Assignment 5

American Options

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COMPUTATIONAL FINANCE: PRICING AND VALUATION

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1 Introduction

An American option is an option contract that allows the holder to exercise the option rights at any time before or at the maturity date. In contrast, the European style option only allow execution on the day of maturity, thus American-style options offer a greater flexibility in cost of a more expensive premium price.

2 Numerical solution methods

2.1 Monte-Carlo

Monte-Carlo methods can be utilized to price American-style options, where paths S_t of the underlying model is simulated. We look at the Black-Scholes model to analyze the numerical method:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1}$$

For an European-style option we integrate until the date of maturity, $t = T$ since it is at maturity date the option is exercised. American-style options are a bit more complicated since we should always check if an early exercise is preferable.

We are looking for a stopping time τ for when and if the option should be exercised early. Since τ depends on S_t we expect τ to be stochastic. We can think of a price limit β such that if the price S_t reaches β we should exercise early. We define τ as:

$$\tau = min(t \leq T, (t, S_t) \in stopping\ \ area) \tag{2}$$

Therefor we define the stopping time for when a sample path S_t touches the early-exercise curve which is defined as the boundary of the stopping region. Since one does not know when the holder of an option will exercise, the price of the option reflects the worst case scenario. This leads to $V(S, 0)$ being given by the supremum over all possible stopping times:

$$V(S, 0) = sup\ E_Q(e^{-rt}\Psi(S_\tau), S_0 = S) \tag{3}$$

where $0 \leq \tau \leq T$, τ is the stopping time and Ψ is the pay-off function.

We analyze two different Monte-Carlo methods to price American options; parametric methods and regression methods.

2.1.1 Parametric methods

In the parametric method we calculate lower and upper bounds defined by:

$$V^{low}(S,0) \leq V(S,0) \leq V^{high}(S,0) \tag{4}$$

If we denote the stopping time τ for a specific level β as $\tau(\beta)$ a lower bound to 4 is given by:

$$V^{low(\beta)}(S,0) = E_Q(e^{-r\tau}\Psi(S_\tau), S_0 = S) \tag{5}$$

The bound is calculated by Monte-Carlo simulations over different paths, where the paths are stopped by the chosen stopping rule

2.1.2 Regression models

One can use regression models by approximating American options as Bermudan options where early exercise is restricted to specific discrete dates before the date of maturity. The time instances when one have the right to early exercise are created artificially by:

$$\Delta t = \frac{T}{M} \quad t_j = j\Delta t \;\; j = 0, 1, 2, ..., M \tag{6}$$

Similarly as for the binomial method we can set the values as:

$$V_j(S) = max(\Psi(S), V_j^{cont}(S)) \tag{7}$$

where the continuation value V_j^{cont} is defined as:

$$V_j^{cont}(S) = e^{-r\Delta t}E_Q(V_{j+1}(S_{j+1}), S_j = S) \tag{8}$$

2.2 Finite difference

Finite difference methods can also be implemented to price American-style options. To approximate the option value a linear complementary problem is introduced using a transformation of variables. The Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV \tag{9}$$

is transformed into the simplest form of the heat equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} \tag{10}$$

where the transformations are as follows:

- $t \longrightarrow \tau = T - t$ *Time to maturity*
- $V(S,t) \longrightarrow y(x,\tau)$ *Option value*
- $S \longrightarrow x$ *Price of underlying asset*

After discretization of the transformed Black-Scholes equation we finally arrive at the complementary problem in linear form, where the following equation has to be solved for w.

$$Aw = b \tag{11}$$

where w is the numerical approximation of y and A is the spatial matrix defined by:

$$A = \begin{bmatrix} 1+2\lambda\theta & -\lambda\theta & 0 \\ -\lambda\theta & \ddots & \vdots \\ 0 & \dots & \dots \end{bmatrix}$$

An iterative procedure and a direct method is discussed to solve this linear complementary problem.

2.2.1 PSOR

The linear complementary problem $Ax = \hat{b}$ is rewritten to $Ax = \hat{b} = b - Ag$. For the standard SOR method an iterative process can be obtained by calculating:

$$x_i^{(k)} = x_i^{(k-1)} + \omega_r \frac{r_i^{(k)}}{a_{ii}} \tag{12}$$

where k denotes the number of iteration, a_{ij} is an element of the matrix A, w_R is a relaxation parameter chosen to improve convergence and $r_i^{(k)}$ is defined as:

$$r_i^{(k)} = \hat{b}_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \tag{13}$$

For the PSOR method we have an outer loop iterating over all k, and an inner loop iterating for all i between 1 and $m-1$. In each iteration in the inner loop we calculate $r_i^{(k)}$ and calculate:

$$x_i^{(k)} = max(0, x_i^{(k-1)} + \omega_r \frac{r_i^{(k)}}{a_{ii}}) \tag{14}$$

2.2.2 Direct method

For an American put we can solve the linear complementary problem $Aw = b$ with a direct method with two different phases. We introduce the following equivalent system $\hat{A}w = \hat{b}$ where $\hat{A}$ is a triangular matrix. In the first phase we start by calculating the RL-decomposition of A, and then set $\hat{A} = L$ and calculate $\hat{b}$ from $R\hat{b} = b$ using a backward loop. In the second phase we forward loop for index i. We start with $i = 1$ and calculate the next component of $\hat{A}w = \hat{b}$ denoting it $\hat{w}_i$. After this we set $w_i = max(\hat{w}_i, g_i)$ where g is the transformed pay-off function.

2.2.3 Splitting operators

Splitting operators is a method used for example to solve problems arising from the Navier Stokes equation, but can be applied to solve the linear complementary problem that arises from American options. The linear complementary problem can be formulated as follows:

$$\lambda = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv, x > 0, t \in [0, T] \tag{15}$$

$$[v - (E - x)]\lambda = 0, x > 0, t \in [0, T] \tag{16}$$

$$v - (E - x) \geq 0, \lambda \geq 0, x > 0, t \in [0, T] \tag{17}$$

$$v = max(E - x, 0), x > 0, t = T \tag{18}$$

$$v = E, x = 0, t \in [0, T] \tag{19}$$

$$v \longrightarrow 0, x \longrightarrow \infty, t \in [0, T] \tag{20}$$

where v is the option price, E is the strike, t is time, sigma is volatility, r is the risk free interest rate, x is the spot price of the underlying asset and λ is an auxiliary variable that forces the option value to be greater than $E - x$.

After discretizating the space using for example finite difference we arrive at

$$\frac{\partial v}{\partial t} + Av - \lambda = 0, t \in [0, T] \tag{21}$$

After discretizating the time we arrive at a linear complementary problem which can be solved using PSOR as discussed above, or for example operator splittings. When using operators splittings we split the linear complementary problems into the black scholes operator in one fractional time step, and the constraint $v_i^{(k)} \geq E - x_i$ in another fractional time step. In the first step we use LU decomposition to solve for an intermediate vector $\hat{v^{(k)}}$. In the second step we project this intermediate vector and updates the auxiliary variable, thus enforcing the constraint.