Transform and Conquer

Transform and Conquer

This group of techniques solves a problem by a *transformation*

- to a simpler/more convenient instance of the same problem (*instance simplification*)
- to a different representation of the same instance (*representation change*)
- to a different problem for which an algorithm is already available (*problem reduction*)

Instance simplification - Presorting

Solve a problem's instance by transforming it into another simpler/easier instance of the same problem

Presorting

- Many problems involving lists are easier when the list is sorted.
  - searching
  - computing the median (selection problem)
  - checking if all elements are distinct (element uniqueness)

Also:

- Topological sorting helps solving some problems for dags.
- Presorting is used in many geometric algorithms.

How fast can we sort ?

Efficiency of algorithms involving sorting depends on efficiency of sorting.

$\lceil \log_2 n! \rceil \approx n \log_2 n$ comparisons are necessary in the worst case to sort a list of size n by any comparison-based algorithm.

About $n\log_2 n$ comparisons are also sufficient to sort array of size n (by mergesort).

Searching with presorting

Problem: Search for a given K in A[0..n-1]

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Presorting-based algorithm:

Stage 1 Sort the array by an efficient sorting algorithm

Stage 2 Apply binary search

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- Good or bad?
- Why do we have our dictionaries, telephone directories, etc. sorted?

Element Uniqueness with Presorting

- Brute force algorithm
  Compare all pairs of elements

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  Efficiency: $O(n^2)$

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  Stage 1: sort by efficient sorting algorithm (e.g. mergesort)

  Stage 2: scan array to check pairs of <u>adjacent</u> elements

Element Uniqueness with Presorting

- Presorting-based algorithm

  Stage 1: sort by efficient sorting algorithm (e.g. mergesort)

  Stage 2: scan array to check pairs of adjacent elements

  Efficiency: $\Theta(n\log n) + O(n) = \Theta(n\log n)$

Gaussian Elimination

Given: A system of n linear equations in n unknowns with an arbitrary coefficient matrix.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

Gaussian Elimination

Given: A system of n linear equations in n unknowns with an arbitrary coefficient matrix.

Transform to: An equivalent system of n linear equations in n unknowns with an upper triangular coefficient matrix.

Solve the latter by substitutions starting with the last equation and moving up to the first one.

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{array}
\quad \longrightarrow \quad
\begin{array}{r}
a'_{11}x_1 + a'_{12}x_2 + \ldots + a'_{1n}x_n = b'_1 \\
a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2 \\
\\
a'_{nn}x_n = b'_n
\end{array}
$$

Gaussian Elimination (cont.)

The transformation is accomplished by a sequence of elementary operations on the system's coefficient matrix (which don't change the system's solution):

for $i \leftarrow 1$ to n-1 do

replace each of the subsequent rows (i.e., rows i+1, …, n) by a difference between that row and an appropriate multiple of the i-th row to make the new coefficient in the i-th column of that row 0

Gaussian Elimination (cont.)

ALGORITHM *GaussElimination*$(A[1..n, 1..n], b[1..n])$

//Applies Gaussian elimination to matrix A of a system's coefficients,
//augmented with vector b of the system's right-hand side values
//Input: Matrix $A[1..n, 1,..n]$ and column-vector $b[1..n]$
//Output: An equivalent upper-triangular matrix in place of A with the
//corresponding right-hand side values in the $(n+1)$st column
for $i \leftarrow 1$ **to** n **do** $A[i, n+1] \leftarrow b[i]$ //augments the matrix
for $i \leftarrow 1$ **to** $n-1$ **do**
    for $j \leftarrow i+1$ **to** n **do**
        for $k \leftarrow i$ **to** $n+1$ **do**
            $A[j, k] \leftarrow A[j, k] - A[i, k] * A[j, i] / A[i, i]$

Example of Gaussian Elimination

$$
\begin{aligned}
2x_1 - 4x_2 + x_3 &= 6 \\
3x_1 - x_2 + x_3 &= 11 \\
x_1 + x_2 - x_3 &= -3
\end{aligned}
$$

Example of Gaussian Elimination

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2x_1 - 4x_2 + x_3 &= 6 \\
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\end{aligned}
$$

$$
\begin{vmatrix}
2 & -4 & 1 & 6 \\
3 & -1 & 1 & 11 \\
1 & 1 & -1 & -3
\end{vmatrix}
$$

Example of Gaussian Elimination

$$\begin{vmatrix} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{vmatrix}$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{array}\right|$$

