Stochastic Optimization of a Multireservoir Hydroelectric System: A Decomposition Approach

Article *in* Water Resources Research · June 1985

DOI: 10.1029/WR021i006p00779

2 authors:

Mario V. F. Pereira
PSR Energy Consulting

Pinto Leontina

WATER RESOURCES RESEARCH, VOL. 21, NO. 6, PAGES 779–792, JUNE 1985

Stochastic Optimization of a Multireservoir Hydroelectric System: A Decomposition Approach

M. V. F. PEREIRA AND L. M. V. G. PINTO

CEPEL, Centro de Pesquisas de Energia Elétrica, Rio de Janeiro, Brazil

A computational scheme that is able to determine at each stage the most economical generation decision for each plant of a hydrothermal system is presented. The algorithm is based on the stochastic extension of Benders decomposition and can be implemented from already existing operation models. The results can be used in the weekly or monthly generation scheduling activities in real-time operation. A case study with 37 reservoirs of the Brazilian system is presented and discussed.

INTRODUCTION

The objective of the optimal operation of a hydrothermal generating system is to determine a generation schedule for each plant in the system that minimizes the expected operation cost along the planning period. The operation cost is composed of fuel costs for the thermal units and penalties for failure in load supply.

The availability of limited amounts of hydroelectric energy, in the form of stored water in the system reservoirs, makes the optimal operation problem very complex because it creates a link between an operating decision in a given stage and the future consequences of this decision. In other words, if we deplete the stocks of hydroelectric energy, and low inflow volumes occur, it may be necessary to use very expensive thermal generation in the future or even fail to supply the load. On the other hand, if we keep the reservoir levels high through a more intensive use of thermal generation and high inflow volumes occur, there may be spillage in the system, which means a waste of energy, and consequently, higher operating costs.

Because it is impossible to have perfect forecasts of the future inflow sequences and, in a certain measure, of the future load itself, the operation problem is essentially stochastic. The existence of multiple interconnected reservoirs and the need for multiperiod optimization characterize the problem as large scale. Finally, the objective function is nonlinear, due not only to the nonlinear thermal costs, but also to the product of outflow and head in the expression of hydroelectric production. Because the worth of energy generated in a hydro plant cannot be measured directly as a function of the plant state alone, but rather in terms of fuel savings from avoided thermal generation, the objective function is also nonseparable.

A recent survey of the literature [*Rosenthal*, 1980] gives a measure of the complexity of the operation problem: none of the 92 scheduling models found in the literature deals simultaneously with all the aspects of the problem (multiple reservoirs, multiple periods, stochastic inflows, and nonseparable benefits).

This paper describes a computational scheme that is able to produce at each stage of the planning period the most economical generation decision for each plant of the hydrothermal sytem. The algorithm is based on a stochastic extension of Benders decomposition and can be easily implemented from already existing operation models.

The results of the algorithm can be used in the allocation of weekly or monthly generation targets for each plant in a hydrothermal system. These targets are then further refined into daily and finally hourly targets in real-time operation. A description of the hierarchical scheme for operations scheduling can be found in the works by *Pereira and Pinto* [1982, 1983, 1984].

In the next section, the operation planning problem is formulated and the different solution approaches are discussed. The Benders decomposition principle is then derived and extended to the stochastic case. A case study with four reservoirs is used to illustrate in detail the application of the algorithm, and a case study with 37 reservoirs of the Brazilian system is then presented and discussed.

OPERATION PLANNING PROBLEM

More formally, the objective of the operation planning problem is to find an operation strategy that for each stage of the planning period, given the system state at the beginning of that stage, produces the generation targets for the system plants.

If the inflows for any given stage t are known at the beginning of the stage, the resulting control problem can theoretically be solved by a stochastic dynamic programming (SDP) recursion of the "hazard-decision" type

$$\alpha_t^*(X_t) = \underset{A_t|X_t}{E} \left\{ \min_{U_t} \left[C_t(U_t) + \frac{1}{\beta} \alpha_{t+1}^*(X_{t+1}) \right] \right\} \tag{1}$$

$$\forall\, t = T, T-1, \cdots, 1$$

s/to

$$X_{t+1} = f_t(X_t, A_t, U_t) \tag{1a}$$

$$g_{t+1}(X_{t+1}) \geq 0 \tag{1b}$$

$$h_t(U_t) \geq 0 \tag{1c}$$

where

- X_t system state vector at the beginning of stage t;
- A_t inflow vector during stage t;
- $E_{A_t|X_t}$ expected value over all possible inflow vectors A_t conditioned by the state vector X_t (which is known at the beginning of the stage);
- U_t decision vector for the stage;
- $C_t(U_t)$ operating cost associated with decision U_t;
- T number of stages in the planning period;

Paper number 5W0167.
0043-1397/85/005W-0167$05.00

$\alpha_t^*(X_t)$ expected operation cost from t to the end of the planning period under the hypothesis of optimal operation;
β discount factor;
$f_t(X_t, A_t, U_t)$ system transition equation;
$g_{t+1}(X_{t+1})$ represents the set of state constraints in $t + 1$;
$h_t(U_t)$ represents the set of constraints on the decisions variables.

The representation of the system state, X_t, should include all variables that may affect the result of the operation. In the case of hydroelectric systems, at least two classes of state variables should be represented: the storage level in the reservoirs, V_t, and some kind of information about the "hydrologic trend" in the system. This information could be given, for example, by the incremental inflows to the reservoirs during the previous stage, A_{t-1}. Therefore

$$X_t = \begin{bmatrix} V_t \\ A_{t-1} \end{bmatrix} \tag{2}$$

The decision vector U_t usually represents the outflow through the turbines, Q_t, and the outflow through the spillway, S_t. In vector form,

$$U_t = \begin{bmatrix} Q_t \\ S_t \end{bmatrix} \tag{3}$$

The immediate operation cost associated to the decision U_t, $C_t(U_t)$ is calculated as follows.

1. The total hydro generation in the system is calculated as

$$GH(U_t) = \sum_{i=1}^{N} \rho_i Q_t(i) \tag{4}$$

where N is the number of hydro plants, and ρ_i is the generation characteristic of the ith plant. It should be noted that the generation characteristic is, in fact, a function of the initial volume, end volume, and outflow, that is,

$$\rho_i = \rho(V_t(i), V_{t+1}(i), Q_t(i), S_t(i)) \tag{5}$$

However, in the derivations that will follow, the generation characteristic will be considered as a constant. The effect of this assumption will be discussed in the case study section.

2. The remaining load is met by the thermal units (load curtailment is represented as an additional thermal unit with very high operating cost). The immediate operation cost $C_t(U_t)$ is then calculated as the result of the thermal optimization subproblem

$$\begin{aligned} C_t(U_t) = \min \ & \sum_{j=1}^{J} CT_j(GT_j) \\ \text{s/to} \ & \sum_{j=1}^{J} GT_j = L_t - GH(U_t) \\ & \mathbf{GT}_j \le GT_j \le \overline{GT_j} \end{aligned} \tag{6}$$

where

J number of thermal units;
GT_j generation of the jth plant;
$CT_j(GT_j)$ plant operating cost;
$\overline{GT_j}$, $\mathbf{GT}_j$ upper and lower bounds on the operation of the jth thermal plant, respectively;
$GH(U_t)$ calculated in (4);
L_t the load for stage t.

The transition equation (1a), $f_t(X_t, A_t, U_t)$, corresponds to the well-known water balance equation

$$V_{t+1}(i) = V_t(i) + A_t(i) - (Q_t(i) + S_t(i)) + \sum_{j \in M_i} (Q_t(j) + S_t(j)) \tag{7}$$

where M_i is the set of plants immediately upstream of plant i.

The constraints on the system state (1b), $g_{t+1}(X_{t+1})$ are usually represented as upper and lower bounds on the volumes

$$\mathbf{V}_{t+1} \le V_{t+1} \le \bar{V}_{t+1} \tag{8}$$

The constraints on the decision variables $h_t(U_t)$ also correspond to upper bounds on the turbined outflow

$$Q_t \le \bar{Q}_t \tag{9}$$

and lower bounds on the total outflow

$$Q_t + S_t \ge \mathbf{U}_t \tag{10}$$

The solution of the recursion equation (1) usually requires the discretization of the state space. Supposing that each of the N reservoir volumes V_t and each of the N incremental inflows in the previous stage A_{t-1} is discretized in M intervals, there will be M^{2N} discretized states in the state space.

It is easy to see that the number of discretized states, and consequently, the computational effort, increases exponentially with the number of state variables. This implies that the explicit resolution of (1) is infeasible even for very small reservoir systems. For example, supposing $M = 20$ intervals:

1 reservoir = $>20^2 = 400$ states
2 reservoirs = $>20^4 = 160{,}000$ states
3 reservoirs = $>20^6 = 64$ million states
4 reservoirs = $>20^8 = 25$ billion states
5 reservoirs = $>20^{10} = 10$ trillion states

This "curse of dimensionality" is further confirmed by a recent survey of dynamic programming applications in water resources. According to *Yakowitz* [1982],

> the largest numerical stochastic dynamic programming solutions within or outside the water resources literature to come to our attention are for problems having at most two or three state variables. Even then, the authors reporting their findings often remark on the ferocity of the computational burden

Therefore it becomes necessary to develop methods able to approximate the solution of the operating problem with a reasonable computational cost. The two main approaches for the solution of the problem are to reduce the number of state variables, retaining the stochastic structure (aggregation/disaggregation methods), or to ignore the stochastic character of inflows, retaining a detailed representation of the generating system (deterministic equivalent methods). Both approaches will be briefly described in the next sections. A very comprehensive and updated state-of-the-art survey of reservoir management techniques can be found in the work by *Yeh* [1982].

