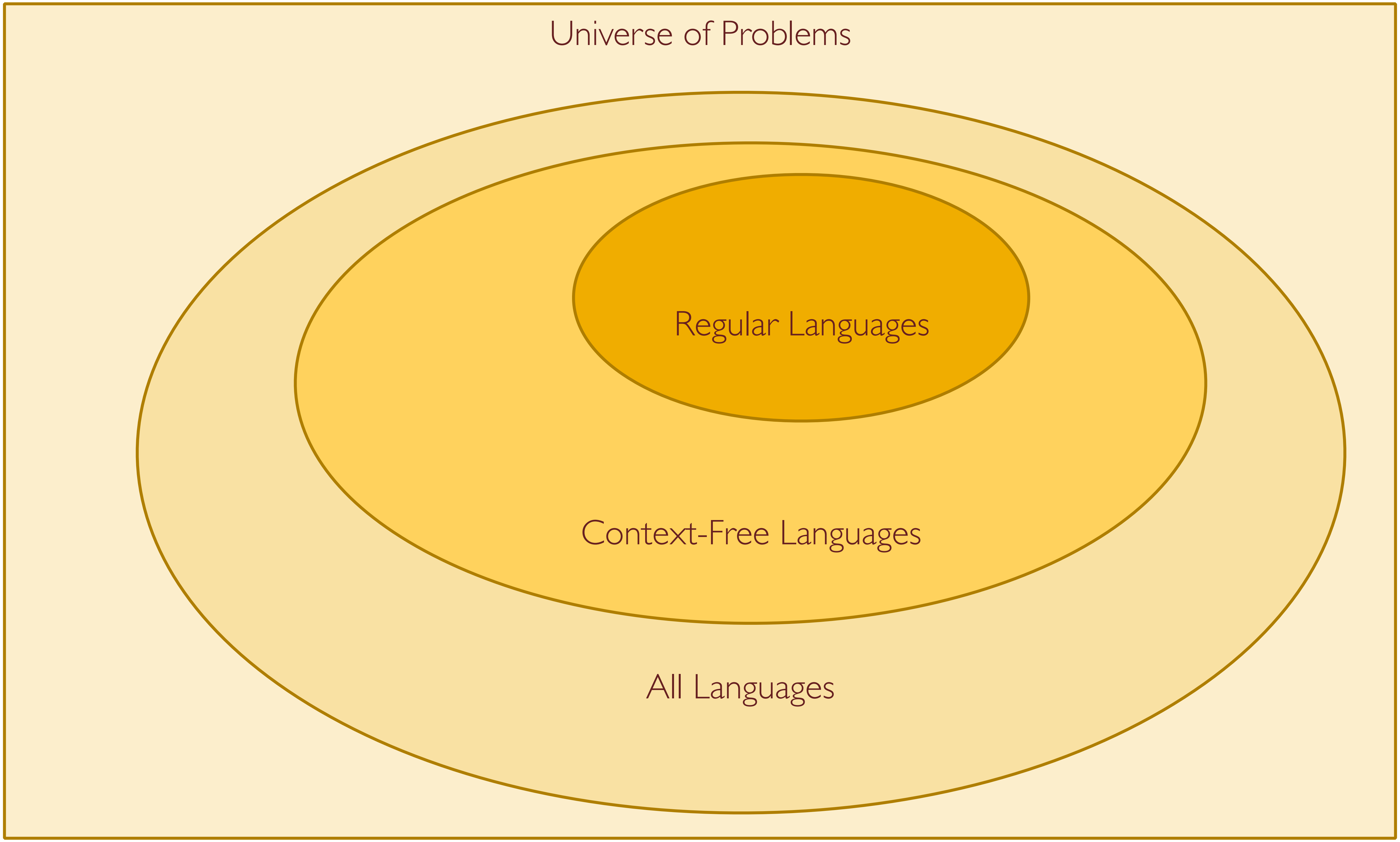
# Computing Theory

COMP 147

# The Space of Problems



# Overview

- Context-free languages:
  - CFLs include languages that were excluded from regular languages due to bounded memory

- Context-free Grammars
  - CFGs are a notation for describing CFLs
  - Languages definable by CFG ⇔ CFLs
  - Analogous to regex for regular languages
  - Recursive definitions

# Overview

- Pushdown Automata (PDA)
  - Automata that also have a stack
  - Stacks provide unbounded memory
  - Languages recognized by PDA = CFLs

- Pumping Lemma for CFLs
  - How do we show a language is not context-free?

# General Grammars

- Grammars define the syntax (structure) of a language
  - Usually defined by rules that can generate legal strings (sentences) in the language
  - The set of strings that can be generated by the rules of the grammar is the language of the grammar

# Context-Free Grammars

- A context-free grammar is a notation for describing languages.
- It is more powerful than finite automata or RE’s, but still cannot define all possible languages.
- Useful for nested structures, e.g., parentheses in programming languages.

# Context-Free Grammars

- Basic idea is to use “variables” to stand for sets of strings (i.e., languages).
- These variables are defined recursively, in terms of one another.
- Recursive rules a“productions”b involve only concatenation.

