Geometry of Syzygies

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1 Introduction

In this project we introduce several important and useful concepts and techniques from commutative and homological algebra which can prove fruitful in the study of projective geometry through means of calculating numerical invariants and conveying other algebraic and geometric information about ideals and their corresponding varieties. There is also some interplay with combinatorics along the way. We build our toolkit in order to consider a conjectural connection which states that under particular assumptions on a projective variety we may bound the degree of the minimal homogeneous generators of its associated ideal in terms of geometric data. This may have computational consequences, and is also deeply linked to the world of cohomology. In the final section we look for examples in a class of polynomial ideals known as toric ideals with the aid of software system Macaulay2.

2 Free Resolutions

2.1 Graded Rings

Definition 2.1. A graded ring R is a ring which splits into a direct sum of abelian groups $R = \bigoplus_{n\in\mathbb{N}} R_n$ such that $R_iR_j \subseteq R_{i+j}$ for all i,$j \geq 0$, and a graded (left) module M over a graded ring R is a (left) R-module which has a direct sum decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ as abelian groups such that $R_iM_j \subseteq M_{i+j}$ for all i,j. Note that this gives each graded component M_d an R_0-module structure.

We say an R-module M is finitely generated if there exist elements $m_1, m_2, \dots, m_n \in M$ such that every element $m \in M$ can be expressed as an R-linear combination $m = r_1m_1 + r_2m_2 + \dots + r_nm_n$.

The example which we shall be considering throughout is the polynomial ring $S = k[x_0, \dots, x_r]$ with each variable in degree 1, so $S = \bigoplus_{d\in\mathbb{N}} S_d$ where S_d is the k-vector space of homogeneous polynomials of degree d.

Definition 2.2. For a graded module M we may define a new module $M(d)$, called the d-th twist of M, by 'shifting' the grading of M by d steps, i.e. $M(d)$ is isomorphic to M as a module and has grading defined by $M(d)_e = M_{d+e}$.

Let M,N be graded modules. Suppose $f : M \to N$ is a morphism of the underlying modules, then we say f has degree d if $f(M_n) \subset N_{n+d} = N(d)_n$.

Many natural maps of graded modules take the grading of one to the grading of the other with a shift of degrees and using our notation we may interpret them as degree preserving maps between one of the modules and the shift of the other. This is convenient for keeping track of graded components and so from now on we use the term graded morphism to refer to a degree 0 morphism of graded modules. Note that in particular a graded morphism $f : M \to N$ restricts to an R_0-linear map of graded components $f : M_d \to N_d$ for each d.

For a graded ring R we define a graded free R-module to be a direct sum of modules of the form $R(d)$, for various d. Notice that $R(d)_{-d} = R_0$, so that $R(d)$ has its generator in degree $-d$, and not d.

2.2 Construction of a free resolution

Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded S-module, where $S = k[x_0, \dots, x_r]$ is the polynomial ring in $r+1$ variables over a field k, with all variables of degree 1. Given homogeneous elements $m_i \in M$ of degrees a_i that generate M as an S-module we may define a map from the graded free module $F_0 = \bigoplus_i S(-a_i)$ onto M by sending the i-th generator of F_0 to m_i. Let $M_1 \subset F_0$ be the kernel of the map $\phi_0 : F_0 \to M$. Then by the Hilbert Basis theorem M_1 is also a finitely generated module. The elements of M_1 are called syzygies on the generators m_i, or simply syzygies of M. Choosing finitely many homogeneous syzygies that generate M_1 we may define a map from a graded free S-module F_1 to F_0 with image M_1. Continuing in this way we can construct a sequence of maps of graded free modules

$$\dots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

called a free resolution of M.

Definition 2.3. Formally, we define a free resolution of an R-module M to be a complex

$$\mathbf{F} : \dots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of free R-modules such that $\operatorname{coker}\varphi_1 = M$ and $\mathbf{F}$ is exact.

We sometimes abuse this notation and say that an exact sequence

$$\mathbf{F} : \dots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

is a resolution of M. A resolution $\mathbf{F}$ is a graded free resolution if R is a graded ring, the F_i are graded free modules, and the maps are homogeneous maps of degree 0 (i.e. graded morphisms). If for some $n < \infty$ we have $F_{n+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq n$, then we say that $\mathbf{F}$ is a finite resolution of length n.

Example 2.4. Let f, g be polynomials (homogeneous or not) of positive degree in S, neither of which divides the other. Let $n = \deg f, m = \deg g$. Then

$$\mathbf{K}(f) : 0 \to S(-n) \xrightarrow{(f)} S$$

gives a graded free resolution of $S/(f)$ since S is an integral domain.

Further, if we set $h = \gcd(f, g), f' = f/h, g' = g/h$, and $p = \deg h$ then

$$\mathbf{K}(f, g) : 0 \to S(-(n+m-p)) \xrightarrow{\begin{pmatrix} g' \\ -f' \end{pmatrix}} S(-n) \bigoplus S(-m) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} S$$

gives a graded free resolution of $S/(f, g)$.

Proof. Once again, we have exactness at $S(-(n+m-p))$ since S is an integral domain, so it remains to show that we have exactness at $S(-n) \bigoplus S(-m)$. First, the composition $\begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} g' \\ -f' \end{pmatrix} = fg' - f'g = (fg - fg)/h = 0$ is zero, so $\operatorname{Im} \begin{pmatrix} g' \\ -f' \end{pmatrix} \subseteq \ker \begin{pmatrix} f & g \end{pmatrix}$ and we have a complex.