Aggregation-Disaggregation Methods

The aggregation-disaggregation methods can be understood from (4). The immediate cost $C_t(U_t)$ associated to a decision U_t is obtained from the total hydro generation and not from the generation of each individual hydro plant. This suggests that the problem can be decomposed in two steps: (1) aggregate the hydro plants into one equivalent energy reservoir and use stochastic dynamic programming to calculate the optimal pro-

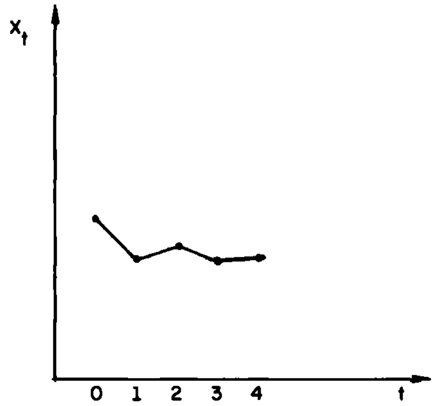

Fig. 1. System trajectory for one inflow sequence.

portion of hydro and thermal generation in the system (strategic problem); and (2) disaggregate the total hydroelectric generation calculated in step 1 into generation targets for each hydroplant in the system (tactical problem).

In order to build the aggregate reservoir it is necessary to assume simplified depletion rules [*Arvaniditis and Rosing*, 1970; *CEPEL/Eletrobrás*, 1977]. Experience with the Brazilian system has shown that this approximation is reasonable when the system has a very large regulation capacity and the basins are hydrologically homogeneous (large cross correlation between annual streamflows of different sites).

Because the cascaded reservoir system is often divided in valleys or basins, the aggregation of hydroplants in each valley as an equivalent energy reservoir leads to the problem of operating reservoirs in parallel. In the Hydro-Québec system, for example, there are six such valleys. In the Norwegian system the number of subsystems in parallel exceeds 20. The Brazilian system is presently divided in four subsystems. Many interesting solution techniques have been developed to handle the dimensionality problem in the case of reservoirs in parallel; they are as follows.

1. *The locally informed strategy proposed by Davis* [1972] *and applied by Pronovost et al.* [1978] *to the Hydro-Québec system.* This technique may be seen as a stochastic extension of the dynamic programming-successive approximation algorithm. It is very fast and easy to program.

2. *The Power Pool model developed by Egeland et al.* [1982] *for the Norwegian system.* Although the reference is recent, this model has been in use for several years. It is also based on successive approximations but with a better representation of the relation between each reservoir and the "rest of the world" than the previous algorithm. The computational requirements are correspondingly heavier. A similar approach was applied by *Araripe Neto et al.* [1984] to the long-term operations scheduling of the Brazilian system, with positive results.

3. *The aggregation-decomposition technique proposed by Turgeon* [1980]. In this approach each reservoir is optimized in conjunction with the aggregate representation of the remaining system. Although computational requirements are heavier, there are no iterations, which results in very acceptable computer times.

The main difficulties with the aggregation-disaggregation methods appear in the disaggregation step, because the system performance in terms of spillage in the plants or loss of peak power due to reservoir depletion is sensitive to the allocation of generation targets. Some disaggregation criteria were suggested, for example, by *Egeland et al.* [1982] and *Pereira and Kelman* [1983].

In summary, the main advantage of the aggregation-disaggregation method is to concentrate the focus on the most important aspect in economical terms, which is the decision about the proportion of thermal generation. The main drawback is on the definition of the disaggregation criteria, that is, on the "fine tuning" of the performance of the hydroelectric system.

Deterministic Equivalent Methods

The deterministic equivalent methods assume that the inflows are known along the whole planning period. In this way, instead of an operation strategy that produces the optimal operating decision U_t^* for each possible state X_t, it is enough to determine a trajectory $\{X_1^*, X_2^*, \cdots, X_t^*, \cdots, X_T^*\}$ which will correspond to the optimal reservoir evolution for the preestablished inflow sequence.

With the hypothesis of deterministic inflows, the dimensionality problem disappears. Although the remaining problem is still very complex, several algorithms are already available to solve it, such as those suggested by *Rosenthal* [1981], *Hanscom et al.* [1980], *Murray and Yakowitz* [1979], and *Ikura and Gross* [1984].

The theoretical basis for using deterministic inflows is the certainty equivalence principle, which establishes that the optimal strategy for the solution of certain classes of stochastic control problems can be obtained by replacing the stochastic components by their expected value [*Bryson and Ho*, 1975]. It should be noted that this method assumes that the deterministic equivalent problem is resolved at each stage, as soon as the new inflow measures become available.

In theoretical terms, the hypothesis of deterministic equivalence is not well suited to the operation planning problem, as is discussed in the work by *Gjelsvik*, [1981]. Because the operation with deterministic inflow is "better" (more flexible) than could be expected from the SDP solution approach (1), the deterministic equivalent methods will tend to use less thermal generation than the amount recommended by the (unknown) stochastic solution [*Read*, 1979]. However, to the authors' knowledge, a quantitative evaluation of this effect has been carried out only for the system of Turkey [*Dagli and Miles*, 1980] and New Zealand [*Boshier and Read*, 1981], with favorable results.

In summary, the main advantage of the deterministic equivalent methods is to allow a correct representation of the hydroelectric system. The disadvantage is to produce an "optimistic" operation which, in case of severe droughts, can lead to high losses.

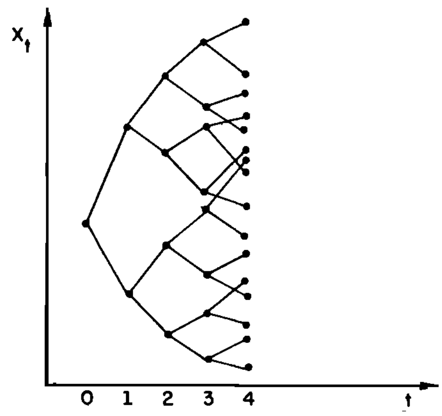

Fig. 2. Multiple inflow scenarios.

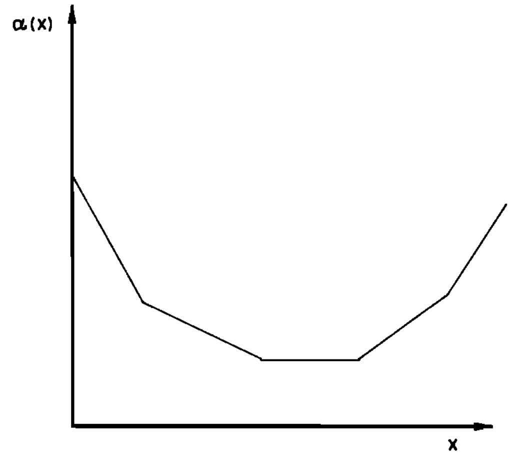

Fig. 3. Example of the function $\alpha(x)$.

OUTLINE OF THE PROPOSED APPROACH

The algorithm proposed in this paper tries to retain the detailed representation of the hydroelectric system while representing at least partially the stochastic characteristics of streamflow. It has been mentioned in the previous section that the deterministic equivalent method tries to calculate an optimal system trajectory, as is illustrated in Figure 1 for a period of four stages and one reservoir. The initial state X_0 is assumed to be known. The transition to X_1, the system state at the end of the first period, depends on the inflow A_1 during the first stage; analogously, the transitions to X_2, X_3, and X_4 depend on the inflows in the respective stages.

Suppose now that instead of only one sequence $A = \{A_1, A_2, A_3, A_4\}$ there are two possible sequences, A^1 and A^2. The solutions of the resulting problem can be represented as a forked path where each possible trajectory represents a possible "scenario." In general, the evolution of the reservoir system would have a treelike structure such as in Figure 2, where each bifurcation corresponds to alternative inflow vectors. Note that any number of branches can be represented at each transition point. It should also be noted that this structure is similar to the SDP formulation, in which, given the initial state X_0, some information about the expected future cost is required.

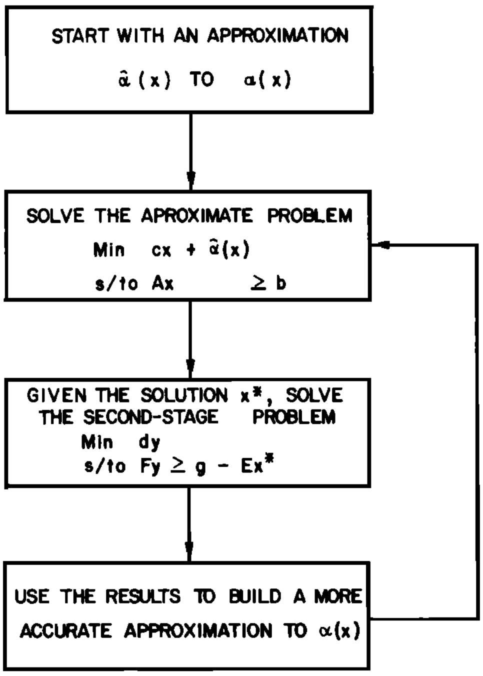

Fig. 4. Simplified scheme of the Benders decomposition algorithm.

