CS5340: Uncertainty Modeling in AI

Tutorial - Extra Question: Solutions

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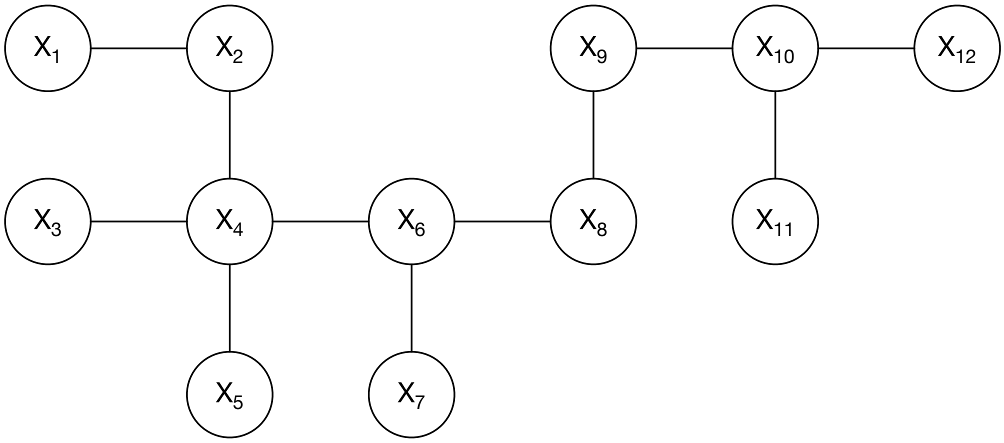
Problem 1. (MRT Inference)
We have previously considered inference for this MRF which models the activity (low or high) at 12 MRT stations. This week, we will repeat the activity but using the *sum product algorithm.*



Recall that each node represents a random variable indicating whether the activity at a particular station is low (0) or high (1) and assume the following factorization:

$$p(x_1, x_2, \ldots, x_{12}) = \frac{1}{Z} \prod_{i \in V} \psi(x_i) \prod_{(i,j) \in E} \psi(x_i, x_j) \tag{1}$$

where V is the set of nodes, E is the set of edges, and that the unary and pairwise factors are given by:

x_i	$\psi(x_i)$
0	10
1	2

Figure 1: Unary Factors

x_i	x_j	$\psi(x_i, x_j)$
0	0	20
0	1	5
1	0	5
1	1	20

Figure 2: Pairwise Factors

Note that the factors are the same across the nodes. Your task is to compute the following conditional probabilities using *the sum-product algorithm.*

Problem 1.a. [2 points] Compute $p(x_{12} = 1 | x_1 = 0, x_7 = 0, x_9 = 1, x_{10} = 0)$.

Solution: **Approach 1:** According to the conditional independence in the MRF.

$$p(x_{12}|x_1, x_7, x_9, x_{10}) = p(x_{12}|x_{10}) = \frac{p(x_{10}, x_{12})}{\sum_{x_{12}} p(x_{10}, x_{12})}$$

Denote message from nodes $M_{E/x_{10},x_{12}} = \{x_1, \ldots x_9, x_{11}\}$ to x_{10} as

$$m(x_{10}) = \sum_{i,j \in M_{E/x_{10},x_{12}}} \psi(x_i)\psi(x_i, x_j)$$

Then

$$p(x_{12}|x_1,x_7,x_9,x_{10}) = \frac{p(x_{10},x_{12})}{\sum_{x_{12}} p(x_{10},x_{12})} = \frac{\sum_{i,j\in M_{E/x_{10},x_{12}}} p(x_1\dots x_{12})}{\sum_{x_{12}}\sum_{i,j\in M_{E/x_{10},x_{12}}} p(x_1\dots x_{12})}$$
$$= \frac{m(x_{10})\psi(x_{10})\psi(x_{10},x_{12})\psi(x_{12})}{\sum_{x_{12}} m(x_{10})\psi(x_{10})\psi(x_{10},x_{12})\psi(x_{12})}$$
$$= \frac{\psi(x_{10},x_{12})\psi(x_{12})}{\sum_{x_{12}}\psi(x_{10},x_{12})\psi(x_{12})} = \frac{\psi(x_{10}=0,x_{12}=1)\psi(x_{12}=1)}{\sum_{x_{12}}\psi(x_{10}=0,x_{12})\psi(x_{12})}$$
$$= \frac{5\times 2}{20\times 10+5\times 2} = \frac{1}{21} = 0.0476$$