Now suppose $\begin{pmatrix} x \\ y \end{pmatrix} \in \ker \begin{pmatrix} f & g \end{pmatrix}$, so $xf + yg = 0$ and by dividing throughout by $h = \gcd(f, g)$ we find $xf' = -yg'$. Then as $f' \mid -yg'$ but $\gcd(f', g') = 1$ we find $f' \mid -y$, so may write $-y = zf'$ for some $z \in S$. Then $xf' = -yg' = zf'g'$, so $x = zg'$ and thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} zg' \\ -zf' \end{pmatrix} = \begin{pmatrix} g' \\ -f' \end{pmatrix} (z) \in \operatorname{Im} \begin{pmatrix} g' \\ -f' \end{pmatrix}$. This shows exactness at $S(-n) \bigoplus S(-m)$, hence the above complex is a free resolution. □

2.3 Application - The Hilbert Function and Polynomial

Definition 2.5. Let M be a finitely generated graded module over $S = k[x_0, \ldots, x_r]$. We define the Hilbert function $H_M : \mathbb{N} \to \mathbb{N}$ as the map $d \mapsto H_M(d) = \dim_k M_d$ for the dimension of the d-th graded component of M as a function of d.

For sufficiently large d the Hilbert function $H_M(d)$ agrees with a polynomial function $P_M(d)$, called the Hilbert Polynomial, which has degree at most r.

Remark 2.6. By their definitions the Hilbert function and Hilbert polynomial are invariants under graded isomorphism of S-modules, and in fact the coefficients of the Hilbert polynomial can give important invariants of the module.

For example, if $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a curve, then the Hilbert polynomial of the homogeneous coordinate ring $S_X = \frac{S}{I(X)}$ of X is given by a linear polynomial $P_M(d) = (\deg X)d + (1 - \text{genus} X)$ where the coefficients $\deg(X)$ and $1 - \text{genus}(X)$ give a topological classification of the embedded curve. [Eis05]

More generally, the degree of the Hilbert polynomial is the dimension of X, and the leading coefficient of $P_M(d)$ multiplied by $(\deg P_M)!$ is equal to the degree of X - the number of points in which X meets a general plane of complementary dimension in $\mathbb{P}^r$. [Eis95]

Proposition 2.7. *If we have a finite free resolution of a finitely generated graded module M we can compute its Hilbert function*

Let $S = k[x_0, \ldots, x_r]$. If M is a graded S-module with finite free resolution

$$\mathbf{F} : 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

with each F_i a finitely generated free module of the form $F_i = \bigoplus_j S(-a_{i,j})$, then the Hilbert function of M is given by $H_M(d) = \sum_{i=0}^{m} (-1)^i \sum_j \binom{r+d-a_{i,j}}{r}$.

Proof. Each map in a graded free resolution is degree preserving by definition, so we may pass to an exact sequence of finite dimensional vector spaces by taking the degree d part of each module in the sequence, and using our knowledge of exact sequences of finite dimensional vectors spaces (i.e. repeated application of the rank-nullity theorem), we get

$$H_M(d) = \dim_{\mathbb{K}} M_d = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{K}} (F_i)_d = \sum_{i=0}^{m} (-1)^i H_{F_i}(d),$$

so it suffices to show that $H_{F_i}(d) = \sum_j \binom{r+d-a_{i,j}}{r}$ where $F_i = \bigoplus_j S(-a_{i,j})$. Decomposing F_i as a direct sum, we get

$$H_{F_i}(d) = \dim_{\mathbb{K}} \bigoplus_j S(-a_{i,j}) = \sum_j \dim_{\mathbb{K}} S(-a_{i,j}) = \sum_j H_{S(-a_{i,j})}(d),$$

so now it suffices to show that $H_{S(-a)}(d) = \binom{r+d-a}{r}$. Shifting back, we have $H_{S(-a)}(d) = \dim_{\mathbb{K}} S(-a)_d = \dim_{\mathbb{K}} S_{d-a} = H_S(d-a)$, and so in order to show that $H_{S(-a)}(d) = H_S(d-a) = \binom{r+(d-a)}{r}$ it is enough to show that $H_S(d) = \binom{r+d}{r}\ \forall d$.

We complete the proof by counting the number of monomials of degree d possible in a polynomial ring in $r+1$ variables:

$$\begin{aligned} H_S(d) &= \dim_{\mathbb{K}} S_d = \dim_{\mathbb{K}} k[x_0,\ldots,x_r]_d = \#\{x_0^{a_0}x_1^{a_1}\ldots x_r^{a_r} \,|\, a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^r a_i = d\} \\ &= \#\{a_i \in \mathbb{Z}_{\geq 0} | \sum_{i=0}^r a_i = d\} = \binom{r+d}{r} \end{aligned}$$

by a "stars and bars" argument. □

Corollary 2.8. *There is a polynomial $P_M(d)$ called the Hilbert polynomial of M such that, if M has a free resolution as above, then $P_M(d) = H_M(d)$ for all $d \geq \max_{i,j}\{a_{i,j}\} - r$.*

Proof. If $d \geq \max_{i,j}\{a_{i,j}\} - r$, then $r+d-a_{i,j} \geq 0\, \forall i,j$, in which case

$$\binom{r+d-a_{i,j}}{r} = \frac{(d+r-a_{i,j})(d+r-a_{i,j}-1)\ldots(d-a_{i,j}+1)}{r!}$$

which is a polynomial of degree r in d. Thus in the desired range all of the terms of $H_M(d) = \sum_{i=0}^m (-1)^i \sum_j \binom{r+d-a_{i,j}}{r}$ become polynomials. □

Example 2.9. Returning to example 2.4, if f, g are polynomials of positive degree in $S = k[x_0,\ldots,x_r]$, neither of which divides the other, and we set $n = \deg f, m = \deg g, p = \deg \gcd(f,g)$, then by proposition 2.7 above

$$\begin{aligned} H_{S/(f)}(d) &= \binom{r+d}{r} - \binom{r+d-n}{r}, \text{ and} \\ H_{S/(f,g)}(d) &= \binom{r+d}{r} - \binom{r+d-n}{r} - \binom{r+d-m}{r} + \binom{r+d-(n+m-p)}{r} \end{aligned}$$

which agree with their corresponding Hilbert polynomials for all $d \geq n-r$, and for all $d \geq n+m-p-r$ respectively. A sharp bound on when the Hilbert function and polynomial coincide is given in theorem 4.2.