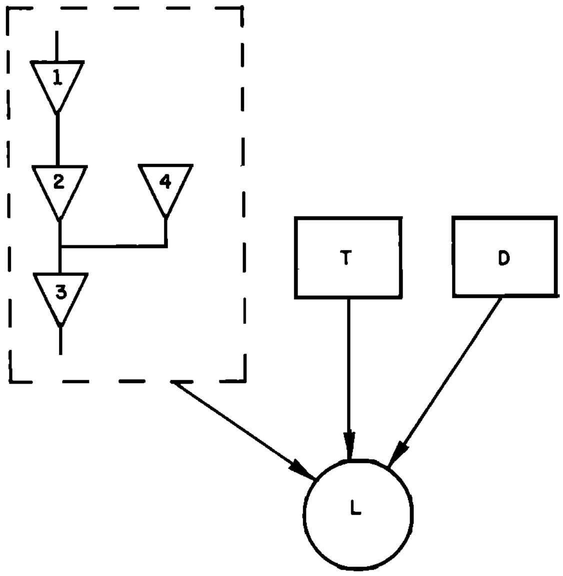

Fig. 5. Four-reservoir hydrothermal system.

The algorithm proposed in this paper tries to solve problems with multiple inflow sequences like the one in Figure 2. It can be seen as an extension of the deterministic equivalent methods, in which the stochasticity of inflow is represented by the different alternatives of inflow vectors at each stage. In this way the "optimistic" trend of the deterministic models is avoided, and the detailed representation of the hydro system is retained.

The resulting optimization problem can be very large, but has a special structure. As is discussed in the next sections, the proposed solution approach decomposes the multiple-period, multiple-scenario problem into smaller one-period subproblems that can be easily solved. The algorithm is based on some results of stochastic programming and on the extension of Benders decomposition to the stochastic case.

Alternative solution approaches can be found in the works by *Gjelsvik* [1982] and *Read* [1979] and in the reservoir scheduling course promoted by the *University of Tennessee* [1982]. The use of multiple inflow sequences has also been suggested by *Grygier and Stedinger* [1983]. The next section outlines the Benders decomposition principle.

BENDERS DECOMPOSITION PRINCIPLE

Let a two-stage deterministic hydrothermal optimization problem be represented as

$$\begin{aligned} \min\ & cx + dy \\ \text{s/to}\ & Ax \geq b \\ & Ex + Fy \geq g \end{aligned} \tag{11}$$

In this problem, the variables x represent decisions about hydro and thermal generation in the first stage (outflows, end volumes, generation level, etc.). The associated cost is represented by cx, and the constraints on system operation (hydraulic constraints, upper and lower bounds on outflows, etc...) are represented by the constraints set $Ax \geq b$.

Given a feasible solution x^* to the first-stage problem

TABLE 1. Characteristics of the Generating System

Hydro Plant	Maximum Storage, $\bar{V}$	Maximum Outflow, $\bar{Q}$	Generation Characteristics, ρ
1	500	100	0.8
2	400	140	0.8
3	500	100	0.8
4	500	240	0.8

Thermal Plant	Maximum Capacity, $\bar{G}$	Unit Cost, C
GT	50	1
D	∞	10

Load is 200.

($Ax^* \geq b$), the operation problem in the second stage can be represented as

$$\begin{aligned} &\min\ dy \\ &\text{s/to}\ \ Fy \geq g - Ex^* \end{aligned} \tag{12}$$

where x^* is known and therefore goes to the right-hand side of the problem. In this problem, the variables y represent decisions about hydro and thermal generation in the second stage. The associated cost is represented by dy and the constraints on system operation are represented by the constraint set $Fy \geq g - Ex$. Note that the first-stage decisions x^* affect the second-stage constraint set. For example, the reservoir volumes at the end of the first-stage (part of the decision variables x) become the initial volume constraints in the second stage. It is also assumed, without loss of generality, that the second-stage problem is always feasible for any given decision x^*. This assumption will be further discussed in the case study.

The objective is to minimize the sum of operation costs $cx + dy$. The decomposition methodology is based on the following observations.

1. The second-stage cost dy^*, where y^* is the optimal solution of problem (12), can be seen as a function $\alpha(x)$ of the first-stage decision x, that is,

$$\begin{aligned} \alpha(x) &= \min\ dy \\ &\text{s/to}\ \ Fy \geq g - Ex \end{aligned} \tag{13}$$

As was mentioned above, it is assumed that the second-stage problem is always feasible for any decision x.

2. The two-stage problem (11) can therefore be rewritten as

$$\begin{aligned} &\min\ cx + \alpha(x) \\ &\text{s/to}\ \ Ax \geq b \end{aligned} \tag{14}$$

where $\alpha(x)$ is the value of the optimal solution of (13) for each x. The structure of problem (14) is similar to a dynamic programming recursion, where cx represents the immediate cost and $\alpha(x)$ gives information about the "future" (second-stage) consequences of decision x. In the case of linear problems, however, it is possible to characterize the function $\alpha(x)$ without state space discretization, therefore avoiding the usual dimensionality problems associated with dynamic programming. It can be shown that $\alpha(x)$ is a convex polihedron, as is illustrated in Figure 3 for the case of one variable.

The Benders decomposition principle [*Benders*, 1962] is a technique for building this polyhedron with any required accuracy based on the iterative solution of stage 1 and 2 problems, as follows.

1. Start with an approximation $\hat{\alpha}(x)$ which is a lower bound to $\alpha(x)$.
2. Solve the approximate problem

$$\begin{aligned} &\min\ cx + \hat{\alpha}(x) \\ &\text{s/to}\ \ Ax \geq b \end{aligned} \tag{15}$$

Note that this problem corresponds to a one-stage hydrothermal optimization problem and can be solved by any specialized algorithm.

3. It is possible to show that

$$\underline{z} = cx^* + \hat{\alpha}(x^*) \tag{16}$$

is a lower bound to the optimal solution of the two-stage problem (11).

4. Solve the second-stage problem

$$\begin{aligned} &\min\ dy \\ &\text{s/to}\ \ Fy \geq g - Ex^* \end{aligned} \tag{17}$$

where x^* is the solution of (15). Note that (17) also corresponds to a one-stage hydrothermal optimization problem and can be solved by any suitable algorithm.

5. Let y^* be the solution of (17). The pair (x^*, y^*) is a feasible solution of the two-stage problem but not necessarily the optimal solution. The objective function value

$$\bar{z} = cx^* + dy^* \tag{18}$$

is therefore an upper bound to the optimal solution value of the two-stage problem.

6. If $\bar{z} - \underline{z}$ is smaller than a given tolerance, the problem is solved and the pair (x^*, y^*) is the optimal solution. Otherwise, generate a new approximation $\hat{\alpha}(x)$ from the solution of (17). This approximation will still be a lower bound to $\alpha(x)$. Go to step 2.

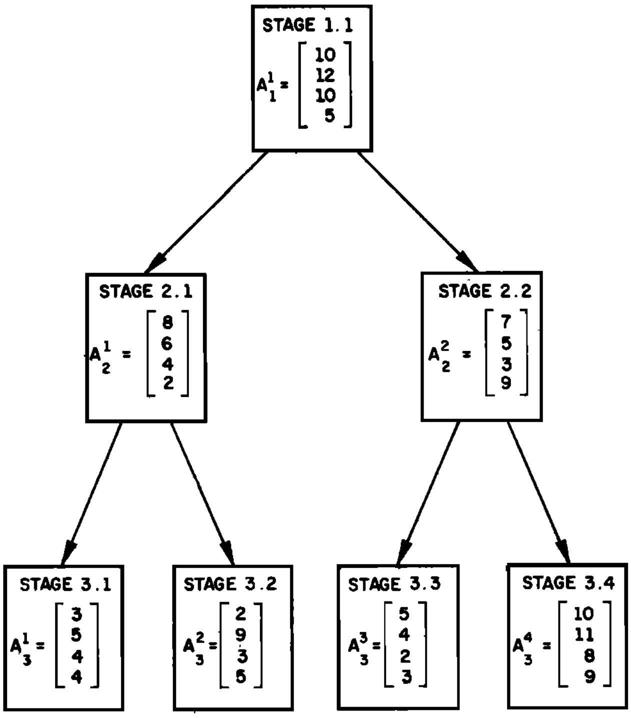

Fig. 6. Inflow vectors for the four-reservoir system.

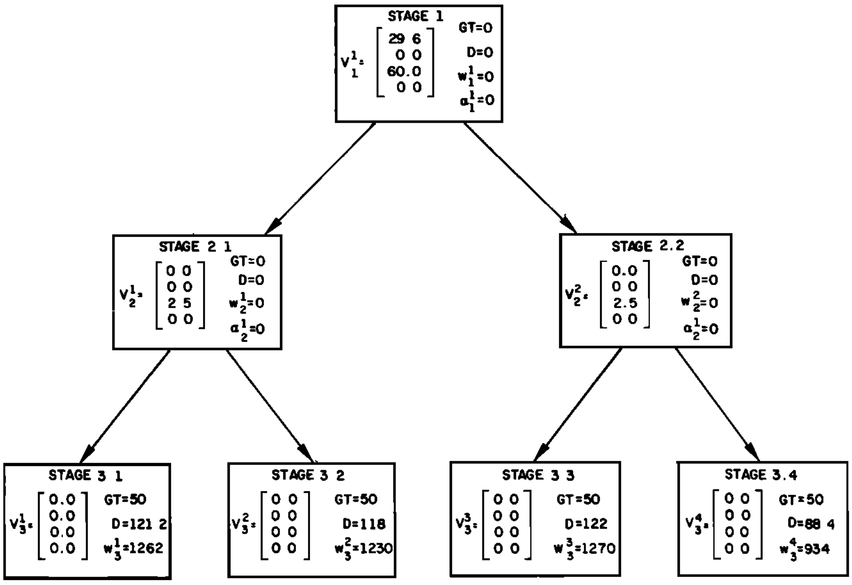

Fig. 7. Results of the first iteration.

