Fabulous Fractions and Musical Scales

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Musical Motivation

Xavier plays the xylophone and Yulia plays Yakety Sax along an 8 note scale. Xavier ascends the scale in steps of 1, and Yulia descends the scale in steps of 5. We label each vertex of the lattice $\mathbb{Z}^2$ with the number of a step in the scale, with the value increasing 1 at a time in the x-direction and 5 at a time in the y-direction. The note to be played at coordinate (x, y) is $x + 5y \bmod 8$

Definition: Tune

A tune consists of the two musicians taking turns to play a number of notes each time. We may write the tune as a sequence of letters, for example:

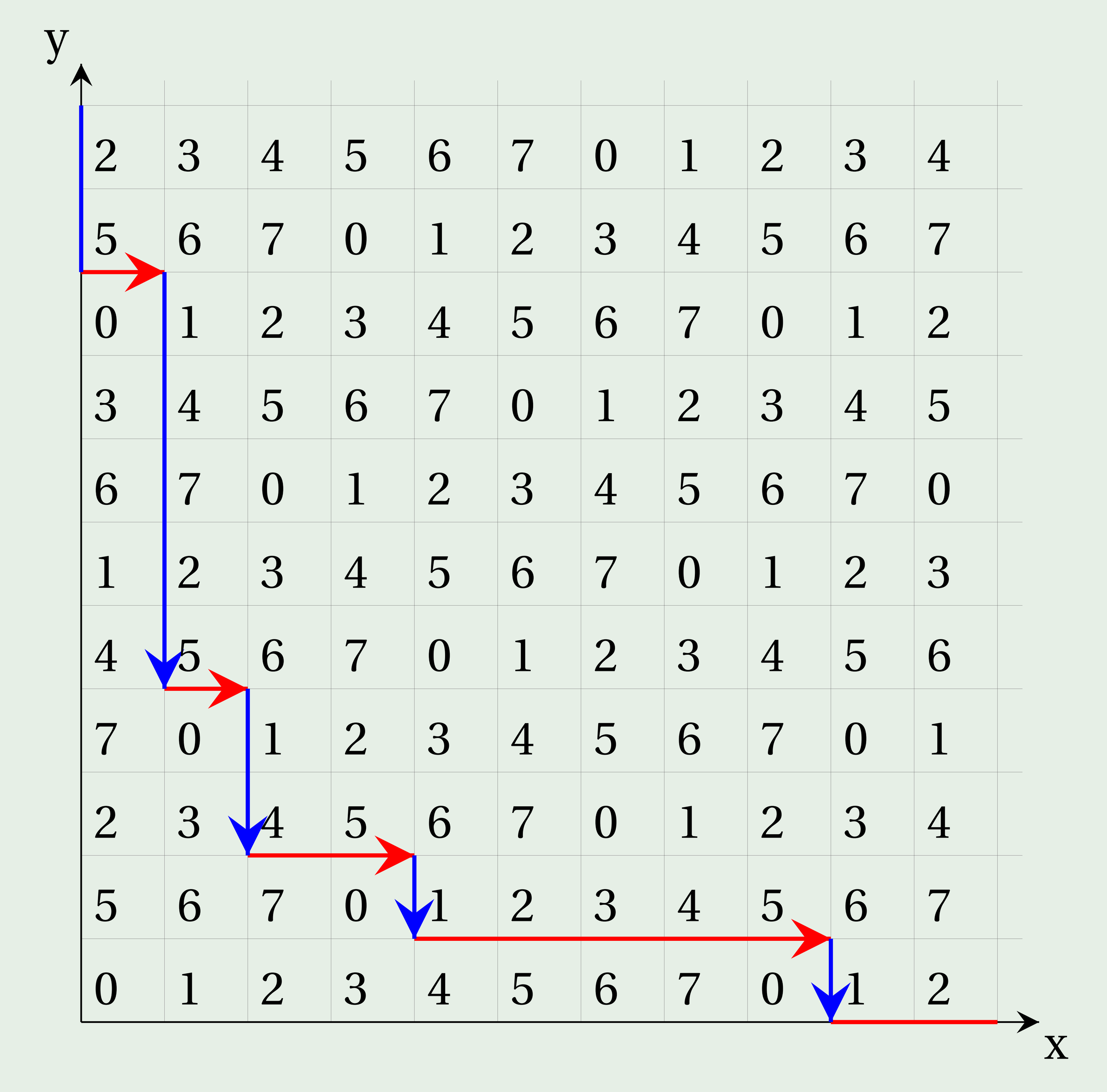
$$x, y, y, y, y, y, x, y, y, x, x, y, x, x, x, x, y$$

This can be seen as a path in the lattice $\mathbb{Z}^2$ where the notes played are those described by the vertices passed along the path. In mathematical terms, we have described the boundary path of a partition, together with the box colouring rule $\frac{1}{8}(1, 5)$.

