E. G. Boring

I. - A color solid in four dimensions

In: L'année psychologique. 1949 vol. 50. pp. 293-304.

PSYCHOLOGIE EXPÉRIMENTALE

I

A COLOR SOLID IN FOUR DIMENSIONS

by Edwin G. Boring
Harvard University.

The system of color qualities has three degrees of freedom, that is to say, all possible colors can be related by ordering them in a tridimensional solid figure [1]. The double pyramid (Ebbinghaus), the double cone (Troland) and the sphere (Wundt) are the familiar forms of the color solid [2].

Such a figure is most useful if it can represent an analysis of color into some set of descriptively adequate parameters. The usual color solid analyzes the colors with respect to the three conventional attributes : hue, brightness and saturation. This figure is a system of cylindrical polar coordinates, designed so that hue, a closed circular attribute, varies circumferentially about the axis, saturation varies radially out from the axis, and brightness varies along the axis or parallel to it, orthogonal to saturation. Such a tridimensional system is based on an *attributive analysis* of color and places every color in relation to the others with respect to hue, brightness and saturation. There are certain difficulties about this system, difficulties that arise when we wish to take account of the unique or principal hues, or when we wish to show the similarity of gray to other unique hues. To these problems we shall return presently.

1. The conception of the color solid which this paper develops has been clarified by correspondence with Dr. F. L. Dimmick and by discussion with my colleague, Walter A. Rosenblith. I am grateful to both of them.
2. On the history of the use of color diagrams, see E. G. Boring. *Sensation and Perception in the History of Experimental Psychology*, 1942, 145-149, 154.

A somewhat different conception of the relations of the colors is given by what we may call *component analysis*, a view which has recently been promoted by F. L. Dimmick and his associates [1]. In this system there are assumed to be seven fundamental components which correspond respectively to the seven unique colors : red, green, yellow, blue, white, black and gray. Dimmick writes the fundamental color equation

Color = (red, green) + (yellow, blue) +
(white, black) + gray . . . (1).

Here the complementaries are paired as mutually exclusive. The first term may be red or green or zero, but not both red and green, for there is no reddish green. The second term may be yellow or blue or zero. Dimmick follows Hering, G. E. Müller and Titchener in assuming that white and black are mutually exclusive complementaries and that the third term may thus also be zero. Hering solved this problem obliquely by assuming that, if all three antagonistic paired processes are in equilibrium, a light sensation will still occur with a brightness depending on the totality of the weights of the active processes [2]. That theory was a *tour de force.* It was Müller who suggested that gray must be a constant addition to the other visual processes, becoming the perceived residual when each pair of the three color processes is in equilibrium [3]. Because, like Hering, he was thinking in terms of physiological processes, he suggested that this constant gray might be contributed by the constant molecular activity of the visual cortex and he called it *cortical gray.* Titchener got away from these unfortunate physiological implications, stressing the belief that the gray is *constant*, an adjective which brings the discussion back to the analytical description of color experience, which is where it belongs [4].

It is obvious that there must be some restriction upon equa-

1. F. L. Dimmick. A reinterpretation of the color-pyramid. *Psychol. Rev.*, 1929, *36*, 83-90. Dimmick and C. H. Holt. Gray and the color pyramid. *Amer. J. Psychol.*, 1929, *41*, 284-290. Dimmick on Color in E. G. Boring, H. S. Langfeld and H. P. Weld. *Foundations of Psychology*, 1948, 269-274. See also the references to Dimmick, *infra*.

2. E. Hering. *Zur Lehre vom Lichtsinne*, 1878, 70-141, esp. 107-121. For a brief statement, see Boring, *op. cit.*, 206-209, 218.

3. G. E. Müller. Zur Psychophysik der Gesichtsempfindungen, *Z. Psychol.*, 1896, *10*, 1-82, 321-413; 1897, *14*, 1-76, 161-196; esp. *10*, 1-4, 30-32, 411 f.; *14*, 40-46. See also Boring, *op. cit.*, 212-214, 219.

