Computing Theory

COMP 147 (4 units)

Chapter 1: Regular Languages
Regular Expressions

Regular Expressions

To describe a regular language L_a
we could give a DFA, D, such that $L(D) = L_a$, or an NFA, N, such that that $L(N) = L_a$.

This is not always convenient.

Regular expressions give us a more compact, more readable, notation for the same languages.

Variations on regex appear everywhere that strings need to be specified: compilers, search tools, editing tools …

Regular Expressions

- The value of a regular expression is a language
  - a regex defines a set of strings over some alphabet Σ (for example $\Sigma = \{0,1\}$)
- The operators allowed in a regex are: union $\cup$, concatenation $\circ$ and star $*$
- Any of these operations on regex produces regex.
- Precedence order: $* \circ \cup$
  - parentheses override precedence
- Example: $(0 \cup 1) \circ 0^*$ ($\circ$ often omitted and sometimes | instead of $\cup$)

Formal Definition

Recursive definition

R is a regular expression if R is

1. a, where $a \in \Sigma$
2. $\boldsymbol{\varepsilon}$
3. $\varnothing$
4. $R_1 \cup R_2$
5. $R_1 \circ R_2$
6. $R_1{}^*$

$\boldsymbol{\varepsilon}$ is the set containing only the empty string

$\varnothing$ is an empty set

where R_1 and R_2 are regular expressions

Shorthand Notation

- R_1R_2 represents $R_1 \circ R_2$
- R^+ represents RR^*
  - $R^+ \cup \varepsilon = R^*$
- R^k represents $\underbrace{RRR\ldots R}_{k\ times}$
- Σ represents all strings of length 1 over Σ
  - Σ represents a, where $a \in \Sigma$

Examples

0^*10^* $\quad \{w \mid w \text{ contains a single } 1\}$

$\Sigma^*1\Sigma^*$ $\quad \{w \mid w \text{ has at least a single } 1\}$

$\Sigma^*(str)\Sigma^*$ $\quad \{w \mid w \text{ contains } str \text{ as a substring}\}$

$1^*(01^+)^*$ $\quad \left\{\begin{array}{c} w \mid \text{every } 0 \text{ in } w \text{ is followed} \\ \text{by at least a single } 1 \end{array}\right\}$

$(\Sigma\Sigma)^*$ $\quad \{w \mid w \text{ is of even length}\}$

Examples

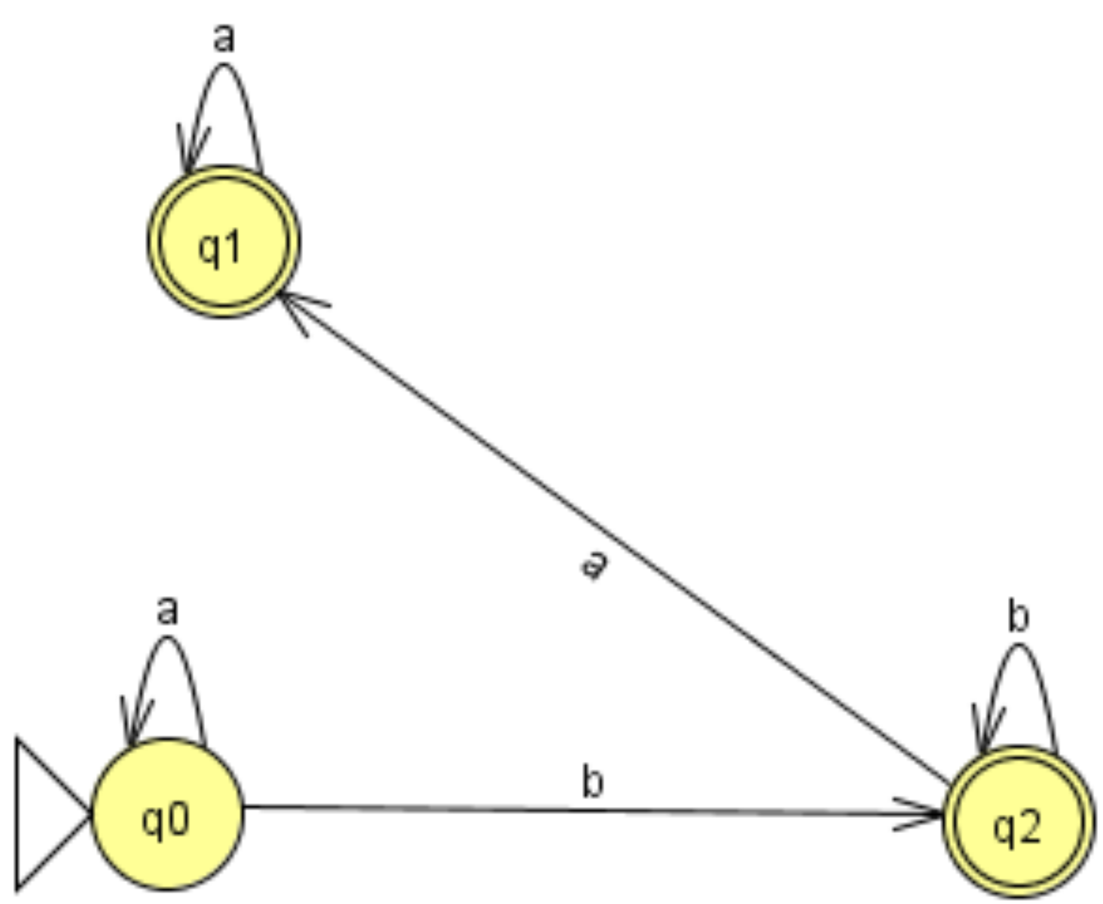
$$0\Sigma^{*}0 \cup 1\Sigma^{*}1 \cup 0 \cup 1$$