The Benders algorithm, expressed in steps 1–6 and illustrated in the flowchart of Figure 4, is very convenient for handling sequential decision processes; the information about the "future" (second-stage) costs is brought as it is needed and it is possible to use upper and lower bounds to control the degree of accuracy. Besides that, all the optimization process is concentrated on the decision variables x, which are the only ones that are really relevant, since, in practice, the optimization will be repeated when the next stage arrives.

The critical point in the decomposition scheme is the modification of $\hat{\alpha}(x)$ from the solution of (17). Associated to the solution of the second-stage problem there is a set of Simplex multipliers (dual variables, shadow prices) which measure the change in system operating cost caused by marginal changes in the trial first-stage decisions. For example, it is possible to measure the incremental benefit in the second-stage resulting from the availability of an additional cubic meter of storage in any of the system reservoirs. These multipliers are then used to form a linear constraint, written only in terms of the first-stage variables x. This constraint, also called a Benders cut, corresponds to a new approximation to $\hat{\alpha}(x)$ and is returned to the first-stage problem. A formal derivation of the Benders decomposition can be found in the appendix.

The extension of the Benders decomposition to a multistage formulation is straightforward and will be illustrated in the case studies. The representation of probabilistic second-stage problems is also discussed in the appendix: it is shown that the decomposition scheme can accomodate treelike structures, as discussed in the previous section, and that the expected value of the multipliers is used to form a Benders cut. The next section illustrates the application of the algorithm to the problem of optimal reservoir management.

Application of the Algorithm to Hydrothermal Optimization

System Description

A simple case-study with four reservoirs will be used to illustrate in some detail the application of the algorithm. A more realistic example with 37 reservoirs will be discussed in a later section.

Figure 5 shows the generating system, composed of four hydroelectric plants, one thermal unit and the "load-shedding" dummy unit. The main characteristics of the generating units and of the load are shown in Table 1.

The vector of initial storages, V_0, corresponds to 10% of the maximum volumes; that is,

$$V_0 = \begin{bmatrix} 50 \\ 40 \\ 50 \\ 50 \end{bmatrix} \tag{19}$$

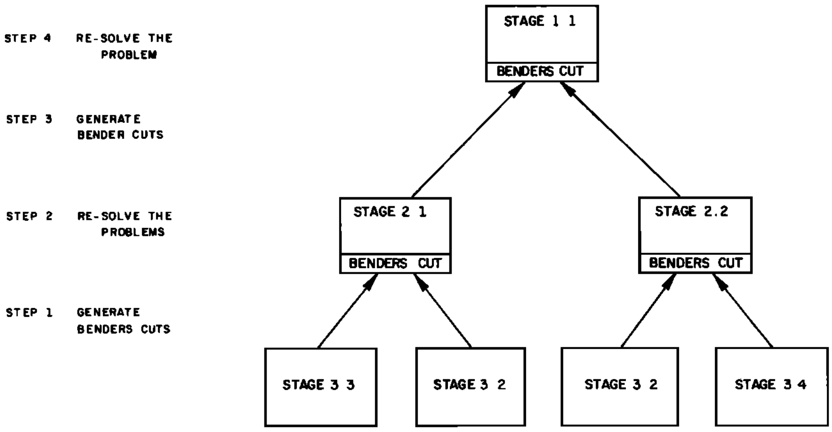

Fig. 8. Scheme for "upward" optimization.

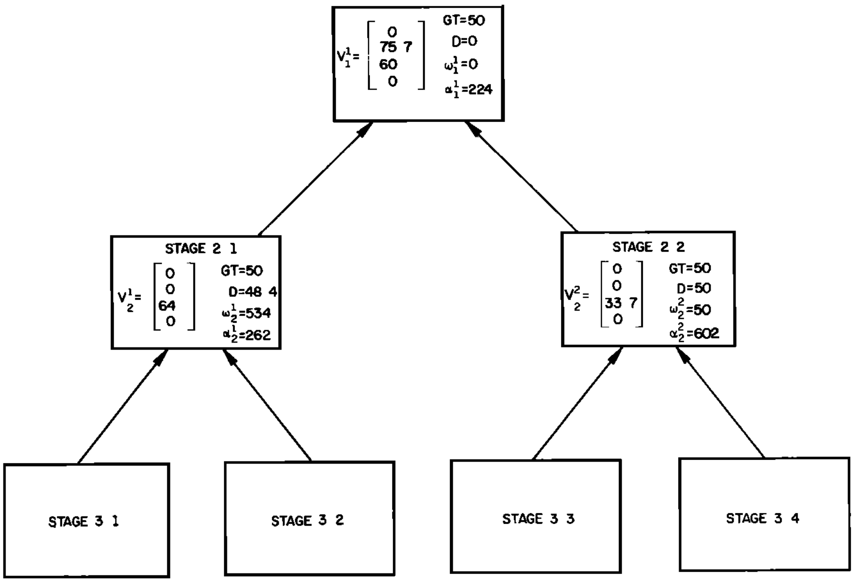

Fig. 9. Results of "upward" optimization.

The optimization interval is composed of three stages. In the first stage, there is only one incremental inflow vector, A_1. In the second stage, there are two equiprobable alternative inflow vectors, $A_2{}^1$ and $A_2{}^2$, and in the third stage, four equiprobable vectors $\{A_3{}^j\}, j = 1, \cdots, 4$ (see Figure 6).

The objective function is to minimize the expected operation cost, composed of fuel costs for the thermal generation GT plus penalties for load-shedding D; that is,

$$\min z = (GT_1{}^1 + 10D_1{}^1) + \frac{1}{2}\sum_{j=1}^{2}(GT_2{}^j + 10D_2{}^j) + \frac{1}{4}\sum_{j=1}^{4}(GT_3{}^j + 10D_3{}^j) \tag{20}$$

where $GT_t{}^j$ and $D_t{}^j$ represent the thermal generation and load-shedding for stage t and inflow alternative j, respectively.

The constraints in each stage are as follows.

1. Water balance equation

$$V_{t-1}(i) + A_t(i) = V_t(i) + Q_t(i) + S_t(i) - \sum_{j \in M_i}(Q_t(j) + S_t(j)) \tag{21}$$

$$t = 1, 2, 3 \qquad i = 1, 2, 3, 4$$

where

- $V_t(i)$ volume of the ith reservoir at the end of stage t;
- $Q_t(i)$ volume turbined by the ith reservoir;
- $S_t(i)$ volume spilled by the ith reservoir;
- M_i set of plants immediately upstream of plant i;
- $A_t(i)$ incremental inflow volume to plant i.

2. Power balance equation

$$\sum_{i=1}^{4}\rho_i Q_t(i) + GT_t + D_t = L_t \tag{22}$$

where

- L_t load in stage t;
- ρ_i generation characteristics of the plant;
- GT_t thermal generation;
- D_t load shedding.

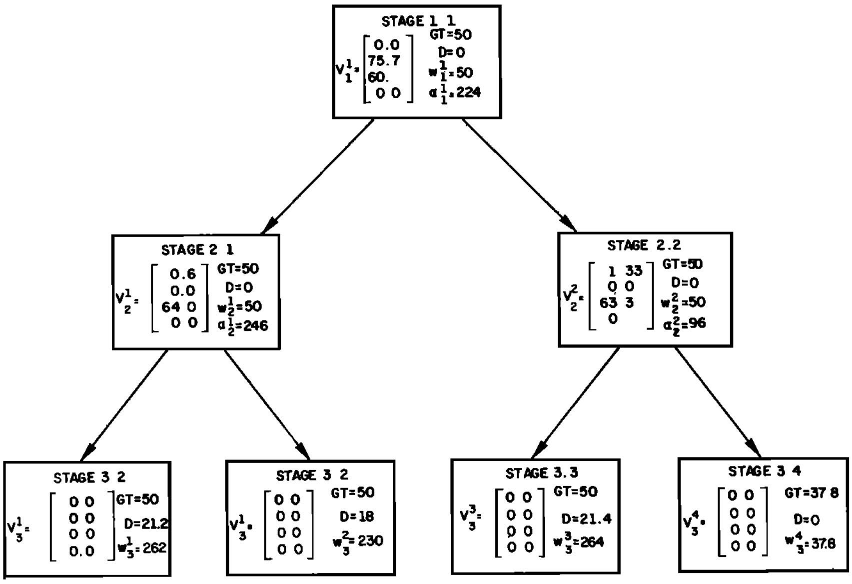

Fig. 10. Results of second iteration.