Approach 2: In the following solutions, we will compute the messages as follows.

$$m_{i\to j}(x_j) = \sum_{x_i\in\{0,1\}}\left(\psi^E(x_i)\psi(x_i,x_j)\prod_{x_k\in\text{neighbors(x}_\text{i})\backslash x_j} m_{k\to i}(x_i)\right) \tag{2}$$

where E is the set of evidence nodes, $\psi^E(x_i)=\delta(x_i=\hat{x}_i)\psi(x_i)$ if $x_i\in E$ and $\psi^E(x_i)=\psi(x_i)$ otherwise.

Node x_{12} is conditionally independent of all other nodes, given x_{10}. Let's compute the message from x_{10} to x_{12}.

$m_{x_{10}\to x_{12}}$		
$x_{12}=0$	$10\times 20+0\times 5$	200
$x_{12}=1$	$10\times 5+0\times 20$	50

$$\tilde{p}(x_{12}=\hat{x}_{12}|x_1=0,x_7=0,x_9=1,x_{10}=0) = \psi(x_{12}=\hat{x}_{12})\times m_{x_{10}\to x_{12}}(x_{12}=\hat{x}_{12}) \tag{3}$$

$\tilde{p}$		
$x_{12}=0$	10×200	2000
$x_{12}=1$	2×50	100

$$p(x_{12}=1|x_1=0,x_7=0,x_9=1,x_{10}=0) = \frac{100}{2000+100} = \frac{1}{21} = 0.0476 \tag{4}$$

Problem 1.b. [2 points] Compute $p(x_1=1|x_3=0,x_4=1,x_6=0)$.

Solution: **Approach 1:** According to the conditional independence in MRF.

$$p(x_1|x_3,x_4,x_6) = p(x_1|x_4) = \frac{p(x_1,x_4)}{\sum_{x_1} p(x_1,x_4)}$$

Denote message from nodes $M_{E/x_1,x_2,x_4} = \{x_3,x_5\dots x_{12}\}$ to x_4 as

$$m(x_4) = \sum_{i,j\in M_{E/x_1,x_2,x_4}} \psi(x_i)\psi(x_i,x_j)$$

Then

$$p(x_1|x_3,x_4,x_6) = \frac{\sum_{x_2}\sum_{i,j\in M_{E/x_1,x_2,x_4}} p(x_1\dots x_{12})}{\sum_{x_2}\sum_{x_1}\sum_{i,j\in M_{E/x_1,x_2,x_4}} p(x_1\dots x_{12})}$$

$$= \frac{\sum_{x_2}\psi(x_1,x_2)\psi(x_2)\psi(x_2,x_4)\psi(x_1)\psi(x_4)m(x_4)}{\sum_{x_1}\sum_{x_2}\psi(x_1,x_2)\psi(x_2)\psi(x_2,x_4)\psi(x_1)\psi(x_4)m(x_4)}$$

$$= \frac{\sum_{x_2}\psi(x_1,x_2)\psi(x_2)\psi(x_2,x_4)\psi(x_1)}{\sum_{x_1}\sum_{x_2}\psi(x_1,x_2)\psi(x_2)\psi(x_2,x_4)\psi(x_1)} = \frac{\sum_{x_2}\psi(x_1=1,x_2)\psi(x_2)\psi(x_2,x_4=1)\psi(x_1=1)}{\sum_{x_1}\sum_{x_2}\psi(x_1,x_2)\psi(x_2)\psi(x_2,x_4=1)\psi(x_1)}$$

$$= \frac{5\times 10\times 5\times 2 + 20\times 2\times 20\times 2}{20\times 10\times 5\times 10 + 5\times 2\times 20\times 10 + 5\times 10\times 5\times 2 + 20\times 2\times 20\times 2} = \frac{7}{47} = 0.1489$$

Solution: **Approach 2:** Node x_1 is conditionally independent of all other nodes except node x_2, given x_4. Let's compute the message from x_4 to x_2.