all strings starting and ending
with the same symbol

Additional Examples

$$a^*b^+a^*$$

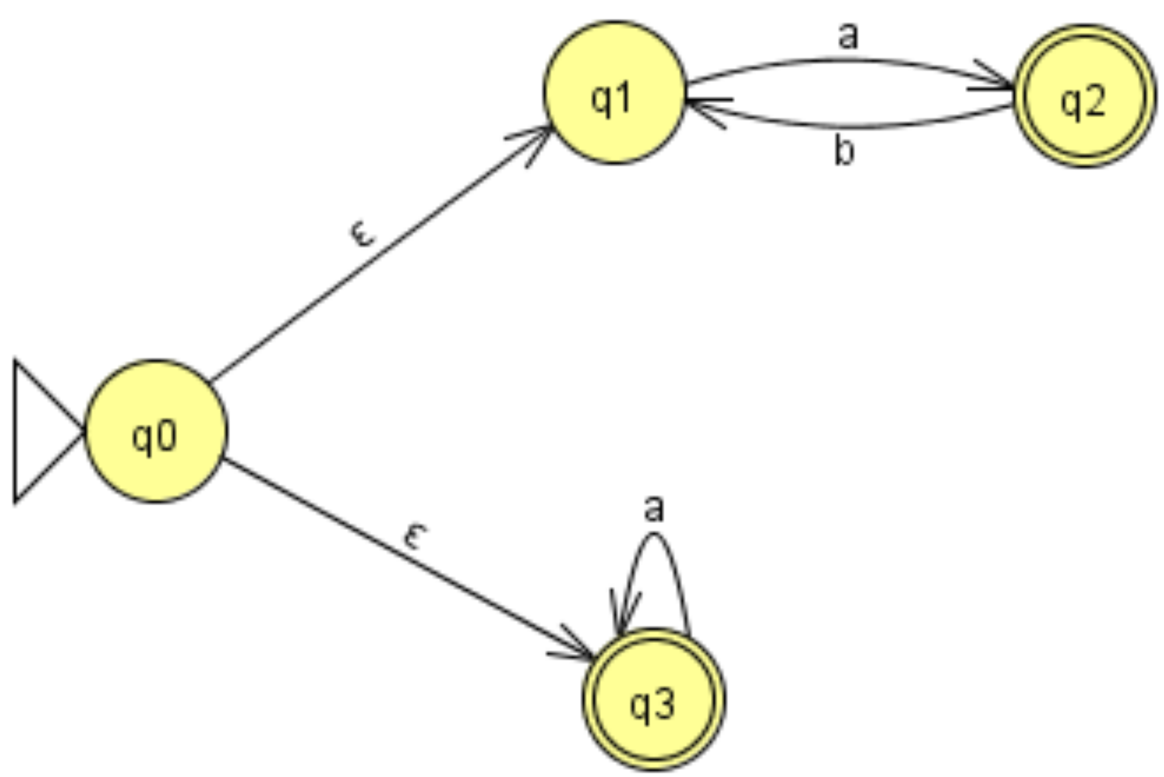
What is the corresponding finite automaton?



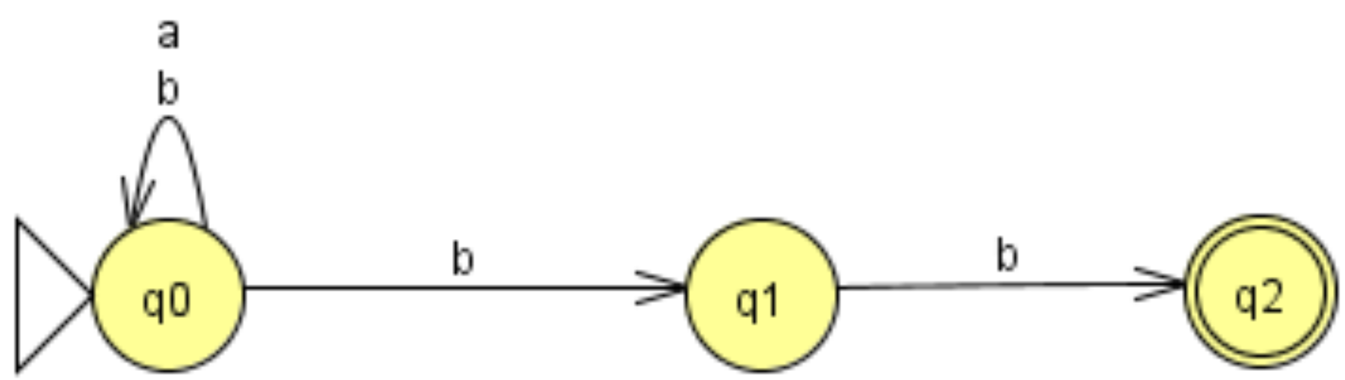
Additional Examples

$$a(ba)^* \cup a^*$$

What is the corresponding finite automaton?



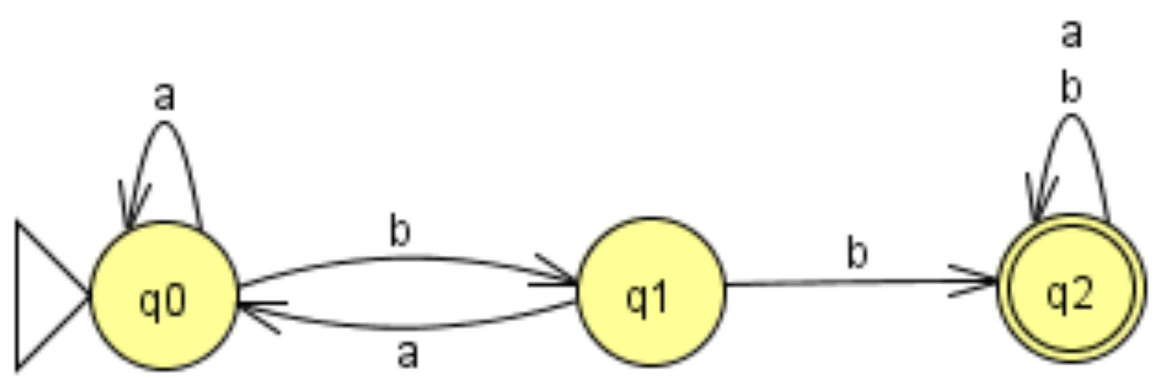
Additional Examples



What is the corresponding regular expression?

$$(a \cup b)^* bb$$

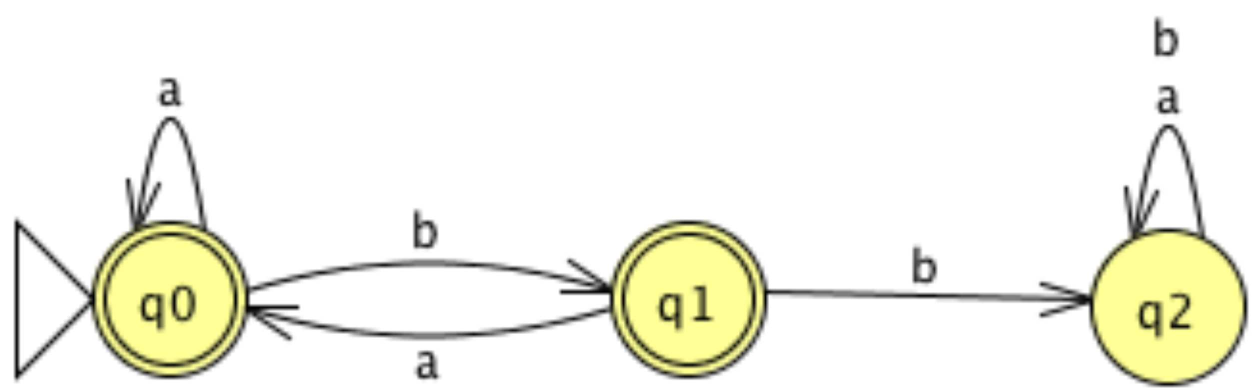
Additional Examples



What is the corresponding regular expression?

$$(a \cup ba)^* bb(a \cup b)^*$$

Additional Examples



What is the corresponding regular expression?

$$(a \cup ba)^*(\epsilon \cup b)$$

Equivalence With Finite Automata

Regular expressions and finite automata are equivalent in their descriptive power.

