
- $\mathbf{u} \ge 0$
- $\mathbf{v} \ge 0$

and if $\frac{1}{2}\, \mathbf{x}^t \mathbf{P}\, \mathbf{x} + \mathbf{p}^t \mathbf{x}$ is convex, then $\mathbf{x}$ solves QP

Optimality conditions for quadratic programming problems

Example. $\min -x_1 - x_2 + \frac{1}{2}(x_1)^2 + (x_2)^2 - x_1x_2$

subject to

$$x_1 + x_2 \leq 3$$
$$-2x_1 - 3x_2 \leq -6$$
$$x_1, x_2 \geq 0$$

Show that $x_1 = 9/5$, $x_2 = 6/5$ is the solution.

KT conditions:

$$x_1 - 1 - x_2 + u_1 - 2u_2 - v_1 = 0$$
$$2x_2 - 1 - x_1 + u_1 - 3u_2 - v_2 = 0$$
$$(x_1 + x_2 - 3)u_1 = 0$$
$$(-2x_1 - 3x_2 + 6)u_2 = 0$$
$$v_1 x_1 = 0,\ v_2 x_2 = 0$$
$$x_1, x_2, u_1, u_2, v_1, v_2 \geq 0$$

Check feasibility:

✓ $x_1 + x_2 \leq 3$? Yes, and $9/5 + 6/5 = 3$, this constraint is binding.

✓ $-2x_1 - 3x_2 \leq -6$? Yes, $-18/5 - 18/5 \leq -6$, this constraint is not binding, hence $u_2 = 0$

✓ $x_1, x_2 \geq 0$? Yes, these constraints are not binding, hence $v_1 = v_2 = 0$

Can we find $u_1 \geq 0$?

Yes, $u_1 = 2/5$

All conditions are satisfied.

Since the objective function is also convex,

$x_1 = 9/5$, $x_2 = 6/5$ is the solution.

Optimality conditions for quadratic programming problems

$f(x_1,x_2) = -x_1 - x_2 + ½ (x_1)^2 + (x_2)^2 - x_1x_2$

What is the matrix **P** and the vector **p**?

- $\partial f(x_1,x_2)/ \partial x_1 = -1 + x_1 - x_2$
- $\partial f(x_1,x_2)/ \partial x_2 = -1 - x_1 + 2 x_2$
- $\partial^2 f(x_1,x_2)/ \partial x_1 \partial x_1 = 1$ $\quad\quad$ $\partial^2 f(x_1,x_2)/ \partial x_2 \partial x_1 = -1$
- $\partial^2 f(x_1,x_2)/ \partial x_1 \partial x_2 = -1$ $\quad\quad$ $\partial^2 f(x_1,x_2)/ \partial x_2 \partial x_2 = 2$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Note:

- the derivative of $½\, \mathbf{x}^t \mathbf{P}\, \mathbf{x} + \mathbf{p}^t\mathbf{x}$ is $\mathbf{P}\, \mathbf{x} + \mathbf{p}$
- the second derivative is **P**

Check that $½\, \mathbf{x}^t \mathbf{P}\, \mathbf{x} + \mathbf{p}^t\mathbf{x} = -x_1 - x_2 + ½ (x_1)^2 + (x_2)^2 - x_1x_2$

3. Optimality conditions for linear programming problems

If we just remove the matrix P from QP, we have the linear program

LP: min $c^t \mathbf{x}$

$$\text{subject to } \mathbf{Ax} \leq \mathbf{a}$$

$$\mathbf{x} \geq \mathbf{0}$$

Since the objective function and the constraints are all linear, the KT necessary and sufficient conditions: If the n-dimensional vector v and the m-dimensional vector u satisfy:

$\mathbf{c} + \mathbf{A}^t \mathbf{u} - \mathbf{v} = \mathbf{0}$ (note: there are n equations here)

$\mathbf{u}^t(\mathbf{Ax} - \mathbf{a}) = \mathbf{0}$ (note: there are m complementarity conditions here)

$\mathbf{v}^t \mathbf{x} = \mathbf{0}$ (note: there are n complementarity conditions here)

$\mathbf{u} \geq \mathbf{0}$

$\mathbf{v} \geq \mathbf{0}$

and $\mathbf{x}$ satisfies the constraints of the LP, then $\mathbf{x}$ is the solution of LP

Example – LP formulation.

- GW Inc. manufactures two types of wooden toys: soldiers and trains.
- A soldier sells for \$27 and uses \$10 worth of raw materials.
- Cost in labor for each soldier is \$14.
- A train sells for \$21 and uses \$9 worth of raw materials.
- Cost in labor for each train is \$10.
- A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor.
- A train requires 1 hour of finishing and 1 hour of carpentry labor.
- Each week, GW can obtain all the needed labor but only 100 finishing hours and 80 carpentry hours.
- Only 40 soldiers per week can be sold.
- **Formulate an LP.**

Optimality conditions for linear programming problems

Example – LP formulation.

