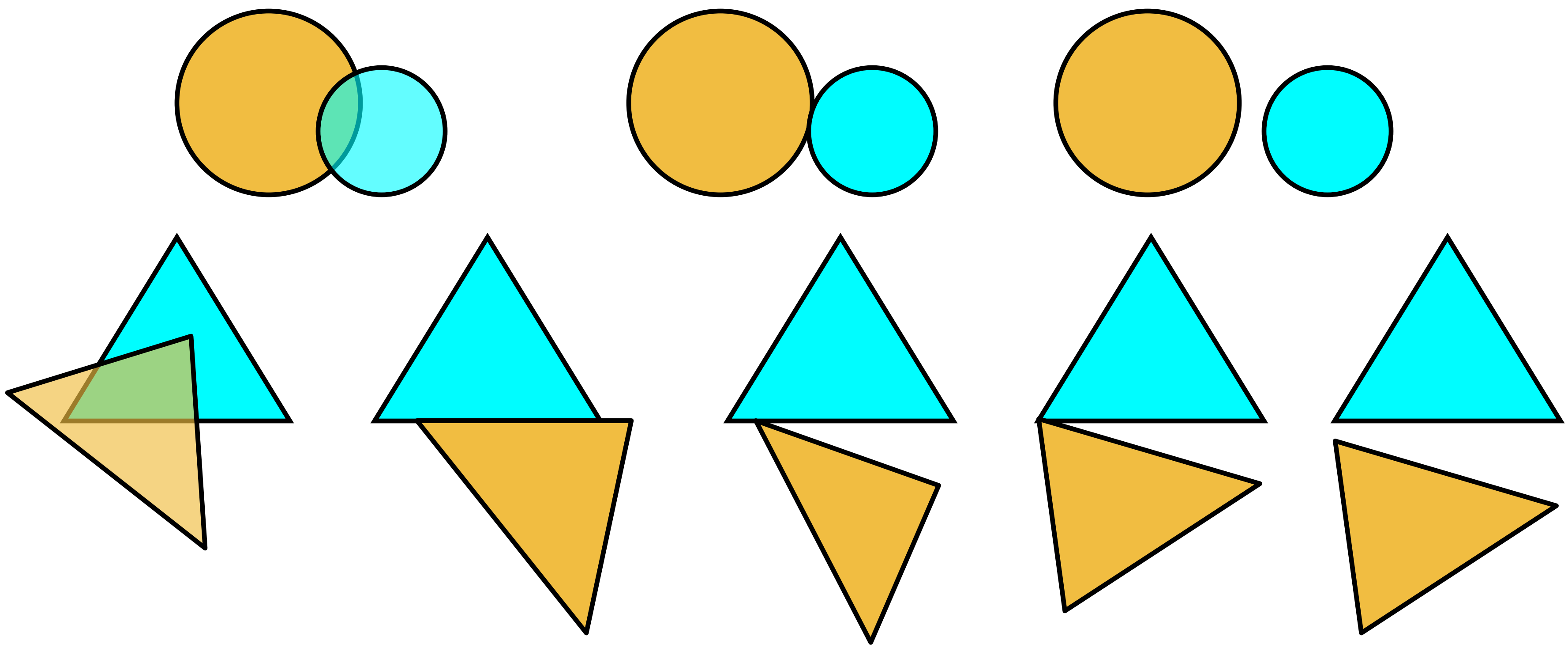
Topology

MAGIC Lab
Modeling, Animation, Graphic, Interaction & Compression

- What is a polygon?
- Set operations
- Interior, boundary, exterior
- Skin, Hair, Wound, Cut, and regularization
- Components, holes
- Polygons and faces
- Loops
- A linear geometric complex

Motivation

- We need a terminology and notation to describe the domain for any particular representation or operation and the precise nature of the result.
  - How would you distinguish these situations?



Symbols and notation

- ! complement, ∪ union, ∩ intersection, ⊗ XOR

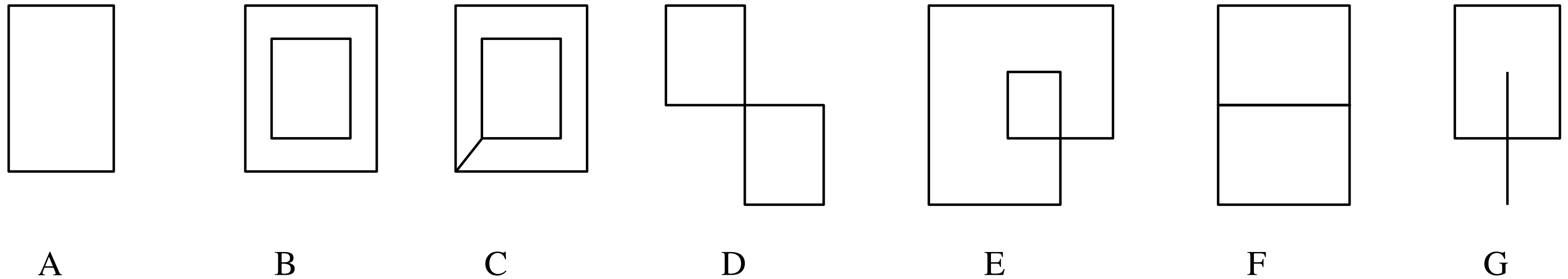
  We often use + for union and no operator for intersection

  ! has highest priority, then intersection: !A(!B+CD)
- ⊂ inclusion, = equality
- ∈ member of, ∉ not member of
- ∅ empty set, Ω whole space (Euclidean)
- Set definition {p: p∈A and p∉B}
- ∀ each, ∃ there is, ⇒ implies, ⇔ iff
- A∩B≠∅ sets interfere, A∩B=∅, A∩C=∅… exclusive
- ⊙ a neighborhood (infinitely small ball) around a point

Define a polygon

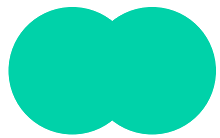
Write the definition of "**polygon**" on a sheet of paper with your name on it.

List the letters corresponding to drawings that represent valid polygons according to your definition.



Set theoretic operations

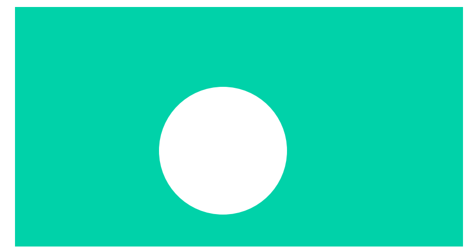
union



$A \cup B = \{s: s \in A$ **or** $s \in B\}$

also written A+B

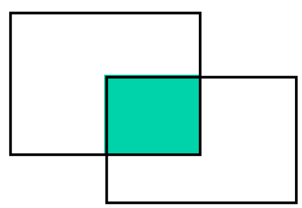
complement



$!S = \{s: s \notin S\}$

! has highest priority

intersection



$A \cap B = \{s: s \in A$ **and** $s \in B\}$

also written AB

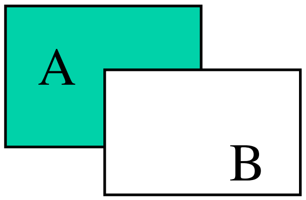
deMorgan Laws

!!A=A

!(A+B) = !A!B

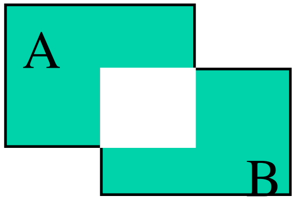
!(AB) = !A+!B

Differences



$A \setminus B = A!B$

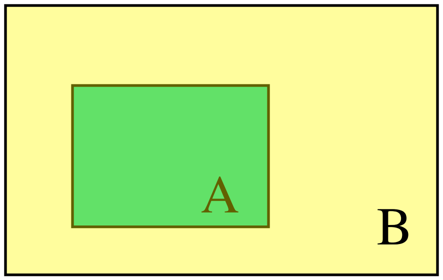
also written A–B



$A \otimes B = A!B + !AB$

called *XOR* and also *symmetric difference*

Inclusion



$A \subset B$

complete the equivalence

$$(A \subset B) \Leftrightarrow (\qquad , p \in A \Rightarrow p \in B)$$

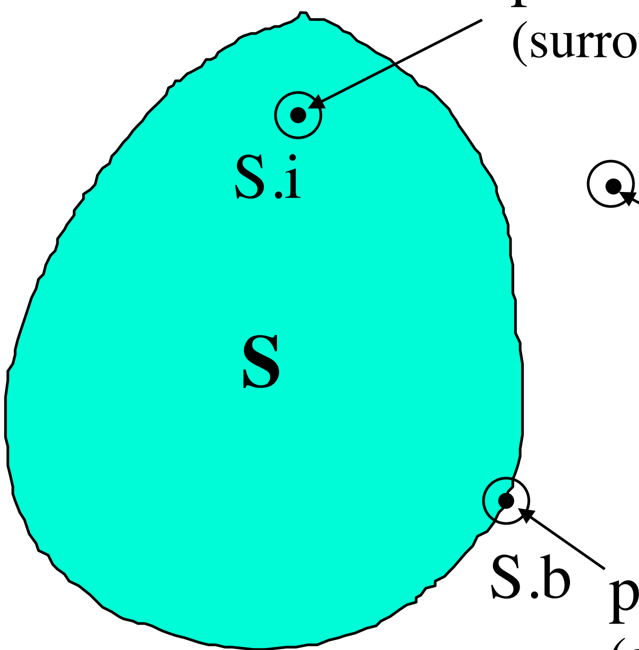
$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

Bounded (finite)

- A set is bounded (or equivalently finite) if it is contained in a ball of finite radius.
  – A line, a ray are not bounded
  – A disk, and edge, are

Boundary of 2D point-sets

- a point and its neighborhood (*tiny* ball around it)