Theorem

A language is regular **if and only if** it can be described by a regular expression.

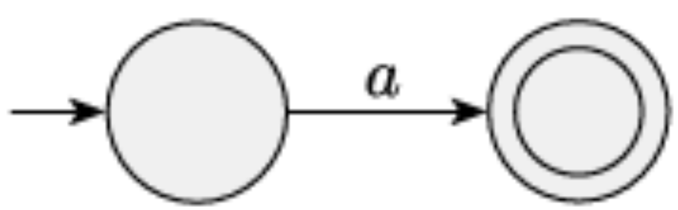
Regex / FA Equivalence

- Theorem: A language is regular if and only if some regular expression describes it.
  - Lemma: If a language is described by a regular expression, then it is regular
    - Proof by construction: show how to build an NFA from a regex
  - Lemma: If a language is regular, then it is described by a regular expression
    - Proof by construction: show how to convert a DFA to a regex

Construct NFA from Regex

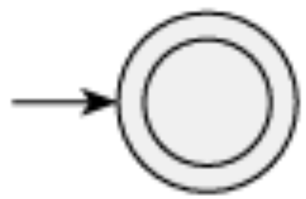
- Convert regex R into NFA N.
  - There are 6 cases (see formal definition of a regex)

- Case 1:
  $R = a \in \Sigma$
  $L(R) = \{ a \}$

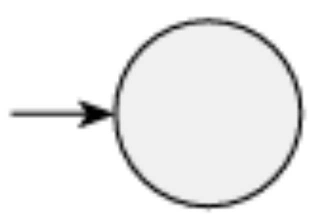
$N =$ 

Construct NFA from Regex

- Case 2:
  $R = \varepsilon, L(R) = \{ \varepsilon \}$

  $N =$ 

- Case 3:
  $R = \emptyset, L(R) = \{ \ \} = \emptyset$

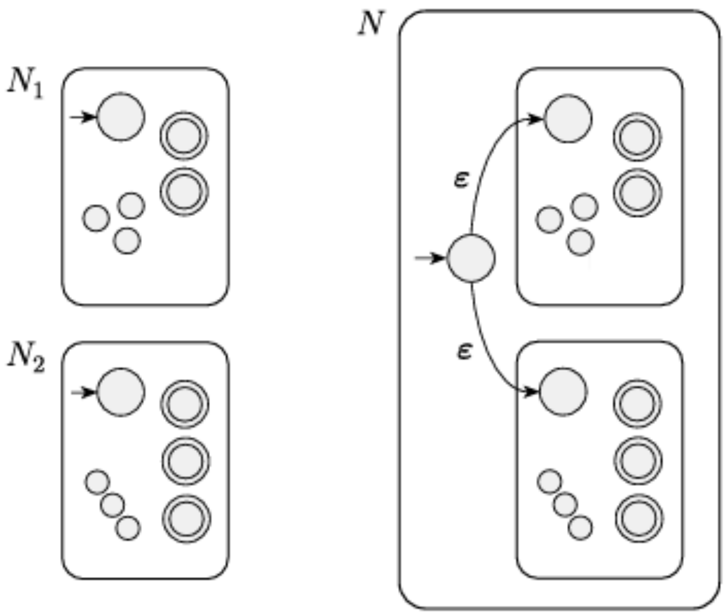
  $N =$ 

Construct NFA from Regex

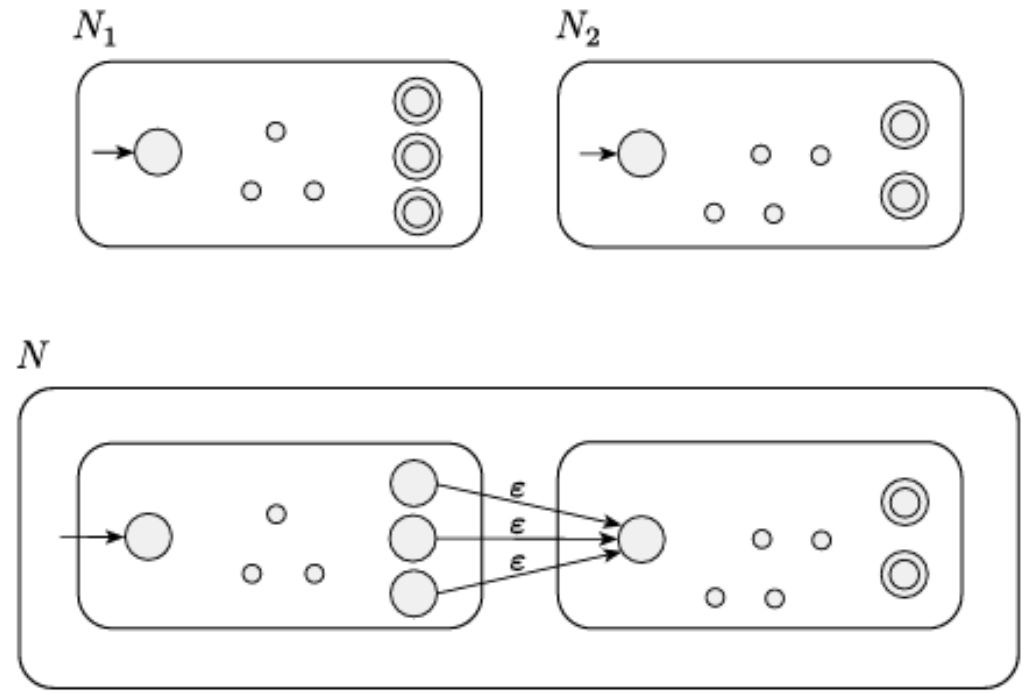
- Case 4:
  $R = R_1 \cup R_2, L(R) = L(R_1) \cup L(R_2)$
- Case 5:
  $R = R_1 \circ R_2, L(R) = L(R_1) \circ L(R_2)$
- Case 6:
  $R = R_1^*, L(R) = L(R_1)^*$
  - In these cases, use the NFA construction techniques from the closure proofs for $\cup$, $\circ$ and $*$.