1. **Decision variables:**

   x_1 = number of soldiers, x_2 = number of trains produced per week

2. **Objective function:** maximize profit

- profit = revenue – cost
- revenue = $27\,x_1 + 21\,x_2$
- cost = $10\,x_1 + 14\,x_1 + 9\,x_2 + 10\,x_2$
- profit = $27\,x_1 + 21\,x_2 - (10\,x_1 + 14\,x_1 + 9\,x_2 + 10\,x_2) = 3x_1 + 2\,x_2$

3. **Constraints:**

- Each week, not more than 100 hours available for finishing: $2\,x_1 + x_2 \leq 100$
- Each week, not more than 80 hours available for carpentry: $x_1 + x_2 \leq 80$
- Demand is limited to 40 soldiers per week: $x_1 \leq 40$
- Non-negativity constraints: $x_1, x_2 \geq 0$

Optimality conditions for linear programming problems

Graphical solution:

Dotted lines: isoprofit lines

Optimal solution:

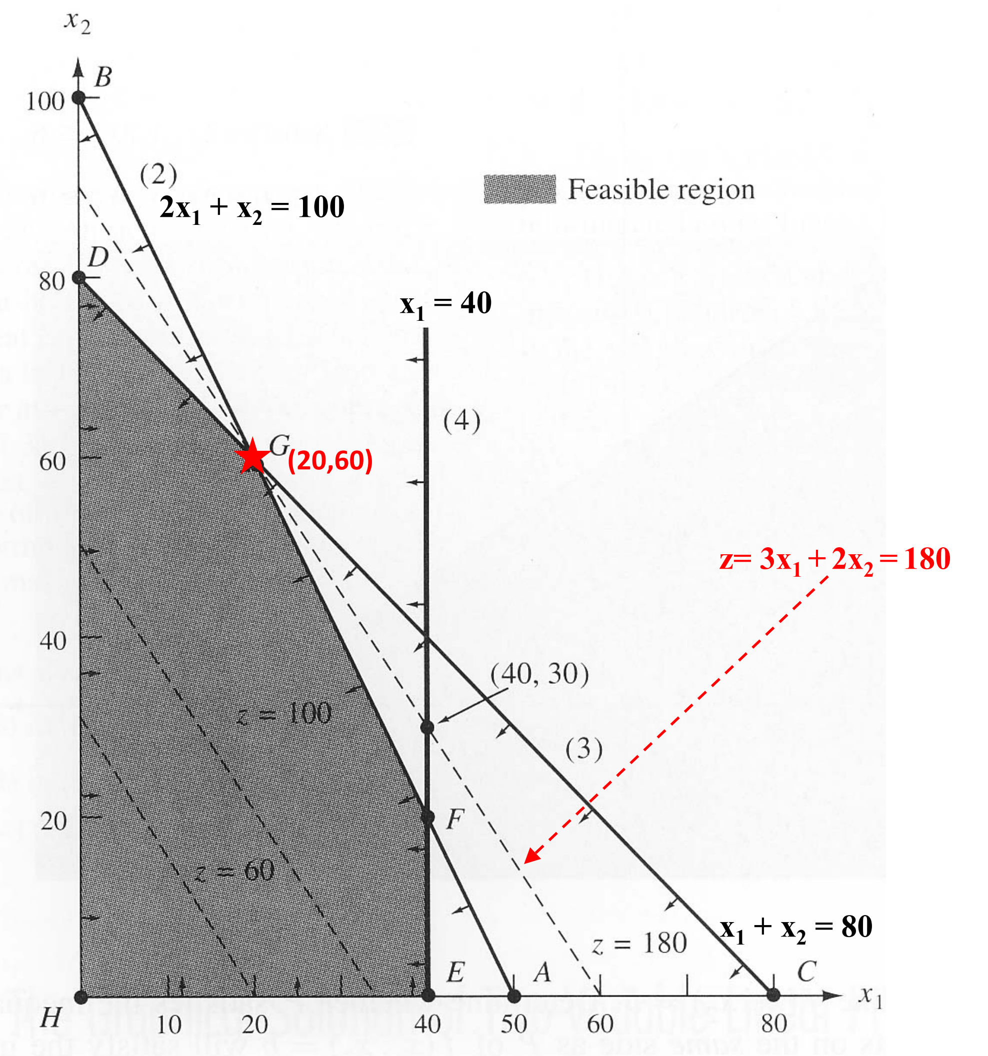
G where $x_1 = 20$, $x_2 = 60$

maximum profit = 3(20) + 2(60) = 180

Feasible region: the set of all points that satisfy the LP constraints

Optimal solution: a point in the feasible region with the best objective function value

If an LP has a solution, there must be a solution at an extreme point



Optimality conditions for linear programming problems

Linear Program:

$$\text{-minimize} - 3x_1 - 2x_2$$

subject to: $2x_1 + x_2 \le 100$ C1

$x_1 + x_2 \le 80$ C2

$x_1 \le 40$ C3

$x_1, x_2 \ge 0$

<u>KT necessary and sufficient conditions:</u>

$$-3 + 2u_1 + u_2 + u_3 - v_1 = 0$$
$$-2 + u_1 + u_2 - v_2 = 0$$
$$u_1(2x_1 + x_2 - 100) = 0$$
$$u_2(x_1 + x_2 - 80) = 0$$
$$u_3(x_1 - 40) = 0$$
$$v_1 x_1 = 0$$
$$v_2 x_2 = 0$$

Optimal solution: $x_1 = 20$, $x_2 = 60$

Check all constraints, C1 and C2 are binding. C3 is not binding, hence $u_3 = 0$.