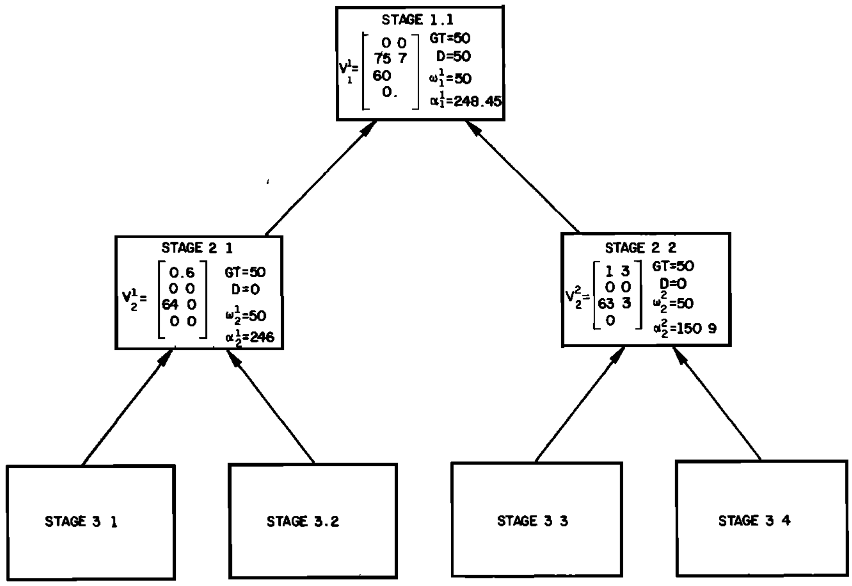

Fig. 11. Second upward optimization.

3. Bounds on volume and maximum outflow

$$0 \le V_t(i) \le \bar{V}(i) \qquad \forall\, i = 1, 2, 3, 4 \quad (23)$$
$$0 \le Q_t(i) \le \bar{Q}(i)$$

It is possible to see that the spillage and load-shedding variables S_t and D_t act as "slack" or "buffer" variables in each stage t, thus ensuring feasibility of the constraint set (21)–(23).

Optimization Procedure

In the first iteration, all constraints concerning future costs (Benders cuts) are relaxed. In other words, the operating decisions in each stage are "myopic." No thermal units are used in the first two stages, leading to very high losses in the third stage.

Figure 7 illustrates the solutions found for each stage and for each inflow alternative. V_t^j is the volume vector at the end of stage t for inflow alternative j; w_t^j is the value of the immediate cost of the solution ($GT + 10D$); and α_t^j is the value of the expected future cost. (In the first iteration, α_t^j is set to zero).

As is seen in (16), the lower bound corresponds to the objective function value in the first stage

$$\underline{z} = w_1^{\,1} + \alpha_1^{\,1} = 0$$

The upper bound is equal to the expected operation cost over all stages:

$$\bar{z} = w_1^{\,1} + \tfrac{1}{2}(w_2^{\,1} + w_2^{\,2}) + \tfrac{1}{4}(w_3^{\,1} + w_3^{\,2} + w_3^{\,3} + w_3^{\,4}) = 1214$$

The optimal solution z^* lies therefore in the range

$$0 \le z^* \le 1214$$

The next step is to bring information "upward," that is, derive Benders cuts from problems in stage t to problems in stage $t-1$, as is illustrated in Figure 8. The Benders cut for stage 2.1 is obtained as follows. From the appendix we have

$$w + \pi^* E(x^* - x) \le \alpha \qquad (24)$$

In this case, w^* is the expected value of the optimal solutions of problems 3.1 and 3.2

$$w^* = \tfrac{1}{2}(w_3^{\,1} + w_3^{\,2})$$

π^* is expected value of the Simplex multiplier vectors associated with the optimal solutions of 3.1 and 3.2:

$$\pi^* = \tfrac{1}{2}(\pi_3^{\,1} + \pi_3^{\,2})$$

The solution vector x^* corresponds to the optimal solution vector of 2.1:

$$x^* = V_2^{\,1}$$

A similar procedure is used to derive the Benders cut for problem 2.2 from the solutions of 3.3 and 3.4. Both problems are then resolved, and the results are used to generate a Benders cut for the first stage problem 1.1. The new solution of problems 1.1, 2.1, and 2.2 are shown in Figure 9. Note that the second- and first-stage decisions include now preventive thermal generations. The value of $\alpha_1^{\,1}$ represents the new lower bound to the expected future generation cost. The new lower bound to the optimal solution is then

$$\underline{z} = w_1^{\,1} + \alpha_1^{\,1} = 274$$

and the updated range for the optimal solution is

$$274 \le z^* \le 1214$$

Since this range is not acceptable, the optimization procedure goes to the second iteration, whose results can be found in Figure 10. The upper bound is updated and the optimal solution range becomes

$$274 \le z^* \le 298.45$$

The second set of Benders cuts is again derived for problems 2.1 and 2.2, and from there to problem 1.1, following the procedure of Figure 8. The results are shown in Figure 11. The new lower bound is

$$\underline{z} = 298.45$$

which is equal to the upper bound. The optimal solution has thus been found in two iterations and corresponds to the strategy in Figure 10. (Remember that the optimal strategy is always associated with the upper bound.)

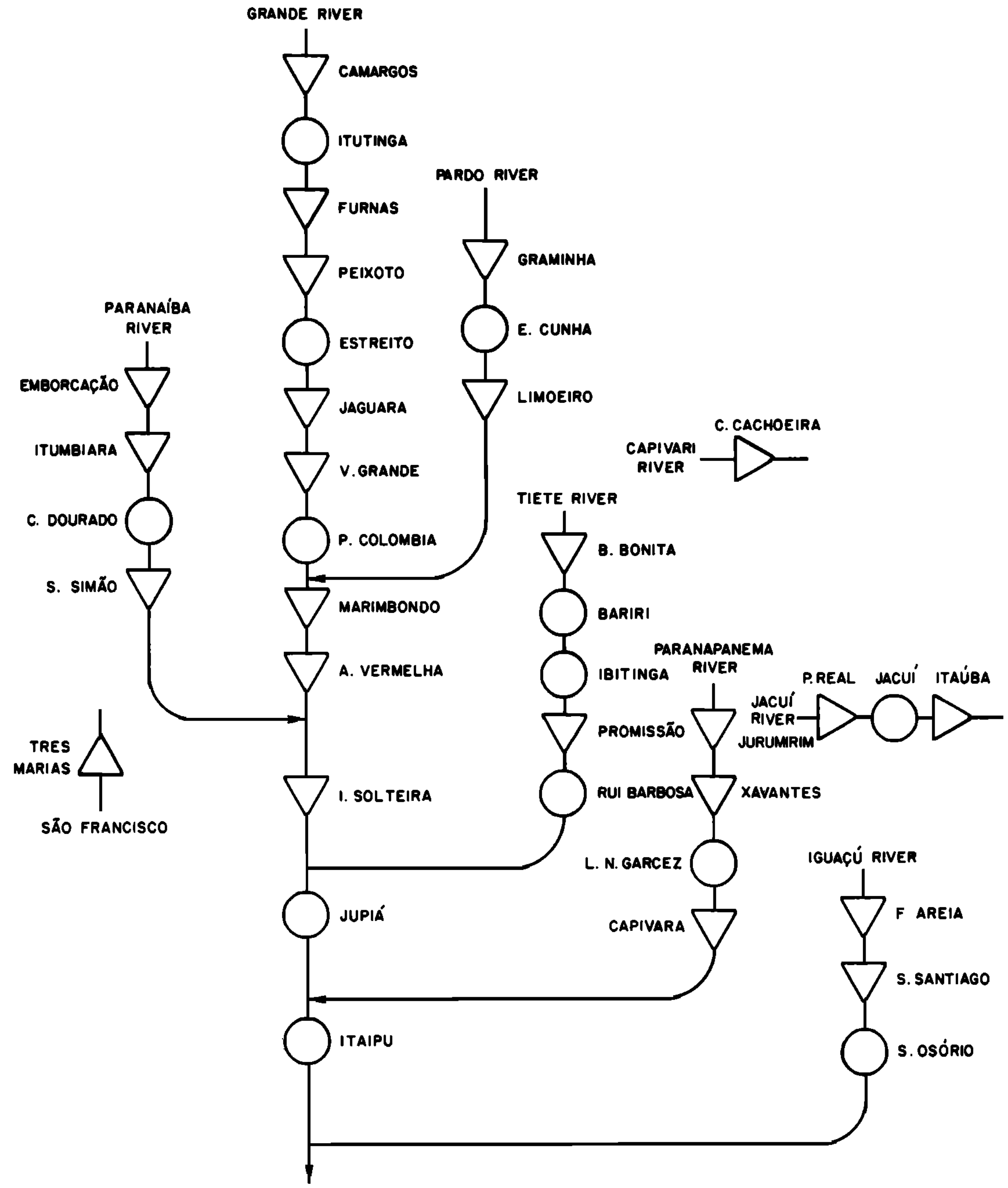

Fig. 12. Schematic representation of part of the south-southeast hydroelectric system.

CASE STUDY

The application of the algorithm will be further illustrated in a case study with 37 reservoirs of the Brasilian system, shown in the schematic diagram of Figure 12. This configuration corresponds to part of the Southeast and South regions of Brazil, where the main load centers are located. Further information about the system can be found in Table 2.

For this illustrative example, the actual thermal plants of the Brazilian system were replaced by 10,000 MWh of thermal generation capability, as well as a dummy load shedding unit. The aggregate thermal unit cost and the deficit unit cost were arbitrarily set as 1 and 10, respectively. The discount rate used was 10% per year, measured in inflation-free currency.

The monthly energy load is 21096 GWh. In contrast with other countries, the load in the Brazilian system does not present significant seasonal variations. For this reason monthly load was considered constant along the study period. Because the Brazilian system is hydro-dominated and characterized by large reservoirs, the supply of the energy demand, that is, the average monthly load, is the most important component in the hydrothermal optimization problem. Therefore modeling of peak supply was not discussed. However, the subproblems can be extended to handle peak supply, through the use of different load levels in the period [*Trinkenreich and Pereira*, 1977; *Pereira and Pinto*, 1983].