4. E. B. Titchener, *A Text-book of Psychology*, 1910, 90 f.; *A Beginner's Psychology*, 1915, 59 f.

tion (1). If all its four terms could vary independently, we should require a four-dimensional figure for the color diagram, a consequence which is contrary to fact. We know that the colors can all find place in a three-dimensional solid, though we are less sure as to whether this solid can be extended indefinitely in size. Müller and Titchener kept the figure within three dimensions by assuming that gray is constant. If the fourth term of the equation is a constant, then there are only three parameters to the system, and the attributive analysis into hue, brightness and saturation works. The relations of yellow and blue to gray and of white and black to gray, are shown in Fig. 1, where the amount of gray is shown as constant and the other factors vary from zero at pure gray in the center up to whatever indeterminate limit may be set by physiological conditions. Similar relations hold for red and green, and also for any duplex or triplex pair of complementaries, like light orange and dark blue-green.

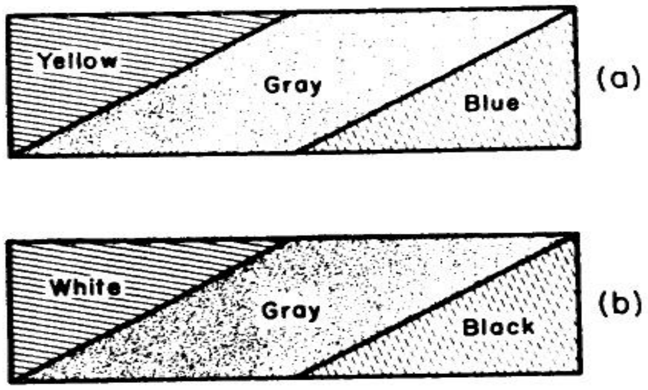

Fig. 1. — Color continua with constant gray.

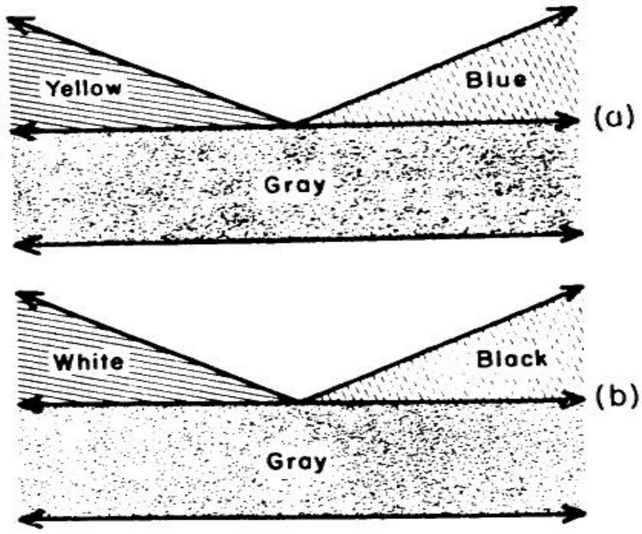

Fig. 2. — Color continua when gray varies inversely with other components.

Dimmick holds, on the other hand, that gray is not constant but varies inversely with the other components. This belief is equivalent to rewriting the color equation as

$$\text{(Red, green)} + \text{(yellow, blue)} + \text{(white, black)} + \text{gray} = 1 \quad \ldots \quad (2).$$

This system has four variables but only three degrees of freedom. Each variable can assume values only between 0 and 1, and the sum is always 1, so that each term shows the proportion that a particular component is of the whole. Complementary pairs, like yellow and blue or white and black, vary inversely with gray as shown in Fig. 2, which should be compared with Fig. 1. It is plain that Fig. 2 is limited at its extremes, where gray becomes zero and the other component 100 per cent.

This kind of component analysis becomes clearer if we examine a series of hues in the region of maximal saturation where gray is zero. A section of this closed continuum is shown in Fig. 3, with red at the center. It should be noted that red varies from 0 to 1 to 0, just as does gray in Fig. 2, and that the same kind of limitation applies to green, yellow and blue.

Fig. 3. — Color continua with each component varying from 0 to 1 and the sum of the components equal to 1.

Among the various requirements of this component theory of color are three which demand special mention here.

(1) The hues (yellow, blue, red, green and their intermediates) must show thresholds at gray, where the hue emerges from gray. These chromatic thresholds are well known and meet the requirements of both Fig. 1 *a* and Fig. 2 *a*. Thus they do not constitute evidence as to which kind of analysis is correct.