Also, $x_1 > 0$, $x_2 > 0$, hence $v_1 = v_2 = 0$

Now find $u_1 \ge 0$ and $u_2 \ge 0$ so that all KT conditions are satisfied.

$$u_1 = 1, u_2 = 1$$

Note that the optimal objective function value:

$$3x_1 + 2x_2 = 3(20) + 2(60) =$$

$$100u_1 + 80u_2 + 40u_3 =$$

$$100(1) + 80(1) + 40(0) = 180$$

4. Dual of a linear programming problem

Primal Linear Program:

maximize $3x_1 + 2x_2$

subject to: $2x_1 + x_2 \leq 100$

$x_1 + x_2 \leq 80$

$x_1 \leq 40$

$x_1, x_2 \geq 0$

Dual linear program:

minimize $100u_1 + 80u_2 + 40u_3$

subject to: $2u_1 + u_2 + u_3 \geq 3$

$u_1 + u_2 \geq 2$

$u_1, u_2, u_3 \geq 0$

Primal LP: Max $c^t x$ st. $Ax \leq b$, $x \geq 0$

Dual LP: Min $b^t u$ st. $A^t u \geq c$, $u \geq 0$

A =

2	1
1	1
1	0

A^t =

2	1	1
1	1	0

Dual of a linear programming problem

Dual linear program:

minimize $100\ u_1 + 80\ u_2\ + 40\ u_3$

subject to: $2\ u_1 + u_2\ + u_3 \geq\ 3$

$u_1 + u_2 \geq\ 2$

$u_1, u_2, u_3 \geq\ 0$

<u>KT necessary and sufficient conditions:</u>

$100 - 2x_1 - x_2 - w_1 = 0$

$80 - x_1 - x_2 - w_2 = 0$

$40 - x_1 - w_3 = 0$

$x_1(\ -2\ u_1 - u_2\ - u_3\ + 3) = 0$

$x_2\ (- u_1 - u_2\ + 2) = 0$

$w_1u_1 = 0\ ,\ w_2u_2\ = 0,\ w_3u_3 = 0$

$x_1 \geq\ 0,\ \ x_2 \geq\ 0,\ \ w_1 \geq\ 0,\ w_2\ \geq\ 0,\ w_3\ \geq\ 0$

Show that the following values satisfy the KT conditions:

- $x_1 = 20,\ x_2 = 60$
- $u_1 = 1,\ u_2 = 1,\ u_3 = 0$
- $w_1 = 0,\ w_2 = 0,\ w_3 = 20$

(must also show that u_1, u_2, u_3 satisfy the dual constraints to conclude that u_1,u_2,u_3 is a solution of dual LP)

5. Standard form Linear Program

Standard form LP: an LP with only equality constraints and non-negative variables

Any LP can be easily converted to standard LP:

Example 1.

Original LP: maximize $3x_1 + 2x_2$

$$\begin{aligned} \text{subject to:}\quad 2x_1 + x_2 &\le 100 \\ x_1 + x_2 &\le 80 \\ x_1 &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Standard LP: maximize $3x_1 + 2x_2$

$$\begin{aligned} \text{subject to:}\quad 2x_1 + x_2 + s_1 &= 100 \\ x_1 + x_2 + s_2 &= 80 \\ x_1 + s_3 &= 40 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

s_1, s_2, s_3 are called slack variables

Standard form Linear Program

Example 2.

Original LP: minimize $100\, u_1 + 80\, u_2 + 40\, u_3$

$$
\begin{aligned}
\text{subject to: } \quad & 2\, u_1 + u_2 + u_3 \geq 3 \\
& u_1 + u_2 \geq 2 \\
& u_1, u_2 \geq 0 \\
& u_3 \text{ unrestricted in sign}
\end{aligned}
$$

Standard LP: minimize $100\, u_1 + 80\, u_2 + 40\, (u_3' - u_3'')$

$$
\begin{aligned}
\text{subject to: } \quad & 2\, u_1 + u_2 + u_3' - u_3'' - s_1 = 3 \\
& u_1 + u_2 - s_2 = 2 \\
& u_1, u_2, u_3', u_3'', s_1, s_2 \geq 0
\end{aligned}
$$

s_1, s_2 are called <u>surplus/excess</u> variables

- Why need to convert LP into standard form? An algorithm for solving LP, the <u>simplex algorithm</u> requires that the LP is in standard form.
- How would you express the complementarity conditions in terms of the slack variables?

Theorem of Complementary Slackness

6. Economic interpretation of the dual problem

DF Company manufactures desks, tables, and chairs.

The resources are as follows:

Resource	Desk	Table	Chair	Available
Lumber	8 board ft	6 board ft	1 board ft	48 board ft
Finishing hour	4 hours	2 hours	1.5 hours	20 finishing hours
Carpentry hour	2 hours	1.5 hours	0.5 hours	8 carpentry hours

A desk sells for \$60, a table for \$30, a chair \$20.