# Example

## CFG for { $0^n1^n$ | n $\geq$ 1}

- Productions:

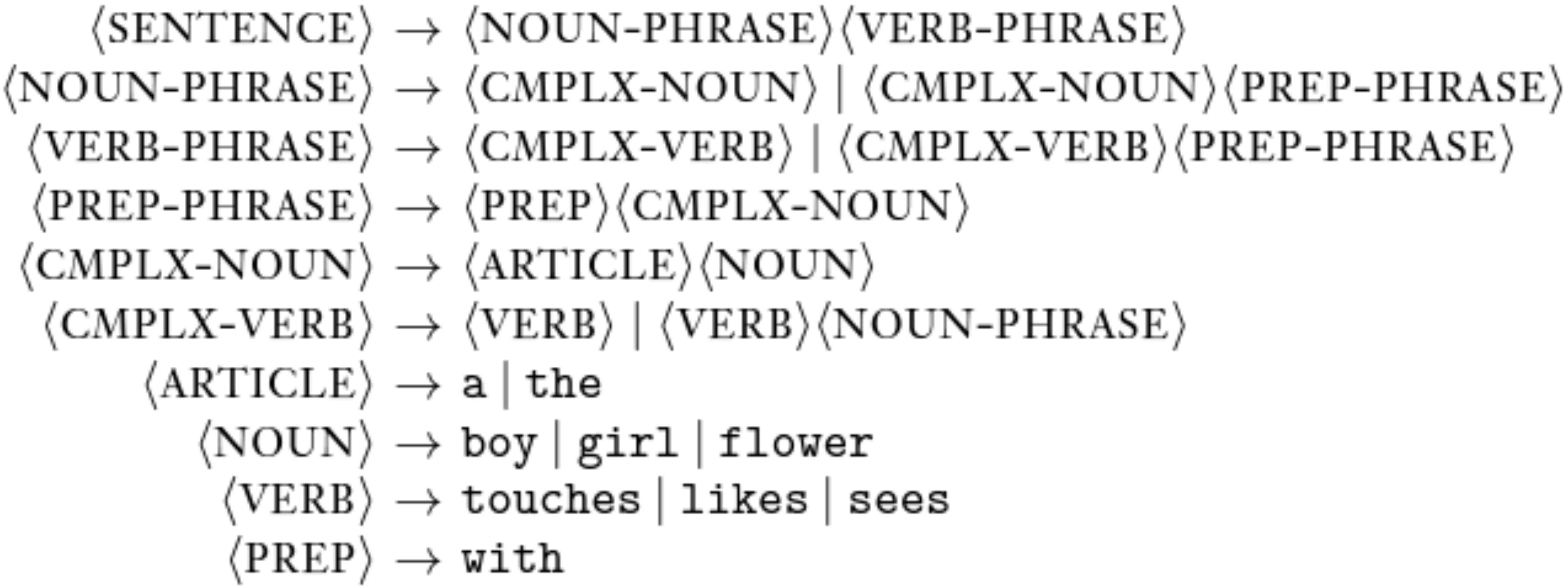
  S -> 01

  S -> 0S1

- Basis: 01 is in the language.

- Induction: if w is in the language, then so is 0w1.

