IS5152 Data-driven Decision Making
SEMESTER 2 2023-2024
Assignment 1

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1. (10 points) A company manufactures two products (1 and 2). Each unit of product 1 can be sold for \$15, and each unit of product 2 for \$25. Each product requires raw material and two types of labor (skilled and unskilled) as shown in the table below:

	Product 1	Product 2
Skilled labor	3 hours	4 hours
Unskilled labor	2 hours	3 hours
Raw material	1 unit	2 units

At present, the company has available 100 hours of skilled labor, 70 hours of unskilled labor, and 30 units of raw material. Because of marketing considerations at least 3 units of product 2 must be produced.

(a) Let x_1 and x_2 be the number of units of product 1 and product 2 to be produced, respectively. Formulate a linear program to maximize total revenue.

(b) Show that the best production level is $x_1 = 24$ and $x_2 = 3$ by checking that all the necessary and sufficient conditions are satisfied.

(c) How much would the company be willing to pay for an additional unit of each type of labor?

(d) What would the company's revenue be if 90 hours of skilled labor were available?

(e) State the dual of the linear program from part (a). What is the solution of this dual linear program?

Answer:

(a)

$$\max \ Z = 15x_1 + 25x_2 \leftrightarrow -\min \ -Z = -15x_1 - 25x_2$$

subject to

$$\begin{aligned} 3x_1 + 4x_2 &\leq 100 \text{ Constraint } 1 \\ 2x_1 + 3x_2 &\leq 70 \text{ Constraint } 2 \\ x_1 + 2x_2 &\leq 30 \text{ Constraint } 3 \\ -x_2 &\leq -3 \text{ Constraint } 4 \\ -x_1 &\leq 0 \text{ Constraint } 5 \\ x_2 &\geq 0 \text{ (redundant)} \end{aligned}$$

(b) Solution: $x_1 = 24$ and $x_2 = 3$.
Check for feasibility:

- C1: $3(24) + 4(3) = 84 < 100$, not binding.

- C2: $2(24)+3(3)=57<700$, not binding.
- C3: $24+2(3)=30$, binding.
- C4: $-x_2=-3$, binding.
- C5: $-x1=-24$, not binding.

$\mathbf{x}=(24,3)$ is feasible. KT necessary and sufficient optimility conditions:

$$\begin{aligned}
-15+3\lambda_1+2\lambda_2+\lambda_3-\lambda_5 &= 0\\
-25+4\lambda_1+3\lambda_2+2\lambda_3-\lambda_4 &= 0\\
\lambda_1(3x_1+4x_2-100) &= 0\\
\lambda_2(2x_1+3x_2-70) &= 0\\
\lambda_3(x_1+2x_2-30) &= 0\\
\lambda_4(-x_2+3) &= 0\\
\lambda_5(x_1) &= 0\\
\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5 &\geq 0
\end{aligned}$$

From complimentarity requirements: $\lambda_1=\lambda_2=\lambda_5=0$. Hence, $\lambda_3=15-3\lambda_1-2\lambda_2+\lambda_5=15\geq 0$. And, $\lambda_4=-25+4\lambda_1+3\lambda_2+2\lambda_3=-25+2(15)=5\geq 0$. We conclude that $x_1=24$ and $x_2=3$ is the best production level to maximize profit. $Z=15x_1+25x_2=(15)(24)+(25)(3)=435=100\lambda_1+700\lambda_2+30\lambda_3-3\lambda_4-\lambda_5=30(15)-3(5)$.

(c) Skilled labor and unskilled labor: $\lambda_1=\lambda_2=\$0$, raw material: $\lambda_3=\$15$.

(d) No change in solution, as the current solution makes used only of 84 hours of skilled labor (and the corresponding shadow price = $\lambda_1=0$).

(e)

$$\min\ W = 100u_1+700u_2+30u_3-3u_4$$

subject to

$$\begin{aligned}
3u_1+2u_2+u_3 &\geq 15\\
4u_1+3u_2+2u_3-u_4 &\geq 25\\
u_1,u_2,u_3,u_4 &\geq 0
\end{aligned}$$

Solution: $u_1=u_2=0$, $u_3=15$, $u_4=5$, $W=30(15)-3(5)=435$.