(2) Black and white must act like the hues and show thresholds at gray, as indicated in Figs. 1 *b*. and 2 *b*. There must be no blackish whites but a series of grayish whites and another series of grayish blacks, separated by pure gray. Casual introspection supports this view, and Dimmick and his associates have supplied definite empirical evidence for it. They have studied the white-gray-black series and have determined thresholds for both white and black at pure gray [1]. This finding supports the propriety of separating the third and fourth terms of equation (1), which treats white and black as mutually exclusive complementaries and separates gray from them. It does not bear on the

1. Dimmick. A note on the series of blacks, grays and whites. *Amer. J. Psychol.*, 1920, *31*, 301 f.; The series of blacks, grays and whites, *Psychol. Rev.*, 1925, *32*, 334-336. Dimmick and G. McMichael. The psychophysical determination of the limits of pure gray. *Amer. J. Psychol.*, 1933, *45*, 313 f. Dimmick, Black and white, *ibid.*, 1941, *54*, 286-289.

correctness of equation (2), which shows that gray decreases when the other components increase.

(3) The empirical test which needs to be made — and here lies a problem for research — is the status of gray in the region of the well saturated hues. Take the series from gray through the reddish grays and the grayish reds to red. There is a chromatic threshold near gray, where reddishness emerges. Can an opposite threshold be determined at the other end where grayishness emerges from the good red? If there can be, then Fig. 2 is a better diagram than Fig. 1, and equation (2) is a proper modification of equation (1).

These considerations raise certain questions about the most useful form for the color solid, as to how it can be made best to represent the facts of color.

The Color Solid.

The component theory of color states (1) that there are seven unique hues (red, green, yellow, blue, white, black and gray); (2) that these seven hues include three pairs of mutually exclusive complementaries (red-green, yellow-blue, white-black) but that gray has no complementary; and (3) that every color is a combination of no more than four components.

It is convenient to employ some special terms. A *unique* color is simplex or pure, having only a single component. A color resolvable into two components, with the other two zero, is *duplex* (grayish red, or the best saturated orange of middle brightness). When only one component is missing, the color is *triplex;* but most colors are *quadruplex,* having four components and lying inside the color solid and not at any of its critical boundaries.

There are certain difficulties with the conventional color solid plotted in cylindrical polar coordinates. The first arise because the true system of colors is symmetrical except that the complementary for gray is missing, whereas the conventional coordinates are not symmetrical. If the dichotomy between chromatic and achromatic colors is abandoned, there is no reason why the angular parameter should sweep through red-yellow-green-blue-red any more than through red-white-green-black-red or yellow-white-blue-black-yellow. The diagram should be symmetrical if the true relations among the colors are.

The second difficulty appears because the limits of any parameter ought to be found at the boundaries of the figure and not in the middle of a geometrically linear continuum. The color circle (red-yellow-green-blue-red) is due to Newton, but it takes no account of the four unique colors in it. Ebbinghaus, to meet this difficulty, changed the circle to a square, with its corners representing the unique hues. In the series red-yellow-green, yellow represents a maximum of yellowishness. As you pass through the oranges toward yellow, the colors become yellower and yellower; but as you go on, yellow diminishes. A yellow-green is not yellower than a yellow though it is farther along in the series; nor is an orange yellower than a yellow-green. The figure should therefore exhibit yellow as a limit, and the red-yellow continuum should not appear as an extension of the yellow-green line. Ebbinghaus' color square and color pyramid accomplish this result for all the unique colors except gray [1].

A proper figure should, however, also have gray in an extreme position, if gray is a unique color. The series yellow-gray-blue should break so as to place gray at a corner, just as the series yellow-green-blue is bent to have green at a corner. Can we build a figure that will have these properties?

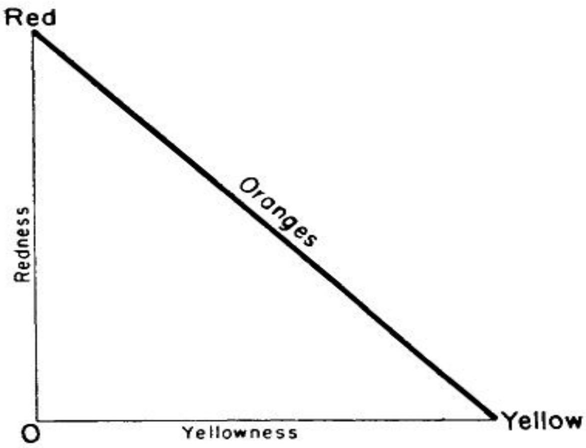

Fig. 4. — Duplex colors of the Red-Yellow linear continuum on orthogonal coordinates.