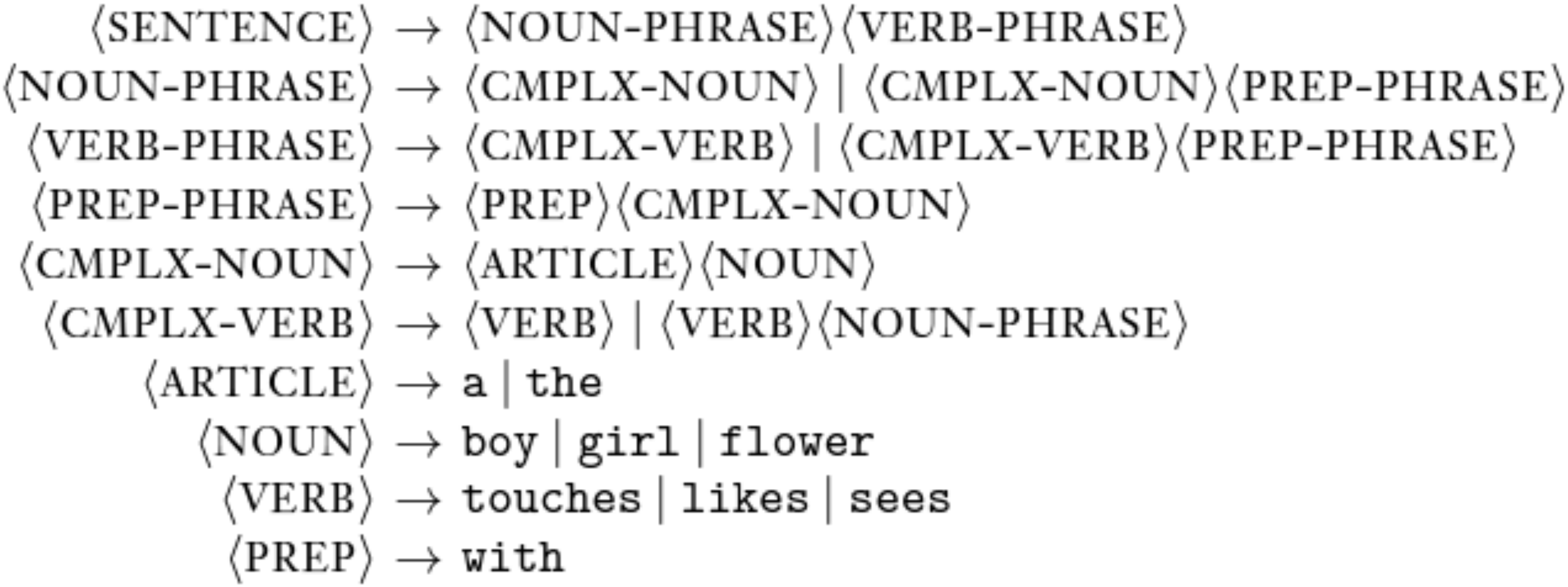
# English Grammar



$$
\begin{array}{rcl}
\langle\text{SENTENCE}\rangle & \to & \langle\text{NOUN-PHRASE}\rangle\langle\text{VERB-PHRASE}\rangle \\
\langle\text{NOUN-PHRASE}\rangle & \to & \langle\text{CMPLX-NOUN}\rangle \mid \langle\text{CMPLX-NOUN}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{VERB-PHRASE}\rangle & \to & \langle\text{CMPLX-VERB}\rangle \mid \langle\text{CMPLX-VERB}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{PREP-PHRASE}\rangle & \to & \langle\text{PREP}\rangle\langle\text{CMPLX-NOUN}\rangle \\
\langle\text{CMPLX-NOUN}\rangle & \to & \langle\text{ARTICLE}\rangle\langle\text{NOUN}\rangle \\
\langle\text{CMPLX-VERB}\rangle & \to & \langle\text{VERB}\rangle \mid \langle\text{VERB}\rangle\langle\text{NOUN-PHRASE}\rangle \\
\langle\text{ARTICLE}\rangle & \to & \texttt{a} \mid \texttt{the} \\
\langle\text{NOUN}\rangle & \to & \texttt{boy} \mid \texttt{girl} \mid \texttt{flower} \\
\langle\text{VERB}\rangle & \to & \texttt{touches} \mid \texttt{likes} \mid \texttt{sees} \\
\langle\text{PREP}\rangle & \to & \texttt{with}
\end{array}
$$

Attempt to generate a sentence.

# English Grammar



$$
\begin{aligned}
\langle\text{SENTENCE}\rangle &\to \langle\text{NOUN-PHRASE}\rangle\langle\text{VERB-PHRASE}\rangle \\
\langle\text{NOUN-PHRASE}\rangle &\to \langle\text{CMPLX-NOUN}\rangle \mid \langle\text{CMPLX-NOUN}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{VERB-PHRASE}\rangle &\to \langle\text{CMPLX-VERB}\rangle \mid \langle\text{CMPLX-VERB}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{PREP-PHRASE}\rangle &\to \langle\text{PREP}\rangle\langle\text{CMPLX-NOUN}\rangle \\
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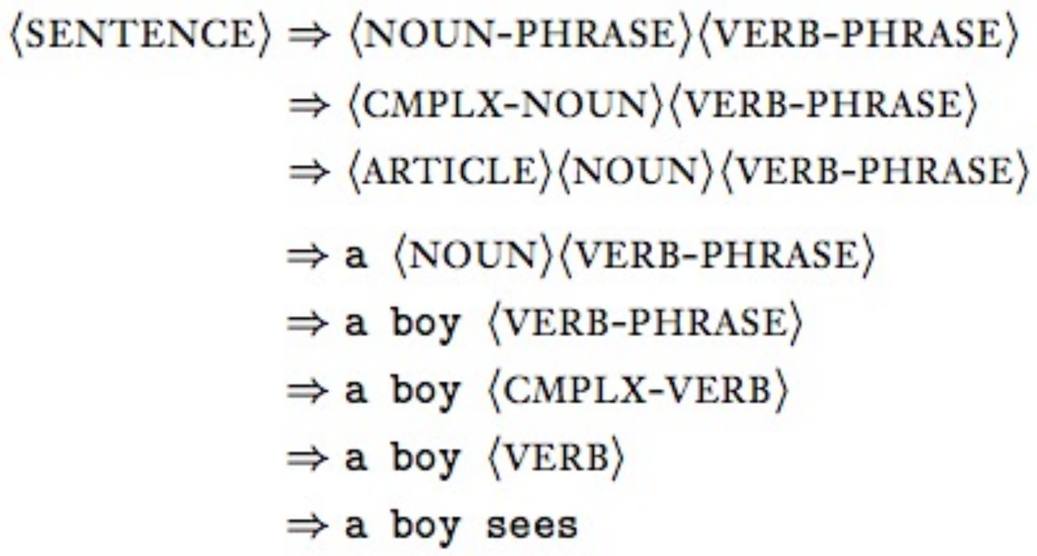
\langle\text{CMPLX-VERB}\rangle &\to \langle\text{VERB}\rangle \mid \langle\text{VERB}\rangle\langle\text{NOUN-PHRASE}\rangle \\
\langle\text{ARTICLE}\rangle &\to \texttt{a} \mid \texttt{the} \\
\langle\text{NOUN}\rangle &\to \texttt{boy} \mid \texttt{girl} \mid \texttt{flower} \\
\langle\text{VERB}\rangle &\to \texttt{touches} \mid \texttt{likes} \mid \texttt{sees} \\
\langle\text{PREP}\rangle &\to \texttt{with}
\end{aligned}
$$



$$
\begin{aligned}
\langle\text{SENTENCE}\rangle &\Rightarrow \langle\text{NOUN-PHRASE}\rangle\langle\text{VERB-PHRASE}\rangle \\
&\Rightarrow \langle\text{CMPLX-NOUN}\rangle\langle\text{VERB-PHRASE}\rangle \\
&\Rightarrow \langle\text{ARTICLE}\rangle\langle\text{NOUN}\rangle\langle\text{VERB-PHRASE}\rangle \\
&\Rightarrow \texttt{a}\ \langle\text{NOUN}\rangle\langle\text{VERB-PHRASE}\rangle \\
&\Rightarrow \texttt{a boy}\ \langle\text{VERB-PHRASE}\rangle \\
&\Rightarrow \texttt{a boy}\ \langle\text{CMPLX-VERB}\rangle \\
&\Rightarrow \texttt{a boy}\ \langle\text{VERB}\rangle \\
&\Rightarrow \texttt{a boy sees}
\end{aligned}
$$

# Context-Free Grammars

- Terminals = symbols of the alphabet of the language being defined.