2.4 Minimal Free Resolutions and Hilbert Syzygy Theorem

Theorem 2.10 (Hilbert Syzygy Theorem)**.** *Any finitely generated graded S-module has a finite graded free resolution*

$$\mathbf{F} : 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

where we may take $m \leq r+1$ the number of variables in S.

Each finitely generated graded S-module has a minimal free resolution which is unique up to isomorphism. The degrees of the generators (i.e. the $a_{i,j}$) of its free modules not only yield the Hilbert function, as is true of any finite free resolution, but form a finer invariant. In this section we give a careful statement of the definition of minimality, and of the uniqueness theorem.

Naively, minimal free resolutions can be described as follows: Given a finitely generated graded module M, choose a *minimal* set of homogeneous generators m_i. Map a graded free module F_0 onto M by sending a basis for F_0 to the set of m_i. Let M_1 be the kernel of the map $F_0 \to M$, and repeat the procedure, starting with a minimal system of homogeneous generators for M_1.

Most of the applications of minimal free resolutions are based on a property that characterises them in a different way, which we will adopt as the formal definition.

Definition 2.11. Let $\mathfrak{m}$ denote the homogeneous maximal ideal $\mathfrak{m} = (x_0, \ldots, x_r) \subset S = k[x_0, \ldots, x_r]$. A complex of graded S-modules

$$\mathbf{F} : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$$

is called minimal if for each i the image of δ_i is contained in $\mathfrak{m}F_{i-1}$. That is, $\mathrm{Im}\delta_i \subset \mathfrak{m}F_{i-1}$ for all i. Informally, we may say a complex of free modules is minimal if its differential is represented by matrices with entries in the maximal ideal $\mathfrak{m}$. In particular, no nonzero constant terms are allowed.

The relation between the above definition and the naive idea of a minimal free resolution is is a consequence of Nakayama's lemma.

Lemma 2.12 (Technical Lemma)**.** *Let R be a commutative ring with identity 1_R, I an ideal of R, and M a (left) R-module, then $M \bigotimes_R R/I \cong M/IM$ are isomorphic as R-modules.*

Proof. The map $M \times R/I \to M/IM, (m, r+I) \mapsto r.m + IM$ is R-bilinear by the R-module structure on M/IM, so by the universal property of tensor products we get an R-linear map $\varphi : M \bigotimes_R R/I \to M/IM$ given by $\varphi : m \otimes r + I \mapsto r.m + IM$, which has inverse $\psi : M/IM \to M \bigotimes_R R/I, m + IM \mapsto m \otimes 1_R + I$. In particular, $\psi \circ \varphi(m \otimes r + I) = \psi(r.m + IM) = r.m \otimes 1_R + I = m \otimes r.1_R + I = m \otimes r + I$ as $r \in R$, and $\varphi \circ \psi(m + IM) = \varphi(m \otimes 1_R + I) = 1_R.m + IM = m + IM$ as M is unital. Thus $\varphi : M \bigotimes_R R/I \xrightarrow{\cong} M/IM$ is an isomorphism of R-modules. □

Remark. We use the above lemma most commonly in the case of the polynomial ring $S = k[x_0, \ldots, x_r]$ and its maximal homogeneous ideal $\mathfrak{m} = (x_0, \ldots, x_r)$ where M is a graded S-module. Then $M \bigotimes_S S/\mathfrak{m} \cong M/\mathfrak{m}M$.

Lemma 2.13 (Nakayama's Lemma)**.** *Suppose M is a finitely generated graded S-module and $m_1, m_2, \ldots, m_n \in M$* generate $M/\mathfrak{m}M$. Then $m_1, m_2, \ldots, m_n$ generate M.

Proof. Let $\bar{M} := M/\sum S \cdot m_i$. Consider

$$\bar{M}/\mathfrak{m}\bar{M} \cong (M/\mathfrak{m}M)/\sum S \cdot (m_i + \mathfrak{m}M) = 0$$

as we assume the m_i generate $M/\mathfrak{m}M$, or more precisely that their images $m_i+\mathfrak{m}M$ are the generators. Therefore $\mathfrak{m}\bar{M} = \bar{M}$. Suppose for contradiction that $\bar{M} \neq 0$, then since $\bar{M}$ is finitely generated (as M is), we may find a nonzero element of least degree in $\bar{M}$. However, this element could not be in $\mathfrak{m}\bar{M}$ since everything here has degree at least one greater. Thus we conclude that $\bar{M} = 0$, and so $M = \sum S \cdot m$ is generated by the m_i. □

Corollary 2.14. *A graded free resolution*

$$\mathbf{F} : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$$

is minimal as a complex if and only if for each i the map δ_i takes a basis of F_i to a minimal set of generators of the image of δ_i (which is equal to the kernel of δ_{i-1} by exactness). That is to say our definition of minimal coincides with the construction of choosing a minimal set of generators at each step.