Construct NFA from Regex

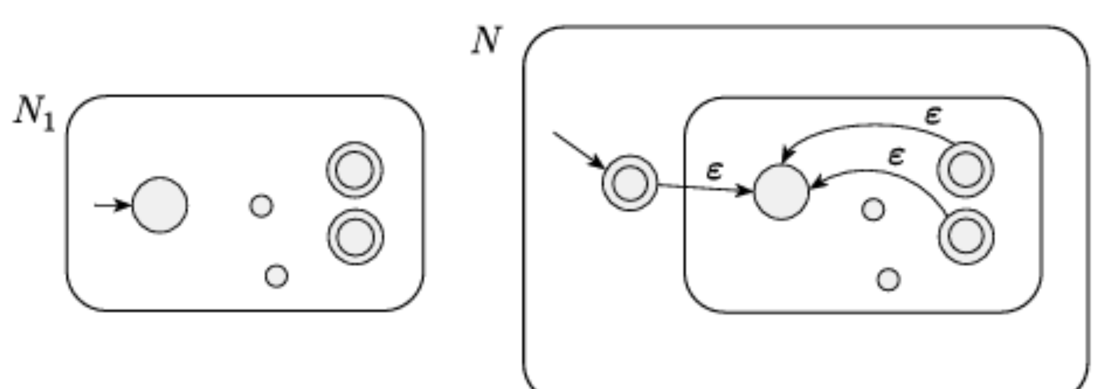
Case 4: $R = R_1 \cup R_2$



Case 5: $R = R_1 \circ R_2$

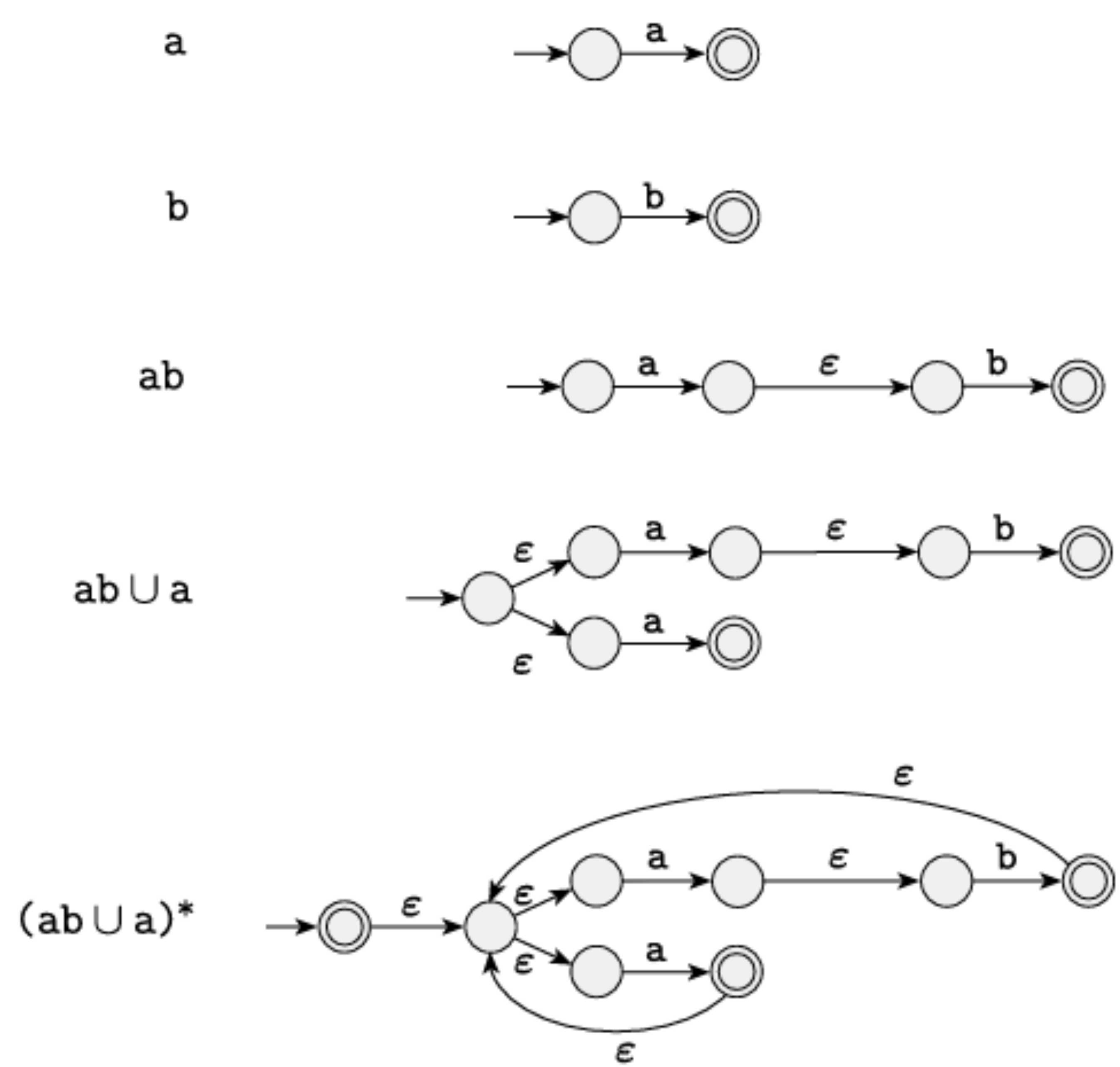


Case 6: $R = R_1{}^*$



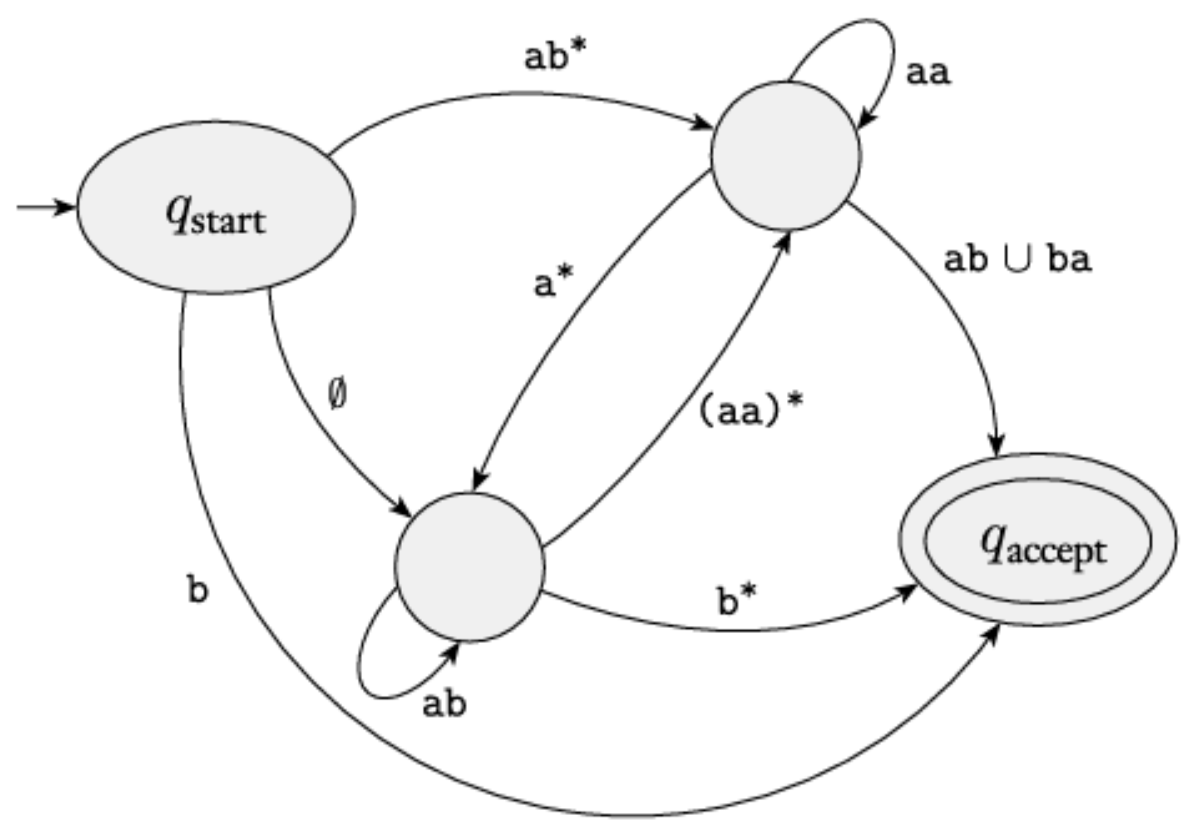
Regex to NFA Example