- Variables = nonterminals = a finite set of other symbols, each of which represents a language.

- Start symbol = the variable whose language is the one being defined.

# Context-Free Grammars

- A production or substitution rule  has the form variable head -> string of variables and terminals body.

- Convention:
  - A, B, C,…  and also S are variables.
  - a, b, c,… are terminals.
  - …, X, Y, Z are either terminals or variables

# Context-free Grammar

A CFG over $\Sigma = \{ \#, 0, 1 \}$:

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

3 *substitution rules* (also called productions)

symbols left of → are *variables*

1st rule identifies the *start symbol*

symbols right of → are strings of *variables* or *terminals*

$\Sigma$ = set of terminals

# Generating strings from a CFG

- A CFG over $\Sigma = \{ \#, 0, 1 \}$:

  $A \rightarrow 0A1$

  $A \rightarrow B$

  $B \rightarrow \#$

  - Start with start symbol
  - Loop:
    - Replace any variable with the RHS of a rule for that variable

Attempt to generate some strings.

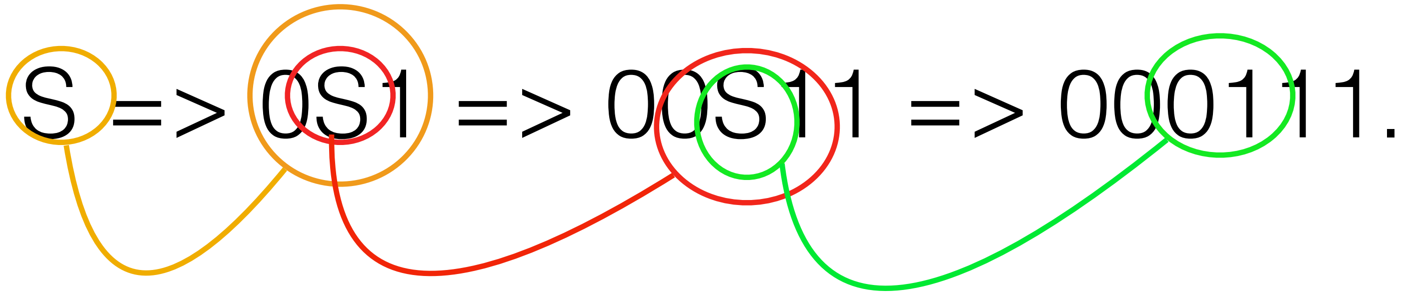
# Derivations

- We derive  strings in the language of a CFG by starting with the start symbol, and repeatedly replacing some variable A by the body of one of its productions.
  - That is, the “productions for A” are those that have head A.

# Derivations

- We say αAβ => αγβ if A -> γ is a production.

- Example:
  S -> 01;
  S -> 0S1.



- S => 0S1 => 00S11 => 000111.

# Language of a Grammar

- If G is a CFG, then L(G), the language of G, is $\{w \mid S \Rightarrow^* w\}$.

- Example:
  G has productions S -> ε and S -> 0S1.

- $L(G) = \{0^n1^n \mid n \geq 0\}$.

# Derivations

S->ab | aSb | SS

Then, which of the following strings is **not** in the language defined by this CFG?

a) aababbab

b) aaabbabaabbb

c) aaabaabbab

d) abababab

# Derivations

- Consider the following Grammar:  S->aSb | ab | SS
- Derivation for aabbab

# Example: Leftmost Derivations

- Leftmost derivation: forcing the leftmost variable
- Balanced-parentheses grammar:

  S -> SS | aSb | ab

- $S \Rightarrow_{lm} SS \Rightarrow_{lm} aSbS \Rightarrow_{lm} aabbS \Rightarrow_{lm} aabbab$
- Thus, $S \Rightarrow^{*}_{lm} aabbab$
- $S \Rightarrow SS \Rightarrow Sab \Rightarrow aSbab \Rightarrow aabbab$ is a derivation, but not a leftmost derivation.

# Example: Rightmost Derivations

- Balanced-parentheses grammar:

  S -> SS | aSb | ab

- $S \Rightarrow_{rm} SS \Rightarrow_{rm} Sab \Rightarrow_{rm} aSbab \Rightarrow_{rm} aabbab$

- Thus, $S \Rightarrow^*_{rm} aabbab$

- S => SS => SSS => SabS => ababS => ababab is neither a rightmost nor a leftmost derivation.

Which of the following is a rightmost derivation of the grammar S -> SS | aSb | ab?