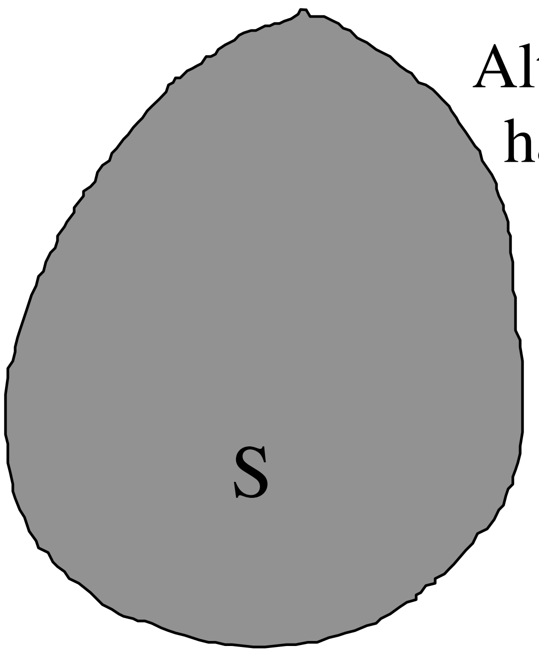


point in the **interior** iS of S
(surrounded by disk entirely in S)

point in the **exterior** eS of S (surrounded by disk entirely out of S)

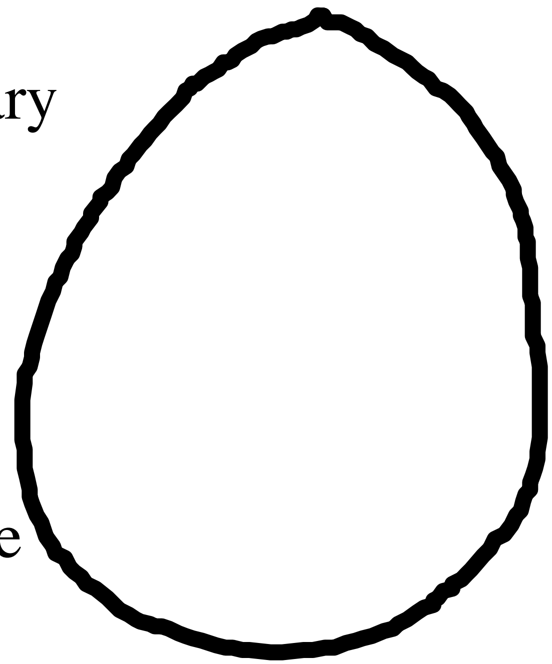
point on the **boundary** bS of S
(all surrounding disks intersect S and !S)
the point is **touching** S and !S

