CS5340: Uncertainty Modeling in AI

Tutorial 4: Solutions

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Problem 1. (Simpson's Paradox)
A dangerous new virus is sweeping the world. Currently, there are two potential drug treatments (A and B) for patients. Dr. Homer Simpson wants to compare the un-cured rate of patients after receiving either treatment A or B, in order to determine the better drug.

The data indicates that there are 240 patients that are not cured among the 1500 patients who received treatment A. There are 105 patients that are not cured among the 550 who received treatment B. *Note: this is a fictitious scenario and we made up these numbers.*

Problem 1.a. Can you help Homer construct a probabilistic graphical model for the above scenario.

Solution:

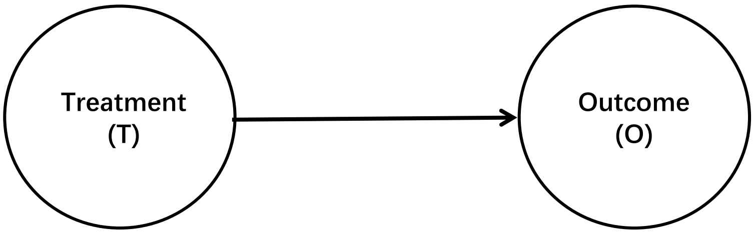


Figure 1: PGM constructed by ourselves.

Define two binary random variables Treatment(T), which takes the value of A and B, and Outcome(O), which takes the value of Cured(0) and Not cured(1). We can define a Bayes Net as Fig.1. The joint distribution $p(T,O) = p(O|T)p(T)$. From the data, we can compute the MLE estimates (assuming Categorical distributions):

$$p(O = 1|T = A) = \frac{240}{1500} = 0.16 \tag{1}$$

$$p(O = 0|T = A) = 1 - \frac{240}{1500} = 0.84 \tag{2}$$

Similarly, for the treatment B

$$p(O = 1|T = B) = \frac{105}{550} = 0.19 \tag{3}$$

$$p(O = 0|T = B) = 1 - \frac{105}{550} = 0.81 \tag{4}$$

For prior distribution $p(T)$, we can compute it as

$$p(T = A) = \frac{1500}{1500 + 550} = 0.73 \tag{5}$$

$$p(T = B) = \frac{550}{1500 + 550} = 0.27 \tag{6}$$

Problem 1.b. The data seems to indicate that treatment A is more effective. Can Homer confirm (just from the data) that one of the treatments results in more cures? *Hint:* Consider what happens when there are unobserved variables that could affect the treatment and the outcome.

Solution:

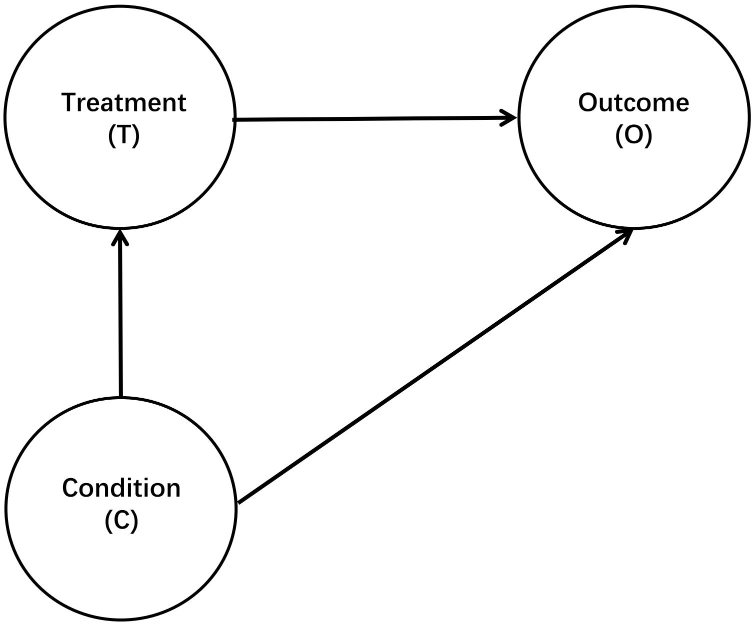


Figure 2: True PGM of the underlying causal process.

We cannot confirm that treatment A is more effective, since we may have some unobserved variables. Suppose the true causal process is represented by Fig.2, where there is another random variable Condition. Condition is a binary random variable that takes value of $Mild(0)$ or $Severe(1)$. It represents the severity of the patient's sickness. If a patient is in the severe condition, doctors tend to give him treatment B (e.g. B has better a treatment outcome, but is more expensive). If a patient is under the mild condition, doctors tend to give him treatment A (cheaper but less effective).