We can get the required symmetry if we use orthogonal coordinates, but we shall need four of them for the four terms of the color equation. That change forces the figure into four dimensions, although the system still has only three degrees of freedom. It is a set of solid figures organized with respect to one another in a four-dimensional orthogonal space. Actually what we come out with is half of a hollow hypersolid, with the origin in the center and the colors arranged in eight bounding tetrahedra. If gray had a complementary, there would be sixteen bounding tetrahedra and the hollow figure would be closed. The analogy to such a figure in visualized tridimensional space is the diagram of Fig. 7, where

1. H. Ebbinghaus. *Grundzüge der Psychologie*, 1897 and later eds., I, bk. 3, sect. 14.

eight bidimensional triangular surfaces appear as the boundaries of a hollow tridimensional octahedron.

Now let us see how this figure is formed.

The duplex linear series of oranges is shown in Fig. 4. The colors lie only in the line. The origin for the two components is external to the locus of the duplex colors. The equation of the line is :

Red + yellow = 1.

We can put four of these series together into a hollow color square, as Ebbinghaus did. Fig. 5 shows two sets of these series, the red-yellow-green-blue set and the yellow-white-blue-black set. The red-white-green-black set is not shown. There are twelve such duplex series (three sets of four), besides the six series which are formed on gray.

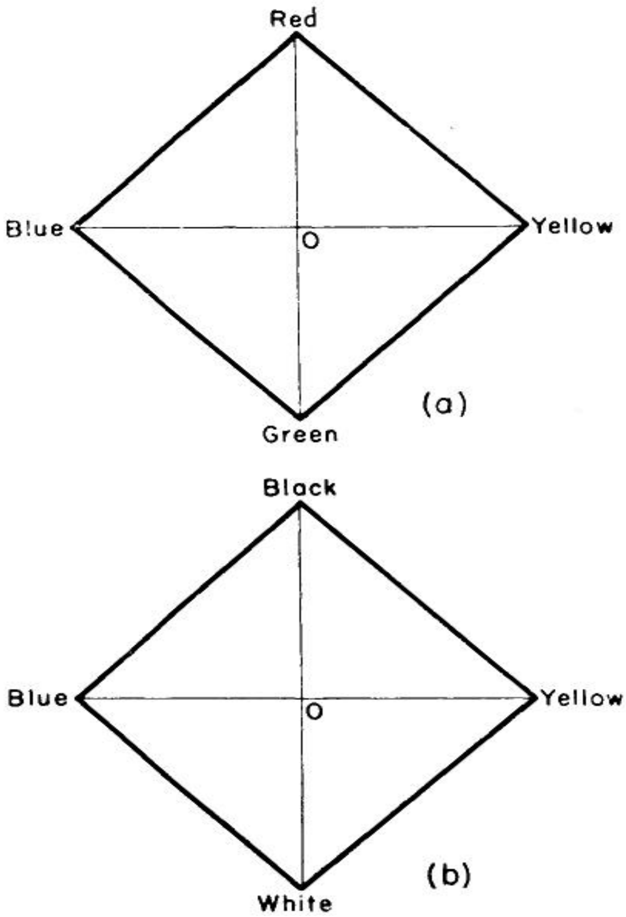

Fig. 5. — Duplex colors : two sets of four linear continua each on orthogonal coordinates. Cf. fig. 4.

A simple continuum for triplex colors is shown for red, yellow and white in Fig. 6. This is a bidimensional triangular continuum referred to an external origin. The triangle lies in a plane whose equation might be written :

Red + yellow + white = 1.

There are twenty such triangles of triplex colors possible for the various triads of the seven components.

If we omit the consideration of gray, we can put the other eight triplex triangles together in a single hollow figure of eight plane surfaces, as in Fig. 7. This would be the proper color figure for three pairs of complementaries with no fourth term in the color equation. This figure suggests already what will happen when we add the fourth dimension. We have here a hollow tridimensional figure, with its origin in its center, bounded by plane triangles which form its surface. When we add gray we shall have a hollow four-dimensional figure, with its origin in the center, bounded by solid tetrahedra which form its solid exterior.