How to **draw** the boundary



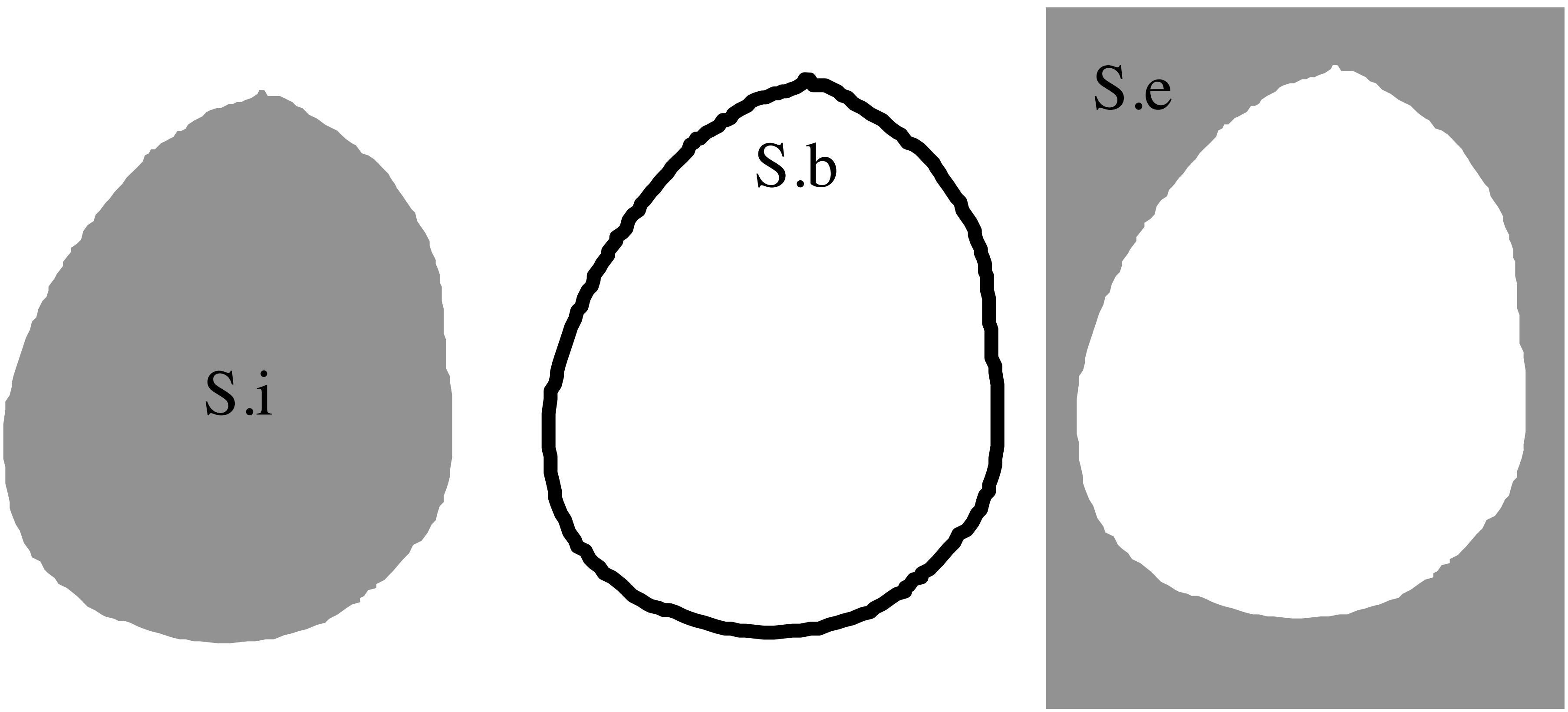
Although the boundary has 0 thickness,

we draw it as a thick line



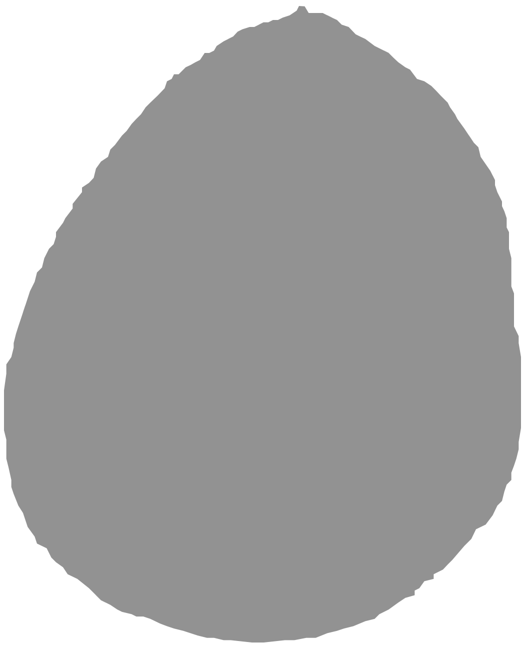
S.b

Interior, boundary, exterior



Open set

A set is said to be **open** when it does **not** include any of its boundary

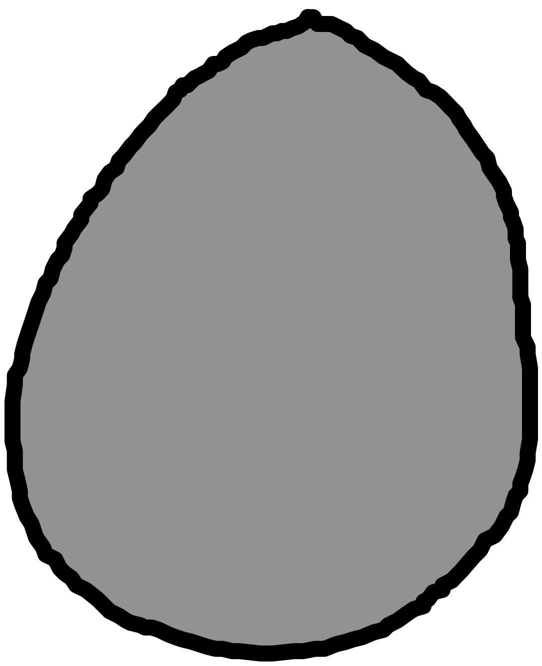


S=S.i

contains no points that touch S.b

Closed set

A set is closed if it contains its boundary

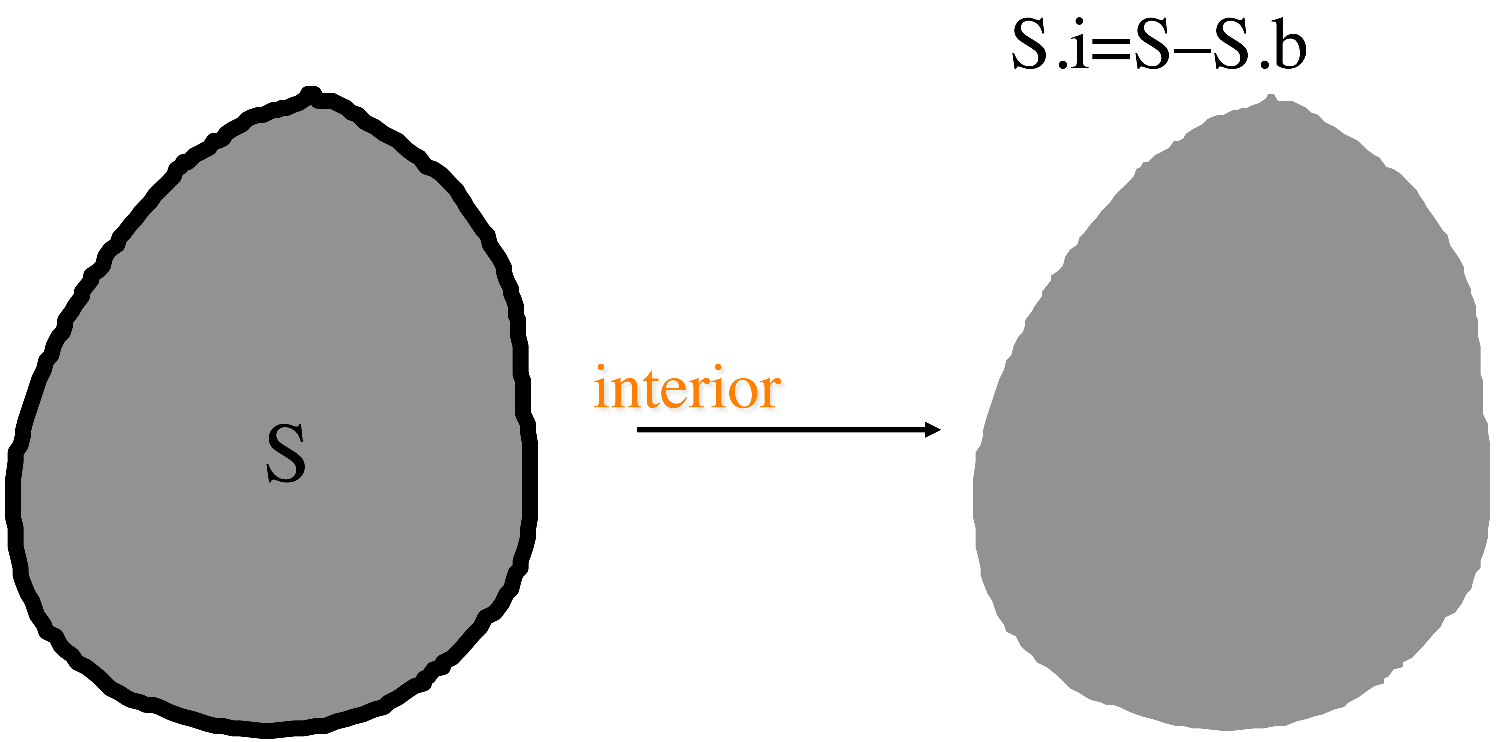


S=S.i+S.b

$S.b \subset S$

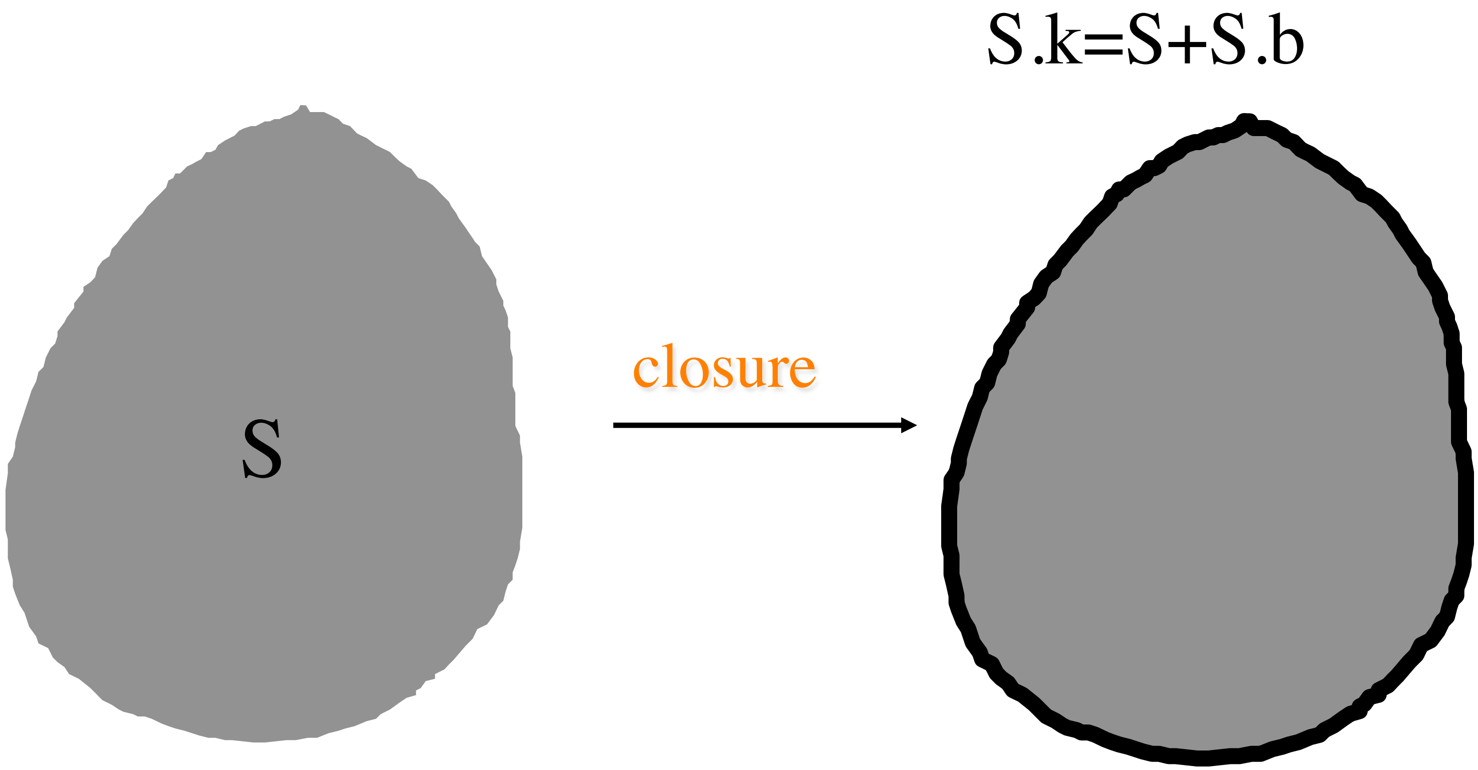
Interior operator

- The **interior** operator returns the difference between the set and its boundary