Suppose among the 1500 patients who received treatment A, there are 1400 patients under the $Mild$ condition with 210 patients that are not cured. There are 100 patients under $Severe$ condition with 30 patients that are not cured. Among the 550 patients who received treatment B, there are 50 patients under the $Mild$ condition with 5 patients that are not cured and 500 patients under the $Severe$ condition with 100 patients that are not cured.

Given this information, we can compute the probability of un-cured patients who received treatment A

or B under different conditions.

$$p(O=1|T=A,C=0)=\frac{210}{1400}=0.15 \tag{7}$$

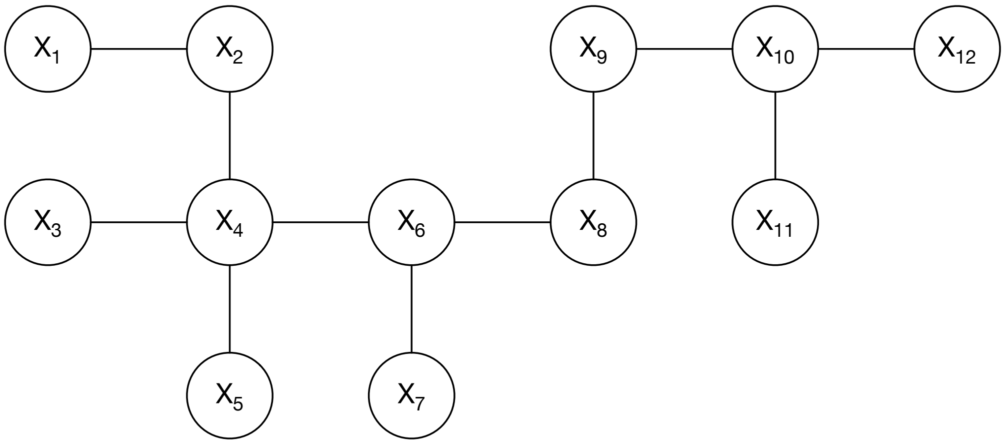
$$p(O=1|T=A,C=1)=\frac{30}{100}=0.3 \tag{8}$$

$$p(O=1|T=B,C=0)=\frac{5}{50}=0.1 \tag{9}$$

$$p(O=1|T=B,C=1)=\frac{100}{500}=0.2 \tag{10}$$

We can see that under each condition, the un-cured rate of patients who received treatment B is smaller than that of patients who received treatment A. The reason is that the majority of patients who received treatment B are under *Severe* condition. Therefore, even we know that B is better, the overall un-cured rate is larger than patients who received A because of the allocation of patients to the different treatment groups.

Problem 2. (MRT Inference)
You are given the following *pairwise* undirected graphical model which models the activity (low or high) at 12 MRT stations.



Each node represents a random variable indicating whether the activity at a particular station is low (0) or high (1). Assume the following factorization:

$$p(x_1, x_2, \ldots, x_{12}) = \frac{1}{Z} \prod_{i \in V} \psi(x_i) \prod_{(i,j) \in E} \psi(x_i, x_j) \tag{11}$$

where V is the set of nodes, E is the set of edges, and that the unary and pairwise factors are given by:

x_i	$\psi(x_i)$
0	10
1	2

Figure 3: Unary Factors

x_i	x_j	$\psi(x_i, x_j)$
0	0	20
0	1	5
1	0	5
1	1	20

Figure 4: Pairwise Factors

Note that the factors are the same across the nodes. Your task is to compute the following conditional probabilities.

Problem 2.a. Compute $p(x_{12} = 1 | x_1 = 0, x_7 = 0, x_9 = 1, x_{10} = 0)$.