Besides the eight triplex continua of Fig. 7 which exclude gray, there are twelve others which include gray but exclude

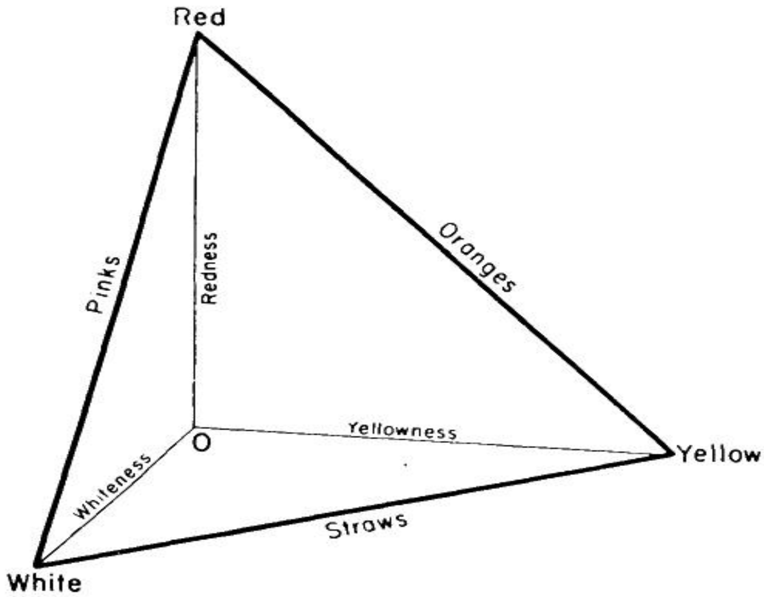

Fig. 6. — Triplex colors of the Red-Yellow-White triangular continuum on orthogonal coordinates.

one pair of complementaries, and these can be grouped in three sets of four. The triangles for the triplex colors based on yellow, blue, white, black and gray are shown in the hollow bottomless square pyramid of Fig. 8. This figure is especially interesting because it is the correct final figure for the dichromatic vision of the color-blind, for whom the red-green component is missing. The equation for dichromatic vision is :

(Yellow, blue) + (white, black) + gray = 1 . . . . (3)

Diagrams similar to Fig. 8 could be formed for red-green-black-white-gray and for red-green-yellow-blue-gray.

We come now to the quadruplex continua. There are eight solid tetrahedra for them. The one for red-yellow-white-gray is shown in Fig. 9. Unlike any of the preceding figures, this one is a solid and its origin, which lies in the fourth dimension, can not be shown. This figure is necessarily a regular solid tetrahedron, with equal equilateral triangles for its faces. It can not, of course, be drawn in a single diagram but it is possible for us to arrive at some conception of its form.

The hollow octahedron for the three pairs of complementaries is shown in Fig. 7. If on each of these eight triangular surfaces as a base were erected a regular tetrahedron with gray at its apex, we should have the desired eight tetrahedra, although they would not in three dimensions be referred to orthogonal axes. It is putting this figure together in four dimens-