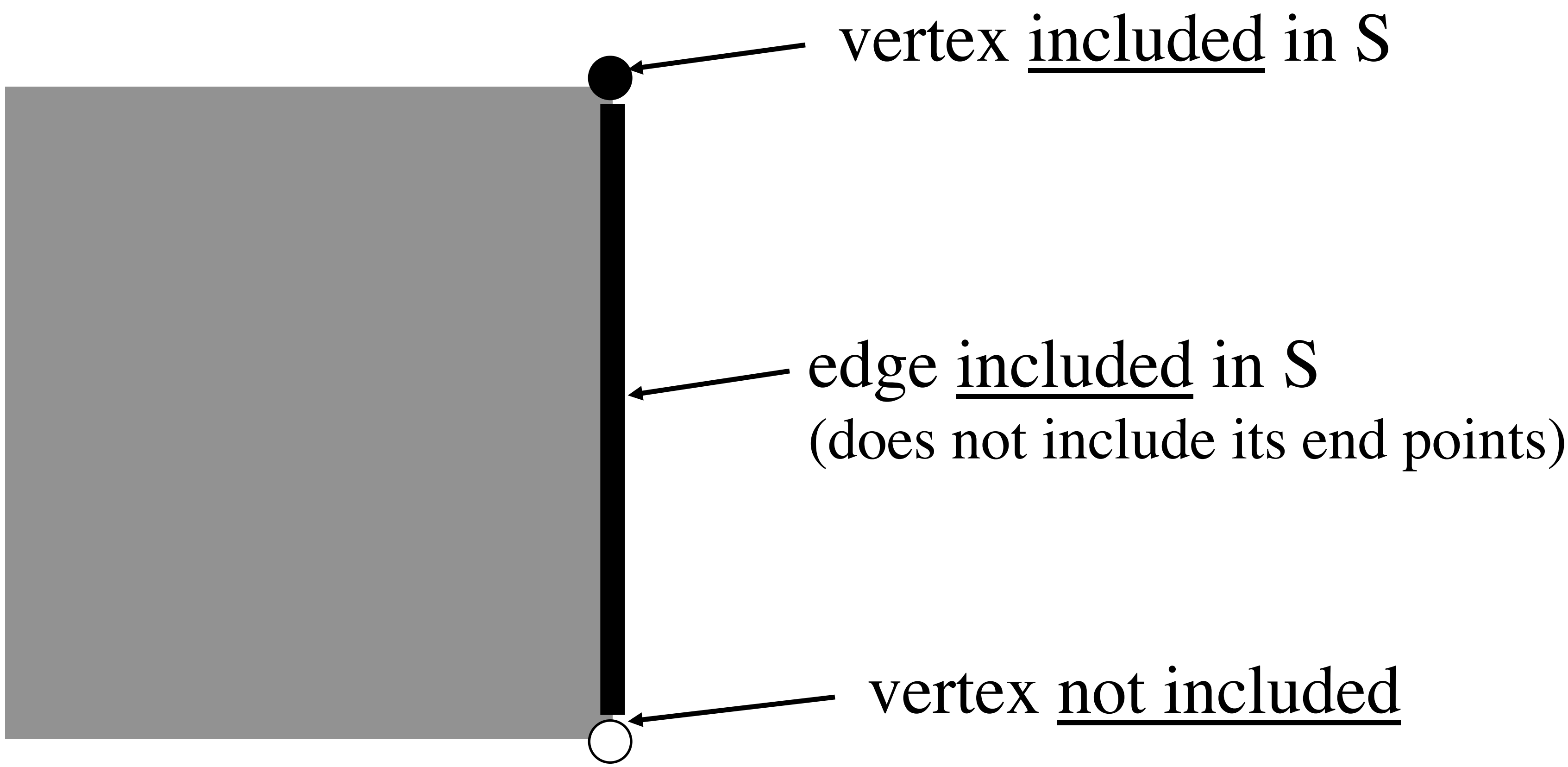
Closure operator

The **closure** operator returns the union of the set with its boundary



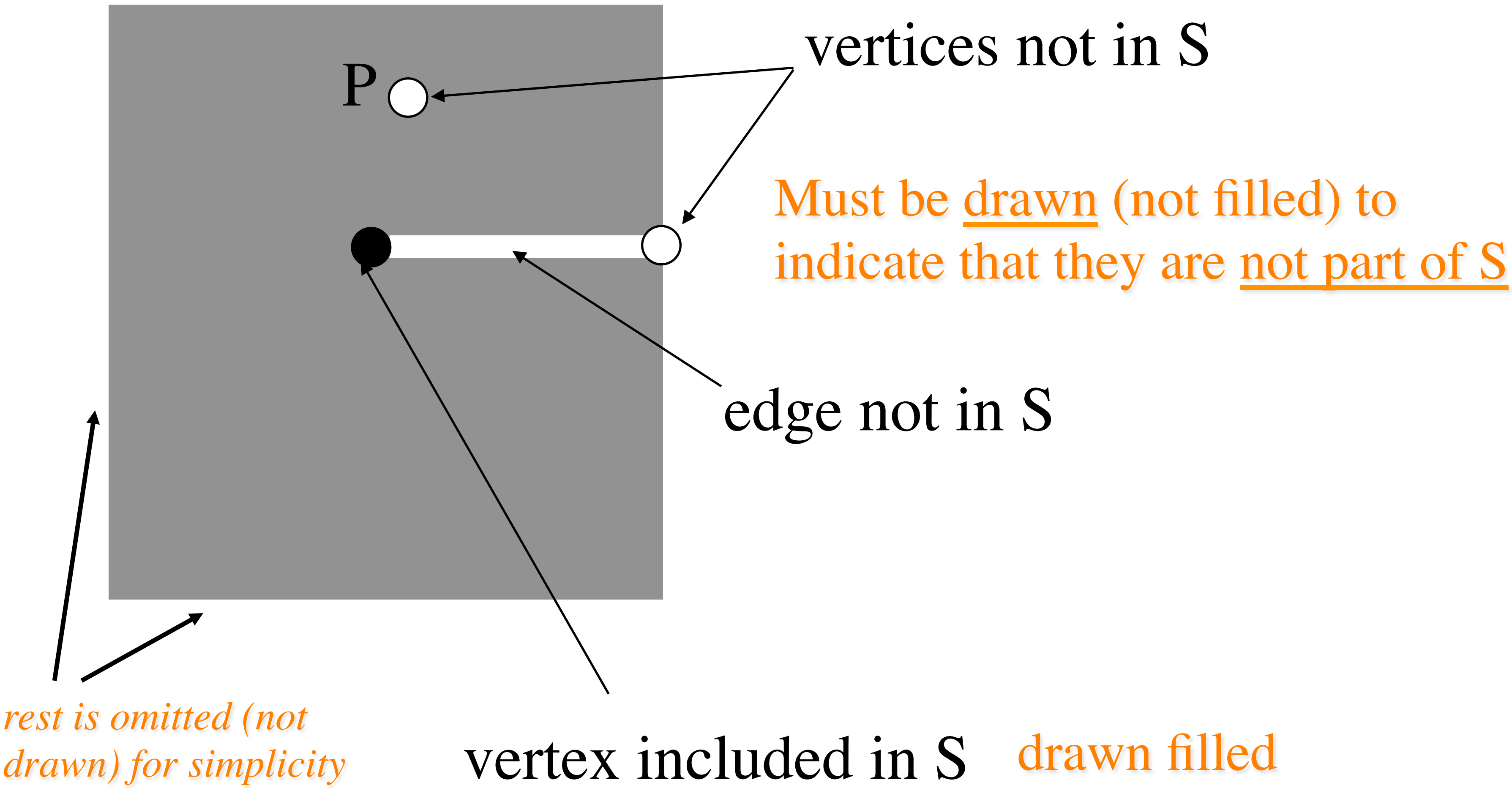
Drawing the boundary

- Use color or **fill** to indicate which portions of the boundary are included in the set S.