Solution: According to the conditional independence assertions in the MRF.

$$p(x_{12}|x_1, x_7, x_9, x_{10}) = p(x_{12}|x_{10}) = \frac{p(x_{10}, x_{12})}{\sum_{x_{12}} p(x_{10}, x_{12})}$$

Denote

$$m(x_{10}) = \sum_{i,j \in M_{E/x_{10}, x_{12}}} \psi(x_i)\psi(x_i, x_j)$$

Then

$$p(x_{12}|x_1, x_7, x_9, x_{10}) = \frac{p(x_{10}, x_{12})}{\sum_{x_{12}} p(x_{10}, x_{12})} = \frac{\sum_{i,j \in M_{E/x_{10}, x_{12}}} p(x_1 \ldots x_{12})}{\sum_{x_{12}} \sum_{i,j \in M_{E/x_{10}, x_{12}}} p(x_1 \ldots x_{12})}$$
$$= \frac{m(x_{10})\psi(x_{10})\psi(x_{10}, x_{12})\psi(x_{12})}{\sum_{x_{12}} m(x_{10})\psi(x_{10})\psi(x_{10}, x_{12})\psi(x_{12})}$$

$$= \frac{\psi(x_{10}, x_{12})\psi(x_{12})}{\sum_{x_{12}} \psi(x_{10}, x_{12})\psi(x_{12})} = \frac{\psi(x_{10}=0, x_{12}=1)\psi(x_{12}=1)}{\sum_{x_{12}} \psi(x_{10}=0, x_{12})\psi(x_{12})}$$

$$= \frac{5 \times 2}{20 \times 10 + 5 \times 2} = \frac{1}{21} = 0.0476$$

Problem 2.b. Compute $p(x_1 = 1 | x_3 = 0, x_4 = 1, x_6 = 0)$.

Solution: According to the conditional independence assertions in MRF.

$$p(x_1|x_3, x_4, x_6) = p(x_1|x_4) = \frac{p(x_1, x_4)}{\sum_{x_1} p(x_1, x_4)}$$

Denote

$$m(x_4) = \sum_{i,j \in M_{E/x_1,x_2,x_4}} \psi(x_i)\psi(x_i, x_j)$$

Then

$$p(x_1|x_3, x_4, x_6) = \frac{\sum_{x_2} \sum_{i,j \in M_{E/x_1,x_2,x_4}} p(x_1 \dots x_{12})}{\sum_{x_2} \sum_{x_1} \sum_{i,j \in M_{E/x_1,x_2,x_4}} p(x_1 \dots x_{12})}$$

$$= \frac{\sum_{x_2} \psi(x_1, x_2)\psi(x_2)\psi(x_2, x_4)\psi(x_1)\psi(x_4)m(x_4)}{\sum_{x_1} \sum_{x_2} \psi(x_1, x_2)\psi(x_2)\psi(x_2, x_4)\psi(x_1)\psi(x_4)m(x_4)}$$

$$= \frac{\sum_{x_2} \psi(x_1, x_2)\psi(x_2)\psi(x_2, x_4)\psi(x_1)}{\sum_{x_1} \sum_{x_2} \psi(x_1, x_2)\psi(x_2)\psi(x_2, x_4)\psi(x_1)} = \frac{\sum_{x_2} \psi(x_1 = 1, x_2)\psi(x_2)\psi(x_2, x_4 = 1)\psi(x_1 = 1)}{\sum_{x_1} \sum_{x_2} \psi(x_1, x_2)\psi(x_2)\psi(x_2, x_4 = 1)\psi(x_1)}$$

$$= \frac{5 \times 10 \times 5 \times 2 + 20 \times 2 \times 20 \times 2}{20 \times 10 \times 5 \times 10 + 5 \times 2 \times 20 \times 10 + 5 \times 10 \times 5 \times 2 + 20 \times 2 \times 20 \times 2} = \frac{7}{47} = 0.1489$$

Problem 2.c. Compute $p(x_{10} = 1 | x_9 = 1, x_{12} = 1, x_2 = 0)$.

Solution: According to the conditional independence assertions in MRF.

$$p(x_{10}|x_9, x_{12}, x_2) = p(x_{10}|x_9, x_{12}) = \frac{p(x_{10}, x_9, x_{12})}{p(x_9, x_{12})}$$

Denote

$$m(x_9) = \sum_{i,j \in M_{E/x_9,x_{10},x_{11},x_{12}}} \psi(x_i)\psi(x_i, x_j)$$