Let x_1, x_2, x_3 be the numbers of desks, tables, and chairs produced.

Linear Program: maximize $60x_1 + 30x_2 + 20x_3$

subject to:

$$\begin{aligned} 8x_1 + 6x_2 + x_3 &\le 48 && \text{(Lumber constraint)} \\ 4x_1 + 2x_2 + 1.5\,x_3 &\le 20 && \text{(Finishing constraint)} \\ 2x_1 + 1.5x_2 + 0.5\,x_3 &\le 8 && \text{(Carpentry constraint)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Economic interpretation of the dual problem

Primal LP: maximize $60x_1 + 30x_2 + 20x_3$

subject to:
$$8x_1 + 6x_2 + x_3 \le 48 \quad \text{(Lumber constraint)}$$
$$4x_1 + 2x_2 + 1.5\,x_3 \le 20 \quad \text{(Finishing constraint)}$$
$$2x_1 + 1.5x_2 + 0.5\,x_3 \le 8 \quad \text{(Carpentry constraint)}$$
$$x_1, x_2, x_3 \ge 0$$

Dual LP: minimize $48u_1 + 20u_2 + 8u_3$

subject to:
$$8u_1 + 4\,u_2 + 2\,u_3 \ge 60$$
$$6u_1 + 2\,u_2 + 1.5\,u_3 \ge 30$$
$$u_1 + 1.5u_2 + 0.5\,u_3 \ge 20$$
$$u_1,\ u_2,\ u_3 \ge 0$$

Solution:

Primal: $x_1 = 2, x_2 = 0, x_3 = 8$

Dual: $u_1 = 0, u_2 = 10, u_3 = 10$

Objective function = 280

Economic interpretation of the dual problem

- u_1 = price paid for 1 board foot of lumber
- u_2 = price paid for 1 finishing hour
- u_3 = price paid for 1 carpentry hour
- Total price to be paid is $48\,u_1 + 20\,u_2 + 8\,u_3$
- A buyer must be willing to pay at least $60 for the combination that includes 8 board ft of lumber, 4 finishing hours, and 2 carpentry hours. Why?

  Hence $8u_1 + 4\,u_2 + 2\,u_3 \geq 60$
- Similarly, the buyer must be willing to pay at least $30 for the combination that includes 6 board ft of lumber, 2 finishing hours, and 1.5 carpentry hours.

  Hence $6u_1 + 2\,u_2 + 1.5\,u_3 \geq 30$
- Also, $u_1 + 1.5u_2 + 0.5\,u_3 \geq 20$

Objective function: minimize $48\,u_1 + 20\,u_2 + 8\,u_3$

Economic interpretation of the dual problem

Dual variables are also referred to as **resource shadow prices**

Another interpretation of dual variables/shadow prices:

The shadow price of the i-th constraint is the amount by which the optimal objective function value is improved if we increase the corresponding right hand side by 1

Primal LP: maximize $60x_1 + 30x_2 + 20x_3$

subject to:

$$
\begin{aligned}
8x_1 + 6x_2 + x_3 &\leq 48 \quad \text{(Lumber)}\\
4x_1 + 2x_2 + 1.5x_3 &\leq 20 \quad \text{(Finishing)}\\
2x_1 + 1.5x_2 + 0.5x_3 &\leq 8 \quad \text{(Carpentry)}\\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Solution:

Primal: $x_1 = 2, x_2 = 0, x_3 = 8$

Dual: $u_1 = 0, u_2 = 10, u_3 = 10$

Objective function = 280

One additional foot of lumber will not increase the revenue, because $u_1 = 0$

If one additional hour of carpentry (now 9 hours are available), then the revenue would increase by u_3 to $(280 + 10) = 290$. Optimal values for x_1, x_2, x_3 would change.

7. Dual of nonlinear programming problem

Consider the primal NLP

$$\text{minimize } f(x_1,x_2,x_3,\ldots, x_n)$$

$$\text{subject to: } g_1(x_1,x_2,x_3,\ldots, x_n) \le b_1$$

$$g_2(x_1,x_2,x_3,\ldots, x_n) \le b_2$$

$$\ldots\ldots$$

$$g_m(x_1,x_2,x_3,\ldots, x_n) \le b_m$$

Its dual is:

$$\max f(x_1,x_2,x_3,\ldots, x_n) + \lambda_1 [g_1(x_1,x_2,x_3,\ldots, x_n) - b_1] + \lambda_2 [g_2(x_1,x_2,x_3,\ldots, x_n) - b_2]$$

$$+ \ldots + \lambda_m [g_m(x_1,x_2,x_3,\ldots x_n) - b_m]$$

st:

$$\partial f(x_1,x_2,x_3,.., x_n)/\partial x_j + \sum_{i=1}^{m} \lambda_i \, \partial g_i(x_1,x_2,x_3.., x_n)/ \partial x_j = 0 \quad , j = 1,2, \ldots n$$

$$\lambda_i \ge 0 , i = 1,2, \ldots. m$$

Dual of nonlinear programming problem

Consider the primal QP

minimize $\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\mathbf{x} + \mathbf{p}^t\,\mathbf{x}$

subject to $\mathbf{A}\mathbf{x} \le \mathbf{a}$

$\mathbf{x} \ge \mathbf{0}$

Its dual is:

maximize $\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} + \mathbf{p}^t\mathbf{x} + \mathbf{u}^t(\mathbf{A}\mathbf{x} - \mathbf{a}) - \mathbf{v}^t\,\mathbf{x}$

st: $\mathbf{P}\,\mathbf{x} + \mathbf{p} + \mathbf{A}^t\,\mathbf{u} - \mathbf{v} = \mathbf{0}$