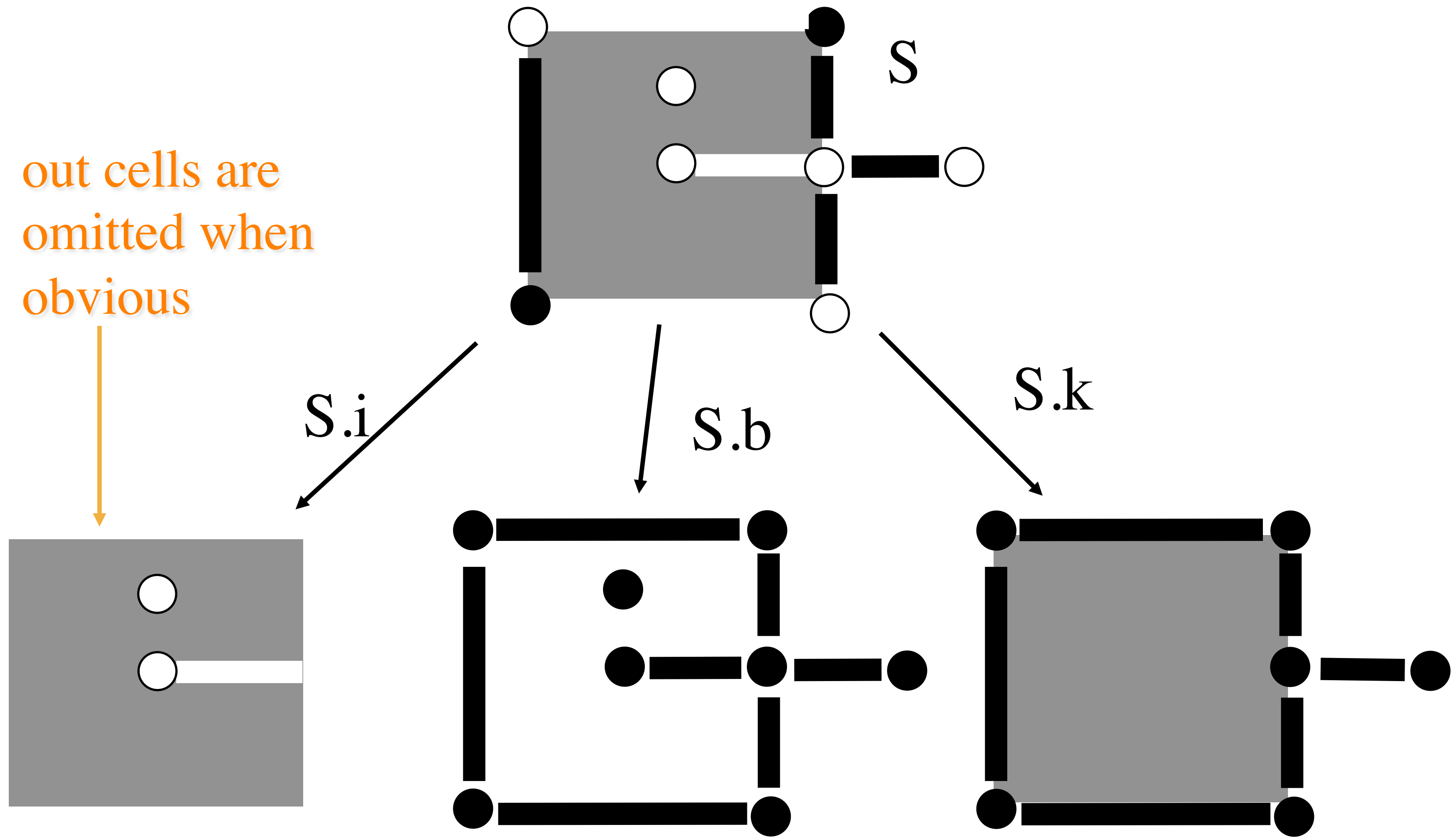
Drawing interior boundaries

- Interior boundaries = boundary portions not touching S.e



Interior, boundary, and closure

out cells are omitted when obvious

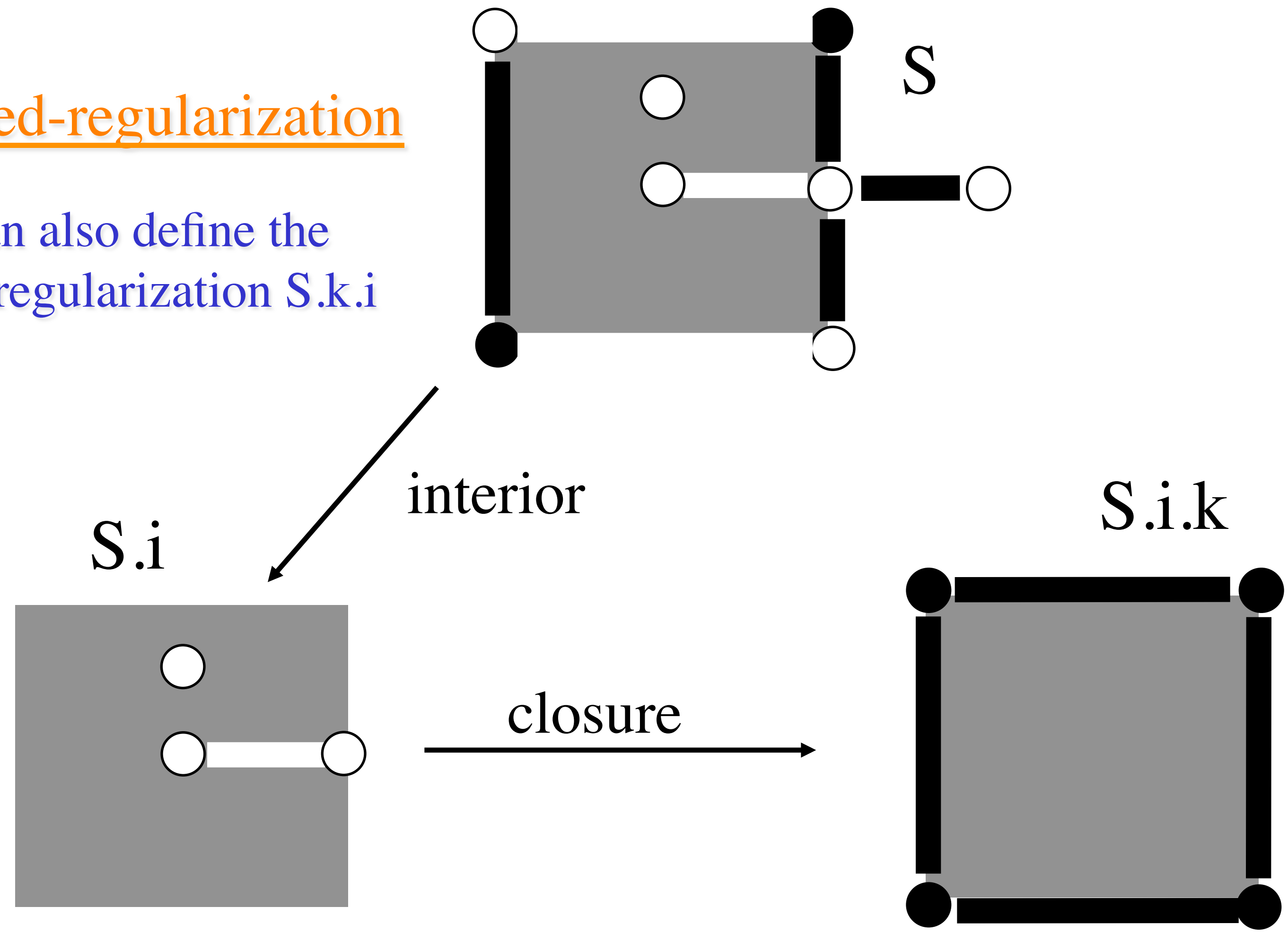


Regularization operator

- The regularization S.r of set S is S.i.k, the closure of its interior

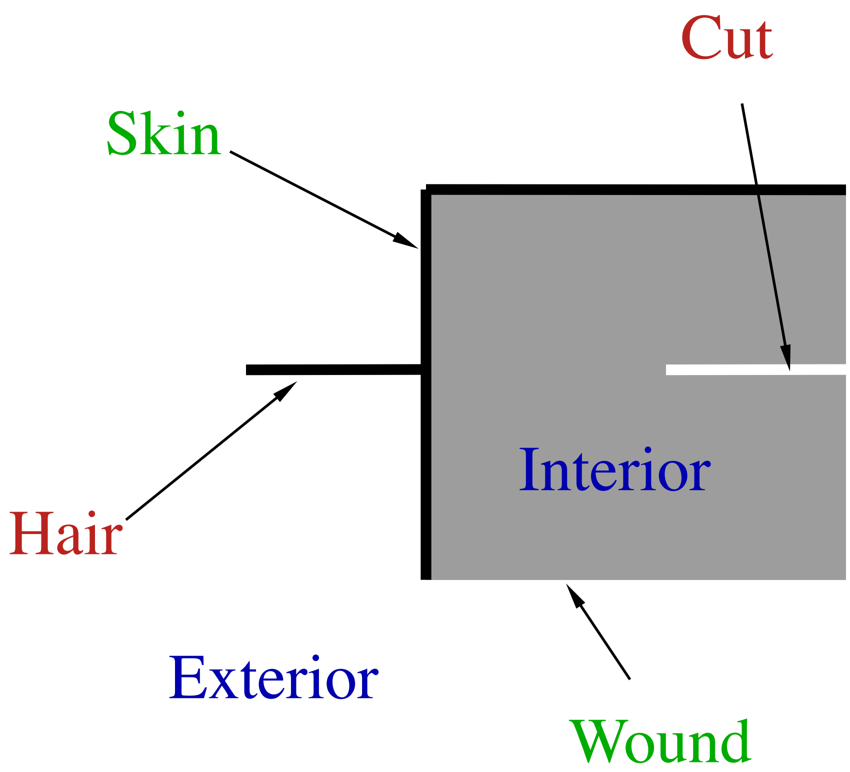
closed-regularization

We can also define the open-regularization S.k.i



Decomposing the boundary

Membrane S.m separates S.i from S.e

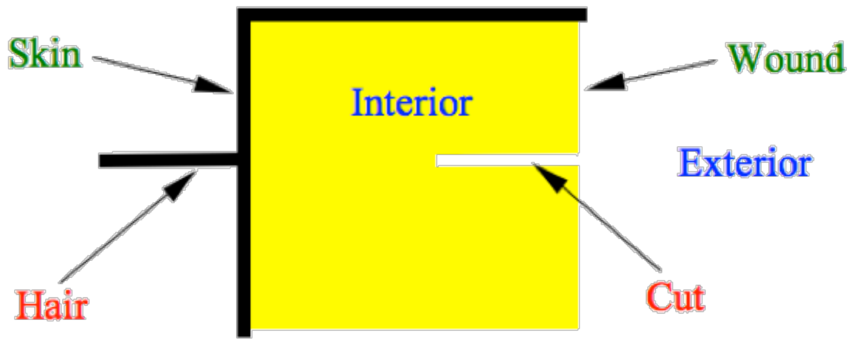


S.b	**Dangle**	**Membrane S.m**
In S	Hair S.h	Skin S.s
In !S	Cut S.s	Wound S.w

Any set S defines a decomposition of Ω into 6 exhaustive & exclusive sets: S.i, S.e, S.s, S.w, S.h, S.c

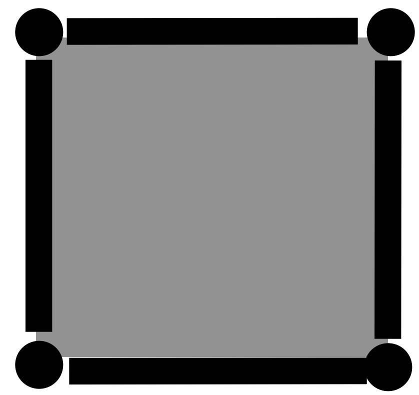
Definitions

- ***Boundary:*** S.b = points touching to S and !S
- ***Interior:*** S.i = S – S.b
- ***Exterior***: S.e = !S – S.b
- ***Membrane:*** S.m = S.i.b∩S.e.b
  - ***Skin:*** S.s = S.m∩S
  - ***Wound:*** S.w = S.m–S
- ***Dangle:*** S.d = S.b–S.m
  - ***Hair:*** S.h = S.d∩S
  - ***Cut:*** S.c = S.d–S



Regularized sets

A set is **closed-regularized** if it is equal to the closure of its interior



S = S.i.k

A set is **open-regularized** if it is equal to the interior of its closure

S = S.k.i

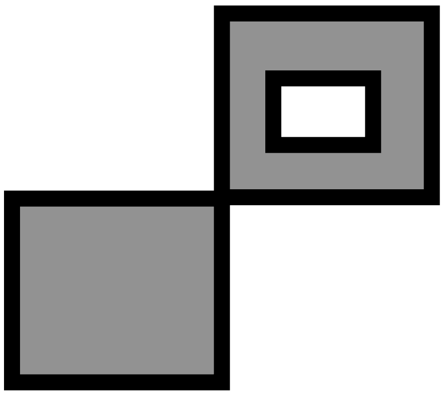
Regularized sets have no *cut* and no *hair*

Connected components

- A set **S** is ***connected*** if from every point **p** in **S** one can walk to every other point **q** in **S** along a curve **C** that lies entirely in **S.**
- A non-empty set **S** has one or more maximally connected components, which we will call the ***components*** of **S**
  - If a connected subset of S contains a component U, then S=U.

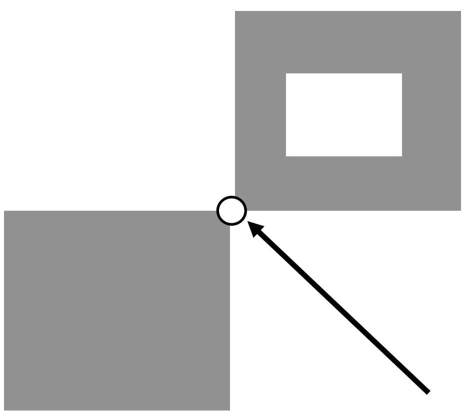
Draw a set S that is not connected but whose closure S.k is.

Connected components



This set is closed and connected

Can join any pair of points by a curve in S

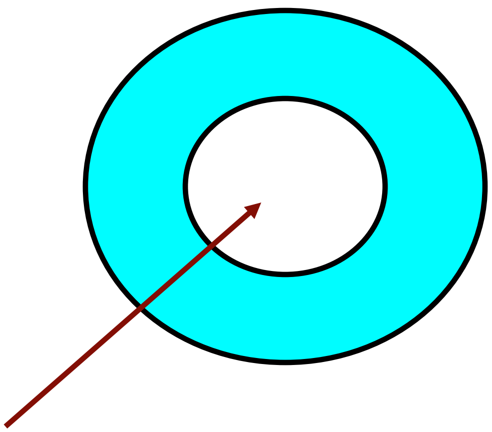


vertex not in S

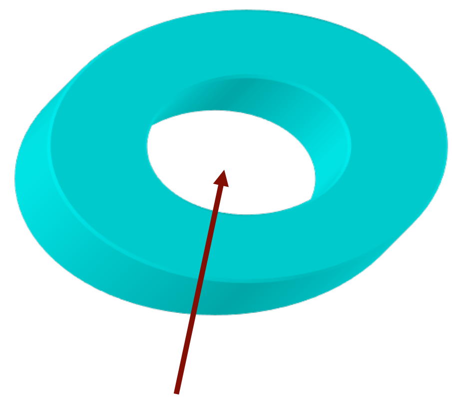
This open set is not connected:
it has 2 components

Holes of a bounded set

- A hole in S is a bounded components of its complement
  - For example a closed cavity in a 3D shape
  - Do not confuse a hole with the concave part where the coffee stays or with the through-hole in the handle of a mug



Hole in a 2D set



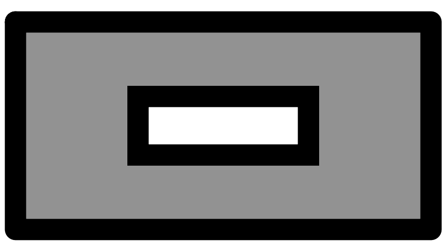
NOT a hole in a 3D set.
S has **genus** 1 (1 **handle**)

Cannot say where the handle is!

