Decision making with multiple objectives

Outline

1. Multi-attribute decision making with no uncertainty: goal programming
2. Analytic hierarchy process
3. Goal programming approach to pattern classification

1. Multi-attribute decision problems

- Problems where the decision maker must choose an action by determining how each possible action affects the relevant attributes.
- For example:

  Ah Goo just graduated from a university and has received 5 job offers.

  In choosing which to accept, he will consider the following attributes of each job:

    - Starting salary of job
    - Location of job
    - Degree of interest he has in doing the work involved in a job
    - Long-term opportunities of the job

Goal programming

- **Goal programming** is an approach that is used for solving a multi-objective optimisation problem as a linear program that balances trade-offs in conflicting objectives.

- To apply goal programming in reaching a decision, identify:

    - A goal in the form of a specific numerical target value you wish that objective to achieve and

    - A penalty in the form of a value for each unit the objective is below the goal if the objective is to maximise or above the goal if the objective is to minimise.

Example 1

- MTV Steel Company produces three sizes of tubes.
- The relevant data are as follows:

Tube type	Selling price ($)	Demand (ft)	Machine Time (min/ft)	Welding Material (oz/ft)	Production Cost ($/ft)	Purchase Cost (from Japan) ($/ft)
A	10	2000	0.50	1	3	6
B	12	4000	0.45	1	4	6
C	9	5000	0.60	1	4	7
Available amount			40 hour	5500 oz		

Example 1 (continued)

- The demand for tubes A, B and C are 2000, 4000, and 5000 units, respectively.
- Each foot of A requires 0.50 minutes of processing time, B 0.45 minutes, and C 0.60 minutes.
- If all demands are to be met, the total processing time is

  $$2000\times(0.50/60) + 4000\times(0.45/60) + 5000\times(0.60/60) \text{ hours} =$$

  96.67 hours.
- Each foot of A requires 1 ounce of welding, B also 1 ounce, and so does C.
- If all demands are to be met, the total welding material needed is

  $$2000\times1 + 4000\times1 + 5000\times1 = \mathbf{11000} \text{ ounces.}$$
- Only **40** hours of machine time and **5500** ounces of welding material are available.

Example 1 (continued)

- MTV is considering to purchase some tubes from suppliers in Japan at the cost of \$6 per foot of A, \$6 per foot of B, and \$7 per foot of C.
- The first objective is to maximise the company's profit.
- However, a second objective arises when the CEO was informed that the government has requested a voluntary effort to reduce the amount of money spent on import.
- These are the two objectives to achieve simultaneously.

Example 1 (continued)

The steps to solve the MTV Steel problem:

- Define the six decision variables:

  AP = the number of feet of A to produce

  BP = the number of feet of B to produce

  CP = the number of feet of C to produce

  AJ = the number of feet of A to buy from Japan

  BJ = the number of feet of B to buy from Japan

  CJ = the number of feet of C to buy from Japan

- The two objectives are:
  - § maximise profit: (7AP + 8BP + 5CP) + (4AJ + 6BJ + 2CJ)
  - § minimise cost of import: 6AJ + 6BJ + 7CJ

$$\begin{bmatrix} 7 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 9 \end{bmatrix} - \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix}$$

Example 1 (continued)

The constraints are:

- **Demand constraints:**

  for type A: $AP + AJ = 2000$

  for type B: $BP + BJ = 4000$

  for type C: $CP + CJ = 5000$

- **Resource constraints:**

  for machine time: $0.5\ AP + 0.45\ BP + 0.6\ CP \leq 2400$

  for welding material: $AP + BP + CP \leq 5500$

- **Logical constraints:**

  $AP, BP, CP, AJ, BJ, CJ \geq 0$

Example 1 (continued)

- The optimal solution of the linear program (LP) to maximise the profit is

AP = 2000	AJ = 0
BP = 0	BJ = 4000
CP = 2333.33	CJ = 2666.67

The total net profit is \$55,000 and the total cost of import is

$$6AJ + 6BJ + 7CJ = 6\times0 + 6\times4000 + 7\times2666.67 = \$42{,}666.67$$

- And the solution of the LP which minimises the cost of import is

AP = 1200	AJ = 800
BP = 4000	BJ = 0
CP = 0	CJ = 5000

with the total cost of import = $6\times800 + 0 + 7\times5000 = \$39{,}800$ and

a total profit of $7AP + 8BP + 5CP + 4AJ + 6BJ + 2CJ = \$53{,}600$.

Example 1 (continued)

To apply goal programming approach to deal with the conflicting objectives identify:

- **The goal**: Since the maximum achievable profit is $55,000, the CEO may choose to set this value as the target to reflect the goal of achieving the highest possible profit. On the other hand, suppose the CEO is satisfied if an attempt is made to achieve an import cost of $40,000 .
- **The penalty**: Two penalties are chosen to reflect the relative trade-offs between the objectives. If the CEO feels that it is twice as important to meet the target of $55,000 for the profit as it is to meet the target of $40,000 for the cost of imports, then
    - profit penalty = 2 for each dollar of profit below $55,000
    - import penalty = 1 for each dollar of import cost above $40,000

Example 1 (continued)

- LP formulation of the goal programming problem:
- Identify the decision variables:

AP, BP, CP, AJ, BJ, CJ as defined before

  - P^+ = dollar amount by which profit exceeds \$55,000
  - P^- = dollar amount by which profit falls under \$55,000
  - I^+ = dollar amount by which import exceeds \$40,000
  - I^- = dollar amount by which import falls under \$40,000

- Identify the objective function: minimise the total penalty for not meeting the two goals:

minimise $2P^- + 1I^+$

Example 1 (continued)

- Identify the constraints:

Demand constraints:

$AP + AJ = 2000$ $BP + BJ = 4000$ $CP + CJ = 5000$

Resource constraints:

for machine time: $0.5\ AP + 0.45\ BP + 0.6\ CP \leq 2400$

for welding material: $AP + BP + CP \leq 5500$

- **Goal constraints:**

profit goal:

$7AP + 8BP + 5CP + 4AJ + 6BJ + 2CJ - P^{+} + P^{-} = 55000$

import goal:

$6AJ + 6BJ + 7CJ - I^{+} + I^{-} = 40000$

- **Logical constraints:** $AP, BP, CP, AJ, BJ, CJ, P^{+}, P^{-}, I^{+}, I^{-} \geq 0$

Example 1 (continued)

Optimal solution of the linear programs:

	Produce			*Import*			*Profit*	*Import cost*
	A	B	C	A	B	C		
Profit Maximising LP	2000	0	2333.33	0	4000	2666.67	55000	42666.69
Import Minimising LP	1200	4000	0	800	0	5000	53600	39800.00
Penalty Minimising LP	2000	3111.11	0	0	888.89	5000	54222.22	40333.33

Profit: not as good

but Import cost: not as bad

Example 2

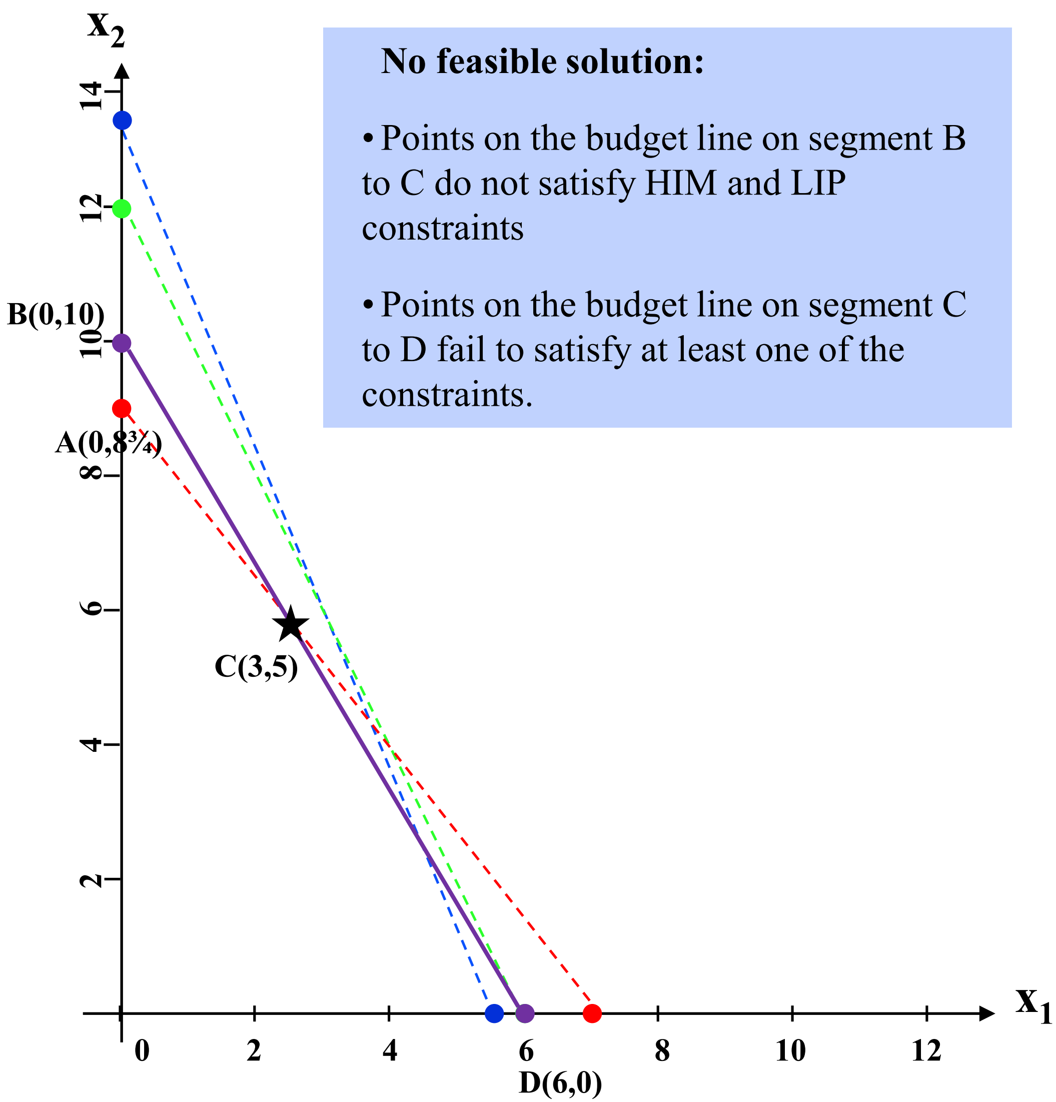
The Must-see advertising agency is trying to determine a TV advertising schedule for Priceless Auto Company. The company has 3 goals:

- Goal 1: its ads should be seen by at least 40 millions high income men (HIM).
- Goal 2: its ads should be seen by at least 60 millions low income people (LIP).
- Goal 3: its ads should be seen by at least 35 millions high income women (HIW).