Proof. Consider the right exact sequence $F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} \mathrm{Im}\delta_i \to 0$. The complex $\mathbf{F}$ is minimal (i.e. $\mathrm{Im}\delta_i \subset \mathfrak{m}F_{i-1}$ for all i) if and only if for each i the induced map $\bar{\delta}_{i+1} : F_{i+1}/\mathfrak{m}F_{i+1} \to F_i/\mathfrak{m}F_i$ is zero. This is because $\delta_{i+1}(\mathfrak{m}F_{i+1}) = \mathfrak{m}\delta_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i$ already so the map is well defined and for $\bar{x} \in F_{i+1}/\mathfrak{m}F_{i+1}$ with representative $x \in F_{i+1}$ we have that $\bar{\delta}_{i+1}(\bar{x}) = 0 \in F_{i+1}/\mathfrak{m}F_{i+1} \Leftrightarrow \delta_{i+1}(x) \in \mathfrak{m}F_{i+1}$.

So $\mathrm{Im}\delta_i \subset \mathfrak{m}F_{i-1}\, \forall i$ iff the map $\bar{\delta}_{i+1} : F_{i+1}/\mathfrak{m}F_{i+1} \to F_i/\mathfrak{m}F_i$ is zero $\forall i$, and this holds if and only if the induced map $\tilde{\delta}_i : F_i/\mathfrak{m}F_i \to \mathrm{Im}\delta_i/\mathfrak{m}\mathrm{Im}\delta_i$ is an isomorphism, by considering the induced right exact sequence $F_{i+1}/\mathfrak{m}F_{i+1} \xrightarrow{\bar{\delta}_{i+1}} F_i/\mathfrak{m}F_i \xrightarrow{\tilde{\delta}_i} \mathrm{Im}\delta_i/\mathfrak{m}\mathrm{Im}\delta_i \to 0$. Finally, by Nakayama's lemma, this occurs if and only if a basis of F_i maps to a minimal set of generators of $\mathrm{Im}\delta_i$ (where the minimality of the set of generators corresponds to injectivity of $\tilde{\delta}_i$). □

Theorem 2.15 (Uniqueness of minimal free resolution)**.** *Let M be a finitely generated graded S-module . If $\boldsymbol{F}$ and $\boldsymbol{G}$ are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $\mathbf{F} \to \mathbf{G}$ inducing the identity map on M. Further, any free resolution of M contains the minimal free resolution as a direct summand.*

Proof. For uniqueness, see [Eis95, Theorem 20.2].

We can construct a minimal free resolution from any resolution, proving the second statement along the way. If $\mathbf{F}$ is a nonminimal complex of free modules, a matrix representing some differential of $\mathbf{F}$ must contain a nonzero element of degree 0. This corresponds to a free basis element of some F_i that maps to an element of F_{i-1} not contained in $\mathfrak{m}F_{i-1}$, so nonzero in the quotient $F_{i-1}/\mathfrak{m}F_{i-1}$. Then by Nakayama's lemma, this element may be taken as a basis element of F_{i-1}. Thus we have found a subcomplex of $\mathbf{F}$ of the form

$$\mathbf{G} : 0 \longrightarrow S(-a) \xrightarrow{c} S(-a) \longrightarrow 0$$

for a nonzero scalar c (such a thing is called a trivial complex) embedded in $\mathbf{F}$ in such a way that $\mathbf{F}/\mathbf{G}$ is again a free complex. As $\mathbf{G}$ has no homology at all (the sequence is exact as multiplication by a nonzero scalar is invertible, so an isomorphism), the long exact sequence in homology corresponding to the short exact sequence of complexes $0 \to \mathbf{G} \to \mathbf{F} \to \mathbf{F}/\mathbf{G} \to 0$ shows that the homology of $\mathbf{F}/\mathbf{G}$ is the same as that of $\mathbf{F}$. In particular, if $\mathbf{F}$ is a free resolution of M, then so is $\mathbf{F}/\mathbf{G}$. Continuing in this way by quotienting out by trivial subcomplexes we eventually reach a minimal complex. If $\mathbf{F}$ was a resolution of M, we have constructed a minimal free resolution. $\square$

Example 2.16. The nonminimal free resolution

$$\mathbf{F} : 0 \to S(-2) \bigoplus S(-1) \xrightarrow{\begin{pmatrix} y & -1 \\ -x & -1 \\ 0 & 1 \end{pmatrix}} S(-1)^3 \xrightarrow{\begin{pmatrix} x & y & x+y \end{pmatrix}} S$$

can be realised as a direct sum of the complex $\mathbf{K}(x,y)$ (as in example 2.4) and a trivial complex $\mathbf{G} : 0 \longrightarrow S(-1) \xrightarrow{1} S(-1) \longrightarrow 0$ via a change of basis $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ at $S(-1)^3$. That is

$$\begin{tikzcd}
\mathbf{F} : & 0 \arrow[r] & S(-2) \bigoplus S(-1) \arrow[r] \arrow[d, "\mathrm{id}"'] & S(-1)^3 \arrow[r] \arrow[d, "P"] & S \arrow[d, "\mathrm{id}"'] \\
\mathbf{K}(x,y) \bigoplus \mathbf{G} : & 0 \arrow[r] & S(-2) \bigoplus S(-1) \arrow[r] & S(-1)^3 \arrow[r] & S
\end{tikzcd}$$

Both $\mathbf{F}$ and $\mathbf{K}(x,y)$ are free resolutions of $S/(x,y)$, but $\mathbf{K}(x,y)$ is minimal and $\mathbf{F}$ is not.