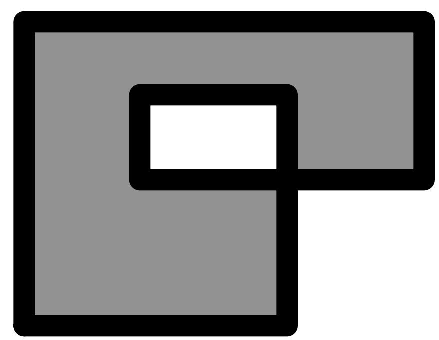
Examples of holes

Simply connected closed set

Open set with no hole



Closed set with one hole

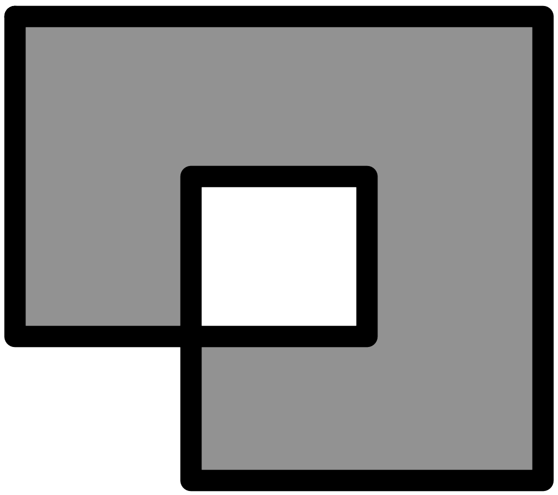


Closed set with one hole

Manifold boundary

Connected set with a manifold boundary

Connected set with a non-manifold boundary



in 2D removing a non-manifold vertex would change the the number of connected components or of holes (which are the connected components of the complement)

Write the definition of a polygon

2-cell (region), not its bounding curve

connected

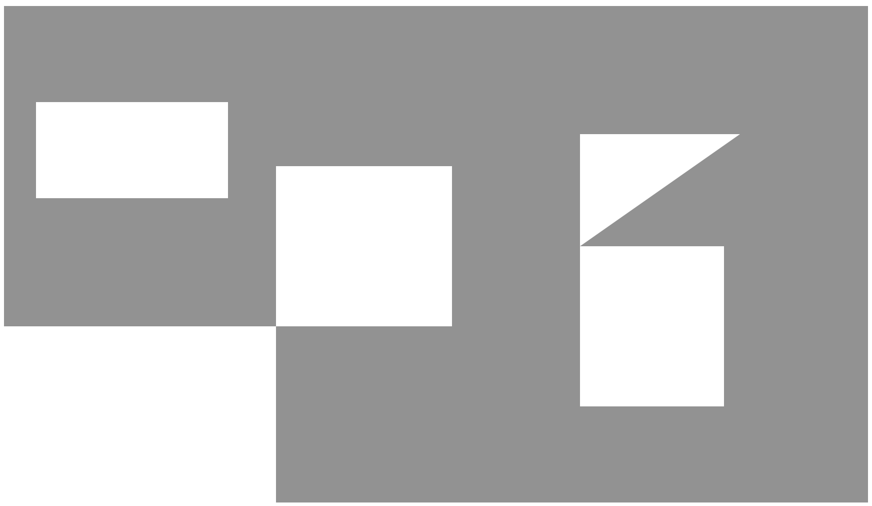
may have holes

needs not be manifold

is bounded by straight line segments (a finite number)

is bounded (finite)

Do not confuse a topological definition with a particular data structure or representation scheme



Definition of a **polygon**

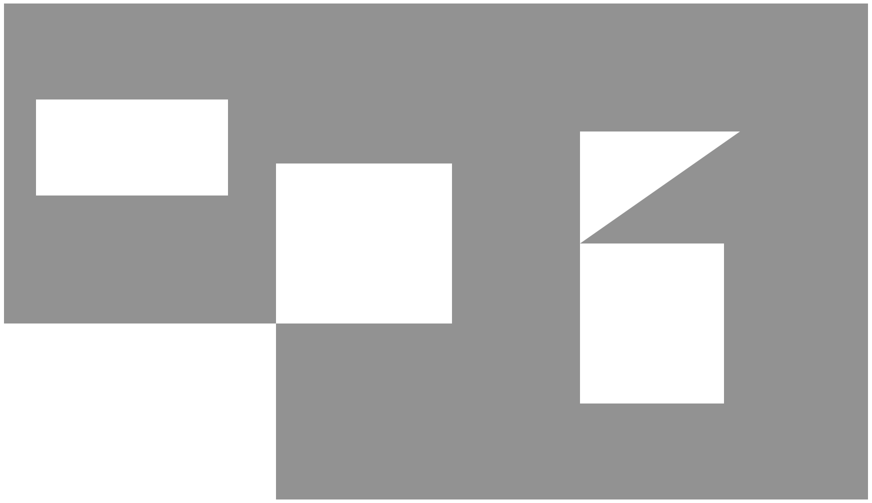
Connected, bounded, **open-regularized**, subset of the plane, with boundary is a subset of a finite union of lines.

A polygon is open: does not contain its boundary

A polygon is regularized: has no cut (S=S.k.i)

A polygon can be non-manifold

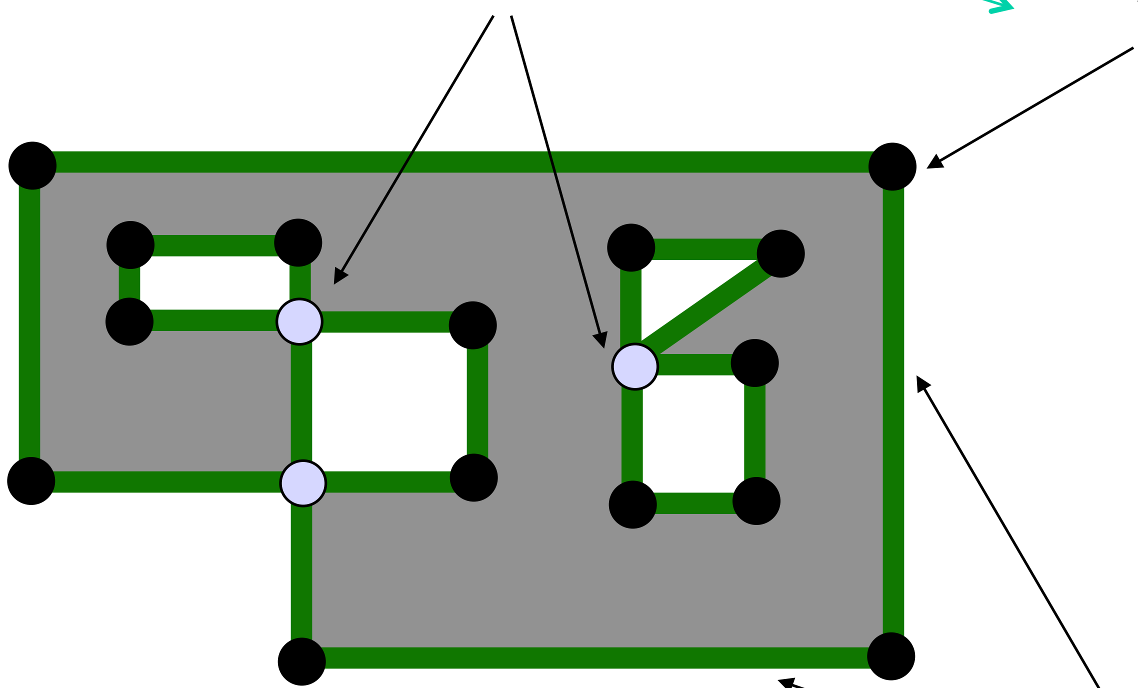
A polygon may have holes



Cells of a polygon

- Given a polygon, do we know what are its edges and vertices?
- Note that this is different from the question: given a set of edges and vertices, what is the polygon they define.
  - Because a representation of a polygon in terms of edges and vertices may be
    - ambiguous
    - invalid
- We will discuss a representation scheme for modeling polygons