Questions

- When do the two musicians play the same note or make a specific harmony?
- Can we generalise this for scales with more notes and for different playing pattern?

Diagram: A tune as a path in the lattice



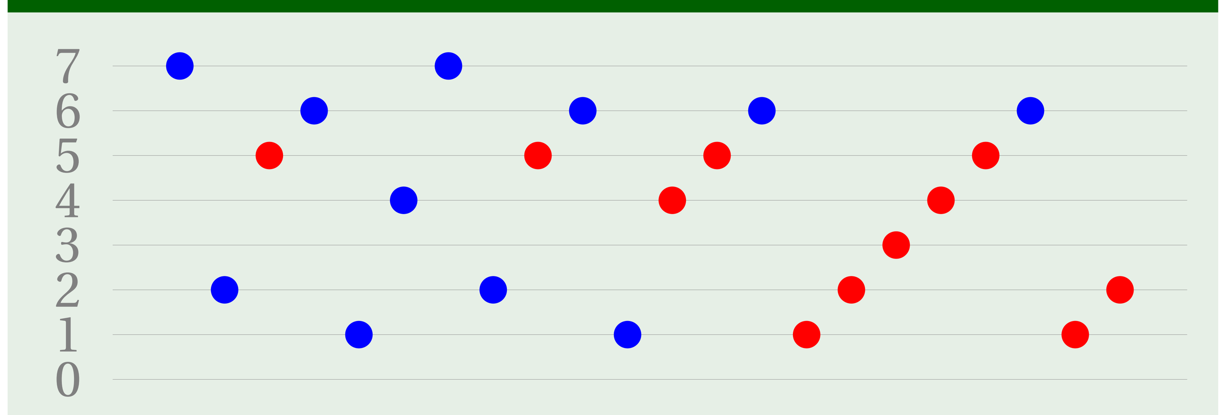
Choosing the right staff for the job

We can translate our boundary path through the lattice onto a musical stave with 8 lines.

- Each line represents one note on our musical scale.
- We proceed along the path determined by our tune, and make a note which note to play according to the labelling of the lattice.
- These notes are added to the staff on the corresponding line in sequence.

The red notes are played by Xavier and Yulia plays the blues.

Diagram: Tune depicted on the stave



Continued fractions

Let r, b be coprime integers with $r > b > 0$. Then the Hirzebruch-Jung continued fraction of r/b is the expression

$$\frac{r}{b} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}} = [a_1, a_2, \dots, a_k]$$