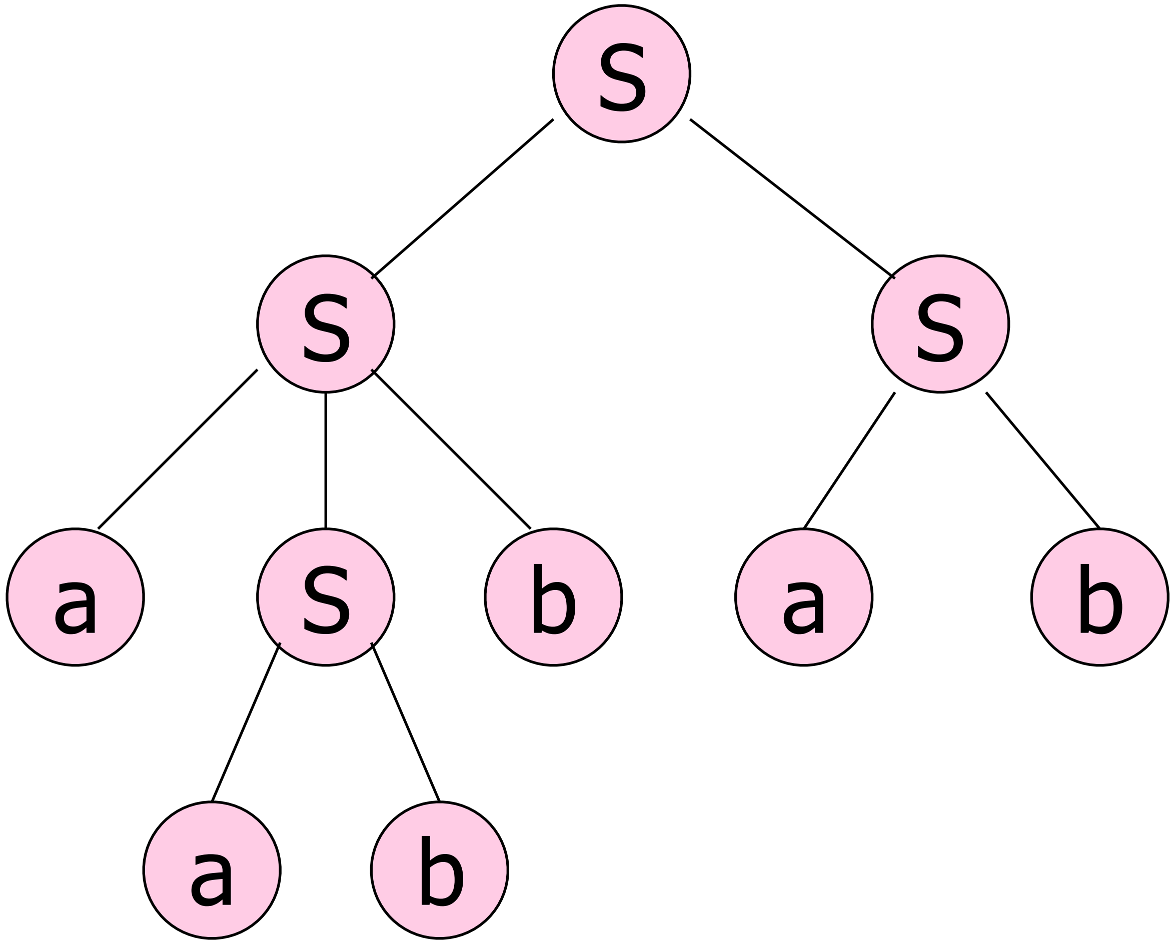
1. S=>SS=>SSS=>abSS=>abaSbS=>abaabbS=>abaabbab
2. S=>SS=>SSS=>SaSbS=>SaabbS=>Saabbab=>abaabbab
3. S=>SS=>SSS=>SSab=>SaSbab=>Saabbab=>abaabbab
4. S=>SS=>abS=>abSS=>abaSbS=>abaabbS=>abaabbab

# Parse Trees

- Parse trees are trees labeled by symbols of a particular CFG.
- Leaves: labeled by a terminal or ε.
- Interior nodes: labeled by a variable.
  - Children are labeled by the body of a production for the parent.
- Root: must be labeled by the start symbol.

# Example: Parse Tree

S -> SS | aSb | ab



# Parse Trees

Which of the following cannot appear as the label of a node a leaf or interior node b of a parse tree?

[✓ 1.Variable S]