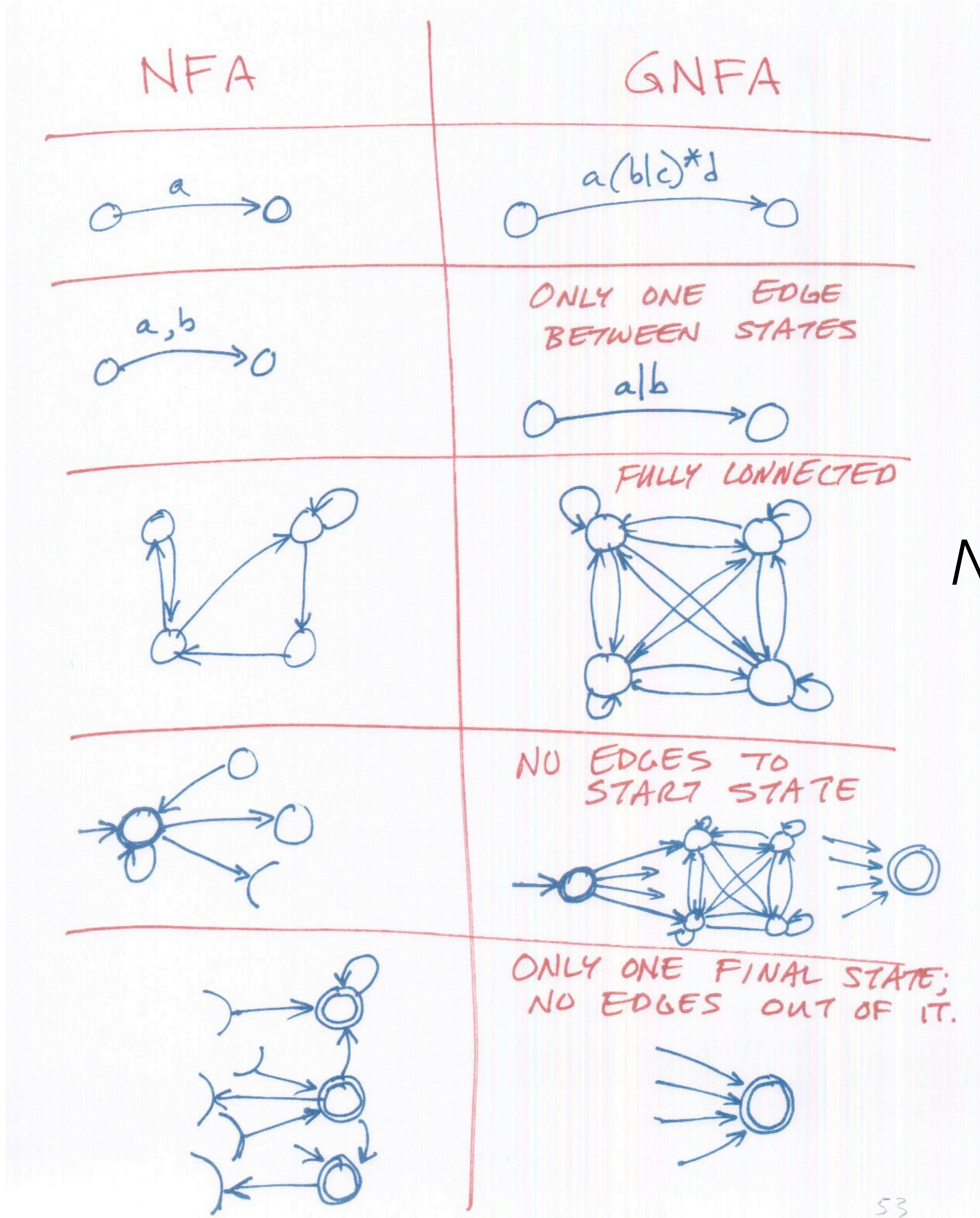
- R = (ab ∪ a)*



Construct Regex from DFA

- Convert DFA D into regex R.
  - We'll use a new type of FA as an intermediate step
  - GNFA: generalized nondeterministic finite automata
  - GNFA allow regex as transition labels