Eliminate the first column
After the first row

Example of Gaussian Elimination

$$\begin{vmatrix} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{vmatrix} \begin{array}{l} \\ -\left(\frac{3}{2}\right) \times row\ 1 \\ -\left(\frac{1}{2}\right) \times row\ 1 \end{array}$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{array}\right| \quad \begin{array}{l} \\ -\left(\dfrac{3}{2}\right) \times [2 \quad -4 \quad 1 \quad 6] \\ -\left(\dfrac{1}{2}\right) \times [2 \quad -4 \quad 1 \quad 6] \end{array}$$

Example of Gaussian Elimination

$$\left|\begin{matrix} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{matrix}\right| \quad \begin{matrix} \\ -\left[3 \quad -6 \quad \frac{3}{2} \quad 9\right] \\ -\left[1 \quad -2 \quad \frac{1}{2} \quad 3\right] \end{matrix}$$

Example of Gaussian Elimination

$$\left|\begin{matrix} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{matrix}\right| \begin{matrix} \\ -[3 \quad -6 \quad \frac{3}{2} \quad 9] \\ -[1 \quad -2 \quad \frac{1}{2} \quad 3] \end{matrix} \quad \Rrightarrow \quad \left|\begin{matrix} 2 & -4 & 1 & 6 \\ & & & \\ & & & \end{matrix}\right|$$

Example of Gaussian Elimination

$$\left|\begin{matrix} 2 & -4 & 1 & 6 \\ 3 & -1 & 1 & 11 \\ 1 & 1 & -1 & -3 \end{matrix}\right| \begin{matrix} \\ -[3 \quad -6 \quad \frac{3}{2} \quad 9] \\ -[1 \quad -2 \quad \frac{1}{2} \quad 3] \end{matrix} \Rrightarrow \left|\begin{matrix} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{matrix}\right|$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{array}\right|$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{array}\right|$$

Eliminate the second column
After the second row

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{array}\right| \begin{array}{l} \\ \\ -\left(\frac{3}{5}\right) \times row\ 2 \end{array}$$

Example of Gaussian Elimination

$$
\left|\begin{matrix} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{matrix}\right| \quad -\left(\frac{3}{5}\right) \times \begin{bmatrix} 0 & 5 & -\frac{1}{2} & 2 \end{bmatrix}
$$

Example of Gaussian Elimination

$$\begin{vmatrix} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{vmatrix} \quad -\begin{bmatrix} 0 & 3 & -\frac{3}{10} & \frac{6}{5} \end{bmatrix}$$

Example of Gaussian Elimination

$$\left|\begin{matrix} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{matrix}\right| \begin{matrix} \\ \\ -\begin{bmatrix} 0 & 3 & -\frac{3}{10} & \frac{6}{5} \end{bmatrix} \end{matrix} \Rightarrow \left|\begin{matrix} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ & & & \end{matrix}\right|$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{2} & -6 \end{array}\right| \begin{array}{c} \\ \\ -\left[0 \quad 3 \quad -\frac{3}{10} \quad \frac{6}{5}\right] \end{array} \Rightarrow \left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{6}{5} & -\frac{36}{5} \end{array}\right|$$

Example of Gaussian Elimination

$$
\left|\begin{array}{cccc}
2 & -4 & 1 & 6 \\
0 & 5 & -\frac{1}{2} & 2 \\
0 & 0 & -\frac{6}{5} & -\frac{36}{5}
\end{array}\right|
$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{6}{5} & -\frac{36}{5} \end{array}\right|$$

$$-\frac{6}{5}x_3 = -\frac{36}{5}$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{6}{5} & -\frac{36}{5} \end{array}\right|$$

$$5x_2 - \frac{1}{2}x_3 = 2$$

$$-\frac{6}{5}x_3 = -\frac{36}{5}$$

Example of Gaussian Elimination

$$\left|\begin{array}{cccc} 2 & -4 & 1 & 6 \\ 0 & 5 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{6}{5} & -\frac{36}{5} \end{array}\right|$$

$$2x_1 - 4x_2 + x_3 = 6$$

$$5x_2 - \frac{1}{2}x_3 = 2$$

$$-\frac{6}{5}x_3 = -\frac{36}{5}$$

Example of Gaussian Elimination

$$2x_1 - 4x_2 + x_3 = 6$$

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Example of Gaussian Elimination

$$2x_1 - 4x_2 + x_3 = 6$$

$$5x_2 - \frac{1}{2}x_3 = 2$$

$$-\frac{6}{5}x_3 = -\frac{36}{5} \qquad \rightarrow x_3 = 6$$

Example of Gaussian Elimination

$$2x_1 - 4x_2 + x_3 = 6$$

$$5x_2 - \frac{1}{2}x_3 = 2 \qquad \rightarrow 5x_2 - \frac{1}{2} \times 6 = 2$$

$$-\frac{6}{5}x_3 = -\frac{36}{5} \qquad \rightarrow x_3 = 6$$

Example of Gaussian Elimination

$$2x_1 - 4x_2 + x_3 = 6$$

$$5x_2 - \frac{1}{2}x_3 = 2 \qquad \rightarrow 5x_2 - \frac{1}{2} \times 6 = 2 \qquad \rightarrow x_2 = 1$$