The agency can purchase 2 types of ads. The cost per minute and the potential audiences of a one-minute ad are:

	HIM	LIP	HIW	Cost
Football Ad	7 million	10 million	5 million	$100,000
Soap opera Ad	3 million	5 million	4 million	$60,000

Example 2 (continued)



No feasible solution:

- Points on the budget line on segment B to C do not satisfy HIM and LIP constraints
- Points on the budget line on segment C to D fail to satisfy at least one of the constraints.

- Total budget is $600,000
- Let

  x_1 = min of ads shown during football games

  x_2 = min of ads shown during soap operas

- The constraints are

  HIM: $7x_1 + 3x_2 \geq 40$

  LIP: $10x_1 + 5x_2 \geq 60$

  HIW: $5x_1 + 4x_2 \geq 35$

  Budget: $100x_1 + 60x_2 \leq 600$

Example 2 (continued)

- Suppose the auto company determines the following penalty:
  - Each million by which it falls short of the HIM goal costs $200,000 in lost sales.
  - Each million by which it falls short of the LIP goal costs $100,000 in lost sales.
  - Each million by which it falls short of the HIW goal costs $50,000 in lost sales.
- We define the following variables:
  - s_i^+ = amount by which we numerically exceed the *i*-th goal.
  - s_i^- = amount by which we are numerically under the *i*-th goal.

These variables are referred to as <u>deviational</u> variables.

Example 2 (continued)

- The linear program to be minimised is

$$\min \; Z = 200\, s_1^- + 100\, s_2^- + 50\, s_3^-$$

subject to

$$7x_1 + 3x_2 + s_1^- - s_1^+ = 40 \text{ (HIM constraint)}$$

$$10x_1 + 5x_2 + s_2^- - s_2^+ = 60 \text{ (LIP constraint)}$$

$$5x_1 + 4x_2 + s_3^- - s_3^+ = 35 \text{ (HIW constraint)}$$

$$100x_1 + 60x_2 \leq 600 \text{ (Budget constraint)}$$

$$x_1, x_2, s_1^-, s_1^+, s_2^-, s_2^+, s_3^-, s_3^+ \geq 0$$

This is an LP with 8 variables, use a computer program to solve it!

- And the optimal solution is

$Z = 250$ (thousand)

$x_1 = 6,\; x_2 = 0$

$s_1^- = 0,\; s_1^+ = 2,\; s_2^- = 0,\; s_2^+ = 0,\; s_3^- = 5,\; s_3^+ = 0$

Preemptive goal programming

- When the decision maker is not able to determine precisely the relative importance of the goals, **preemptive goal programming** may be a useful tool.
- The objective function coefficient for the variable representing goal i will be P_i and we assume that: $\mathbf{P_1 >>> P_2 >>> P_3 \ldots\ldots >>> P_n}$
- For Example 2, we have the preemptive goal programming formulation

$$\min \; Z = P_1 s_1^- + P_2 s_2^- + P_3 s_3^-$$

$$\text{s.t. } 7x_1 + 3x_2 + s_1^- - s_1^+ = 40 \text{ (HIM constraint)}$$

$$10x_1 + 5x_2 + s_2^- - s_2^+ = 60 \text{ (LIP constraint)}$$

$$5x_1 + 4x_2 + s_3^- - s_3^+ = 35 \text{ (HIW constraint)}$$

$$100x_1 + 60x_2 \leq 600 \text{ (Budget constraint)}$$

$$x_1, x_2, s_1^-, s_1^+, s_2^-, s_2^+, s_3^-, s_3^+ \geq 0$$

Given the budget of $600,000:

• First try to satisfy HIM constraint. If it possible to do this, then try to satisfy LIP constraint, etc.

• Stop when a constraint cannot be satisfied and find a solution that least violate this constraint.

Preemptive goal programming (continued)

