CS5340: Uncertainty Modeling in AI

Tutorial 1

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Problem 1. (Two Numbers Game)

Consider the following game involving two teams:
Team 1:

1. Pick 2 different numbers between 0 and 10, inclusive.
2. Write each number on a piece of paper each.
3. Turn the papers face down.

Team 2: Objective is to pick the larger number.

1. Pick one of the pieces of paper.
2. Have a peek at the number.
3. Decides to keep the number or switch.

Problem 1.a. Can Team 2 win more than 50% of the time? If so, what should their strategy be?

Problem 1.b. How can Team 1 minimize the win percentage of Team 2?

Problem 2. (Legal Reasoning)
(Source: Kevin Murphy, Machine Learning, Chapter 2. Original Source: Peter Lee)

Suppose a crime has been committed and blood is found at a scene. The blood type is present in only 1% of the population. The prosecutor claims: “There is a 1% chance that the defendant would have the crime blood type if he were innocent. Thus, there is a 99% chance that he is guilty!” Is the prosecutor correct? If not, what is wrong with this argument?
Hint: Let the event A be the event ‘person has blood of this type’ and event B be the event ‘person is innocent’.

Problem 3. (Conjugate Distributions)

Problem 3.a. (*Beta-Binomial*) Show that the Beta distribution is conjugate to the Binomial distribution. Suppose we have $x \sim \mathrm{Bin}(n, \pi), \pi \sim \mathrm{Beta}(\alpha, \beta)$, then

$$p(x|n, \pi) = \tbinom{n}{x}\pi^x(1-\pi)^{n-x} \tag{1}$$

$$p(\pi|\alpha, \beta) = \frac{1}{B(\alpha, \beta)}\pi^{\alpha-1}(1-\pi)^{\beta-1} \tag{2}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \tag{3}$$

Problem 3.b. (*Normal with unknown mean*, *Challenge*) Show that the (univariate) Normal distribution is conjugate to the (univariate) Normal distribution with unknown mean, but known variance. Let the known variance be σ^2 and denote the observed data $\{x_1, \ldots, x_n\}$ as $\mathcal{X}$. The prior and likelihood distributions are given by

$$p(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \tag{4}$$

$$p(\mathcal{X}|\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\} \tag{5}$$

Problem 4. (Variance of a Sum)
(Source: Kevin Murphy, Machine Learning, Chapter 2.)

We learnt that the expectation of a sum is equal to the sum of the expectations. In this exercise, we consider the variance:

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Show that the variance of a sum of two random variables is:

$$\mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2\mathrm{Cov}[X, Y]$$

where $\mathrm{Cov}[X, Y]$ is the covariance of X and Y,

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Extra: What happens to the variance sum formula above when the random variables X and Y are independent?