For example,

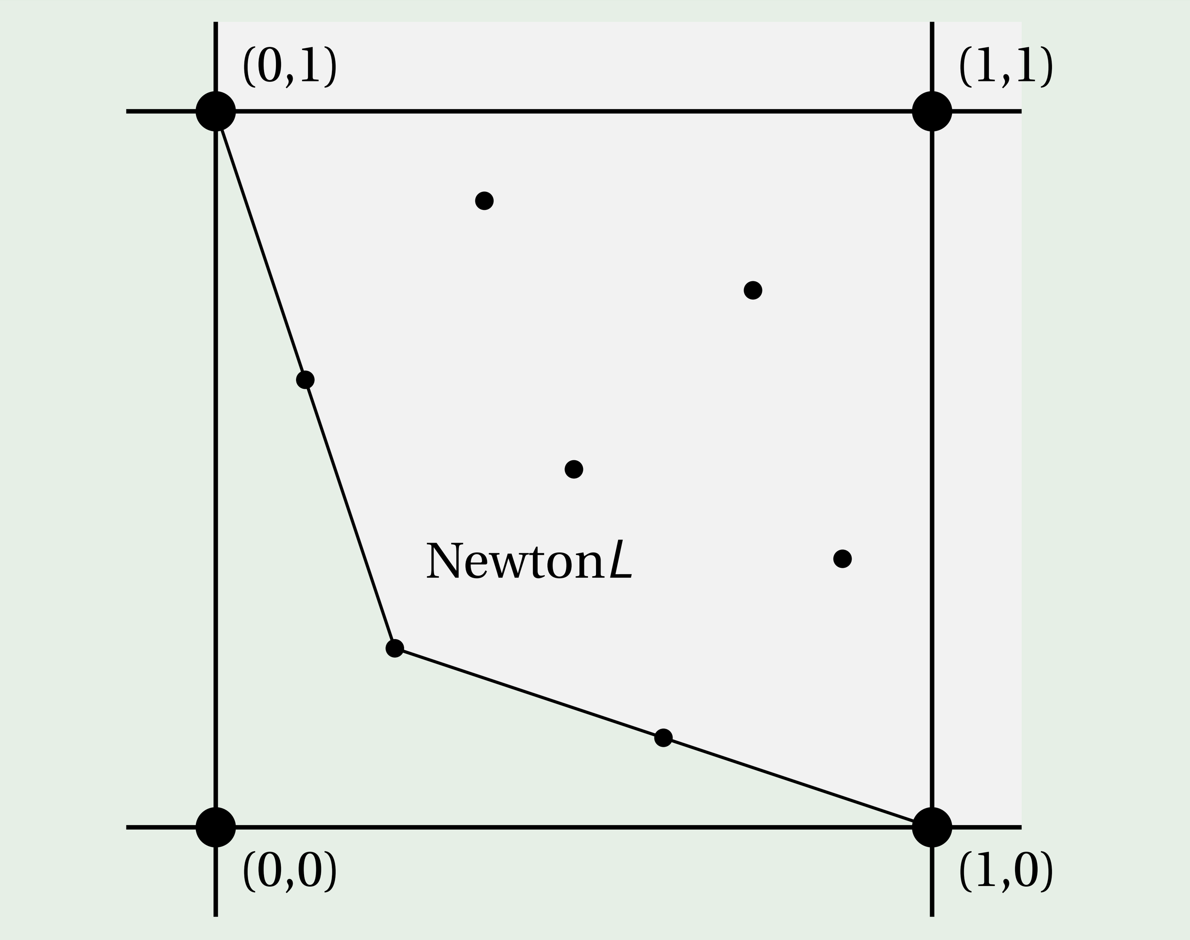
$$\frac{8}{5} = 2 - \cfrac{1}{3 - \frac{1}{2}} = [2, 3, 2]$$

More generally, when $b = 1$ we have $\frac{r}{1} = r = [r]$, and when $b = r - 1$ we have $\frac{r}{r-1} = 2 - \frac{1}{\left(\frac{r-1}{r-2}\right)} = \ldots = [2, \ldots, 2]$.

Lattices and Newton polygons

Consider the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, b) \subset \mathbb{R}^2$. Define the Newton polygon of L to be the convex hull $\text{Newton}L$ in $\mathbb{R}^2$ of all the nonzero lattice points in the positive quadrant.

Diagram: Newton polygon



Proposition: Continued fraction in the lattice

Write

$$e_0 = (0, 1), e_1 = \frac{1}{r}(1, b), e_2, \ldots, e_k, e_{k+1} = (1, 0)$$

for the lattice points on the boundary of $\text{Newton}L$. Then

1. Any two consecutive lattice points e_i, e_{i+1} for $i = 0, \ldots, k$ form an oriented basis of L.
2. Any three consecutive lattice points e_{i-1}, e_i, e_{i+1} for $i = 1, \ldots, k$ satisfy a relation
$$e_{i-1} + e_{i+1} = a_i e_i$$
for some integer $a_i \geq 2$.
3. The integers $a_1, a_2, \ldots, a_k$ in (2.) are the entries in the continued fraction:
$$\frac{r}{b} = [a_1, a_2, \ldots, a_k]$$

Consequences

The meaning of (1.) above is that once we choose a consecutive pair of the special lattice points e_i, e_{i+1}, then any other lattice point v in L can be expressed as a combination