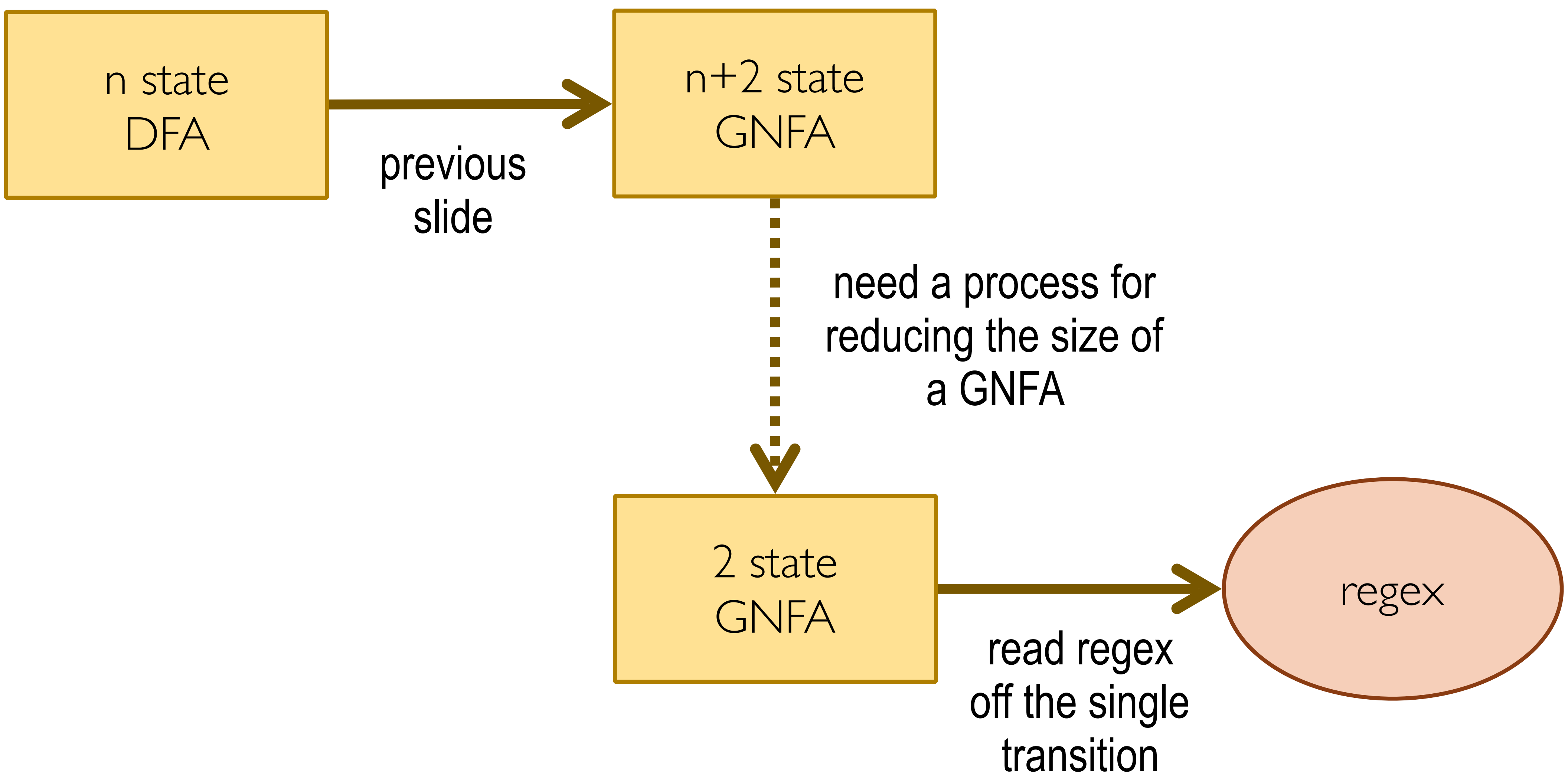
Not necessary in our case

DFA to Regex (step 1: DFA to GNFA)

- Add new start state with **ε** transition to old start state.
- Add new accept state with **ε** transitions from all old accept states
- For any pair of states that has multiple transitions, replace with transition labeled with union of previous labels

(Example on white board)

DFA to Regex



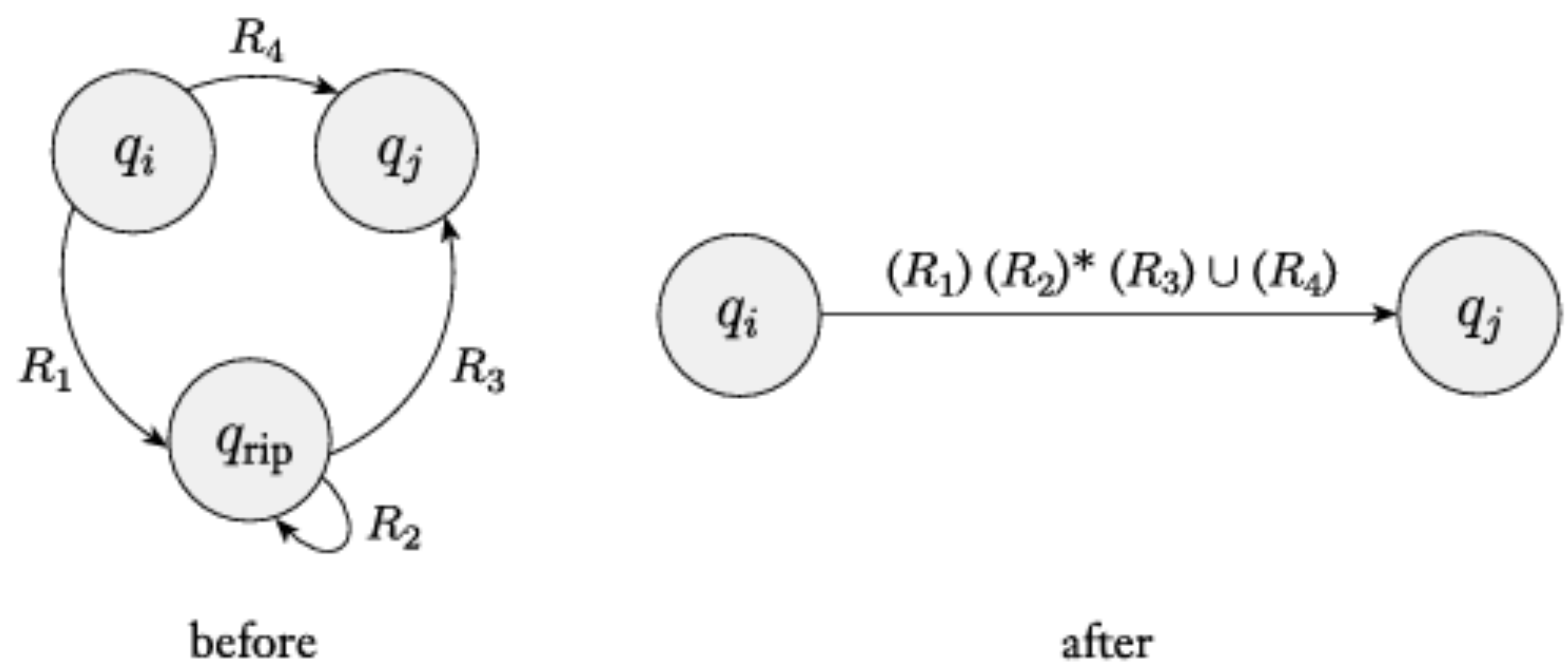
n state
DFA
previous
slide
n+2 state
GNFA
need a process for
reducing the size of
a GNFA
2 state
GNFA
read regex
off the single
transition
regex

Ripping a state from a GNFA

- Select a state at random (do not select start or accept states)
- Let's call it q_{rip}
- Rip the state out of the GNFA
- Remove q_{rip} and all edges to/from it
- Modify the other transition edges so that the machine accepts the same language