Then

$$p(x_{10}|x_9, x_{12}, x_2) = \frac{p(x_{10}, x_9, x_{12})}{p(x_9, x_{12})} = \frac{\sum_{x_{11}} \sum_{i,j \in M_{E/x_9,x_{10},x_{11},x_{12}}} p(x_1 \dots x_{12})}{\sum_{x_{10}} \sum_{x_{11}} \sum_{i,j \in M_{E/x_9,x_{10},x_{11},x_{12}}} p(x_1 \dots x_{12})}$$

$$= \frac{\sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}$$

$$= \frac{\sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}$$

$$= \frac{\sum_{x_{11}} \psi(x_9 = 1, x_{10} = 1)\psi(x_{10} = 1)\psi(x_{10} = 1, x_{11})\psi(x_{11})\psi(x_{10} = 1, x_{12} = 1)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9 = 1, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12} = 1)}$$

$$= \frac{20 \times 2 \times 5 \times 10 \times 20 + 20 \times 2 \times 20 \times 2 \times 20}{5 \times 10 \times 20 \times 10 \times 5 + 20 \times 2 \times 5 \times 10 \times 20 + 5 \times 10 \times 5 \times 2 \times 5 + 20 \times 2 \times 20 \times 2 \times 20}$$

$$= \frac{48}{83} = 0.5783$$

Problem 2.d. Compute $p(x_6 = 0|x_4 = 1, x_8 = 1, x_{10} = 0)$.

Solution: According to the conditional independence assertions in MRF.

$$p(x_6|x_4, x_8, x_{10}) = p(x_6|x_4, x_8) = \frac{\sum_{x_7} p(x_4, x_6, x_8, x_7)}{\sum_{x_7} \sum_{x_7} p(x_4, x_6, x_8, x_7)}$$

Then

$$p(x_6 = 0|x_4 = 1, x_8 = 1) = \frac{\sum_{x_7} \psi(x_4 = 1, x_6 = 0)\psi(x_8 = 1, x_6 = 0)\psi(x_7)\psi(x_7, x_6 = 0)\psi(x_6 = 0)}{\sum_{x_6} \sum_{x_7} \psi(x_4 = 1, x_6)\psi(x_8 = 1, x_6)\psi(x_7)\psi(x_7, x_6)\psi(x_6)}$$

$$= \frac{10 \times 5 \times 5 \times 2 \times 5 + 5 \times 5 \times 10 \times 20 \times 10}{10 \times 5 \times 5 \times 2 \times 5 + 5 \times 5 \times 10 \times 20 \times 10 + 20 \times 20 \times 10 \times 5 \times 2 + 20 \times 20 \times 2 \times 20 \times 2} = \frac{35}{83} = 0.4217$$

Problem 2.e. Compute $p(x_8 = 1|x_1 = 0, x_6 = 0, x_9 = 1, x_{12} = 1)$.

Solution: According to the conditional independence assertions in MRF.

$$p(x_8|x_1, x_6, x_9, x_{12}) = p(x_8|x_6, x_9) = \frac{p(x_8, x_6, x_9)}{\sum_{x_8} p(x_8, x_6, x_9)}$$

Then

$$p(x_8 = 1|x_6 = 0, x_9 = 1) = \frac{\psi(x_8 = 1, x_6 = 0)\psi(x_9 = 1, x_8 = 1)\psi(x_8 = 1)}{\sum_{x_8} \psi(x_8, x_6 = 0)\psi(x_9 = 1, x_8)\psi(x_8)}$$

$$= \frac{5 \times 20 \times 2}{5 \times 20 \times 2 + 20 \times 5 \times 10} = \frac{1}{6} = 0.1667$$

Problem 2.f. Compute $p(x_2 = 0|x_1 = 0, x_3 = 1, x_4 = 1, x_7 = 1, x_{11} = 0)$.

Solution: According to the conditional independence assertions in MRF.

$$p(x_2|x_1, x_3, x_4, x_7, x_{11}) = p(x_2|x_1, x_4) = \frac{p(x_1, x_2, x_4)}{\sum_{x_2} p(x_1, x_2, x_4)}$$

Then

$$\begin{aligned}
p(x_2|x_1, x_4) &= \frac{\psi(x_2 = 0, x_1 = 0)\psi(x_4 = 1, x_2 = 0)\psi(x_2 = 0)}{\sum_{x_2} \psi(x_2, x_1 = 0)\psi(x_4 = 1, x_2)\psi(x_2)} \\
&= \frac{20 \times 5 \times 10}{5 \times 20 \times 2 + 20 \times 5 \times 10} = \frac{5}{6} = 0.8333
\end{aligned}$$

Problem 3. (Image Denoising)

For this problem, you will be working on Image Denoising, taking a noisy image and making it a clean one. Please refer to the provided Image-Denoising-Pre.ipynb notebook. You can download the notebook and relevant images in a zipfile from NUS Canvas (in the MRF module under Home).

To use the notebook, you have to install jupyter (`https://jupyter.org/install`) and a python distro; we use anaconda (`https://www.anaconda.com/products/individual`) but you can use whichever distribution you like.