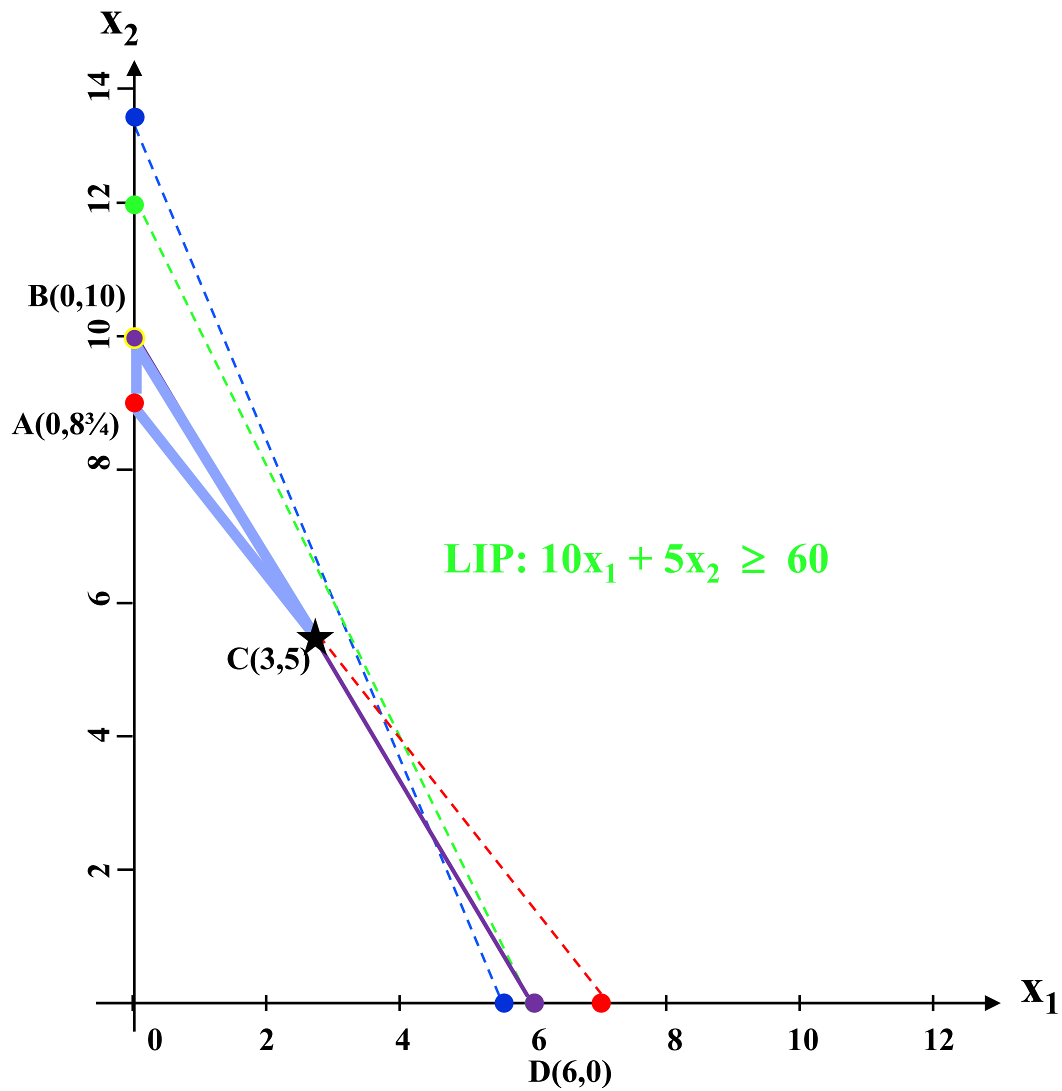
- Since the problem involves 2 variables, we will use the **graphical approach** to solve it.
- Suppose that
  - HIW has the highest priority,
  - LIP the second highest and
  - HIM the lowest.
- The triangle ABC contains points that satisfy HIW and budget constraints.
- No point in ABC satisfies the LIP constraint, but point C(3,5) is the closest.

  A(0,8¾): $10x_1 + 5x_2 = 43.75$

  B(0,10): $10x_1 + 5x_2 = 50$

  C(3,5): $10x_1 + 5x_2 = 55$ (nearest to 60)

- For this set of of priorities the solution is to purchase 3 football ads and 5 soap opera ads.



Preemptive goal programming (continued)

Optimal solution found by preemptive goal programming for all possible combinations of priorities:

Priorities			*Optimal values of*		*Deviation from*		
1st	2nd	3rd	x_1	x_2	HIM	LIP	HIW
HIM	LIP	HIW	6	0	0	0	5
HIM	HIW	LIP	5	5/3	0	5/3	10/3
LIP	HIM	HIW	6	0	0	0	5
LIP	HIW	HIM	6	0	0	0	5
HIW	HIM	LIP	3	5	4	5	0
HIW	LIP	HIM	3	5	4	5	0

2. The analytic hierarchy process

- Sometimes it is difficult to choose between alternatives:

  Objective 1: High starting salary (SAL)

  Objective 2: Quality of life where job is located (QL)

  Objective 3: Interest in work (IW)

  Objective 4: Job location near family (NF)

- A job may meet objectives 2-4, but offer low starting salary.
- **Analytic hierarchy process (AHP)** provides a powerful tool for decision making in situations involving multiple objectives.
- AHP generates a weight w_i for the *i*th objective.