Vertices and edges of a polygon



Non-smooth points of S.b that are not crossings

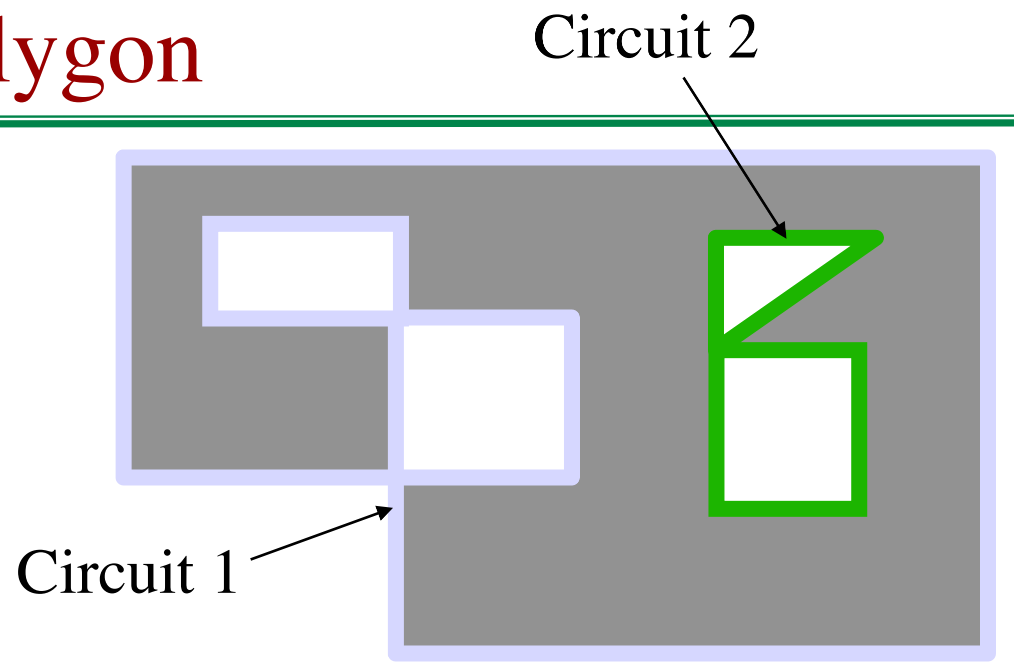
Connected components of the boundary **without vertices and crossings**

Edges, vertices, crossings are exclusive

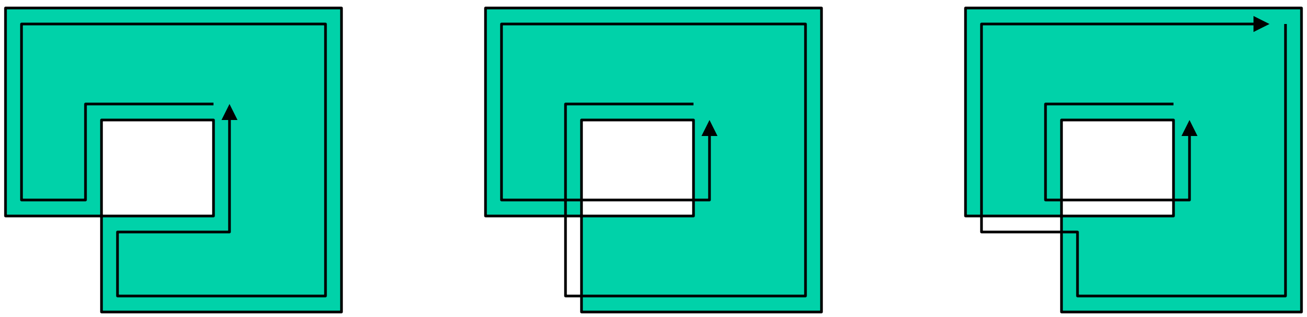
Circuits of a polygon

The circuits of S are the components of S.b

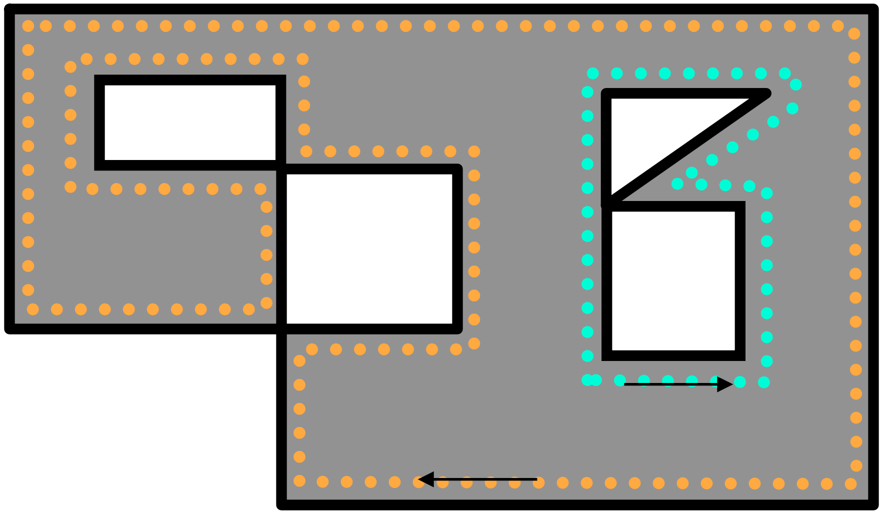
Circuits are exclusive



Can they be represented as a circular list of vertex ids?
The same vertex may appear more than once, but the circuit should not cross itself



Computing the circuits



Edges = roads
Vertices = turns
Crossings = intersections

Each tour visits a loop
- Pick unvisited edge
- Walk along interior sidewalk
  - keep road on your left
  - never cross a street

Each circuit may be represented by a circular list of vertex ids (with replications).

S.b is the union of all its circuits (which are disjoint)

Definition of a **Face**

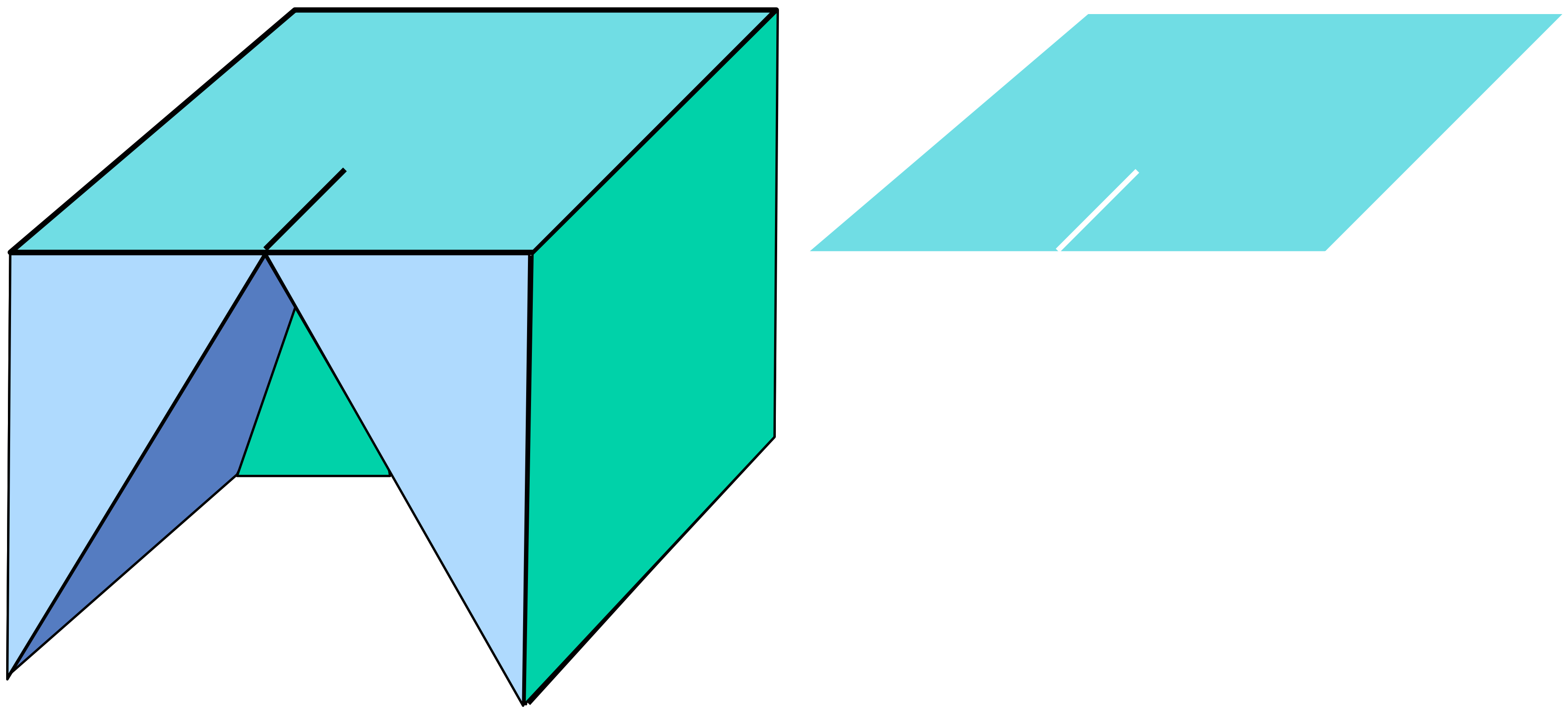
Open, connected, subset of the plane,
with boundary in the union of a finite number of lines

Needs **not** be **regularized**: may have a cut

Needs **not** be **bounded**: may be infinite

Why distinguish faces & polygons

- Want polyhedra boundaries to be decomposed into an **exclusive** set of **vertices**, **edges**, and **faces**
- Polyhedra are bounded by faces, not by polygons

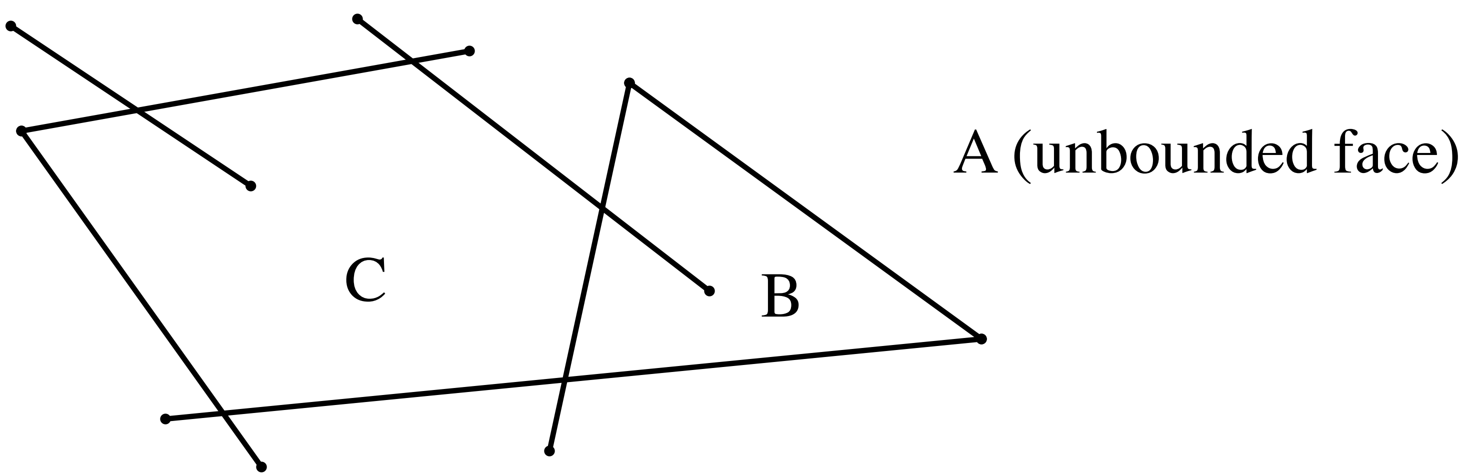


Faces from edges of arrangement

- Consider the union U of edges in the plane Ω
- The connected components of Ω–U are **faces**