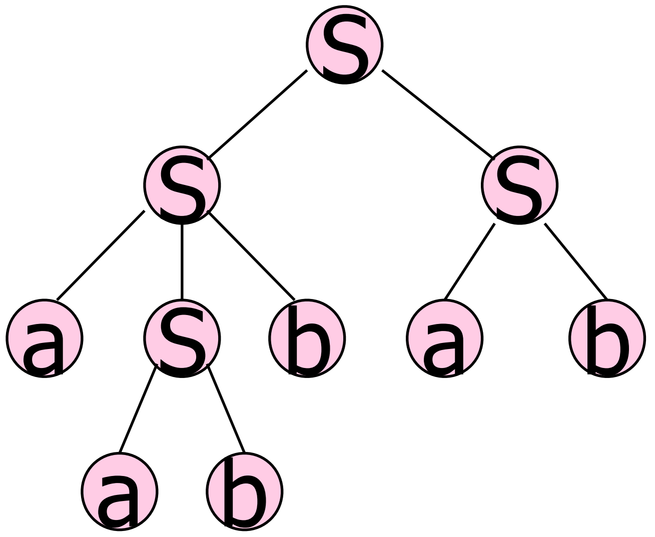
2.Terminal a

3.Terminal b

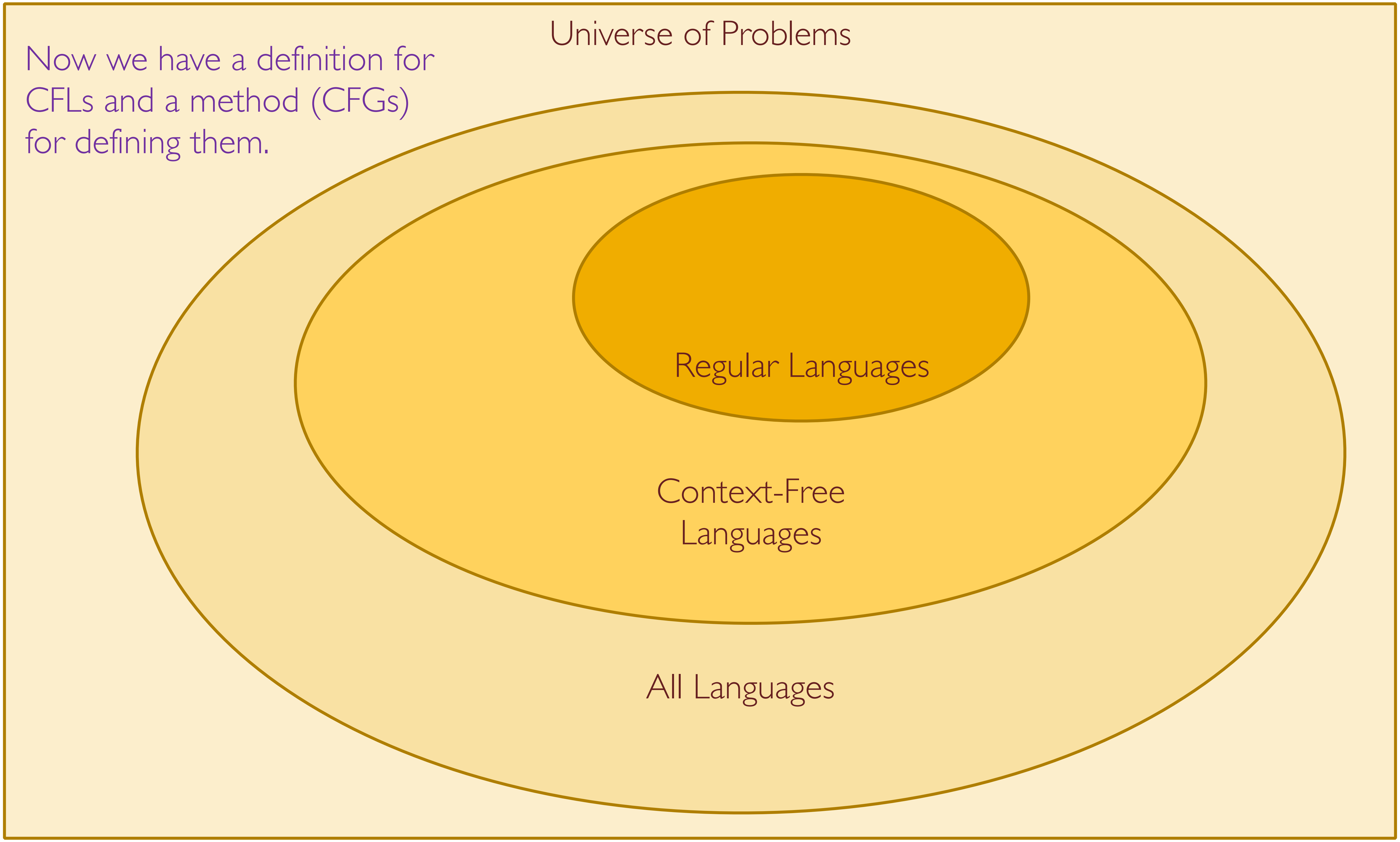
4.Terminal String ab

# Yield of a Parse Tree

- The concatenation of the labels of the leaves in left-to-right order
  - That is, in the order of a preorder traversal.
- is called the yield of the parse tree.
- Example: yield of is aabbab



# The Space of Problems

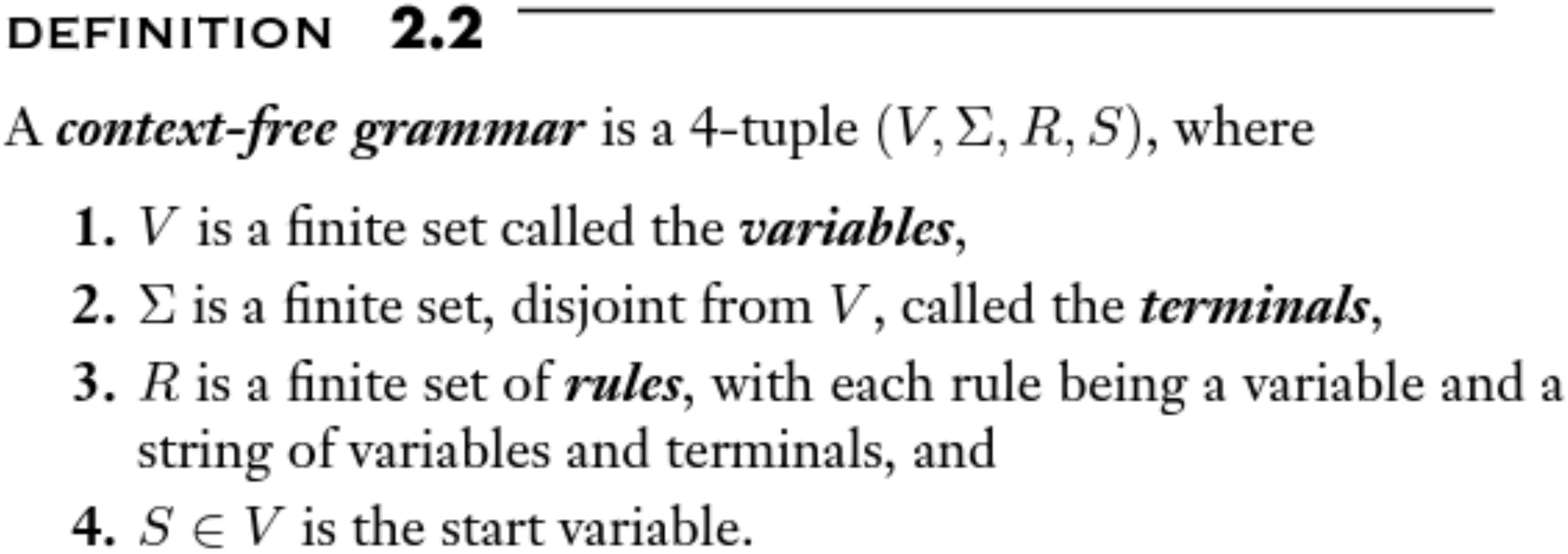


Now we have a definition for CFLs and a method (CFGs) for defining them.