For us, the most important aspect of uniqueness of minimal free resolutions is that if $\mathbf{F} : \ldots \to F_1 \to F_0$ is the minimal free resolution of a finitely generated graded S-module M, the number of generators of each degree required for the free modules F_i is determined by M only. The easiest way to state a precise result is to use the functor Tor; this useful tool from homological algebra is introduced in, for example, [Eis95, Section 6.1].

Proposition 2.17. *If $\mathbf{F} : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$ is the minimal free resolution of a finitely generated graded S-module M and $\mathbb{K}$ is the residue field $S/\mathfrak{m}$, then any minimal set of homogeneous generators of F_i contains exactly $\dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K}, M)_j$ generators of degree j. Thus we may express $F_i = \bigoplus_{j=0}^{\infty} S(-j)^{\beta_{i,j}}$ where $\beta_{i,j} := \dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K}, M)_j$ are called the graded Betti numbers of M.*

Proof. By definition $\operatorname{Tor}_i^S(\mathbb{K}, M)_j$ is the $\mathbb{K}$-vector space which is the degree j component of the graded vector space that is the i-th homology of the complex $\mathbb{K} \bigotimes_S \mathbf{F}$.

Claim 1: As $\mathbf{F}$ is minimal, the maps in the complex $\mathbb{K} \bigotimes_S \mathbf{F}$ are all zero.

Proof of claim 1: Consider a general map $\mathrm{id}_{\mathbb{K}} \otimes \delta_i : \mathbb{K} \bigotimes_S F_i \to \mathbb{K} \bigotimes_S F_{i-1}$ in the complex $\mathbb{K} \bigotimes_S \mathbf{F}$. As the simple tensors generate $\mathbb{K} \bigotimes_S F_i$ we need only consider their images. Recalling the identification $\mathbb{K} = S/\mathfrak{m}$, for a simple tensor $(a+\mathfrak{m}) \otimes f \in \mathbb{K} \bigotimes_S F_i$ we have $\mathrm{id}_{\mathbb{K}} \otimes \delta_i : (a+\mathfrak{m}) \otimes f \mapsto (a+\mathfrak{m}) \otimes \delta_i(f)$. By minimality we have $\delta_i(f) \in \mathfrak{m}F_{i-1}$, so we may write $\delta_i(f) = tf'$ for some $t \in \mathfrak{m} \subset S, f' \in F_{i-1}$, and then

$$\begin{aligned} \mathrm{id}_{\mathbb{K}} \otimes \delta_i((a+\mathfrak{m}) \otimes f) &= (a+\mathfrak{m}) \otimes \delta_i(f) = (a+\mathfrak{m}) \otimes tf' \\ &= (ta+\mathfrak{m}) \otimes f' = (0+\mathfrak{m}) \otimes f' = 0 \in \mathbb{K} \bigotimes_S F_{i-1}. \end{aligned}$$

Thus the maps in $\mathbb{K} \bigotimes_S \mathbf{F}$ are all zero by the minimality of $\mathbf{F}$. Equivalently, this might more readily be seen through the identification $\mathbb{K} \bigotimes_S F_i = S/\mathfrak{m} \bigotimes_S F_i \cong F_i/\mathfrak{m}F_i$ for each i, by lemma 2.12.

Therefore taking homology gives $\operatorname{Tor}_i^S(\mathbb{K}, M) = \mathbb{K} \bigotimes_S F_i \cong F_i/\mathfrak{m}F_i$, and then by Nakayama's lemma, $\dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K}, M)_j = \dim_{\mathbb{K}}(F_i/\mathfrak{m}F_i)_j$ gives the number of degree j generators that F_i requires. $\square$

Corollary 2.18. *If M is a finitely generated graded S-module, then the projective dimension of M is equal to the length of the minimal free resolution of M.*

Proof. The projective dimension $\mathrm{pd}(M)$ of M is defined to be the minimal length of a projective resolution of M (that is, a resolution by projective modules). The minimal free resolution of M is a projective resolution, since all free modules are projective, so $\mathrm{pd}(M) \leq$ the length of the minimal free resolution of M. To show that the length of the minimal free resolution is at most the projective dimension, note that $\mathrm{Tor}_i^S(\mathbb{K}, M) = 0$ when i is greater than the projective dimension of M (Tor is actually defined in terms of projective resolutions). Then by the above proposition, if $\mathrm{Tor}_i^S(\mathbb{K}, M) = 0$ then F_i has no generators of any degree for the minimal free resolution, so the minimal free resolution will have length less than i too. $\square$

Fact. *The Quillen-Suslin theorem states that every finitely generated projective module over a polynomial ring is free. [Lang 2002]*

Remark. If we allow the variables to have different degrees, $H_M(t)$ becomes, for large t, a polynomial with coefficients that are periodic in t. (See example 5.4)

2.5 Graded Betti Numbers and Betti Diagrams

We have seen in proposition 2.7 that the numerical invariants associated to free resolutions suffice to describe Hilbert functions, and will soon show that the numerical invariants of minimal free resolutions contain more information. As we will be dealing with them a lot we introduce a compact way to display them, called a Betti diagram.