$$-\frac{6}{5}x_3 = -\frac{36}{5} \qquad \rightarrow x_3 = 6$$

Example of Gaussian Elimination

$2x_1 - 4x_2 + x_3 = 6$ $\quad\rightarrow 2x_1 - 4 \times 1 + 6 = 6$

$5x_2 - \frac{1}{2}x_3 = 2$ $\quad\rightarrow 5x_2 - \frac{1}{2} \times 6 = 2$ $\quad\rightarrow x_2 = 1$

$-\frac{6}{5}x_3 = -\frac{36}{5}$ $\quad\rightarrow x_3 = 6$

Example of Gaussian Elimination

$$2x_1 - 4x_2 + x_3 = 6 \qquad \rightarrow 2x_1 - 4 \times 1 + 6 = 6 \qquad \rightarrow x_1 = 2$$

$$5x_2 - \frac{1}{2}x_3 = 2 \qquad \rightarrow 5x_2 - \frac{1}{2} \times 6 = 2 \qquad \rightarrow x_2 = 1$$

$$-\frac{6}{5}x_3 = -\frac{36}{5} \qquad \rightarrow x_3 = 6$$

Example of Gaussian Elimination

Solve

$$
\begin{aligned}
2x_1 - 4x_2 + x_3 &= 6 \\
3x_1 - x_2 + x_3 &= 11 \\
x_1 + x_2 - x_3 &= -3
\end{aligned}
$$

Gaussian elimination

$$
\begin{array}{rrrrl}
2 & -4 & 1 & 6 & \\
3 & -1 & 1 & 11 & \text{row2 – (3/2)*row1} \\
1 & 1 & -1 & -3 & \text{row3 – (1/2)*row1}
\end{array}
\Rightarrow
\begin{array}{rrrrl}
2 & -4 & 1 & 6 & \\
0 & 5 & -1/2 & 2 & \\
0 & 3 & -3/2 & -6 & \text{row3–(3/5)*row2}
\end{array}
$$

$$
\Downarrow
$$

$$
\begin{array}{rrrr}
2 & -4 & 1 & 6 \\
0 & 5 & -1/2 & 2 \\
0 & 0 & -6/5 & -36/5
\end{array}
$$

Backward substitution

$$
\begin{aligned}
x_3 &= (-36/5) \ / \ (-6/5) = 6 \\
x_2 &= (2+(1/2)*6) \ / \ 5 = 1 \\
x_1 &= (6 – 6 + 4*1)/2 = 2
\end{aligned}
$$

Pseudocode and Efficiency of Gaussian Elimination

Stage 1: Reduction to the upper-triangular matrix

```
for i ← 1 to n-1 do
     for j ← i+1 to n do
          for k ← i to n+1 do
               A[j, k] ← A[j, k] - A[i, k] * A[j, i] / A[i, i]
```

Stage 2: Backward substitution

```
for j ← n downto 1 do
     t ← 0
     for k ← j +1 to n do
          t ← t + A[j, k] * x[k]
     x[j] ← (A[j, n+1] - t) / A[j, j]
```

Pseudocode and Efficiency of Gaussian Elimination

Stage 1: Reduction to the upper-triangular matrix

```
for i ← 1 to n-1 do
     for j ← i+1 to n do
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```

Stage 2: Backward substitution

```
for j ← n downto 1 do
      t ← 0
      for k ← j +1 to n do
            t ← t + A[j, k] * x[k]
      x[j] ← (A[j, n+1] - t) / A[j, j]
```

Efficiency: $\Theta(n^3) + \Theta(n^2) = \Theta(n^3)$

Discussion

Solve the following system by Gaussian elimination.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + x_2 + x_3 &= 3 \\
x_1 - x_2 + 3x_3 &= 8.
\end{aligned}
$$

Discussion

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 3 & 8 \end{bmatrix} \quad \begin{array}{l} \\ \text{row 2 - } \frac{2}{1}\text{row 1} \\ \text{row 3 - } \frac{1}{1}\text{row 1} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -2 & 2 & 6 \end{bmatrix} \quad \begin{array}{l} \\ \\ \text{row 3 - } \frac{-2}{-1}\text{row 2} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Then, by backward substitutions, we obtain the solution as follows:

$x_3 = 8/4 = 2,\ x_2 = (-1 + x_3)/(-1) = -1,$ and $x_1 = (2 - x_3 - x_2)/1 = 1.$