The edges define an arrangement

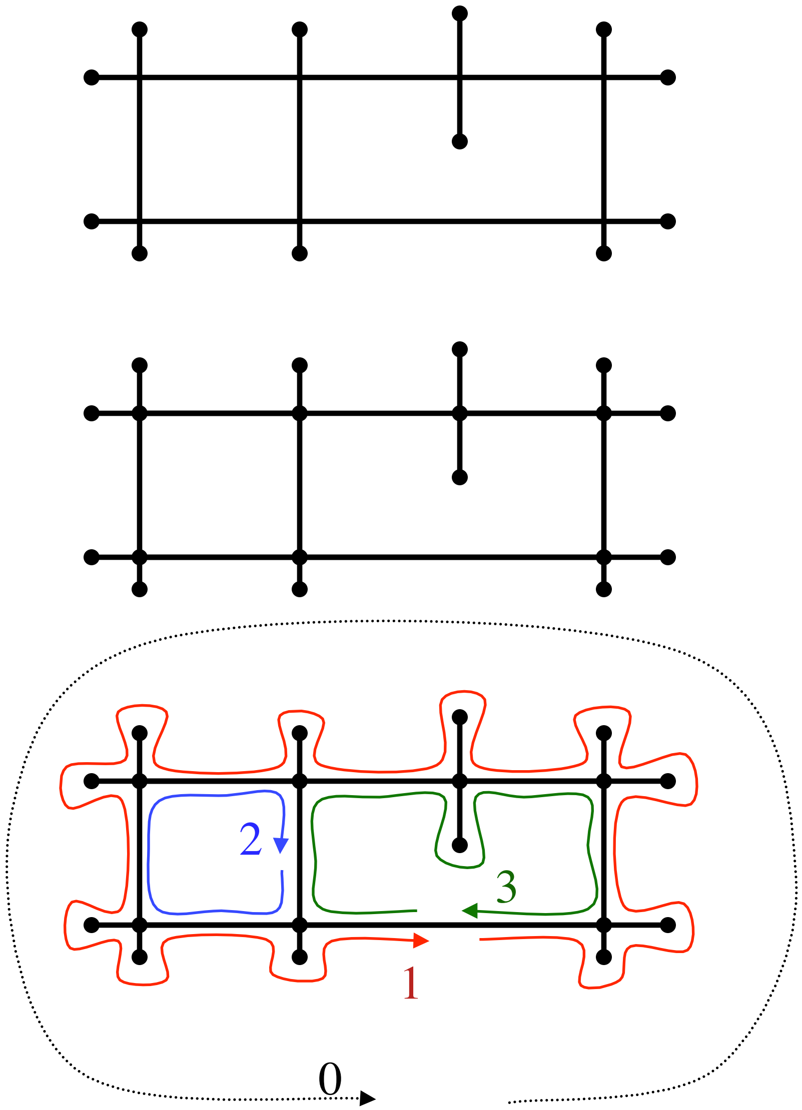
The faces are the 2-cells of the arrangement

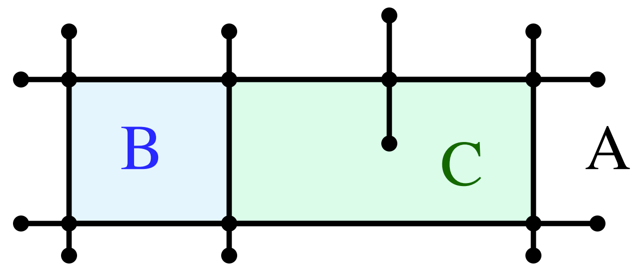


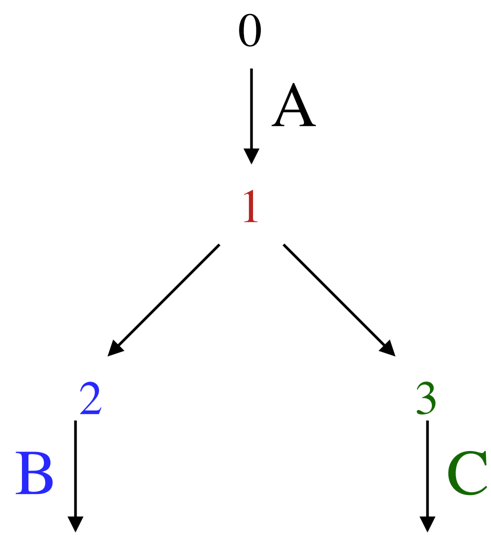
How to compute the arrangement

- Split all edges are their intersections and at their intersections with vertices of other edges
- Build circuits for each face
  - Keep edges to your left and never cross them
- Build an **inclusion tree** of circuits
  - Assume a sidewalk at infinity around everything
  - Each node is a loop that contains all its children
    - Cast ray inwards and use parity of # of intersections
- Identify faces and their boundaries
  - Nodes at even graph-distance from the root are the outer boundaries of faces
  - Their children are the boundaries of their holes

Inclusion tree





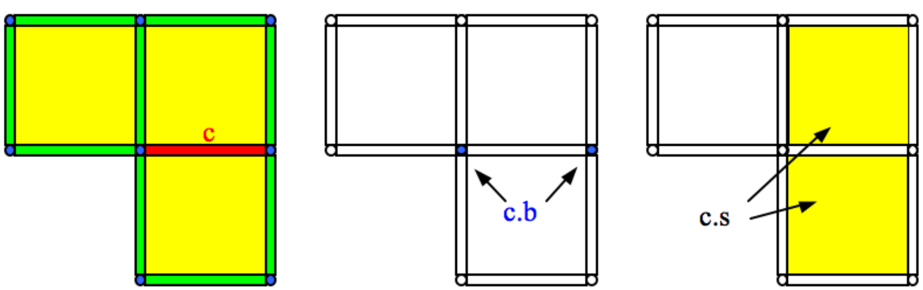


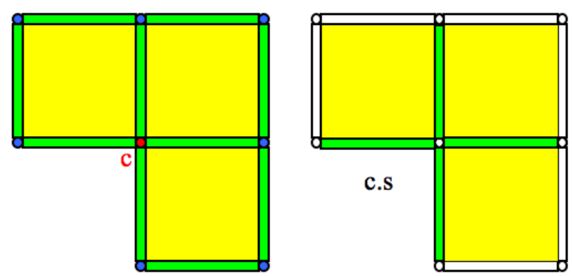
Linear Complex K

Exclusive and exhaustive collection of cells: vertices (0-cells), edges (1-cells), faces (2-cells), and in 3D volumes (3-cells).

The boundary of any cell c of K is the union of cells of K.

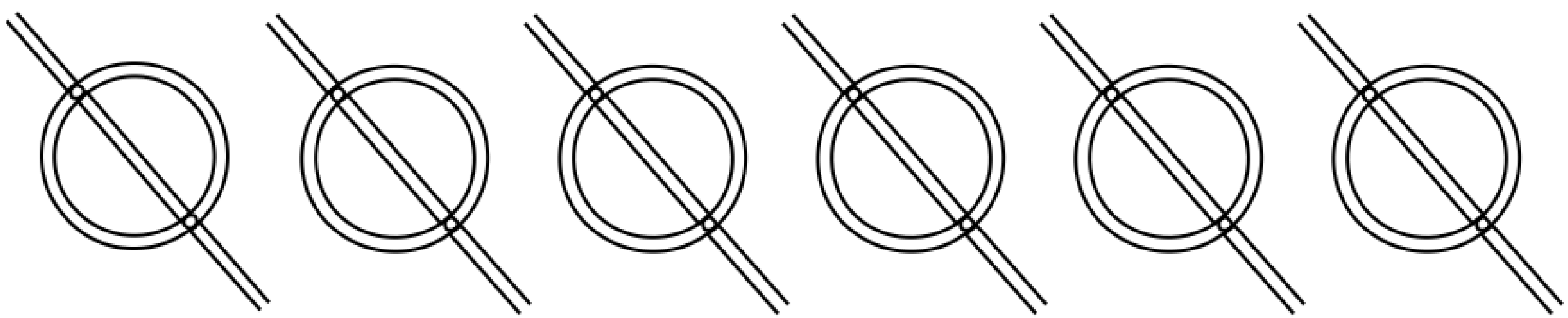
Each cell c of K can be asked for is set c.p, its dimension c.d, its boundary c.b, and its star c.s.





Test 1

Let S=Disk⊗Line. The series of figures, below, show the SGC, G, defined by S (i.e., S=G.p). From left to right, identify (fill in) the cells for SGCs yielding the following pointsets: S, S.e, S.b, S.i, S.k, and S.r. Make sure that you correctly classify the vertices (0-cells).



Test 2

Consider the SGC, G, shown below left. Let S=G.p. In the subsequent figures, from left to right, mark (i.e. fill in) the cells of SGCs whose point sets are in the skin, wound, membrane, hair, cut, and dangle of S.