# Last time

- Consider the following Grammar:  S -> 0S1| 01.
- Give derivations and Parse Trees for string 0011

# CFG: Formal Definition



**DEFINITION 2.2**

A ***context-free grammar*** is a 4-tuple $(V, \Sigma, R, S)$, where

1. $V$ is a finite set called the ***variables***,
2. $\Sigma$ is a finite set, disjoint from $V$, called the ***terminals***,
3. $R$ is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Example:
S->ab | aSb | SS

Then, which of the following strings is not in the language defined by this CFG?

a) aababbab

b) aaabbabaabbb

c) aaabaabbab

d) abababab

Which of the following is a rightmost derivation of the grammar S -> SS | aSb | ab?

1. S=>SS=>SSS=>abSS=>abaSbS=>abaabbS=>abaabbab
2. S=>SS=>SSS=>SaSbS=>SaabbS=>Saabbab=>abaabbab
3. S=>SS=>SSS=>SSab=>SaSbab=>Saabbab=>abaabbab
4. S=>SS=>abS=>abSS=>abaSbS=>abaabbS=>abaabbab

# Last time

- Consider the following Grammar:  S -> 0S1| 01.
- Give derivations and Parse Trees for string 0011

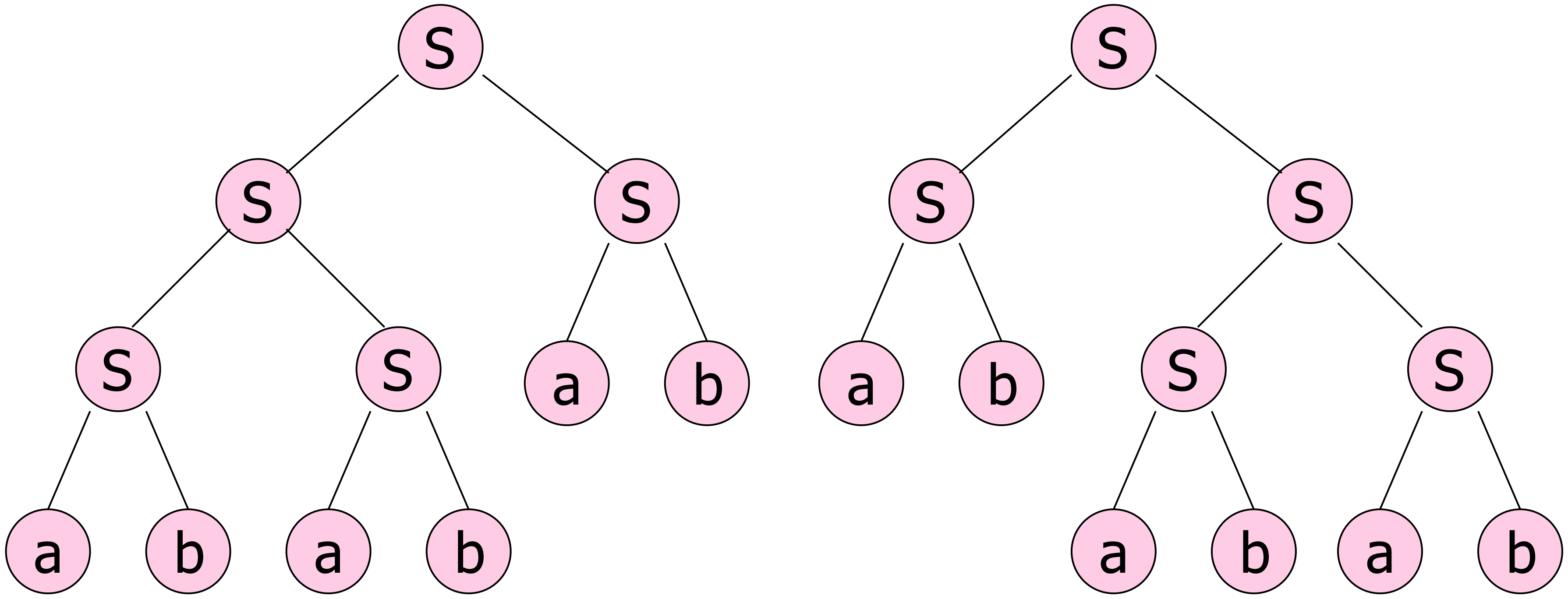
# Ambiguous Grammars

- A CFG is ambiguous  if there is a string in the language that is the yield of two or more parse trees (or two different leftmost (or rightmost) derivations)

- Example: S -> SS | aSb | ab

- Two parse trees for ababab on next slide.

# Ambiguous Grammars



# Example

- S -> S + S | S * S | 1 | 2 | 3 | 4
- Can you give 2 parse trees for 2 + 3 * 4?

# Ambiguity is a Property of Grammars, not Languages

- For the balanced-parentheses language, here is another CFG, which is unambiguous.

- B -> aRB | ε

- R -> b | aRR

B, the start symbol, derives balanced strings.

R generates certain strings that have one more right paren than left.

# Example: Unambiguous Grammar

- B -> aRB | ε            R -> b | aRR
- Construct a unique leftmost derivation for a given balanced string of parentheses by scanning the string from left to right.
  - If we need to expand B, then use B -> aRB if the next symbol is "a"; use **ε** if at the end.
  - If we need to expand R, use R -> b if the next symbol is "b" and aRR if it is "a".