$m_{x_4\to x_2}$		
$x_2 = 0$	$0\times 20 + 2\times 5$	10
$x_2 = 1$	$0\times 5 + 2\times 20$	40

Now, let's compute the message from x_2 to x_1.

$m_{x_2\to x_1}$		
$x_1 = 0$	$10\times 20\times 10 + 2\times 5\times 40$	2400
$x_1 = 1$	$10\times 5\times 10 + 2\times 20\times 40$	2100

$$\tilde{p}(x_1=\hat{x}_1|x_3=0,x_4=1,x_6=0) = \psi(x_1=\hat{x}_1)\times m_{x_2\to x_1}(x_1=\hat{x}_1) \tag{5}$$

$\tilde{p}$		
$x_1 = 0$	10×2400	24000
$x_1 = 1$	2×2100	4200

$$p(x_1=1|x_3=0,x_4=1,x_6=0) = \frac{4200}{24000+4200} = \frac{7}{47} = 0.1489 \tag{6}$$

Problem 1.c. [2 points] Compute $p(x_{10}=1|x_9=1,x_{12}=1,x_2=0)$.

Solution: **Approach 1:** According to the conditional independence in MRF.

$$p(x_{10}|x_9,x_{12},x_2) = p(x_{10}|x_9,x_{12}) = \frac{p(x_{10},x_9,x_{12})}{p(x_9,x_{12})}$$

Denote message from nodes $M_{E/x_9,x_{10},x_{11},x_{12}} = \{x_1,\dots x_8\}$ to x_9 as

$$m(x_9) = \sum_{i,j\in M_{E/x_9,x_{10},x_{11},x_{12}}} \psi(x_i)\psi(x_i,x_j)$$

Then

$$p(x_{10}|x_9,x_{12},x_2) = \frac{p(x_{10},x_9,x_{12})}{p(x_9,x_{12})} = \frac{\sum_{x_{11}}\sum_{i,j\in M_{E/x_9,x_{10},x_{11},x_{12}}} p(x_1\dots x_{12})}{\sum_{x_{10}}\sum_{x_{11}}\sum_{i,j\in M_{E/x_9,x_{10},x_{11},x_{12}}} p(x_1\dots x_{12})}$$

$$= \frac{\sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}$$

$$= \frac{\sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9)\psi(x_9, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12})\psi(x_{12})m(x_9)}$$

$$= \frac{\sum_{x_{11}} \psi(x_9 = 1, x_{10} = 1)\psi(x_{10} = 1)\psi(x_{10} = 1, x_{11})\psi(x_{11})\psi(x_{10} = 1, x_{12} = 1)}{\sum_{x_{10}} \sum_{x_{11}} \psi(x_9 = 1, x_{10})\psi(x_{10})\psi(x_{10}, x_{11})\psi(x_{11})\psi(x_{10}, x_{12} = 1)}$$

$$= \frac{20 \times 2 \times 5 \times 10 \times 20 + 20 \times 2 \times 20 \times 2 \times 20}{5 \times 10 \times 20 \times 10 \times 5 + 20 \times 2 \times 5 \times 10 \times 20 + 5 \times 10 \times 5 \times 2 \times 5 + 20 \times 2 \times 20 \times 2 \times 20}$$

$$= \frac{48}{83} = 0.5783$$

Solution: **Approach 2:**

Node x_{10} is conditionally independent of all other nodes except node x_{11}, given x_9 and x_{12}. Let's compute messages from x_{11}, x_9, and x_{12} to x_{10}.