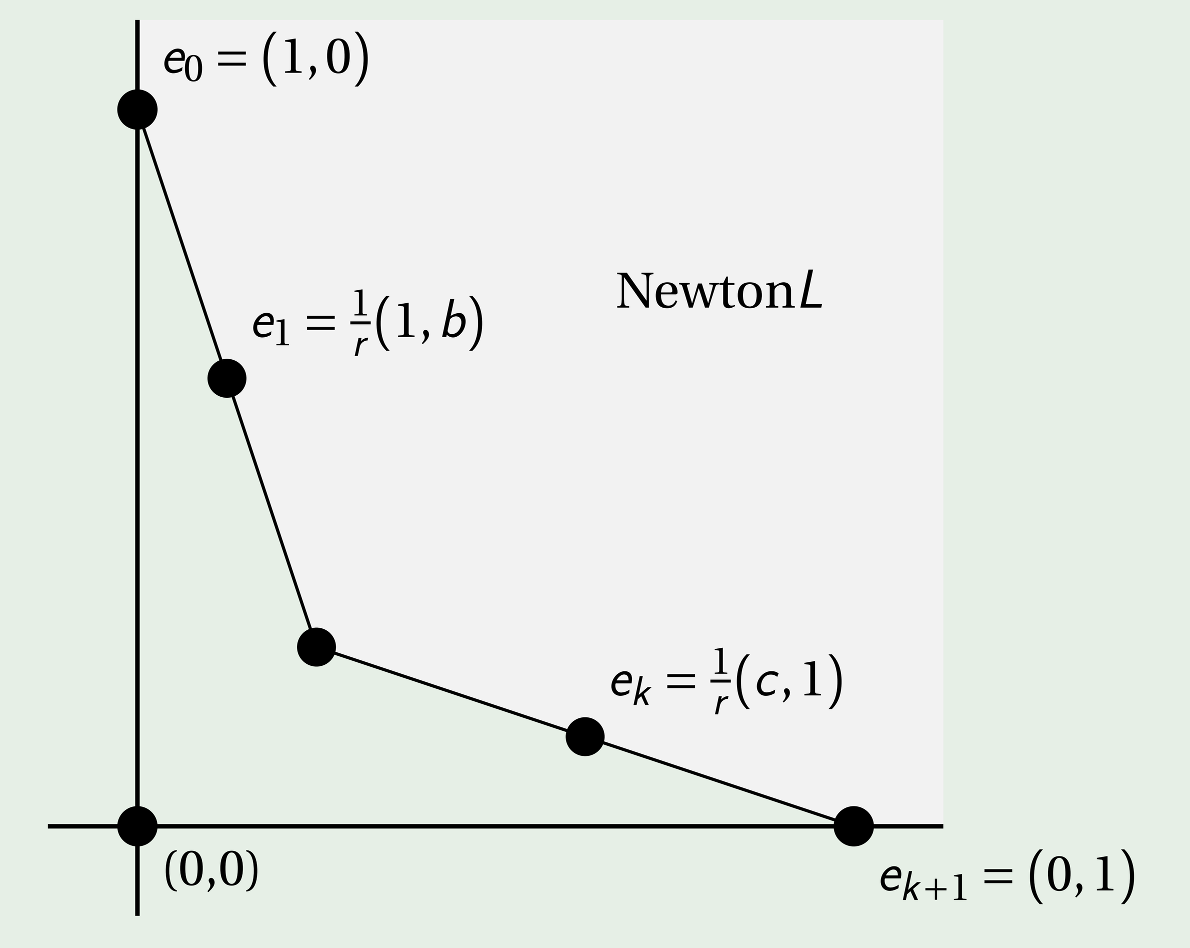
$$v = me_i + ne_{i+1}$$

for some integers m, n.

In the setting above, $e_k = \frac{1}{r}(c, 1)$ where $r > c > 0$ and $bc \equiv 1 \mod r$, and by reflecting the lattice L in the line $x = y$ (i.e. swapping the x and y coordinates) to obtain the lattice $L' = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, c)$ we determine the following continued fraction expression:

$$\frac{r}{c} = [a_k, a_{k-1}, \ldots, a_1]$$

Diagram: Newton polygon boundary points



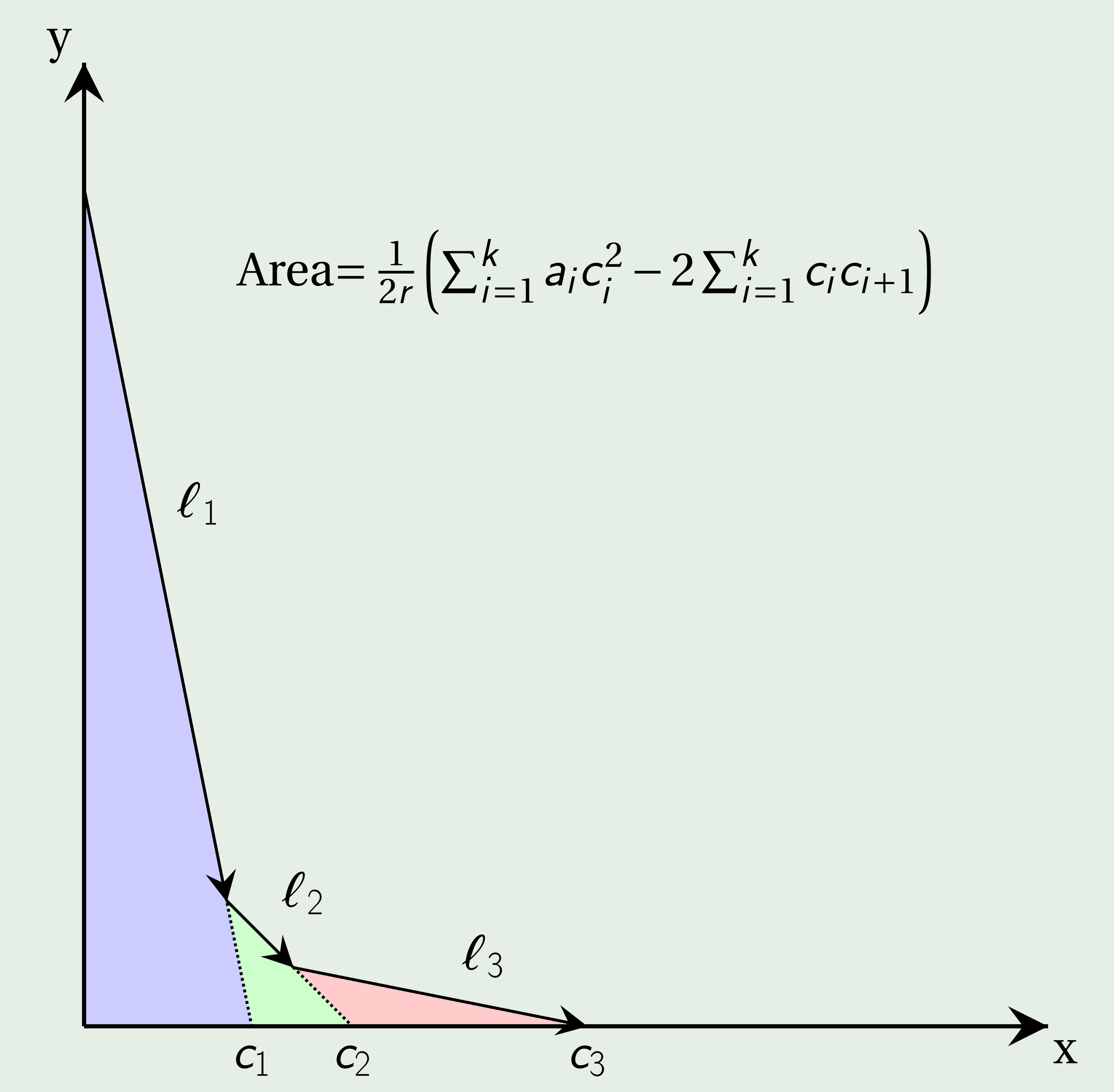
Decomposing a tune into phrases

A tune is split into a number of phrases. A phrase is a repeating pattern of notes which start and end on the same note with no other repeated notes in between. We can measure the complexity of the tune as the area enclosed by its boundary path (i.e. the size of the associated partition). Can we compute this area knowing only the number of repetitions of each phrase appearing in the tune?

Phrases form a lattice

We linearly approximate a phrase by a line with negative slope connecting two vertices in $\mathbb{Z}^2$ which correspond to the same note on the scale. These phrases form a lattice to which our previous proposition applies. Therefore, there are a finite number of basic phrases which generate the rest.

Diagram: Linear approximation of a tune's path



Result: Continued fraction in area formula

If for each $i = 1, 2, \ldots, k$ we assign a number c_i to denote how many repetitions of the basic phrase e_i appears in the tune, then the area cut out by the convex arrangement of lines approximating the boundary path is given by the quadratic form in the diagram above, which may also be expressed as

$$q(\underline{c}) = \frac{1}{2r} \cdot \underline{c} \begin{pmatrix} a_1 & -1 & 0 & 0 \\ -1 & a_2 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_k \end{pmatrix} \underline{c}^T$$

where $\underline{c} = (c_1, c_2, \ldots, c_k)$ and the a_i arise in the continued fraction associated to the lattice of phrases. This quadratic form also appears as the intersection pairing of exceptional curves in the resolution of a cyclic quotient singularity.

Further directions: What can we determine if zither playing Zachary wants to join the musical ensemble?