The system operation was calculated for a 5-month period with two inflow "branches" per period. The inflow scenarios were produced by an adapted multivariate stochastic streamflow model [*Pereira et al.*, 1984, model M_2].

TABLE 2. Plant Data for Part of the Brazilian South-Southeast System

	Name	Maximum Storage, 10^6 m^3	Minimum Storage, 10^6 m^3	Maximum Outflow, m^3/s	Nominal Head, m	Nominal Power, MWh
1	Camargos	792.0	120.0	252.9	22.5	50
2	Itutinga	12.0	12.0	227.5	27.5	55
3	Furnas	22,950.0	5733.0	1709.6	87.9	1326
4	Peixoto	4080.0	1875.0	1273.2	43.0	483
5	Estreito	1340.0	1340.0	1978.8	63.9	1116
6	Jaguara	450.0	450.0	1068.7	46.3	436
7	Volta Grande	2150.0	2150.0	1565.1	27.5	380
8	P. Colombia	1450.0	1450.0	1578.5	23.8	331
9	Graminha	555.0	51.0	100.8	93.0	83
10	E. Cunha	14.0	14.0	137.4	90.0	109
11	Limoeiro	25.0	25.0	145.9	25.7	33
12	Marimbondo	6150.0	890.0	2839.6	60.7	1521
13	A. Vermelha	11,025.0	5856.0	2988.0	53.5	1411
14	Emborcação	17,636.0	4621.0	920.1	125.9	1022
15	Itumbiara	17,027.0	4573.0	3260.0	80.1	2305
16	C. Dourada	460.0	460.0	1753.5	29.0	449
17	S. Simão	12,580.0	7000.0	2698.7	71.3	1698
18	B. Bonita	3135.0	569.0	878.0	18.9	146
19	Bariri	544.0	544.0	765.4	22.3	150
20	Ibitinga	985.0	985.0	795.7	19.0	133
21	Promissão	7296.0	5168.0	1353.1	22.6	270
22	N. Avanhadav	2700.0	2700.0	1219.3	28.5	306
23	Ilha Solteira	21,166.0	8300.0	8301.9	44.7	3276
24	Jupia	3680.0	3680.0	7055.2	22.7	1413
25	Jurumirim	6636.0	3598.0	351.9	3.9	100
26	Xavantes	8795.0	5754.0	658.1	7.6	416
27	L. N. Garcez	48.0	48.0	474.1	18.0	75
28	Capivara	10,541.0	4817.0	1647.5	45.0	654
29	Itaipu	29,000.0	29,000.0	11,831.3	118.0	12,326
30	Foz do Areia	6070.0	2020.0	2251.8	130.5	2594
31	Santo Santiago	6750.0	2670.0	2248.0	101.9	2022
32	Salto Osório	1270.0	825.0	1462.5	70.1	905
33	Passo Real	3546.0	289.0	420.1	39.5	146
34	Jacui	29.0	29.0	234.6	93.1	193
35	Itauba	620.0	620.0	657.8	90.1	523
36	Capivari-Cachoeira	179.0	23.0	40.1	737.0	261
37	Três Marias	19,528.0	4250.0	976.0	47.0	405

The system constraints at each stage are the same as the previous case study (water balance equation, load balance equation, bounds on volume, and outflow). The total number of constraints is 3584, and the number of decision variables (not counting slack variables) is 3616. In the decomposition scheme, each one-stage subproblem has 112 constraints.

Convergence was obtained in 14 iterations (2% tolerance). The evolution of upper and lower bounds is shown in Figure 13. A standard LP package [*Land and Powell*, 1973] was used to solve the subproblems. Average CPU time per subproblem was 2 s (IBM 4341). No effort was made to use the optimal solution of a previous iteration as a starting point; that is, all subproblems were solved "from scratch." Core usage for each subproblem was 250 kbytes (note that only one subproblem is in core at each time).

Two additional runs were made with different inflow sequences (different "seeds" in the streamflow generator). Both converged in 12 iterations (2% tolerance). One of the optimal solution values was almost identical to the original run, while the other was about 25% higher. However, end volumes and the thermal decision for the first stage were similar in all three cases.

The sensitivity of the computational effort to the number of scenarios was tested with a four-stage problem with three branches per stage. This results in a problem with 4480 constraints. Convergence was reached with 15 iterations, with the same average CPU time per subproblem.

As was mentioned previously, all the derivations assumed that the generation characteristics of the plants were constant. The nonlinear effect of head variation can be taken into account through a successive linearization scheme that has been successfully applied in existing operation models [*Trinkenreich and Pereira*, 1977; *Pereira and Pinto*, 1983], and Benders decomposition can be extended to the nonlinear case [*Geoffrion*, 1972]. This subject is presently being investigated.

CONCLUSIONS

The analytical solution of the optimal hydrothermal scheduling problem is well beyond present computational capabilities. Therefore it becomes necessary to develop methods able to approximate this solution with a reasonable computer cost.

The methodology described in this paper is able to determine generation targets for each plant of a interconnected reservoir system so as to minimize the expected operation cost, composed of fuel costs for thermal generation plus penalties for load curtailment. The algorithm is based on stochastic programming and Benders decomposition. The results of the algorithm can be used in the allocation of weekly or monthly generation targets for each plant on a hydrothermal system. These targets are then further refined into daily and finally hourly targets for real-time operation [*Pereira and Pinto*, 1982, 1983, 1984].

In computational terms, the proposed algorithm presents the following advantages.

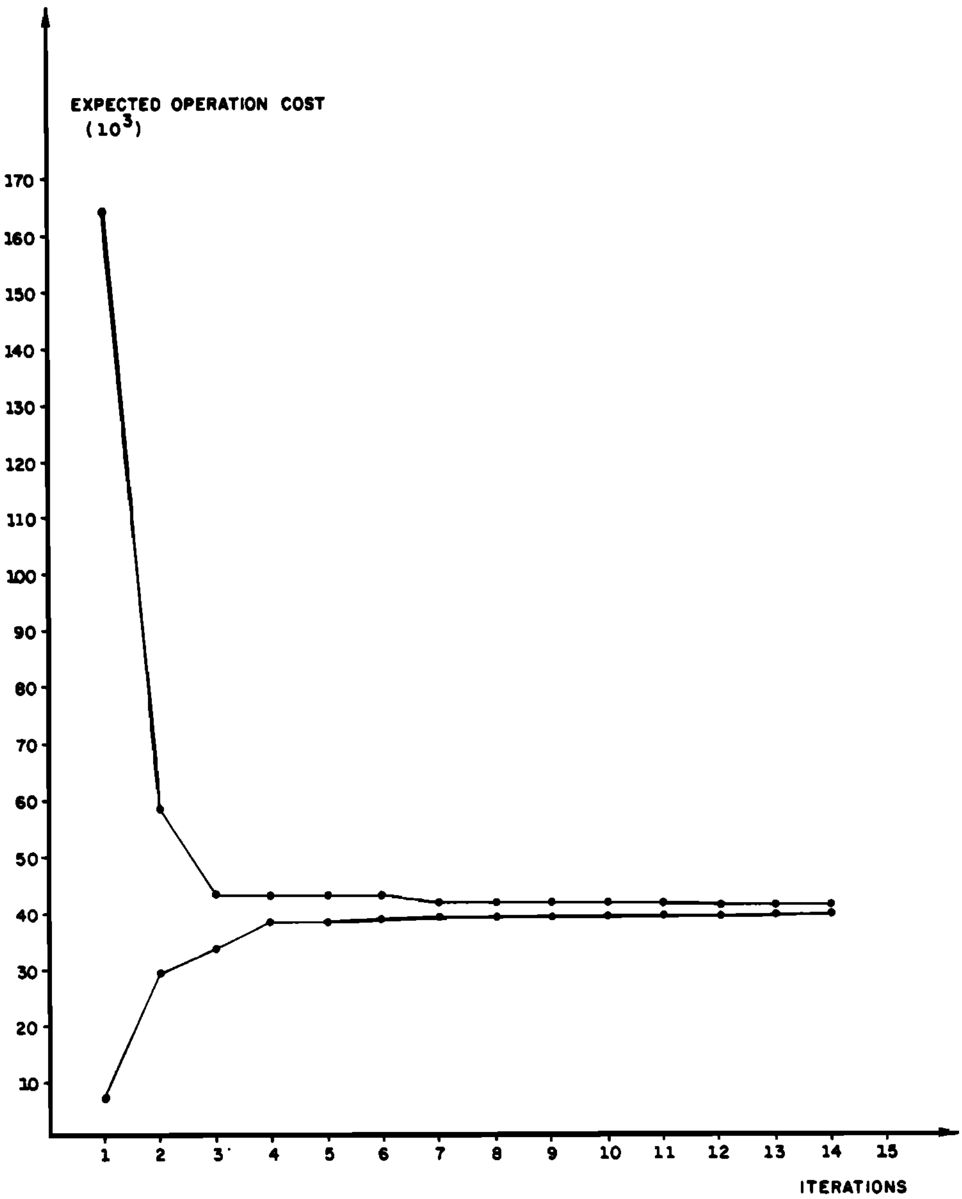

Fig. 13. Upper and lower bounds in the south-southeast system case study.