$\mathbf{u}, \mathbf{v} \ge \mathbf{0}$

Multiply this constraint by **x** and then simplify the objective function

An equivalent formulation:

\- minimize $\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} + \mathbf{u}^t\mathbf{a}$

st: $\mathbf{P}\,\mathbf{x} + \mathbf{p} + \mathbf{A}^t\,\mathbf{u} \ge \mathbf{0}$

$\mathbf{u} \ge \mathbf{0}$

$$\mathbf{x}^t\,\mathrm{P}\,\mathbf{x} + \mathbf{x}^t\,\mathbf{p} + \mathbf{x}^t\,\mathrm{A}^t\mathbf{u} - \mathbf{x}^t\mathbf{v} = \mathbf{0}$$
$$-\mathbf{x}^t\,\mathrm{P}\mathbf{x} = \mathbf{p}^t\mathbf{x} + \mathbf{u}^t\mathrm{A}\mathbf{x} - \mathbf{v}^t\mathbf{x}$$

$$\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} + \mathbf{p}^t\mathbf{x} + \mathbf{u}^t(\mathbf{A}\mathbf{x} - \mathbf{a}) - \mathbf{v}^t\,\mathbf{x} =$$
$$\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} - \mathbf{x}^t\,\mathrm{P}\mathbf{x} + \mathbf{u}^t(-\mathbf{a}) =$$
$$-\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} - \mathbf{u}^t(\mathbf{a})$$

Dual of nonlinear programming problem

Example

minimize $\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\mathbf{x} + \mathbf{p}^t\,\mathbf{x}$

subject to $\mathbf{A}\mathbf{x} \le \mathbf{a}$

$\mathbf{x} \ge \mathbf{0}$

Dual:

\- minimize $\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\,\mathbf{x} + \mathbf{u}^t\mathbf{a}$

st: $\mathbf{P}\,\mathbf{x} + \mathbf{p} + \mathbf{A}^t\,\mathbf{u} \ge \mathbf{0}$

$\mathbf{u} \ge \mathbf{0}$

Primal: minimize $x_1^2 + x_2^2$

subject to: $-x_1 - x_2 \le -4$

$x_1 + 2x_2 \le 8$

$x_1 \ge 0$

$x_2 \ge 0$

$\frac{1}{2}\,\mathbf{x}^t\,\mathbf{P}\mathbf{x} + \mathbf{p}^t\,\mathbf{x} = x_1^2 + x_2^2$

$$\mathbf{P}\mathbf{x} + \mathbf{p} = \begin{bmatrix} 2\,x_1 \\ 2\,x_2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\mathbf{A}^t\mathbf{u} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 + u_2 \\ -u_1 + 2u_2 \end{bmatrix}$$