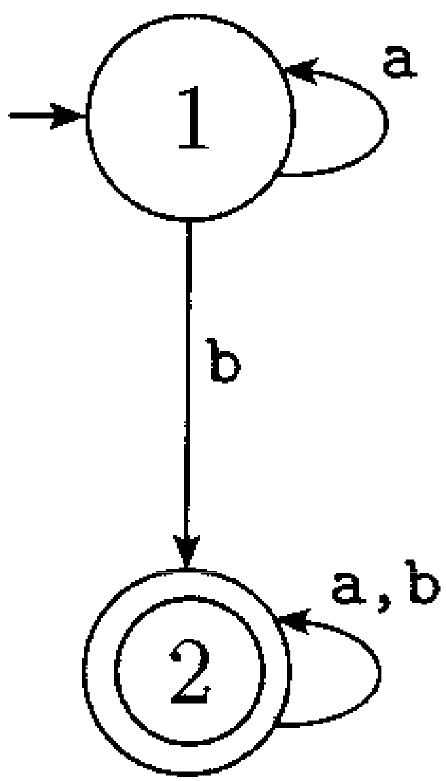
Ripping a state from a GNFA

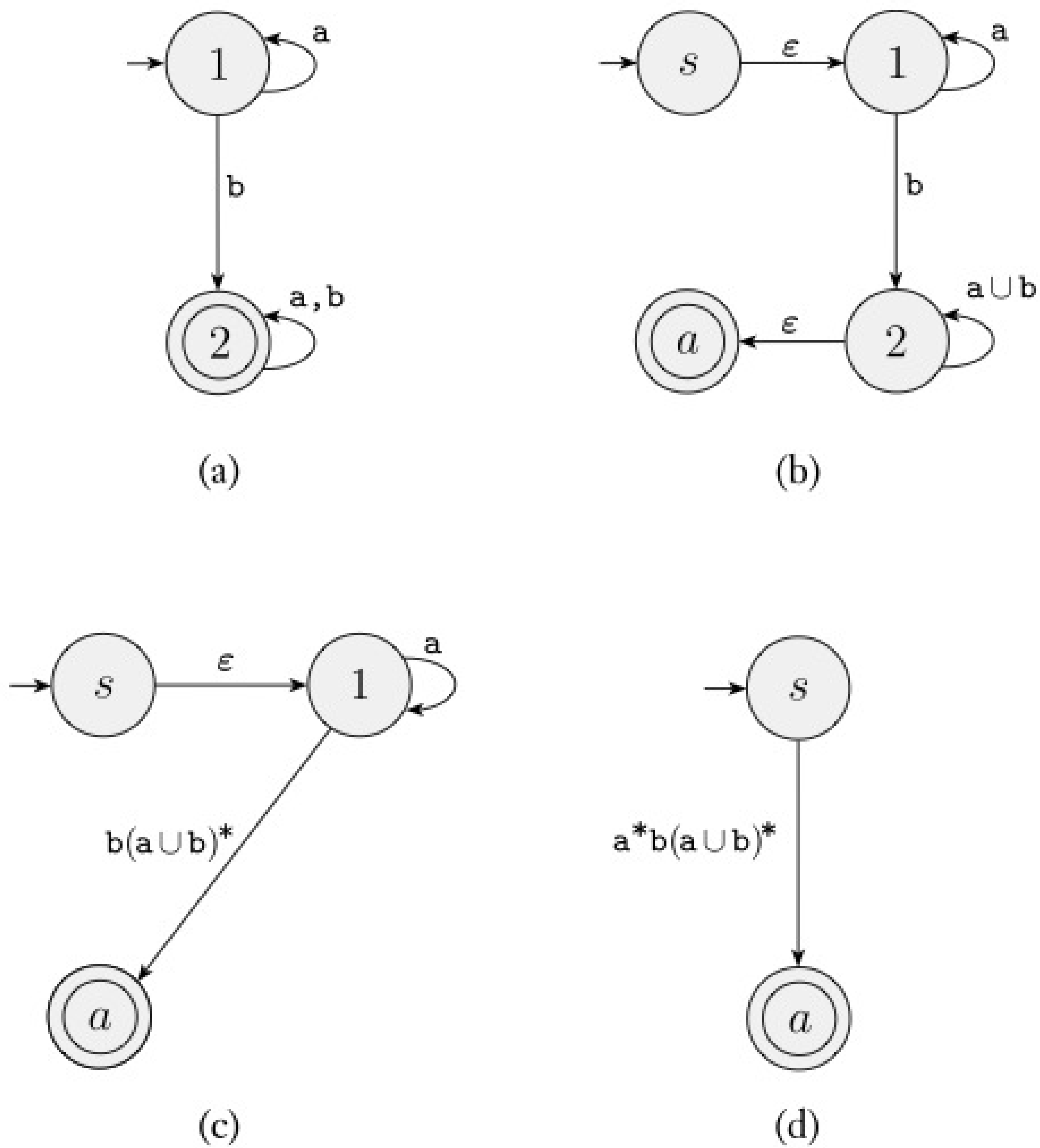
- To reduce the size of the GNFA, we'll rip out states, one at a time, and repair the transitions
  - Example:



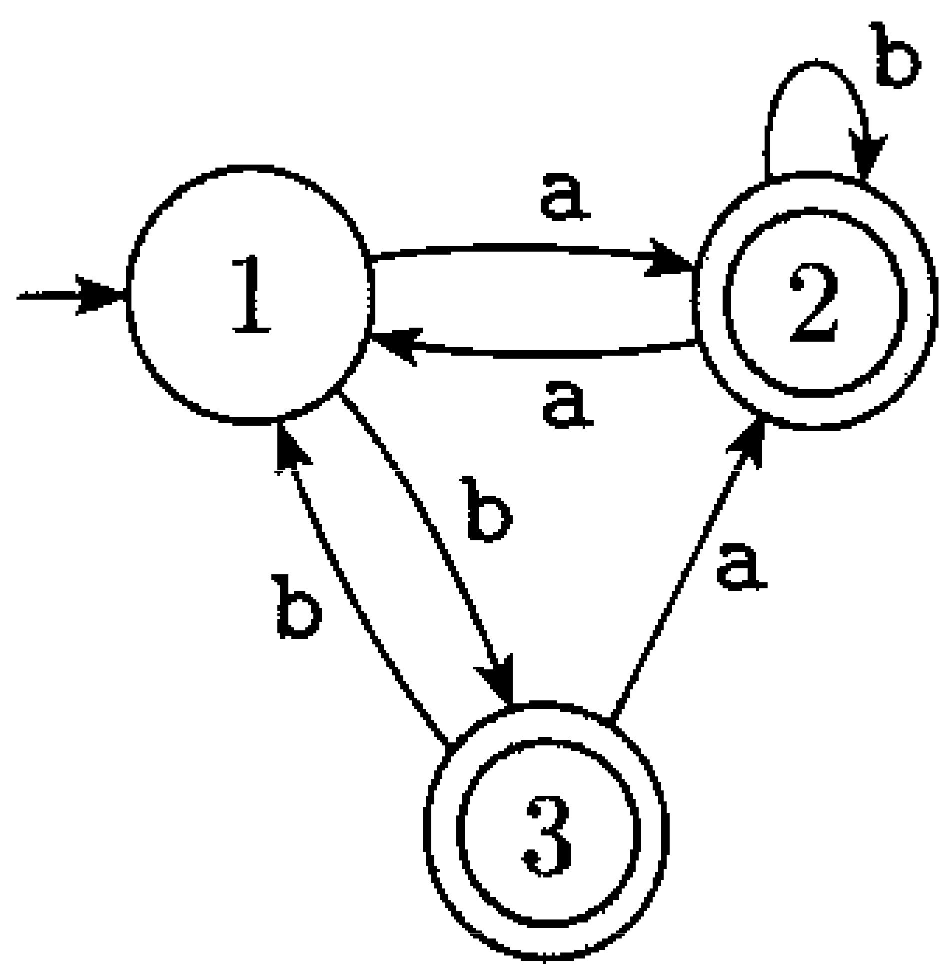
  - If we formalize this into an algorithm, we'll complete our proof