Definition 2.19. Suppose that $\mathbf{F}$ is a free complex

$$\mathbf{F} : 0 \longrightarrow F_s \longrightarrow \ldots \longrightarrow F_i \rightarrow \ldots \rightarrow F_0$$

where $F_i = \displaystyle\bigoplus_{j=0}^{\infty} S(-j)^{\beta_{i,j}}$, that is to say that F_i requires $\beta_{i,j}$ minimal generators of degree j. The Betti diagram of $\mathbf{F}$ has the form:

	0	1	...	s
i	$\beta_{0,i}$	$\beta_{1,i+1}$	...	$\beta_{s,i+s}$
$i+1$	$\beta_{0,i+1}$	$\beta_{1,i+2}$	...	$\beta_{s,i+s+1}$
...	...	...	...	...
j	$\beta_{0,j}$	$\beta_{1,j+1}$	...	$\beta_{s,j+s}$

It consists of a table with $s+1$ columns, labeled $0, 1, \ldots, s$, corresponding to the free modules $F_0, \ldots, F_s$. It has rows labeled with consecutive integers corresponding to the degrees. We sometimes omit the row and column labels when they are clear from context. The i-th column specifies the degrees of the generators of F_i. Thus, for example, the row labels at the left of the diagram correspond to the possible degrees of a generator of F_0. For clarity we sometimes replace a 0 in the diagram by a dash "-" and an indefinite value by an asterisk "*". Note that the entry in the j-th row of the i-th column is $\beta_{i,i+j}$ rather than $\beta_{i,j}$ which will instead be found in position $(j-i, i)$. This choice will be explained below.

If $\mathbf{F}$ is the minimal free resolution of a module M, we refer to the Betti diagram of $\mathbf{F}$ as the Betti diagram of M, and the $\beta_{i,j}$ of $\mathbf{F}$ are called the graded Betti numbers of M, sometimes denoted as $\beta_{i,j}(M)$.

Recall from proposition 2.17 when $\mathbf{F}$ is minimal and the graded $\mathbb{K}$-vector space $\operatorname{Tor}_i^S(M, \mathbb{K})$ is the homology of the complex $\mathbf{F} \bigotimes_{\mathbf{F}} \mathbb{K}$, that since $\mathbf{F}$ is minimal, the differentials in this complex are zero, so $\beta_{i,j} = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_j$. In other words, when $\mathbf{F}$ is minimal its free modules are of the form $F_i = \bigoplus_{j=0}^{\infty} S(-j)^{\beta_{i,j}}$where $\beta_{i,j} = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_j$. Note that the Tor functor is symmetric over commutative rings such as S.

As the graded Betti numbers of a finitely generated graded S-module M are defined in terms of the minimal free resolution of M, we see that they are invariant under graded isomorphisms of M.

For example, the number $\beta_{0,j}$ is the number of elements of degree j required among the minimal generators of M. We will often consider the case where M is the homogeneous coordinate ring S_X of a (nonempty) projective variety X. As an S-module, S_X is then generated by the element $1 \in S_X$, so we will have $\beta_{0,0} = 1$ and $\beta_{0,j} = 0$ for $j \neq 1$.

On the other hand, $\beta_{1,j}$ is the number of independent forms (polynomials) of degree j needed to generate the ideal I_X of X. If S_X is not the zero ring (that is, $X \neq \emptyset$), there are no elements of the ideal of X in degree 0, so $\beta_{1,0} = 0$. This is the case $i = d = 0$ of the following:

Proposition 2.20. *Let $\{\beta_{i,j}\}$ be the graded Betti numbers of a finitely generated S-module. If for a given i there exists d such that $\beta_{i,j} = 0$ for all $j < d$, then $\beta_{i+1,j+1} = 0$ for all $j < d$. That is to say, if F_i has no minimal generators of degree less than d, then F_{i+1} has no minimal generators of degree less than $d+1$ by the minimality of the resolution.*

Proof. Suppose that the minimal free resolution is $\mathbf{F} : \ldots \longrightarrow F_1 \xrightarrow{\delta_1} F_0$. By corollary 2.14, any generator of F_{i+1} must map to a nonzero element of the same degree in $\mathfrak{m}F_i$. To say that $\beta_{i,j} = 0$ for all $j < d$ means that all generators - and thus all nonzero elements - of F_i have degree $\geq d$. Thus all nonzero elements of $\mathfrak{m}F_i$ have degree $\geq d+1$, so F_{i+1} can have generators only in degree $\geq d+1$ and

$\beta_{i+1,j+1} = 0$ for $j < d$ as claimed. $\square$

The above proposition gives a first hint as to why it is convenient to write the Betti diagram in the form we chose to, with $\beta_{i,i+j}$ in the j-th row of the i-th column: it says that if the i-th column of the Betti diagram has zeros above the j-th row, then the $(i+1)$-st column also has zeros above the j-th row. This also allows a more compact display of Betti numbers than if we had written $\beta_{i,j}$ in the j-th row of the i-th column. A deeper reason for this choice will be made clear in the description of the Castelnuovo-Mumford regularity in chapter 5.

The formula for the Hilbert function given previously in proposition 2.7 has a convenient expression in terms of graded Betti numbers (since the graded Betti numbers store information more concisely than the previous "$a_{i,j}$" notation).

Corollary 2.21. *If $\{\beta_{i,j}\}$ are the graded Betti numbers of a finitely generated graded S-module M, then the alternating sums $B_j := \sum_{i=0}^{\infty}(-1)^i\beta_{i,j}$ determine the Hilbert function of M via the formula*

$$\dim_k M_d = H_M(d) = \sum_{j=0}^{\infty} B_j \binom{r+d-j}{r}.$$

Moreover, the values of the B_j can be deduced inductively from the Hilbert function $H_M(d)$ via the formula

$$B_j = H_M(j) - \sum_{k:k<j} B_k \binom{r+j-k}{r}.$$

Note that the alternating sums B_j are actually finite with the number of terms in the sum being bounded above by the length of the minimal free resolution of M.