Dual

\- minimize $x_1^2 + x_2^2 - 4\,u_1 + 8\,u_2$

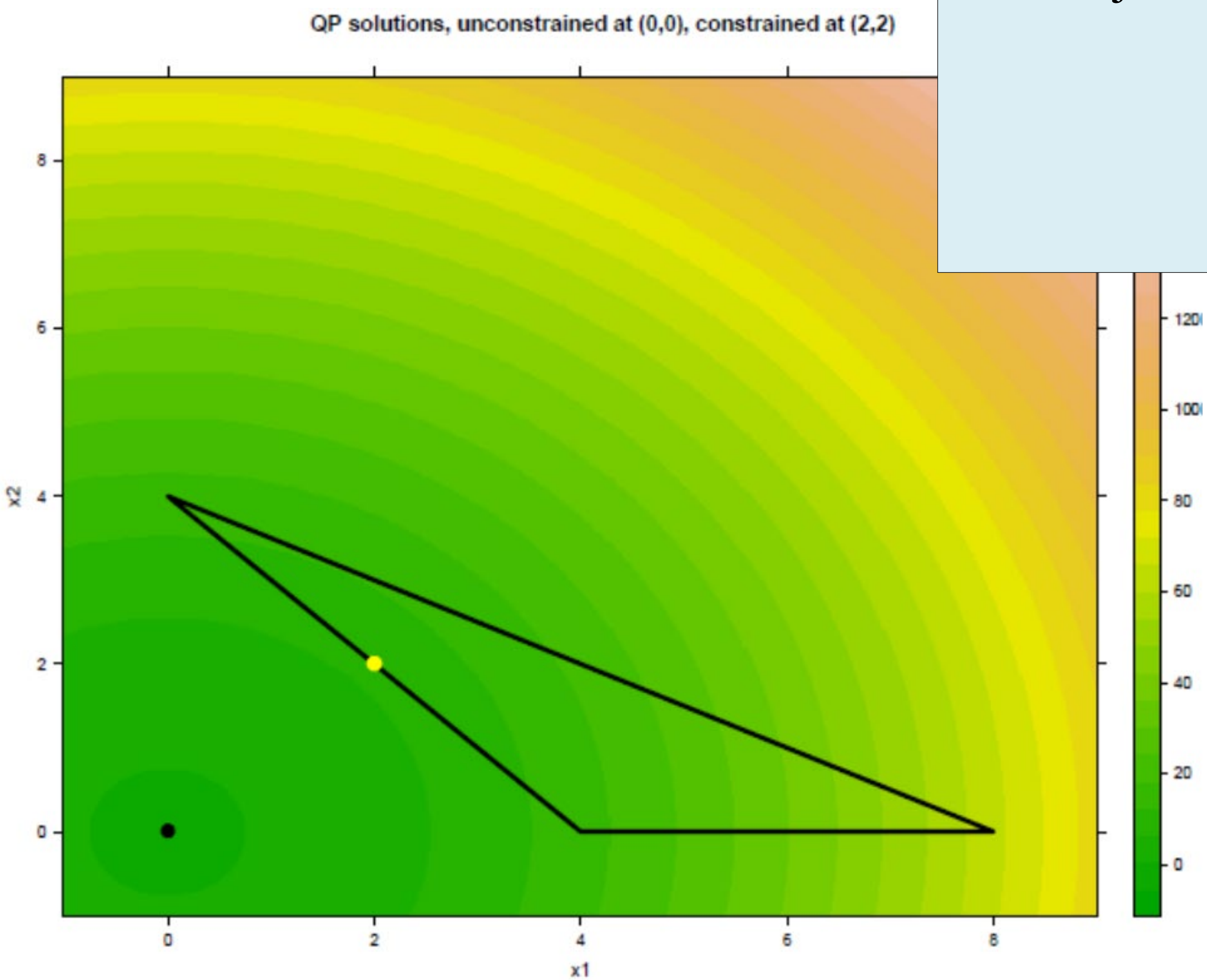
subject to: $2\,x_1 - u_1 + u_2 \ge 0$

$2\,x_2 - u_1 + 2\,u_2 \ge 0$

$u_1 \ge 0$

$u_2 \ge 0$

Dual of nonlinear programming problem



Primal: minimize $x_1^2 + x_2^2$

$$
\begin{aligned}
\text{subject to: } -x_1 - x_2 &\le -4 \\
x_1 + 2x_2 &\le 8 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$

Dual of nonlinear programming problem

Primal: minimize $x_1^2 + x_2^2$

subject to: $-x_1 - x_2 \le -4$

$x_1 + 2x_2 \le 8$

$x_1 \ge 0$

$x_2 \ge 0$

Dual

$-$ minimize $x_1^2 + x_2^2 - 4u_1 + 8u_2$

subject to: $2x_1 - u_1 + u_2 \ge 0$

$2x_2 - u_1 + 2u_2 \ge 0$

$u_1 \ge 0$

$u_2 \ge 0$

KT conditions:

$2x_1 - u_1 + u_2 - v_1 = 0$ ♣

$2x_2 - u_1 + 2u_2 - v_2 = 0$ ♣

$u_1(-x_1 - x_2 + 4) = 0$ ♦

$u_2(x_1 + 2x_2 - 8) = 0$ ♠

$x_1 v_1 = 0, \quad x_2 v_2 = 0$ ♠

Solution:

- x_1 and x_2 cannot be both 0. Why?
- Suppose $x_1 > 0$, $x_2 > 0$, and $x_1 + 2x_2 < 8$
- Then $v_1 = 0$, $v_2 = 0$, $u_2 = 0$ ♠
- Hence, $u_1 = 2x_1 = 2x_2 > 0$ ♣
- And $x_1 + x_2 = 4$, $x_1 = x_2 = 2$, ♦
- Therefore, $u_1 = 4$
- Dual objective function value $= -(x_1^2 + x_2^2 - 4u_1 + 8u_2) = -(4 + 4 - 16 + 0) = 8$

8. Application of optimization in decision making

The Multisurface Method (MSM)

MSM is a method based on linear program for finding a piece-wise linear discriminant function.

We first consider the case when the samples from the sets A and B are **linearly separable**.

The linear program is to find **w**, α and β such that:

(LP) max $\alpha - \beta$

subject to

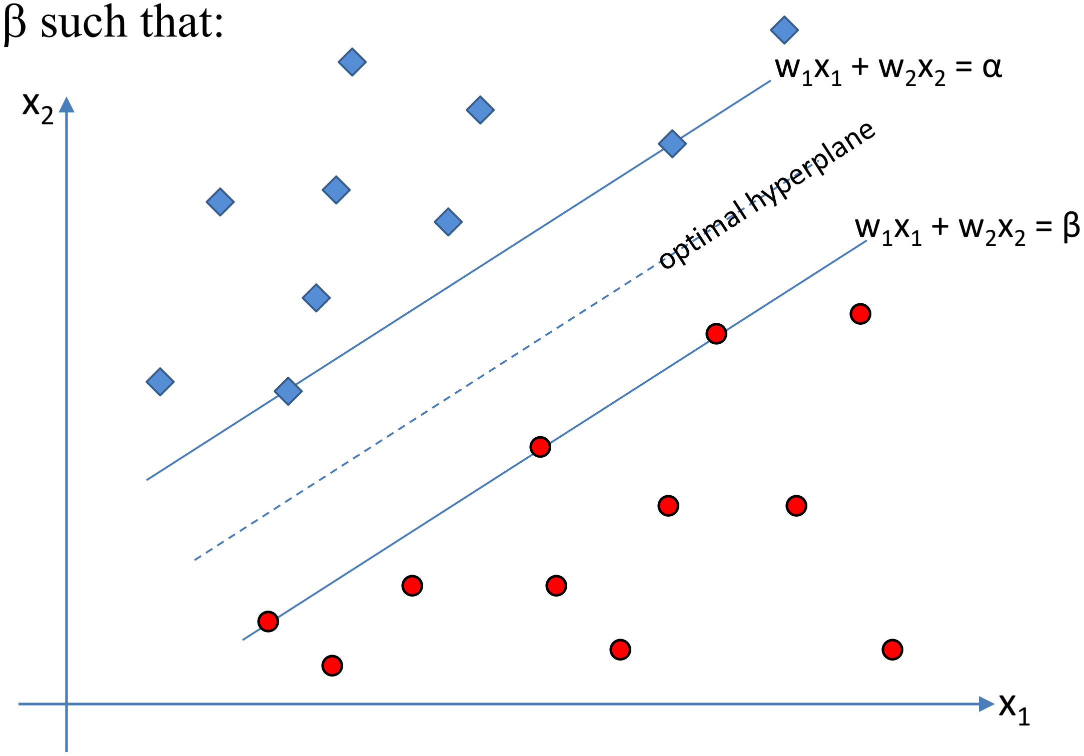
$\mathbf{w}^T\mathbf{x_i} \geq \alpha$ for $\mathbf{x_i} \in A$

$\mathbf{w}^T\mathbf{x_i} \leq \beta$ for $\mathbf{x_i} \in B$

$-1 \leq \mathbf{w} \leq 1$

The optimal hyperplane is

$\mathbf{w}^T\mathbf{x} = (\alpha + \beta)/2$



<u>The Multisurface Method (MSM)</u>

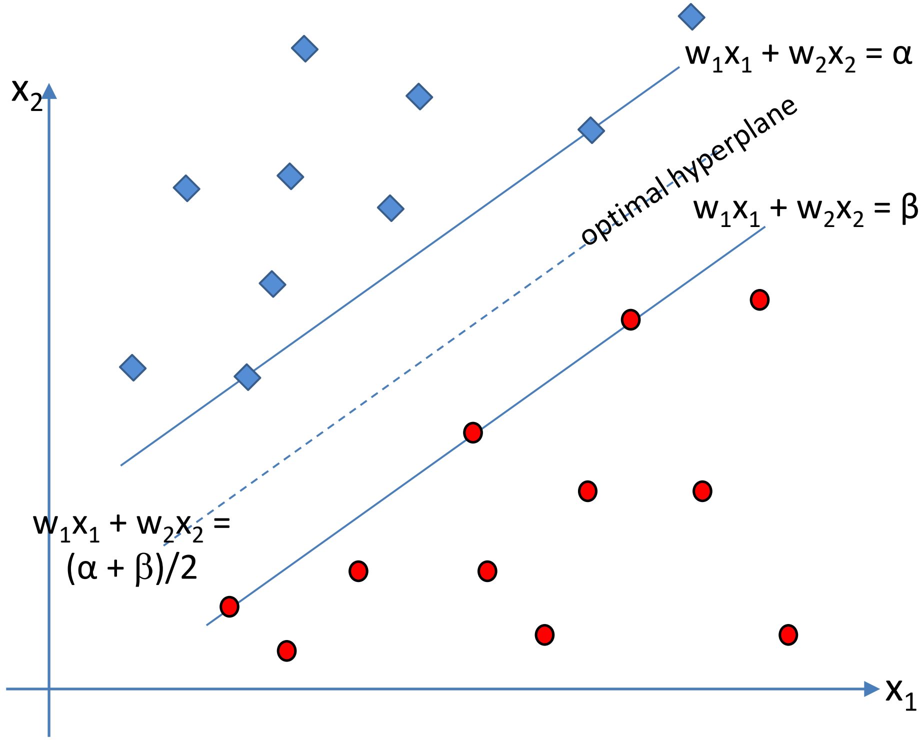
(LP) max α - β

subject to

$\mathbf{w}^T\mathbf{x_i} \geq \alpha$ for $\mathbf{x_i} \in A$

$\mathbf{w}^T\mathbf{x_i} \leq \beta$ for $\mathbf{x_i} \in B$

$\boxed{-1 \leq \mathbf{w} \leq 1}$



Without the constraint $-1 \leq \mathbf{w} \leq 1$, when the samples are linearly separable, the LP will be <u>unbounded</u>:

Let ($\mathbf{w}^*$, α^* , β^*) be feasible for the LP and $Z = \alpha^* - \beta^*$, then for any $k > 0$,

($k\mathbf{w}^*$, $k\alpha^*$, $k\beta^*$) is also feasible, and the objective function value is $k\alpha^* - k\beta^* = k Z$ can have arbitrarily large value.