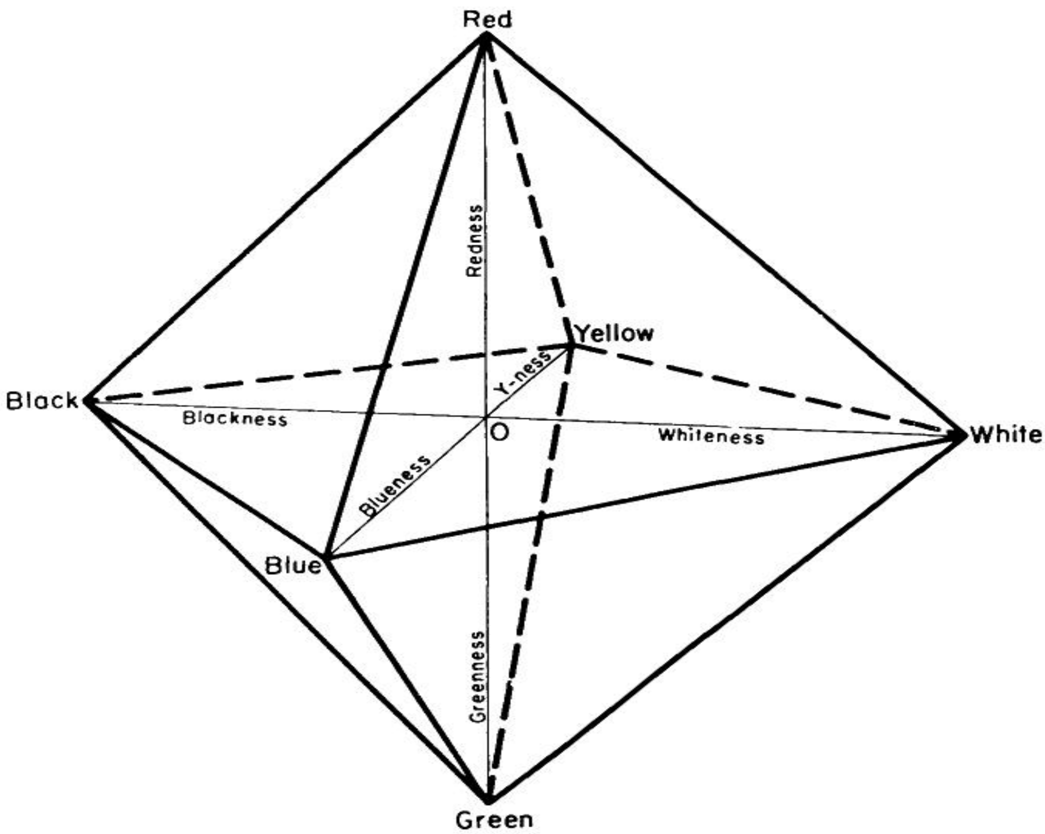

Fig. 7. — Triplex colors for three pairs of complementaries with Gray = 0 This is a hollow octahedron, with the origin of the three orthogonal axes in the center, and eight triangular surfaces. Cf. fig. 6.

ions which enables us to bring these eight gray apices, which bristle in eight directions for three dimensions, together into a single point for pure gray.

We must now warn ourselves against an easy error. It is tempting to think that the tetrahedra might be erected inside the octahedron and the eight gray apices then brought together at the center, thus reconstituting the Ebbinghaus double pyramid which we have just abandoned as unsatisfactory. Any such attempt would meet with failure. The eight Ebbinghaus tetrahedra are right tetrahedra, not regular ones. These regular tetrahedra formed as in Fig. 9 would jut inside the octahedron but not join, nor would the quadruplex colors included in their solid structure be referred to orthogonal axes. The similarity of the two figures is dangerously misleading.

On the other hand, Dr. F. L. Dimmick has just sent me a very ingenious model in which he shows the nature of such

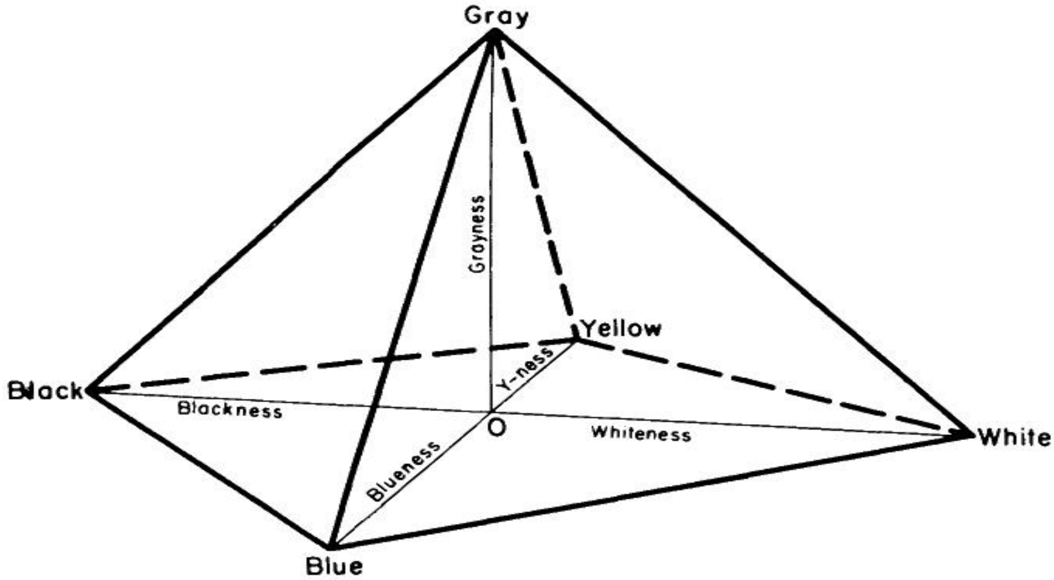

Fig. 8. — Triplex colors with two pairs of complementaries and gray and with Red-Green omitted.