# The Parsing Process

Remaining Input:

Steps of leftmost derivation:

B

aabbab

Next symbol

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

abbab

Next symbol

Steps of leftmost derivation:

B

aRB

B -> aRB | ε        R -> b | aRR

# The Parsing Process

Remaining Input:

bbab

Next
symbol

Steps of leftmost derivation:

B

aRB

aaRRB

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

bab

Next symbol

Steps of leftmost derivation:

B

aRB

aaRRB

aabRB

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

ab

Next symbol

Steps of leftmost derivation:

B

aRB

aaRRB

aabRB

aabbB

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

b

↑

Next symbol

Steps of leftmost derivation:

B

aRB

aaRRB

aabRB

aabbB

aabbaRB

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

Next symbol

Steps of leftmost derivation:

B

aRB

aaRRB

aabRB

aabbB

aabbaRB

aabbabB

B -> aRB | ε

R -> b | aRR

# The Parsing Process

Remaining Input:

Next
symbol

Steps of leftmost derivation:

B

aRB

aaRRB

aabRB

aabbB

aabbaRB

aabbabB

aabbab

B -> aRB | ε

R -> b | aRR

# LL(1) Grammars

- As an aside, a grammar such
  B -> aRB | ε R -> b | aRR,

- where you can always figure out the production to use in a leftmost derivation by scanning the given string left-to-right and looking only at the next one symbol is called LL(1).
  - “Leftmost derivation, left-to-right scan, one symbol of lookahead.”

# LL(1) Grammars – (2)

- Most programming languages have LL(1) grammars.

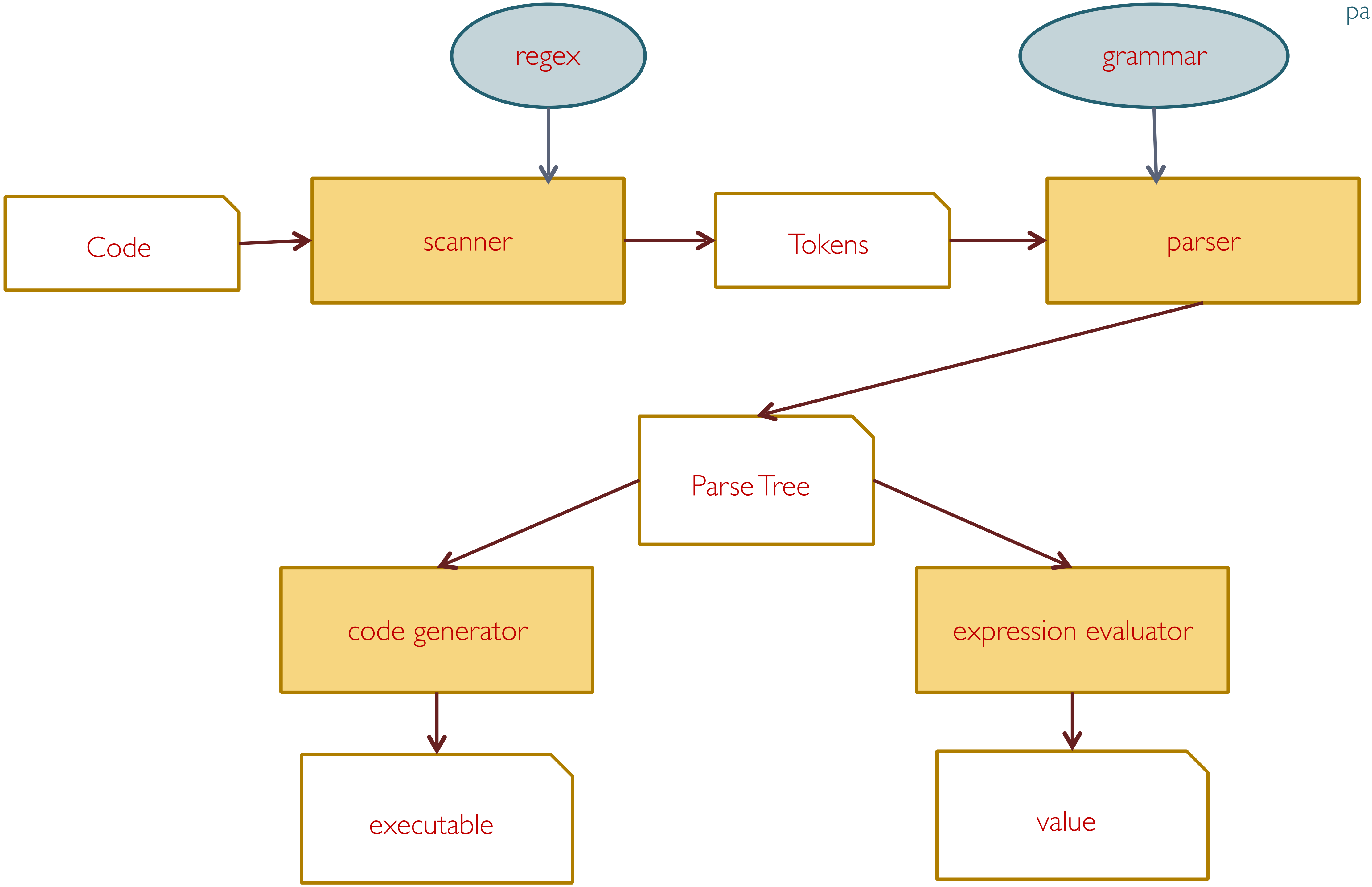
- LL(1) grammars are never ambiguous.

# Inherent Ambiguity

- It would be nice if for every ambiguous grammar, there were some way to “fix” the ambiguity, as we did for the balanced-parentheses grammar.

- Unfortunately, certain CFL’s are inherently ambiguous, meaning that every grammar for the language is ambiguous.

# Compilers



scanners and parsers are defined by languages