2. (10 points) Find the solution of the quadratic programming problem:

$$\min\ 2x_1^2-x_2$$

subject to

$$\begin{aligned}
2x_1-x_2 &\leq 1\\
x_1+x_2 &\leq 1\\
x_1,x_2 &\geq 0
\end{aligned}$$

Show that all the necessary and sufficient optimality conditions are satisfied. Explain why the objective function is convex.

KT conditions:

$$\begin{aligned}
4x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\
-1 - \lambda_1 + \lambda_2 - \lambda_4 &= 0 \\
\lambda_1(2x_1 - x_2 - 1) &= 0 \\
\lambda_2(x_1 + x_2 - 1) &= 0 \\
\lambda_3\, x_1 &= 0 \\
\lambda_4\, x_2 &= 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0
\end{aligned}$$

Guess x_1 as small as possible and x_2 as large as possible subject to feasibility requirements: let $x_1 = 0$ and $x_2 = 1$, then $\lambda_1 = \lambda_4 = 0$. We have:

$$\begin{aligned}
-1 - \lambda_1 + \lambda_2 - \lambda_4 &= 0 \\
\lambda_2 &= 1
\end{aligned}$$

and

$$\begin{aligned}
4x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\
0 + 0 + 1 - \lambda_3 &= 0 \\
\lambda_3 &= 1
\end{aligned}$$

Objective function:

$$\begin{aligned}
f(x_1, x_2) &= 2x_1^2 - x_2 \\
\nabla f(x_1, x_2) &= \begin{pmatrix} 4x_1 \\ -1 \end{pmatrix} \\
H(x_1, x_2) &= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= 4x_1^2 \geq 0
\end{aligned}$$

Since the objective function is convex, then the optimal solution is $(0, 1)$.

3. (10 points) The ABC Company produces two products. The total profit achieved from these products is described by the following equation:

$$\text{Total profit} = -0.1X_1^2 - 0.2X_2^2 + 8X_1 + 12X_2 + 1500$$

where

X_1 = thousands of units of product 1.

X_2 = thousands of units of product 2.

Every 1000 units of product 1 require one hour of time in the shipping department, and every 1000 units of product 2 require 30 minutes in the shipping department. Each unit of product requires one pound of a special ingredient, of which 55000 pounds will be available in the next production period. Additionally, 80 hours of shipping labor will be available in the next production period. Demand for both products is unlimited.

(a) Formulate a quadratic programming (QP) model to maximize the total profit.

$$QP: \qquad \min 0.1X_1^2 + 0.2X_2^2 - 8X_1 - 12X_2 - 1500$$

subject to

$$\begin{aligned} X_1 + 0.5X_2 &\leq 80 \\ X_1 + X_2 &\leq 55 \\ X_1, X_2 &\geq 0 \end{aligned}$$

(b) Use the Kuhn-Tucker (KT) conditions to find a solution of this QP.

KT conditions: Feasibility and

$$\begin{aligned} 0.2X_1 - 8 + u_1 + u_2 - v_1 &= 0 \\ 0.4X_2 - 12 + 0.5u_1 + u_2 - v_2 &= 0 \\ u_1(X_1 + 0.5X_2 - 80) &= 0 \\ u_2(X_1 + X_2 - 55) &= 0 \\ x_1 v_1 &= 0 \\ x_2 v_2 &= 0 \\ u_1, u_2, v_1, v_2 &\geq 0 \end{aligned}$$

If $u_1 = u_2 = v_1 = v_2 = 0$, then $X_1 = 40$ and $X_2 = 30$, not feasible ($X_1 + X_2 = 70 > 55$). Let $u_1 = v_1 = v_2 = 0$ and $u_2 > 0$, then

$$\begin{aligned} X_1 + X_2 - 55 &= 0 \\ 0.2X_1 - 8 + u_2 &= 0 \\ 0.4X_2 - 12 + u_2 &= 0 \end{aligned}$$

The solution to the above linear system of equations is $X_1 = 30, X_2 = 25$ and $u_2 = 2$. All the KT conditions are satisfied: produce 30 thousand units of product 1 and 25 thousand units of product 2.

(c) What is the maximum total profit?

Total profit = $-0.1(900) - 0.2(625) + 8(30) + 12(25) + 1500 = -90 - 125 + 240 + 300 + 1500 = 1825$.

(d) What conditions are required on the objective function and the constraints to guarantee that the solution found in (b) results in maximum profit?

The objective function and the constraints in QP must be convex to guarantee optimal solution.