This is a hollow, bottomless quadrilateral pyramid with the origin of the three orthogonal axes in the center. This figure is like one half of fig. 7, lacking the other half because gray has no complementary. This is the correct figure for the dichromatic vision of the color-blind.

a figure. He abandons the dimension for the axes of coordinates and their origin and erects the eight tetrahedra, like the one of Fig. 9, on the inside surface of the octahedron of Fig. 7, bringing the eight gray apices together in a central point. Geometry is such that the eight regular tetrahedra stick out from the central point at gray with spaces between them, forming a kind of sunburst. Identical triangular faces on different tetrahedra thus appear facing each other but with a space between them. You can imagine the figure by imagining that these interstices are nonexistent. If the tetrahedra are flexibly joined, then pairs of faces can be moved into juxtaposition, but such movement increases the other spaces. The figure does not, of course, accomplish the impossible by putting into three dimensions a solid shape that can exist only in four dimensions, but it does illustrate well the difficulty of building a tridimensional solid that would show the quantitative relations of the four-term color equation.

A way of visualizing a truly four-dimensional figure is to select one axis as a primary parameter and to show successive cuts of the figure at right angles to this axis. Let us take the

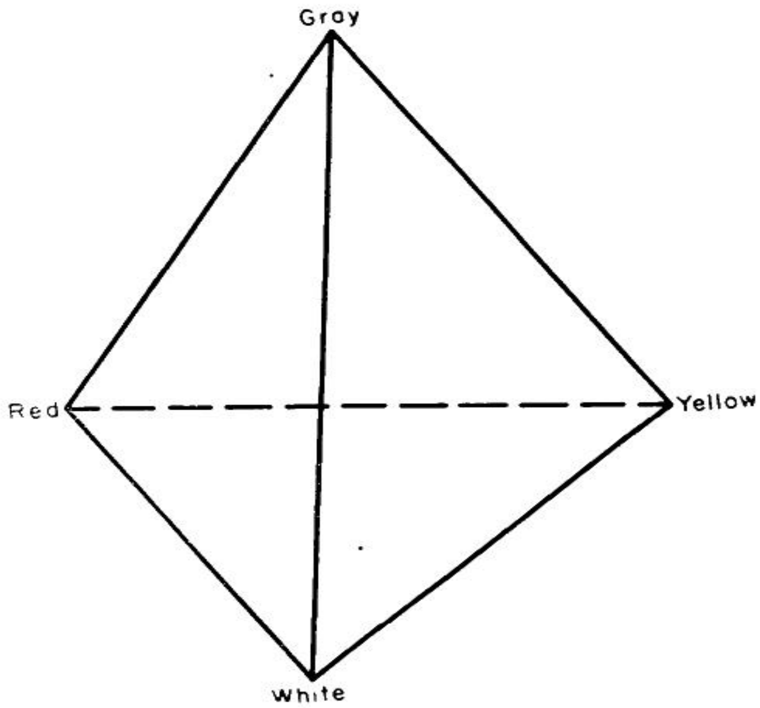

Fig. 9. Quadruplex colors of the Red-Yellow-White-Gray solid tetrahedral continuum.

The origin and the four orthogonal axes are not shown because they lie in a fourth dimension. There are eight of these tetrahedra, one corresponding to each triangular face of fig. 7.

gray axis. Then, if we cut across this axis for gray = 0, we have the octahedron of Fig. 7. As gray increases from zero, the cut is still octahedral, but the octahedron gets smaller and smaller as required by equation (2). Finally, when gray = 1, the octahedron vanishes, for we have reached the apex for pure gray. In a similar manner we could take cuts across the figure perpendicular to the red-green axis, and get successive figures like Fig. 8. As the cuts progressed from red to the origin to green, the figure would get larger and larger, become largest at the origin, and then shrink again.

Another way to visualize a four-dimensional figure is to consider the fourth axis as time and to see the figure based on the other three parameters as expanding or shrinking in time. The only difficulty here is in imagining time as orthogonal to the three Euclidean dimensions.

It is plain that this correct orthogonal figure lacks didactic or demonstrational value. It can not be constructed and shown to students. This discussion serves, nevertheless, as a warning against trying to quantify the Ebbinghaus pyramid. If colors are to be measured psychologically without regard to their stimuli — and surely that quantitative knowledge would be a good thing to have at hand — then the psychologist can use component analysis if he likes, but he had better discard the color solid generated by cylindrical polar coordinates, and stick to the basic formula

$$(\text{Red, green}) + (\text{yellow, blue}) + (\text{white, black}) + \text{gray} = 1.$$