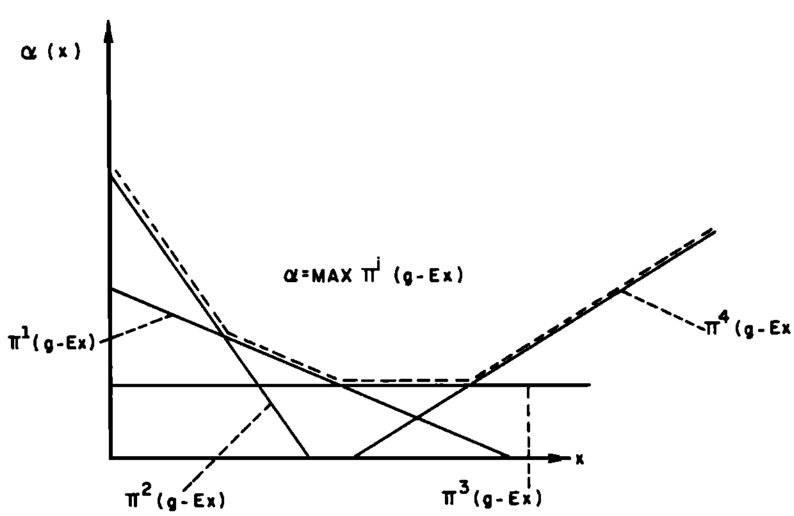

Fig. 14. Geometrical interpretation of the function $\alpha(x)$ in the Benders decomposition scheme.

1. The algorithm supplies at each iteration an upper and lower bound to the optimal solution, allowing an efficient trade-off between computational effort and accuracy.

2. The algorithm can accommodate any kind of streamflow model, because the transition probabilities at the "bifurcations" can be specified at will. This is specially interesting for the Brazilian systems, for which the usual lag-one autoregressive models do not give good results [*Pereira et al.*, 1984].

3. The algorithm decomposes the optimization problem into successive one-stage problems, which correspond to already existing operating models [*Trinkenreich and Pereira*, 1977; *Pereira and Pinto*, 1983].

4. Because the one-stage subproblems at each level can be solved independently, it is possible to apply parallel processing techniques, thus benefitting from the present trend in hardware improvement. Also, each onestage subproblem is small enough to be handled by modern microprocessor core memories.

A case study with four reservoirs was used to illustrate in detail the application of the algorithm. Its performance was further assessed with an example with 37 reservoirs.

Appendix: Derivation of Benders Decomposition

As is discussed in the text, Benders decomposition is a technique for approximating the function

$$\begin{aligned} \alpha(x) &= \min\ dy \\ \text{s/to}\ \ Fy &\geq g - Ex \end{aligned} \tag{A1}$$

in the two-stage optimization problem (11).

Let the dual of problem (A1) be written as

$$\begin{aligned} &\max\ \pi(g - Ex) \\ &\text{s/to}\ \ \pi F \leq d \end{aligned} \tag{A2}$$

where π is a row vector of dual variables (Simplex multipliers, shadow prices). The feasible region $\pi F \leq d$ does not depend on the first-stage decision x. This region is a convex polyhedron and can be characterized by the set of its vertices or corner points.

Let Π be the set of p vertices that characterize the region $\pi F \leq d$; that is, $\Pi = \{\pi^1, \pi^2, \cdots, \pi^p\}$. Because the optimal solution of a linear programming problem is always in a vertex of the feasible region, (A2) can, in principle, be solved by enumeration

$$\begin{aligned} &\max\ \pi^i(g - Ex) \\ &\pi^i \in \Pi \end{aligned} \tag{A3}$$

Problem (A3) can, in turn, be rewritten as

$$\begin{aligned} &\min\ \alpha \\ &\text{s/to}\ \ \alpha \geq \pi^1(g - Ex) \\ &\qquad\ \ \alpha \geq \pi^2(g - Ex) \\ &\qquad\qquad \vdots \\ &\qquad\ \ \alpha \geq \pi^p(g - Ex) \end{aligned} \tag{A4}$$

where α is a scalar variable. Because α is greater than or equal to each $\pi^i(g - Ex)$, it will be greater than or equal to their maximum. Because the objective function is to minimize α, the constraint will be met in the equality.

It is known from linear programming theory that the primal and dual problems have the same value at the optimum, that is, min [primal] = max [dual]. Because (A2) is the dual of (A1), one can conclude that the constraints in (A4) define the function $\alpha(x)$ in the two-stage optimization problem. It also becomes easy to see that $\alpha(x)$ corresponds to the value of a convex function defined by linear constraints, as is illustrated in Figure 14. The two-stage optimization problem can then be written as (see problem (14) in the text)

$$\begin{aligned} &\min\ cx + \alpha \\ &\text{s/to}\ \ Ax \geq b \\ &\qquad \pi^1(g - Ex) - \alpha \leq 0 \\ &\qquad \pi^2(g - Ex) - \alpha \leq 0 \\ &\qquad\qquad \vdots \\ &\qquad \pi^p(g - Ex) - \alpha \leq 0 \end{aligned} \tag{A5}$$

Problem (A5) is written in terms of the variable x plus the scalar variable α. Although the set of constraints $\{\pi^i(g - Ex) \leq \alpha\}$ may be very large, only a few of them will be active (met in the equality) in the optimal solution. This suggests the use of relaxation techniques to solve the problem, in which constraints of the type $\pi^i(g - Ex) - \alpha \leq 0$ are successively added to the problem as they are needed in the solution process.

Outline of the Algorithm

For notational convenience, the set of constraints $\pi^i(g - Ex) - \alpha \leq 0$ of (A5) will be represented by ξ^*. This problem is then written as

$$\begin{aligned} &\min\ cx + \alpha \\ &\text{s/to}\ \ Ax \geq b \end{aligned} \tag{A6}$$

and α is a feasible solution of ξ^* where

$$\xi^* = \{\pi^i(g - Ex) - \alpha \leq 0\} \qquad \forall\ \pi^i \in \Pi$$

The Benders decomposition algorithm is executed in the following steps.

1. Initialize $i = 1$. Relax all constraints in ξ^*; that is, set $\xi^1 = \phi$ (empty set). Initialize the upper bound to the optimal solution as $\bar{z} = +\infty$.

2. Solve the relaxed optimization problem

$$\begin{aligned} &\min\ cx + \alpha \\ &\text{s/to}\ \ Ax \geq b \end{aligned} \tag{A7}$$

α is a feasible solution of ξ^i.

Note that for the first iteration, the value of α will not affect the problem. Problem (A7) will then correspond to the first-stage problem.

3. Let (x^i, α^i) be the optimal solution of (A7). It can be seen that

$$\underline{z} = cx^i + \alpha^i \tag{A8}$$

is a lower bound to the optimal solution, since (A7) is a relaxed version of (A5).

4. Solve the second-stage problem, given the first-stage decision x^i

$$\begin{aligned} w^i &= \min\ dy \\ \text{s/to}\ \ Fy &\geq g - Ex^i \end{aligned} \tag{A9}$$

5. Let y^i be the optimal solution of (A9). The pair $\{x^i, y_i\}$ is a feasible solution of the global problem, but not necessarily the optimal one. An upper bound can therefore be calculated as

$$\bar{z} = \min\ \{\bar{z}, cx^i + dy^i\} \tag{A10}$$

6. If $\bar{z} - z$ is smaller than a preestablished tolerance, the problem is solved: the optimal solution corresponds to the pair (x, y) associated with the upper bound $\bar{z}$. Otherwise, improve the lower bound estimate of the function $\alpha(x)$ by adding a new constraint to (A7); that is

$$\xi^{i+1} = \xi^i U\{\pi^i(g - Ex) - \alpha \leq 0\} \tag{A11}$$

for some $\pi^i \in \Pi$. This constraint is easily obtained if we observe that the Simplex multiplier vector associated with the constraints of (A9) is a basic feasible solution of the dual problem (A2), and therefore is a vertex π^i of the feasible region of (A2). The Simplex multiplier vector can be calculated as

$$\pi^i = d_B B^{-1} \tag{A12}$$

where d_B is the cost vector corresponding to the basic variables in the optimal solution and B^{-1} is the inverse of the optimal basis.

7. Increment $i = i + 1$ and go to step 2.

Steps 1–7 describe the Benders decomposition algorithm. More details can be found in the work by *Lasdon* [1970]. In practice, the Benders cut $\pi^i(g - Ex) - \alpha \leq 0$ described in step 7 is slightly rewritten as follows: let w^i be the optimal solution value of the second-stage problem (A9). Let π^i be the associated Simplex multiplier vector. From the equality of primal and dual optimal solutions we have

$$w^i = \pi^i(g - Ex^i) \tag{A13}$$

If we put $\pi^i g$ in evidence, we obtain

$$\pi^i g = w^i + \pi^i E x^i \tag{A14}$$

Substituting (A14) into expression (A11) we obtain an alternative expression for the Benders cut

$$w^i + \pi^i E(x^i - x) - \alpha \leq 0 \tag{A15}$$

This expression is useful for the stochastic extension of the Benders algorithm described next.