Application of optimization in decision making

<u>The Multisurface Method (MSM):</u> Nonlinearly separable case

When the samples are linearly non-separable, a sequence of linear programs are solved.

The linear program at each iteration is:

(LP) max α - β

subject to

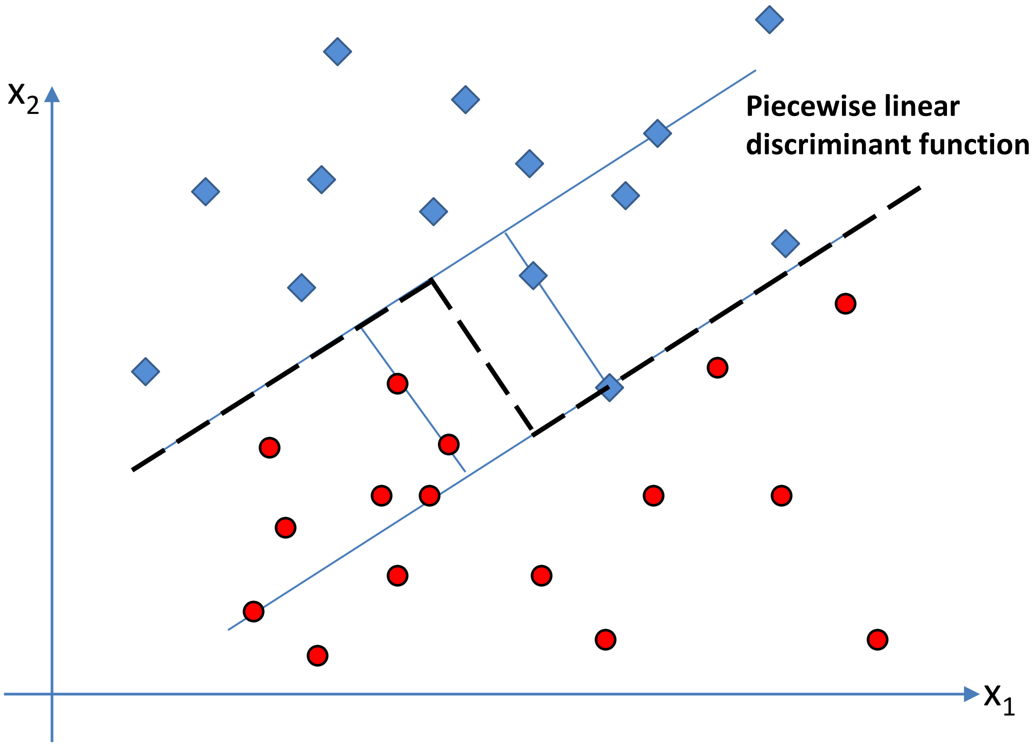
$\mathbf{w}^T\mathbf{x_i} \geq \alpha$ for $\mathbf{x_i} \in A$

$\mathbf{w}^T\mathbf{x_i} \leq \beta$ for $\mathbf{x_i} \in B$

$-1 \leq \mathbf{w_i} \leq 1, i = 1, 2, \ldots, N$

$w_d = 1$ or $w_d = -1$ $\quad d = 1,2,\ldots, N$

The constraint $w_d = 1$ or $w_d = -1$ is added to ensure that the solution of the LP is non-trivial ($\mathbf{w} \neq 0$) when the samples are non-linearly separable.



Application of optimization in decision making

The Multisurface Method (MSM):

(Note: the method is for a 2 class problem, N =the dimensionality of the samples)

- Step 1. Let A be the whole sample set containing samples from set group 1, and B the set containing samples from group 2.
- Step 2: Solve linear program 2*N linear programs LP.
- Step 3: Of the 2N solutions (= hyperplanes), find one that gives the minimum total number of incorrectly classified samples.
- Step 4: If there are incorrectly classified samples, let A be the set of samples from group 1 that are incorrectly classified and B the set of samples from group 2 that are incorrectly classified. Go to Step 2.

(Note: The algorithm terminates when there are no more incorrectly classified samples, or when the number of minimum number of such samples is below a threshold, or if the maximum number of hyperplanes has been reached)

References.

1. W.L. Winston, 3rd Edition, Sections 12.8, 12.9, 6.5, 6.6, 6.7, 3.1, 3.1

   or W.L. Winston, 4th Edition, Chapter 11 and Chapter 6.

2. O.L. Mangasarian, R. Setiono, W.H. Wolberg, Pattern recognition via linear programming: Theory and application to medical diagnosis, in Large Scale Numerical Optimization, 1989, T.F. Coleman and Y. Li Editors, SIAM Press, pages 22-30.

   (This paper presents the method formally and describes its application in medical diagnosis. More information about the method and application is available at U Wisconsin-Madison http://www.cs.wisc.edu/~olvi/uwmp/cancer.html)

Data set: https://archive.ics.uci.edu/ml/datasets/Breast+Cancer+Wisconsin+(Original)