The analytic hierarchy process (continued)

- Let us assume that the weights are

  $w_1 = 0.5115 \quad w_2 = 0.0986 \quad w_3 = 0.2433 \quad w_4 = 0.1466$

  and that there are 3 job offers with the following score:

Objective	Job 1	Job 2	Job 3
SAL: salary	0.571	0.286	0.143
QL: quality of life	0.159	0.252	0.589
IW: interest in work	0.088	0.669	0.243
NF: near family	0.069	0.426	0.506

- Choose the job with the highest overall score:
  - job 1: .5115(.571) + .0986(.159) + .2433(.088) + .1466(.069) = .339
  - **job 2: .5115(.286) + .0986(.252) + .2433(.669) + .1466(.426) = .396**
  - job 3: .5115(.143) + .0986(.589) + .2433(.243) + .1466(.506) = .265

The analytic hierarchy process (continued)

- To obtain weights for each of the n objectives, first generate the pair-wise comparison matrix. For example:

$$\begin{array}{c} \\ SAL \\ QL \\ IW \\ NF \end{array} \begin{array}{c} \begin{array}{cccc} SAL & QL & IW & NF \end{array} \\ \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1/5 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & 2 \\ \frac{1}{4} & 2 & \frac{1}{2} & 1 \end{bmatrix} \end{array}$$

- Interpretation of entries in a pair-wise comparison matrix:
  - $A_{ij} = 1$: objectives i and j are of equal importance
  - $A_{ij} = 3$: objective i is weakly more important than objective j
  - $A_{ij} = 5$: objective i is strongly more important than objective j
  - $A_{ij} = 7$: objective i is very strongly more important than objective j
  - $A_{ij} = 9$: objective i is absolutely more important than objective j
  - $A_{ij} = 2,4,6,8$: intermediate values

The analytic hierarchy process (continued)

- Note that $A_{ij} \times A_{ji} = 1$.
- A perfectly consistent decision maker will have a comparison matrix

$$A = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \ldots. & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \ldots. & w_2/w_n \\ . & . & \ldots. & . \\ w_n/w_1 & w_n/w_2 & \ldots. & w_n/w_n \end{bmatrix}$$

- For example: if $w_1 = 1/2$ and $w_2 = 1/6$, then objective 1 is three times as important as objective 2, so

  $a_{12} = w_1/w_2 = 3$

- We want to recover $w_1, w_2, \ldots, w_n$ from the matrix A.

The analytic hierarchy process (continued)

Steps to estimate the decision maker's weights:

- Divide each entry by the sum of the entries in the same column.
- For our example, we get

$$A_{norm} = \begin{pmatrix} .5128 & .5000 & .5000 & .5333 \\ .1026 & .1000 & .1250 & .0667 \\ .2564 & .2000 & .2500 & .2667 \\ .1282 & .2000 & .1250 & .1333 \end{pmatrix}$$

- Estimate w_i as the average of the entries in row A:

$$w_1 = (.5128 + .5000 + .5000 + .5333)/4 = .5115$$

$$w_2 = (.1026 + .1000 + .1250 + .0667)/4 = .0986$$

$$w_3 = (.2564 + .2000 + .2500 + .2667)/4 = .2433$$

$$w_4 = (.1282 + .2000 + .1250 + .1333)/4 = .1466$$

The analytic hierarchy process (continued)

Checking for consistency of the comparisons:

- Compute Aw^T :

$$Aw^T = \begin{pmatrix} 1 & 5 & 2 & 4 \\ \frac{1}{5} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & 2 \\ \frac{1}{4} & 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} .5115 \\ .0986 \\ .2433 \\ .1466 \end{pmatrix} = \begin{pmatrix} 2.0775 \\ 0.3959 \\ 0.9894 \\ 0.5933 \end{pmatrix}$$

- For a perfectly consistent decision maker, the i-th entry in $A\mathbf{w}^T = n \times$ i-th entry of $\mathbf{w}^T$
- Compute the average of the ratio between the entries of $A\mathbf{w}^T$ and $\mathbf{w}^T$:

  [(2.0075/.5115) + (.3959/.0986) + (.9894/.2433) + (.5933/.1466)]/4 = 4.05

- Compute the **Consistency Index (CI)** = (ave ratio - n)/(n - 1) = (4.05 - 4)/3 = 0.017.
- Compute **CI/RI** = 0.017/0.90 = 0.019.
- This value is less than 0.10, the degree of consistency is satisfactory.

The analytic hierarchy process (continued)

The value of **Random Index**:

n	RI
2	0
3	0.58
4	0.90
5	1.12
6	1.24
7	1.32
8	1.41
9	1.45
10	1.51

The values give the average CI if the entries in A were chosen at random subject to the constraint that all diagonal entries must be 1 and $A_{ij} \times A_{ji} = 1$

Finding the score of an alternative for an objective

(similar to obtaining the weights for the objectives)

- Suppose for the objective 2 quality of life, the pairwise comparison matrix is as follows

$$
\begin{array}{c}
\\
\text{Job 1} \\
\text{Job 2} \\
\text{Job 3}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Job 1} & \text{Job 2} & \text{Job 3} \end{array} \\
\begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/3 \\ 3 & 3 & 1 \end{bmatrix}
\end{array}
$$

- Normalise the entries by dividing by column sum:

$$
\begin{array}{c}
\\
\text{Job 1} \\
\text{Job 2} \\
\text{Job 3}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Job 1} & \text{Job 2} & \text{Job 3} \end{array} \\
\begin{bmatrix} 1/6 & 1/9 & 1/5 \\ 1/3 & 2/9 & 1/5 \\ 1/2 & 6/9 & 3/5 \end{bmatrix}
\end{array}
$$

- Then, job 1 QL score: $(1/6 + 1/9 + 1/5)/3 = 0.159$
  job 2 QL score: $(1/3 + 2/9 + 1/5)/3 = 0.252$
  job 3 QL score: $(1/2 + 6/9 + 3/5)/3 = 0.589$