Stochastic Extension of the Benders Algorithm

The Benders algorithm is able to handle stochastic problems, that is, in which the second-stage problem depends on the outcome of random variables. As an illustration, suppose that vector g in (A1) can assume two values, g_1 and g_2, with associated probabilities p_1 and p_2 (naturally, $p_1 + p_2 = 1$). In this case, the optimization problem is to find the strategy that minimize the expected value of the operation cost; that is

$$\begin{aligned} \min\ & cx + p_1 d y_1 + p_2 d y_2 \\ \text{s/to}\ & Ax \geq b \\ & Ex + F y_1 \geq g_1 \\ & Ex + F y_2 \geq g_2 \end{aligned} \tag{A16}$$

Problem (A16) corresponds to the following decision process: in the first stage, determine a feasible decision x^* such that $Ax^* \geq b$; in the second stage, look for the solutions $y_1{}^*$ and $y_2{}^*$ that optimize

$$\begin{aligned} \min\ & p_1 d y_1 + p_2 d y_2 \\ \text{s/to}\ & F y_1 \geq g_1 - Ex^* \\ & F y_2 \geq g_2 - Ex^* \end{aligned} \tag{A17}$$

Note that problem (A17) can be decomposed into two independent optimizations:

$$\begin{aligned} w_1 = \min\ & d y_1 \\ \text{s/to}\ & F y_1 \geq g_1 - Ex^* \end{aligned} \tag{A18$'$}$$

and

$$\begin{aligned} w_2 = \min\ & d y_2 \\ \text{s/to}\ & F y_2 \geq g_2 - Ex^* \end{aligned} \tag{A18$''$}$$

where the solutions of (A18′) and (A18″) are weighted by the probabilities p_1 and p_2.

As in the deterministic case, the solution of each scenario (A18′) and (A18″) in the second stage is a function of the first-stage decision x. The stochastic decision problem (A17) can thus be rewritten as

$$\begin{aligned} \min\ & cx + \bar{\alpha}(x) \\ \text{s/to}\ & Ax \geq b \end{aligned} \tag{A19}$$

where $\bar{\alpha}(x)$ is the expected value of the solutions of the (A18′) and (A18″) for each specified x.

Again, there is a great similarity between (A19) and the recursion equation of dynamic programming (stochastic, in this case). Therefore it becomes interesting to see that a Benders cut can be constructed from the expected value of the Simplex multipliers associated to each scenario. By straightforward algebraic manipulation it can be shown that the Benders cut at the iteration i is

$$\bar{w}^i + \bar{\pi}^i E(x^i - x) - \alpha \leq 0 \tag{A20}$$

where $\bar{w}^i = p_1 w_1{}^i + p_2 w_2{}^i$ is the expected value of the solutions of the second stage problems, and $\bar{\pi}^i = p_1 \pi_1{}^i + p_2 \pi_2{}^i$ is the expected value of the associated multiplier vectors. This expression is used in the case studies discussed in the text. A similar approach has independently been derived by *Birge* [1980].

Acknowledgments. This work received technical and financial support from the Operations Department of Eletrobras. Special thanks go to G. C. Oliveira of CEPEL for his most useful comments on the composition of this paper and for developing the version of the stochastic streamflow generator used in the case studies. The contributions of the anonymous referees is also gratefully acknowledged and has substantially improved the quality of the paper. The first author is currently on loan to the Electric Power Research Institute (EPRI), Palo Alto, California, and thanks R. Iverson of EPRI for his support of our technical activities.

References

Araripe Neto, T. A., M. V. F. Pereira, and J. Kelman, A risk-constrained stochastic dynamic programming approach to the operation planning of hydrothermal systems, paper presented at the IEEE Summer Power Meeting, Inst. of Elec. and Electr. Eng., Seattle, July 1984.

Arvanitidis, N. V., and J. Rosing, Composite representation of a multireservoir hydroelectric power system, *IEEE Trans. Power Appar. Syst., PAS-89*(2), 327–335, 1970.

Benders, J. F., Partitioning procedures for solving mixed variables programming problems, *Numer. Math., 4*, 238–252, 1962.

Birge, J. R., Solution methods for stochastic dynamic linear programs, *Rep. 80/29*, Syst. Optimiz. Lab., Dept. of Oper. Res., Stanford Univ., December 1980.

Boshier, J. F., and E. G. Read, Stochastic single reservoir models for long term scheduling of hydrothermal power systems, Planning Division internal report, Plann. Div., New Zealand Ministry of Energy, Wellington, 1982.

Bryson, A. E., Jr., and Y. C. Ho, *Applied Optimal Control*, John Wiley, New York, 1975.

CEPEL/Eletrobras, The equivalent system model—General description (in Portuguese), *CEPEL Techn. Rep. 144/83*, CEPEL/Electrobras, Rio de Janeiro, Brazil.

Dagli, C. H., and J. F. Miles, Determining operating policies for a water resources systems, *J. Hydrol.*, *47*, 297–306, 1980.

Davis, R. E., Stochastic dynamic programming for multi-reservoir hydro optimization, *Tech. Memo. 15*, Syst. Contr. Inc., Palo Alto, Calif., April 1972.

Egeland, O., J. Hegge, E. Kylling, and J. Nes, The extended power pool model—operation planning of a multi-river and multi-reservoir hydro-dominated power production system—A hierarchical approach, paper presented at 1982 CIGRÉ Meeting, Int. Conf. on Large High Voltage Syst., Paris, September 1982.

Geoffrion, A. M., Generalized Benders decomposition, *J. Optimiz. Theory Appl.*, *20*, 237–260, 1972.

Gjelsvik, A., Stochastic long-term optimization in hydroelectric power systems, *Tech. Rep. TR 2669, EFI* (Norwegian Research Institute of Electricity Supply), Tronheim, February 1981.

Gjelsvik, A., Stochastic seasonal planning in multireservoir hydroelectric power systems by differential dynamic programming, *Model. Identif. Contr.*, *3*(3), 313–349, 1982.

Grygier, J. C., and J. R. Stedinger, Algorithms for optimizing hydropower system operation, paper presented at the 6th Canadian Hydrothermal Conference, Can. Soc. for Civ. Eng., Ottawa, Can., June 2–3, 1983.

Hanscom, M. A., L. Lafond, L. Lasdon, and G. Provonost, Modelling and resolution of the midterm generation planning problem for a large hydroelectric system, *Manage. Sci.*, *28*, 659–668, 1980.

Ikura, Y., and G. Gross, Efficient large-scale hydro system scheduling with forced spill conditions, paper presented at the Winter Meeting of the Inst. of Elec. and Electr. Eng. Power Engineering Society, Dallas, Tex., February 1984.

Land, A., and S. Powell, *Fortran Codes for Mathematical Programs*, John Wiley, New York, 1973.

Lasdon, L., *Optimization Theory for Large Systems*, MacMillan, New York, 1970.

Murray, D. M., and S. J. Yakowitz, Constrained differential dynamic programming and its application to multireservoir control, *Water Resour. Res.*, *15*(5), 1017–1027, 1979.

Pereira, M. V. F., and J. Kelman, Probabilistic criteria for reservoir simulation (in Portuguese), *CEPEL Tech. Rep. 070/83*, CEPEL, Rio de Janeiro, Brazil, 1983.

Pereira, M. V. F., and L. M. V. G. Pinto, A decomposition approach to the economic dispatch of hydrothermal systems, *IEEE Trans. Power Appar. Syst.*, *PAS-101*(10), 3851–3860, 1982.

Pereira, M. V. F., and L. M. V. G. Pinto, Application of decomposition techniques to the mid- and short-term scheduling of hydrothermal systems, *IEEE Trans. Power Appar. Syst.*, *PAS-101*(11), 1983.

Pereira, M. V. F., and L. M. V. G. Pinto, Operating planning of large-scale hydroelectric systems, in *Eighth Power System Computation Conference*, *Buttersworth*, London, 1984.

Pereira, M. V. F., G. C. Oliveira, C. G. C. Costa, and J. Kelman, Stochastic streamflow modeling for hydroelectric systems, *Water Resour. Res.*, *20*(3), 379–390, 1984.

Provonost, R., and J. Boulva, Long-range operation planning of a hydrothermal system—modelling and optimization, paper presented at *Conference of the Electrical Canadian Association*, Canadian Electrical Association, Toronto, 1978.

Read, E. G., Optimal operation of power systems, Ph. D. dissertation, Univ. Canterbury, Christchurch, New Zealand, 1979.

Rosenthal, R. D., The status of optimization models for the operation of multireservoir systems with stochastic inflows and nonseparable benefits, *Rep. 75*, Tenn. Water Res. Cent., Knoxville, May, 1980.

Rosenthal, R. E., Non-linear network flow algorithm for maximization of benefits in a hydroelectric power systems, *Oper. Res.*, *29*, 763–786, 1981.

Trinkereich, J., and M. V. F. Pereira, Subsystems interchange model, (in Portuguese), paper presented at *IV National Seminar on Production and Transmission of Electrical Energy*, Eletrobras, Rio de Janeiro, 1977.

Turgeon, A., Optimal operation of multireservoir systems with stochastic inflows, *Water Resour. Res.*, *16*(2), 275–283, 1980.

University of Tennessee, Scheduling reservoir releases for optimal power generation, Sci. Prog., Coll. for Bus. Admin., Univ. of Tenn., Knoxville, June 1982.

Yakowitz, S., Dynamic programming application in water resources, *Water Resour. Res.*, *18*(4), 673–696, 1982.

Yeh, W. W-G., State of the art review—Theories and applications of system analysis techniques to the optimal management and operation of reservoir systems, *Rep. UCLA-ENG-82-52*, Water Resour. Progr., School of Eng. and Appl. Sci., Univ. Calif., Los Angeles, Calif., June 1982.

M. V. F. Pereira and L. M. V. G. Pinto, CEPEL, Centro de Pesquisas de Energia Elétrica, Cidade Universitaria, Ilha do Fundão, Caixa Postal 2754, 20.000 Rio de Janeiro, RJ Brasil.

(Received August 15, 1983;
revised February 1, 1985;
accepted February 12, 1985.)