Proof. Recall from proposition 2.7 we know that if $F_i = \bigoplus_j S(-a_{i,j})$ then $H_M(d) = \sum_{i=0}^{m}(-1)^i \sum_j \binom{r+d-a_{i,j}}{r}$.

By grouping generators of the same degree we may write $F_i = \bigoplus_{j=0}^{\infty} S(-j)^{\beta_{i,j}}$ and then the sum becomes $\sum_j \binom{r+d-a_{i,j}}{r} = \sum_{j=0}^{\infty} \beta_{i,j}\binom{r+d-j}{r}$ for each i, hence

$$H_M(d) = \sum_{i=0}^{m}(-1)^i \sum_j \binom{r+d-a_{i,j}}{r} = \sum_{i=0}^{\infty}(-1)^i \sum_{j=0}^{\infty} \beta_{i,j}\binom{r+d-j}{r}.$$

Conversely, to compute the B_j from the Hilbert function $H_M(d)$ we proceed as follows: As $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is finitely generated there is a number $j_0 \in \mathbb{Z}$ such that

$H_M(d) = 0$ for $d < j_0$, i.e. take $j_0 = \min_{i=1,\dots n} \deg m_i$ where $\{m_i\}_{i=1}^n$ is a minimal set of homogeneous generators for M, so $M = \bigoplus_{d \geq j_0} M_d$. It follows that $\beta_{0,j} = 0$ for all $j < j_0$, and from the above proposition 2.20 it further follows that if $j < j_0$ then $\beta_{i,j} = 0$ for all i. Thus $B_j = \sum_{i=0}^{\infty}(-1)^i \beta_{i,j} = 0$ for all $j < j_0$. Initially, we have $B_{j_0} = H_M(j_0)$ and inductively, assuming we know the values of B_k for $k < j$, then as $\binom{r+j-k}{r} = 0$ when $j < k$, only the values of B_k with $k \leq j$ enter into the formula for $H_M(j) = \sum_{k=0}^{\infty} B_j \binom{r+j-k}{r} = \sum_{k=0}^{j} B_j \binom{r+j-k}{r}$, and knowing $H_M(j)$ we can solve for B_j to obtain the required formula. Note that, conveniently, B_j occurs with coefficient $\binom{r}{r} = 1$ in the sum. $\square$

3 Proof of Hilbert's Syzygy Theorem

In this chapter we introduce a fundamental construction of resolutions based on simplicial complexes. This construction gives free resolutions of monomial ideals, but does not always yield minimal resolutions. It includes Koszul complexes, which we use to establish basic bounds on syzygies of all modules, including the Hilbert Syzygy theorem.

3.1 Monomial Ideals and Simplicial Complexes

We now introduce a beautiful method of writing down graded free resolutions of monomial ideals due to Bayer, Peeva, and Sturmfels. So far we have used $\mathbb{Z}$-gradings of modules only, but we can think of the polynomial ring $S = k[x_0, \dots, x_r]$ as $\mathbb{Z}^{r+1}$-graded, with a monomial $x_0^{a_0} \dots x_r^{a_r}$ having degree $(a_0, \dots, a_r) \in \mathbb{Z}^{r+1}$, and the free resolutions we write down will also be $\mathbb{Z}^{r+1}$-graded. We begin by introducing the basics of the theory of finite simplicial complexes.

Definition 3.1. A finite simplicial complex Δ is a finite set $\mathcal{N}$, called the set of vertices (or nodes) of Δ, and a collection $\mathcal{F}$ of subsets of $\mathcal{N}$, called the faces of Δ, such that if $A \in \mathcal{F}$ is a face and $B \subset A$, then B is also in $\mathcal{F}$. Maximal faces are called facets.

A simplex is a simplicial complex in which every subset of $\mathcal{N}$ is a face, i.e. $\mathcal{F} = \mathcal{P}(\mathcal{N})$.

For any vertex set $\mathcal{N}$ we may form the void simplicial complex, which has no faces at all, i.e $\mathcal{F} = \emptyset$. However, if Δ has any faces at all, then the empty set is necessarily a face of Δ, that is $\emptyset \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$.

By contrast, we call the simplicial complex whose only face is $\emptyset$ the irrelevant simplicial complex on $\mathcal{N}$, i.e. where $\mathcal{F} = \{\emptyset\}$.

Remark (Stanley-Reisner correspondence). We may associate to any simplicial complex Δ with vertex set $\mathcal{N} = \{x_0, \ldots, x_r\}$ the squarefree monomial ideal in $S = k[x_0, \ldots, x_r]$ whose generators are the monomials with support equal to a non-face of Δ. [Miller and Sturmfels 2005, Chapter 1]

Under this correspondence, the irrelevant simplicial complex corresponds to the irrelevant ideal $(x_0, \ldots, x_r)$, while the void simplicial complex corresponds to the unit ideal $(1) \subset S$.

Remark 3.2 (Geometric realisation). Suppose $\Delta = (\mathcal{N}, \mathcal{F})$ is a simplicial complex, we consider $\bigoplus_{\#\mathcal{N}} \mathbb{R}$. Any simplicial complex Δ has a geometric realisation, that is, a topological space that is a union of simplexes corresponding to the faces of Δ. It may be constructed by realising the set of vertices of Δ as a linearly independent set in a sufficiently large real vector space, and realising each face of Δ as the convex hull of its vertex points; the realisation of Δ is then the union of these faces.

An orientation of a simplicial complex consists of an ordering of the vertices of Δ. Thus a simplicial complex may have many orientations - this is not the same as an orientation of the underlying topological space.