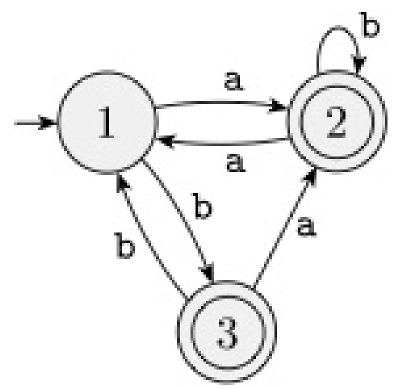
Examples on Board




1
a
b
2
a,b
(a)
s
ε
1
a
b
2
a∪b
ε
a
(b)
s
ε
1
a
b(a∪b)*
a
(c)
s
a*b(a∪b)*
a
(d)

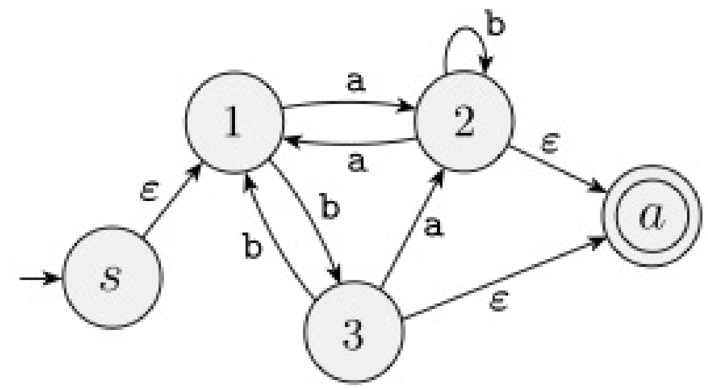
Examples on Board





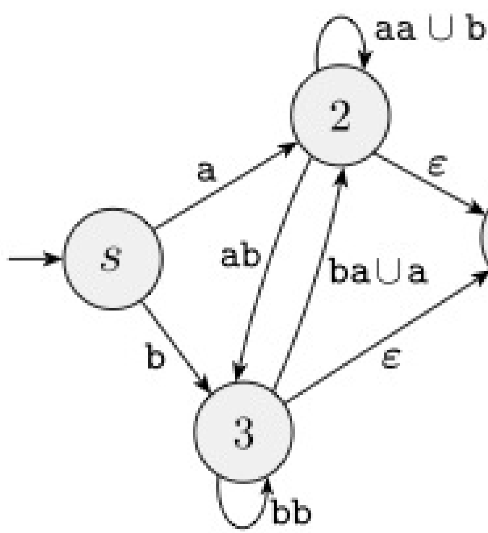
1
2
3
a
a
b
b
b
a

(a)



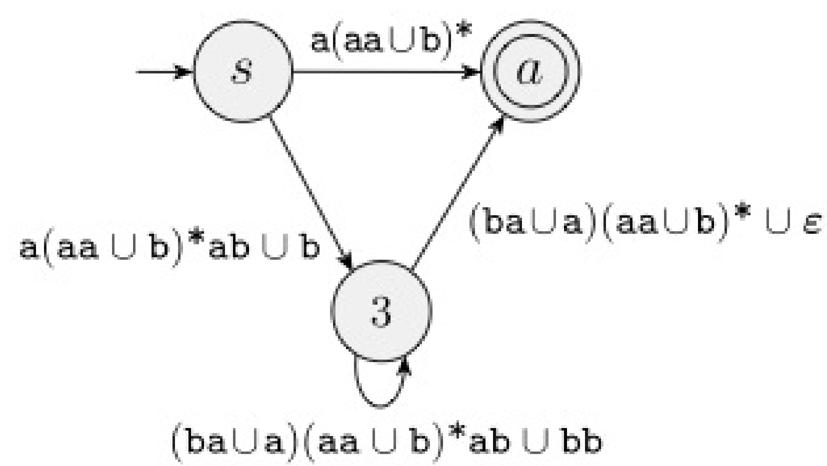
s
1
2
3
a
ε
a
a
b
b
b
a
ε
ε

(b)



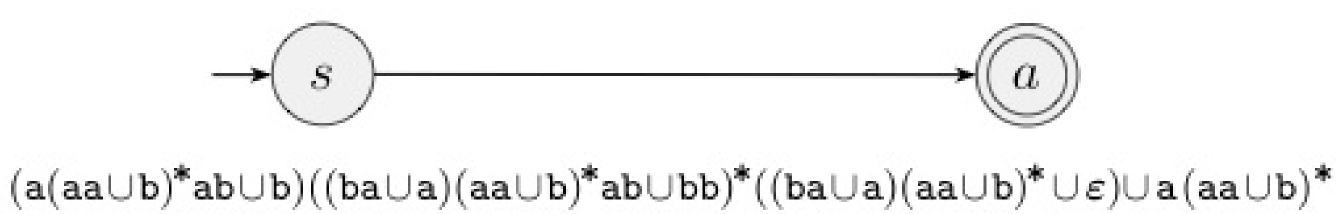
aa ∪ b
2
a
s
ab
ba∪a
ε
a
b
ε
3
bb

(c)



s
a(aa∪b)*
a
a(aa ∪ b)*ab ∪ b
(ba∪a)(aa∪b)* ∪ ε
3
(ba∪a)(aa ∪ b)*ab ∪ bb

(d)



s
a
(a(aa∪b)*ab∪b)((ba∪a)(aa∪b)*ab∪bb)*((ba∪a)(aa∪b)*∪ε)∪a(aa∪b)*

(e)