See the table on page 23

3a. Goal programming approach to pattern classification

- **Reference:** H. Nakayama, Y.B. Yun, T. Asada and M. Yoon, MOP/GP models for machine learning, European Journal of Operational Research 166 (2005) 756-768.
- **Definition:**
  - η_i : **interior deviation**, deviation from the hyperplane of a point x_i that is properly classified. ☺
  - ξ_i : **exterior deviation**, deviation from the hyperplane of a point x_i that is improperly classified. ☹
- Objectives:

  i. Minimize the maximum exterior deviation (decrease errors as much as possible)

  ii. Maximize the minimum interior deviation (maximize the margin)

  iii. Maximize the weighted sum of interior deviations

  iv. Minimize the weighted sum of exterior deviations

3a. Goal programming approach to pattern classification

- Considering objective (i), the linear programming is as follows:

i. Minimize the maximum exterior deviation ξ_i (decrease errors as much as possible)

(LP) Minimize d

Subject to: $y_i(\mathbf{x}_i^T\mathbf{w} + b) = \eta_i - \xi_i$

$$d \geq \xi_i$$

$$\eta_i ,\ \xi_i \geq 0,\ i = 1,2,3, \ldots.. L$$

- For sample with $y_i = +1$, we want :

$$\mathbf{x}_i^T\mathbf{w} + \mathbf{b} \geq \mathbf{0}$$

- If $\eta_i \geq 0$ and $\xi_i = 0$, then the sample is correctly classified (on the right side of the hyperplane)
- If $\eta_i = 0$ and $\xi_i > 0$, then the sample is incorrectly classified (on the wrong side of the hyperplane)

(d is the maximum of all ξ_i $\Rightarrow$ minimize the infinity norm of the vector of exterior deviations/errors)

3a. Goal programming approach to pattern classification

- Considering objectives (iii) and (iv), the linear goal programming is as follows:

iii. Maximize the weighted sum of interior deviations $\boldsymbol{\eta_i}$

iv. Minimize the weighted sum of exterior deviations $\boldsymbol{\xi_i}$

$$\text{(GP)} \qquad \text{Minimize} \quad \sum_{i=1}^{L} (h_i \xi_i - k_i \eta_i)$$

Subject to: $y_i(\mathbf{x}_i^T \mathbf{w} + b) = \eta_i - \xi_i$

$$\eta_i, \ \xi_i \geq 0, \ i = 1,2,3, \ldots.. L$$

- h_i and k_i are positive constants, $y_i = +1$ or -1.
- If $h_i > k_i$ for $i = 1,2, \ldots. L$, then we have $\eta_i \xi_i = 0$ for every i at the solution of (GP)

For a sample with $y_i = +1$

- If it is correctly classified, then

$$\mathbf{x_i}^T\mathbf{w} + b \geq 0$$

In this case, $\mathbf{x_i}^T\mathbf{w} + b = \eta_i - \xi_i$

where $\eta_i \geq 0$ and $\xi_i = 0$,i.e. interior deviation is non-negative and exterior deviation is zero.

- If it is incorrectly classified, then

$$\mathbf{x_i}^T\mathbf{w} + b < 0$$

In this case $\mathbf{x_i}^T\mathbf{w} + b = \eta_i - \xi_i$

where the interior deviation $\eta_i = 0$ and the exterior deviation $\xi_i > 0$

3a. Goal programming approach to pattern classification

- SVM hard margin:

$$(\mathrm{SVM}_{\mathrm{hard}}) \quad \text{Minimize} \quad \|\boldsymbol{w}\|$$

$$\text{subject to} \quad y_i(\boldsymbol{w}^{\mathrm{T}} z_i + b) \geqq 1, \quad i = 1, \ldots, l.$$

- SVM soft margin:

$$(\mathrm{SVM}_{\mathrm{soft}}) \quad \text{Minimize} \quad \|\boldsymbol{w}\| + C \sum_{i=1}^{l} \xi_i$$

$$\text{subject to} \quad y_i(\boldsymbol{w}^{\mathrm{T}} z_i + b) \geqq 1 - \xi_i,$$
$$\xi_i \geqq 0, \quad i = 1, \ldots, l.$$

- SVM total:

$$(\mathrm{SVM}_{\mathrm{total}}) \quad \text{Minimize} \quad \|\boldsymbol{w}\| + \sum_{i=1}^{l} (h_i \xi_i - k_i \eta_i)$$

$$\text{subject to} \quad y_i(\boldsymbol{w}^{\mathrm{T}} z_i + b) = 1 + \eta_i - \xi_i,$$
$$\xi_i, \ \eta_i \geqq 0, \ i = 1, \ldots, l.$$

3b. Goal programming approach to pattern classification

- Reference: D.F. Jones, A. Collins, C. Hand, A classification model based on goal programming with non-standard preference functions with applications to the prediction of cinema-going behaviour, European Journal of Operational Research 177 (2007) 515-524.
- Data: n_1 samples from class A and n_2 samples from class B
- Each class A sample has a score of a_{ij} associated with the j-th criteria (≡ input)
- Each class B sample has a score of b_{ij} associated with the j-th criteria (≡ input)
- Goal programming to distinguish the two classes:

(GP) Minimize $\sum_{i=1}^{n_1}(n_i^{(a)}) + \sum_{i=1}^{n_2}(p_i^{(b)})$

Subject to:

$$\sum_{j=1}^{m} a_{ij}x_j + n_i^{(a)} - p_i^{(a)} = x_0, \quad i = 1,\ldots,n_1,$$

$$\sum_{j=1}^{m} b_{ij}x_j + n_i^{(b)} - p_i^{(b)} = x_0, \quad i = 1,\ldots,n_2,$$

$$\sum_{j=1}^{m} x_j = 1,$$

$$-\alpha \leqslant x_j \leqslant \alpha, \quad j = 1,\ldots,m,$$

- Decision variables: $x_0, x_1, \ldots. x_m$
- The discriminant line is given by

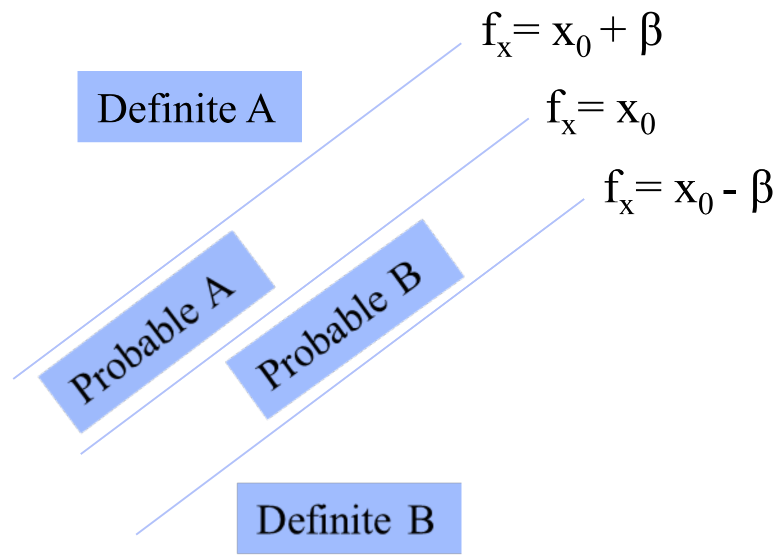
$$x_1y_1 + x_2y_2 + \ldots. x_my_m = x_0$$

where $y_1, y_2, .. y_m$ are input attributes of a data sample.

- α is a user defined parameter
- Perfect classification:
  - objective function value = 0
  - Each class A sample should lie on the positive side of the discriminant line, $n_i^{(a)} = 0,\ p_i^{(a)} \geq 0$
  - Each class B sample …..

3b. Goal programming approach to pattern classification

- Small modification to the model to reduce the number of samples that lie exactly on the discriminant line:



(GP) Minimize $\sum_{i=1}^{n_1} \left(n_i^{(a)}\right) + \sum_{i=1}^{n_2} \left(p_i^{(b)}\right)$

Subject to:

$$\sum_{j=1}^{m} a_{ij}x_j + n_i^{(a)} - p_i^{(a)} = x_0 + \beta, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} b_{ij}x_j + n_i^{(b)} - p_i^{(b)} = x_0 - \beta, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} x_j = 1,$$

$$-\alpha \leqslant x_j \leqslant \alpha, \quad j = 1, \ldots, m.$$

Possible classification:

Table 1
Certainty classes

Achieved value (f_x)	Classification
$f_x < x_0 - \beta$	Definite B
$x_0 - \beta \leqslant f_x < x_0$	Probable B
$f_x = x_0$	Unclassified
$x_0 < f_x \leqslant x_0 + \beta$	Probable A
$x_0 + \beta < f_x$	Definite A

3b. Goal programming approach to pattern classification

- Jones and Tamiz methodology to allow penalty weight changes at $x_0 - \beta$, x_0 , $x_0 + \beta$:

(GP) Minimize $$W_a \sum_{i=1}^{n_1} \left(\underline{u_1 n_{i1}^{(a)}} + u_2 n_{i2}^{(a)} + u_3 n_{i3}^{(a)} \right) + W_b \sum_{i=1}^{n_2} \left(v_1 p_{i1}^{(b)} + v_2 p_{i2}^{(b)} + \underline{v_3 p_{i3}^{(b)}} \right)$$

Subject to:

$$\sum_{j=1}^{m} a_{ij} x_j \underline{+ n_{i1}^{(a)}} - p_{i1}^{(a)} = x_0 - \beta, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} a_{ij} x_j + n_{i2}^{(a)} - p_{i2}^{(a)} = x_0, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} a_{ij} x_j + n_{i3}^{(a)} - p_{i3}^{(a)} = x_0 + \beta, \quad i = 1, \ldots, n_1,$$