$m_{x_{11} \to x_{10}}$	
$x_{10} = 0$	$10 \times 20 + 2 \times 5 = 210$
$x_{10} = 1$	$10 \times 5 + 2 \times 20 = 90$

$m_{x_9 \to x_{10}}$	
$x_{10} = 0$	$0 \times 20 + 2 \times 5 = 10$
$x_{10} = 1$	$0 \times 5 + 2 \times 20 = 40$

$m_{x_{12} \to x_{10}}$	
$x_{10} = 0$	$0 \times 20 + 2 \times 5 = 10$
$x_{10} = 1$	$0 \times 5 + 2 \times 20 = 40$

$$\tilde{p}(x_{10} = \hat{x}_{10} | x_9 = 1, x_{12} = 1, x_2 = 0) = \psi(x_{10} = \hat{x}_{10}) \times m_{x_{11} \to x_{10}}(x_{10} = \hat{x}_{10}) \tag{7}$$

$$\times\, m_{x_9 \to x_{10}}(x_{10} = \hat{x}_{10}) \times m_{x_{12} \to x_{10}}(x_{10} = \hat{x}_{10}) \tag{8}$$

$\tilde{p}$		
$x_{10} = 0$	$10 \times 210 \times 10 \times 10$	210000
$x_{10} = 1$	$2 \times 90 \times 40 \times 40$	288000

$$p(x_{10} = 1 | x_9 = 1, x_{12} = 1, x_2 = 0) = \frac{288000}{210000 + 288000} = \frac{48}{83} = 0.5783 \tag{9}$$

Problem 1.d. [2 points] Compute $p(x_6 = 0 | x_4 = 1, x_8 = 1, x_{10} = 0)$.

Solution: **Approach 1:** According to the conditional independence in MRF.

$$p(x_6 | x_4, x_8, x_{10}) = p(x_6 | x_4, x_8) = \frac{\sum_{x_7} p(x_4, x_6, x_8, x_7)}{\sum_{x_7} \sum_{x_7} p(x_4, x_6, x_8, x_7)}$$

Similar to previous questions, we can ignore the message from outer nodes to x_4 and x_8 (since the messages to this two nodes appear in both denominator and numerator, so they can be eliminated). Then

$$p(x_6 = 0|x_4 = 1, x_8 = 1) = \frac{\sum_{x_7} \psi(x_4 = 1, x_6 = 0)\psi(x_8 = 1, x_6 = 0)\psi(x_7)\psi(x_7, x_6 = 0)\psi(x_6 = 0)}{\sum_{x_6} \sum_{x_7} \psi(x_4 = 1, x_6)\psi(x_8 = 1, x_6)\psi(x_7)\psi(x_7, x_6)\psi(x_6)}$$

$$= \frac{10 \times 5 \times 5 \times 2 \times 5 + 5 \times 5 \times 10 \times 20 \times 10}{10 \times 5 \times 5 \times 2 \times 5 + 5 \times 5 \times 10 \times 20 \times 10 + 20 \times 20 \times 10 \times 5 \times 2 + 20 \times 20 \times 2 \times 20 \times 2} = \frac{35}{83} = 0.4217$$

Solution: Approach 2: Node x_6 is conditionally independent of all other nodes except node x_7, given x_4 and x_8. Let's compute messages from x_7, x_4, and x_8 to x_6.

$m_{x_7 \to x_6}$	
$x_6 = 0$	$10 \times 20 + 2 \times 5 = 210$
$x_6 = 1$	$10 \times 5 + 2 \times 20 = 90$

$m_{x_4 \to x_6}$	
$x_6 = 0$	$0 \times 20 + 2 \times 5 = 10$
$x_6 = 1$	$0 \times 5 + 2 \times 20 = 40$

$m_{x_8 \to x_6}$	
$x_6 = 0$	$0 \times 20 + 2 \times 5 = 10$
$x_6 = 1$	$0 \times 5 + 2 \times 20 = 40$

$$\tilde{p}(x_6 = \hat{x}_6|x_4 = 1, x_8 = 1, x_{10} = 0) = \psi(x_6 = \hat{x}_6) \times m_{x_7 \to x_6}(x_6 = \hat{x}_6) \tag{10}$$
$$\times\, m_{x_8 \to x_6}(x_6 = \hat{x}_6) \times m_{x_8 \to x_6}(x_6 = \hat{x}_6) \tag{11}$$

$\tilde{p}$		
$x_6 = 0$	$10 \times 210 \times 10 \times 10$	210000
$x_6 = 1$	$2 \times 90 \times 40 \times 40$	288000

$$p(x_6 = 0|x_4 = 1, x_8 = 1, x_{10} = 0) = \frac{210000}{210000 + 288000} = \frac{35}{83} = 0.4217 \tag{12}$$

Problem 1.e. [2 points] Compute $p(x_8 = 1|x_1 = 0, x_6 = 0, x_9 = 1, x_{12} = 1)$.