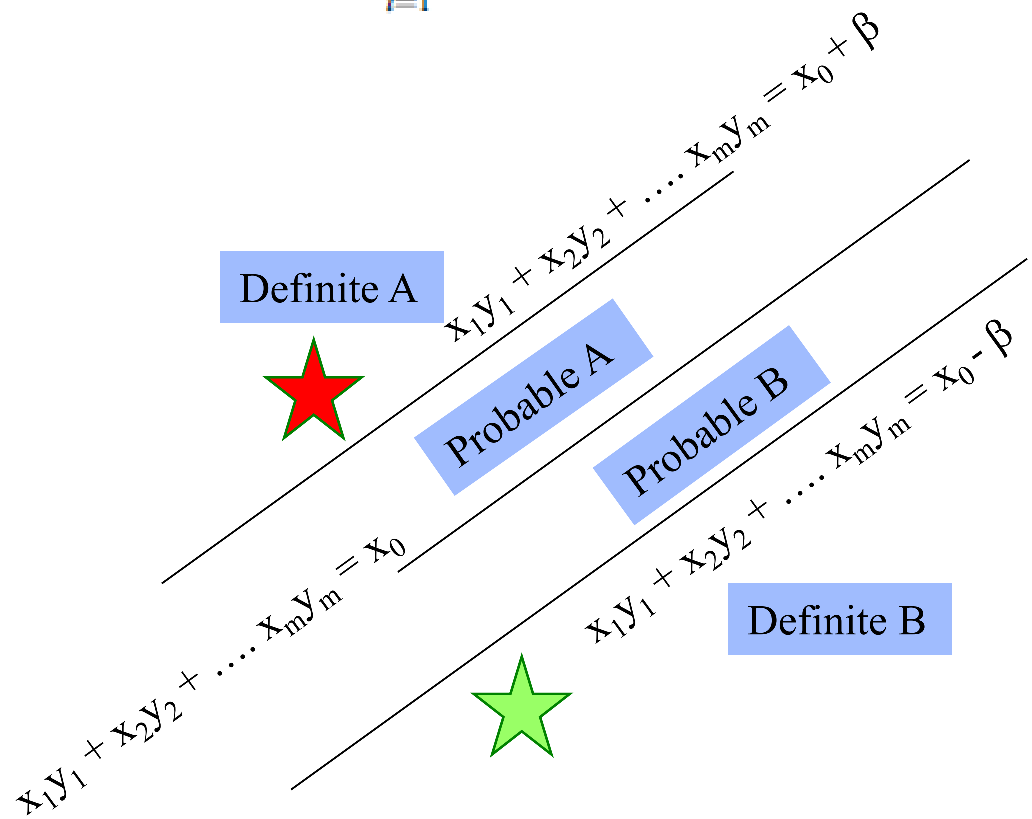
$$\sum_{j=1}^{m} b_{ij} x_j + n_{i1}^{(b)} - p_{i1}^{(b)} = x_0 - \beta, \quad i = 1, \ldots, n_2,$$

$$\sum_{i=1}^{m} b_{ij} x_j + n_{i2}^{(b)} - p_{i2}^{(b)} = x_0, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} b_{ij} x_j + n_{i3}^{(b)} \underline{- p_{i3}^{(b)}} = x_0 + \beta, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} x_j = 1,$$

$$-\alpha \leqslant x_j \leqslant \alpha, \quad j = 1, \ldots, m.$$



- u_1: penalization of A misclassified as definite B
- v_3: penalization of B misclassified as definite A

3b. Goal programming approach to pattern classification

- Jones and Tamiz methodology to allow penaltyweight changes at x_0 - β, x_0 , x_0 + β:

(GP) Minimize

Subject to:

$$W_a \sum_{i=1}^{n_1} \left(u_1 n_{i1}^{(a)} + u_2 n_{i2}^{(a)} + \underline{u_3 n_{i3}^{(a)}} \right) + W_b \sum_{i=1}^{n_2} \left(v_1 p_{i1}^{(b)} + v_2 p_{i2}^{(b)} + v_3 p_{i3}^{(b)} \right)$$

$$\sum_{j=1}^{m} a_{ij} x_j + n_{i1}^{(a)} - p_{i1}^{(a)} = x_0 - \beta, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} a_{ij} x_j + n_{i2}^{(a)} - p_{i2}^{(a)} = x_0, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} a_{ij} x_j \underline{+ n_{i3}^{(a)}} - p_{i3}^{(a)} = x_0 + \beta, \quad i = 1, \ldots, n_1,$$

$$\sum_{j=1}^{m} b_{ij} x_j + n_{i1}^{(b)} - p_{i1}^{(b)} = x_0 - \beta, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} b_{ij} x_j + n_{i2}^{(b)} - p_{i2}^{(b)} = x_0, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} b_{ij} x_j + n_{i3}^{(b)} - p_{i3}^{(b)} = x_0 + \beta, \quad i = 1, \ldots, n_2,$$

$$\sum_{j=1}^{m} x_j = 1,$$

$$-\alpha \leqslant x_j \leqslant \alpha, \quad j = 1, \ldots, m.$$

- W_a and W_b allow for differing importance to be given to the different classes.
- The weight u_1: penalization of 'A as definite B'
- The weight v_3: penalization of 'B as definite A'
- The weights u_2 and v_2 correspond to the increase per unit penalization beyond the discriminant line
- The weight $\underline{u_3}$ corresponds to the increase per unit penalization of deviations away from the 'A as definite A'.
- The weight v_1 correspond to the increase per unit penalization of deviations away from 'B as definite B' classes.

Reference:

Operations Research, Application and Algorithms by Wayne L. Winston, Duxbury Press, 1994, Chapter 14, 3rd Ed.

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