Solution: Approach 1: According to the conditional independence in MRF.

$$p(x_8|x_1, x_6, x_9, x_{12}) = p(x_8|x_6, x_9) = \frac{p(x_8, x_6, x_9)}{\sum_{x_8} p(x_8, x_6, x_9)}$$

Similar to previous questions, we can ignore the message from outer nodes to x_6 and x_9 (since the messages to this two nodes appear in both denominator and numerator, so they can be eliminated). Then

$$p(x_8 = 1|x_6 = 0, x_9 = 1) = \frac{\psi(x_8 = 1, x_6 = 0)\psi(x_9 = 1, x_8 = 1)\psi(x_8 = 1)}{\sum_{x_8} \psi(x_8, x_6 = 0)\psi(x_9 = 1, x_8)\psi(x_8)}$$

$$= \frac{5 \times 20 \times 2}{5 \times 20 \times 2 + 20 \times 5 \times 10} = \frac{1}{6} = 0.1667$$

Approach 2: Node x_8 is conditionally independent of all other nodes, given x_6 and x_9. Let's compute messages from x_6 and x_9 to x_8.

$m_{x_6 \to x_8}$		
$x_8 = 0$	$10 \times 20 + 0 \times 5$	200
$x_8 = 1$	$10 \times 5 + 0 \times 20$	50

$m_{x_9 \to x_8}$		
$x_8 = 0$	$0 \times 20 + 2 \times 5$	10
$x_8 = 1$	$0 \times 5 + 2 \times 20$	40

$$\tilde{p}(x_8 = \hat{x}_8 | x_1 = 0, x_6 = 0, x_9 = 1, x_{12} = 1) = \psi(x_8 = \hat{x}_8) \times m_{x_6 \to x_8}(x_8 = \hat{x}_8) \tag{13}$$
$$\times\, m_{x_9 \to x_8}(x_8 = \hat{x}_8) \tag{14}$$

$\tilde{p}$		
$x_8 = 0$	$10 \times 200 \times 10$	20000
$x_8 = 1$	$2 \times 50 \times 40$	4000

$$p(x_8 = 1 | x_1 = 0, x_6 = 0, x_9 = 1, x_{12} = 1) = \frac{4000}{20000 + 4000} = \frac{1}{6} = 0.1667 \tag{15}$$

Problem 1.f. [2 points] Compute $p(x_2 = 0 | x_1 = 0, x_3 = 1, x_4 = 1, x_7 = 1, x_{11} = 0)$.

Solution: Approach 1: According to the conditional independence in MRF.

$$p(x_2 | x_1, x_3, x_4, x_7, x_{11}) = p(x_2 | x_1, x_4) = \frac{p(x_1, x_2, x_4)}{\sum_{x_2} p(x_1, x_2, x_4)}$$

Similar to previous questions, we can ignore the message from outer nodes to x_1 and x_4 (since the messages to this two nodes appear in both denominator and numerator, so they can be eliminated). Then

$$p(x_2 | x_1, x_4) = \frac{\psi(x_2 = 0, x_1 = 0)\psi(x_4 = 1, x_2 = 0)\psi(x_2 = 0)}{\sum_{x_2} \psi(x_2, x_1 = 0)\psi(x_4 = 1, x_2)\psi(x_2)}$$
$$= \frac{20 \times 5 \times 10}{5 \times 20 \times 2 + 20 \times 5 \times 10} = \frac{5}{6} = 0.8333$$

Solution: Approach 2:

Node x_2 is conditionally independent of all other nodes, given x_1 and x_4. This will result in a similar structure and, therefore, exactly the same computations as the previous question. However, here we need to compute the probability for $x_2 = 0$ whereas in the previous question it was for $x_8 = 1$. The answer can then be computed as $1 - \frac{1}{6} = \frac{5}{6} = 0.8333$.