Definition 3.3 (Labeling by monomials)**.** We will say that Δ is labeled (by monomials of S) if there is a monomial of S associated to each vertex of Δ. We then label each face A of Δ by the least common multiple of the labels of the vertices in A, so $m_A := \operatorname{lcm}_{a \in A}(m_a)$. By convention, the label of the empty face is $m_\emptyset = 1$.

Let Δ be an oriented labeled simplicial complex, and write $I \subset S$ for the ideal generated by the monomials $m_j = x^{\alpha_j}$ labeling the vertices of Δ. That is, $I = (m_j : j \in \mathcal{N}) \subset S$. We will associate to Δ a graded complex of free S-modules

$$\mathcal{C}(\Delta) = \mathcal{C}(\Delta; S) : \ldots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta} F_0$$

where F_i is the free S-module whose basis consists of the set of faces of Δ having i elements, so $F_i = \bigoplus_{f_j \in \mathcal{F}, |f_j| = i} S(-a_{i,j})$. The complex will thus be finite with $F_i = 0$ for $i > |\mathcal{N}|$. The complex $\mathcal{C}(\Delta)$ is sometimes a resolution of S/I.

The differential δ for $\mathcal{C}(\Delta)$ is given by its action on the basis elements, which correspond to faces $A \in \mathcal{F}$.

$$\delta(A) := \sum_{n \in A} (-1)^{\operatorname{pos}(n,A)} \frac{m_A}{m_{A \setminus n}} (A \setminus n)$$

where $\operatorname{pos}(n, A)$, the position of the vertex n in A, is the number of elements preceding n in the ordering of A, and $A \setminus n$ denotes the face obtained from A by removing n. So for $n \in A$, $\operatorname{pos}(n, A)$ may start at zero and be at most $\#A - 1$.

If Δ is not void (i.e. $\mathcal{F} \neq \emptyset$) then $F_0 = S$, with generator corresponding to the face of Δ which is the empty set. Further the generators of F_1 correspond to the vertices of Δ, and each of these generators maps by δ to its labeling monomial (remembering the convention that $m_\emptyset = 1$). That is $\delta(n) = m_n \cdot \emptyset$ for vertices n of Δ. Therefore $H_0(\mathcal{C}(\Delta)) = \operatorname{coker}(F_1 \xrightarrow{\delta} S) = S/I$ since $\operatorname{im}\delta_1 = (m_n : n \in \mathcal{N}_\Delta) = I_\Delta$.

We set the degree of the basis element corresponding to a face A equal to the exponent vector of the monomial that is the label of A, so if $m_A = x^{\alpha_A}$, then $\deg A := \alpha_A \in \mathbb{N}^{\#\mathcal{N}}$.

Claim. With respect to this grading, $\mathcal{C}(\Delta)$ is a $\mathbb{Z}^{r+1}$-graded free complex.

Proof. First, we prove that $\delta^2 = 0$ in order to show that $\mathcal{C}(\Delta)$ is a complex. Let A be a face of Δ and $i, j \in A$ vertices. If j precedes i in A in the orientation on Δ then $\operatorname{pos}(j, A \setminus i) = \operatorname{pos}(j, A)$, and if i precedes j in the ordering of A then $\operatorname{pos}(j, A \setminus i) = \operatorname{pos}(j, A) - 1$. Thus

$$
\begin{aligned}
\delta\,(\delta A) &= \sum_{i \in A} (-1)^{\operatorname{pos}(i,A)} \frac{m_A}{m_{A\setminus i}} \delta(A \setminus i) \\
&= \sum_{i \in A} (-1)^{\operatorname{pos}(i,A)} \frac{m_A}{m_{A\setminus i}} \left(\sum_{j \in A \setminus i} (-1)^{\operatorname{pos}(j,A\setminus i)} \frac{m_{A\setminus i}}{m_{A\setminus i,j}} A \setminus i, j \right) \\
&= \sum_{i,j \in A, j<i} (-1)^{\operatorname{pos}(i,A)+\operatorname{pos}(j,A)} \frac{m_A}{m_{A\setminus i,j}} A \setminus i, j \\
&\quad - \sum_{i,j \in A, i<j} (-1)^{\operatorname{pos}(i,A)+\operatorname{pos}(j,A)} \frac{m_A}{m_{A\setminus i,j}} A \setminus i, j \\
&= 0
\end{aligned}
$$

For the differential to be degree-preserving we require

$$
\deg A = \deg \frac{m_A}{m_{A\setminus i}} A \setminus i
$$

for each vertex $i \in A$. As we defined the degree of a face A in terms of the degree of its labeling monomial m_A we find that

$$
\begin{aligned}
\deg \frac{m_A}{m_{A\setminus i}} A \setminus i &= \deg \frac{m_A}{m_{A\setminus i}} + \deg A \setminus i \\
&= \deg m_A - \deg m_{A\setminus i} + \deg m_{A\setminus i} = \deg m_A = \deg A
\end{aligned}
$$

as required. □

If we take $S = \mathbb{K}$ and label all the vertices of Δ with $1 \in \mathbb{K}$ then $\mathcal{C}(\Delta;\mathbb{K})$ is, up to a shift in homological degree, the usual reduced chain complex of Δ with coefficients in $\mathbb{K}$. [Hatcher 2001]

$$\mathcal{C}(\Delta;\mathbb{K}) : \ldots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta} F_0,$$
$$F_i = \mathbb{K}^{\oplus \#\{i\text{-faces of } \Delta\}},\ \delta A = \sum_{n \in A} (-1)^{\mathrm{pos}(n,A)} A \setminus n.$$

The homology of $\mathcal{C}(\Delta;\mathbb{K})$ is written $H_i(\Delta;\mathbb{K})$ and is called the reduced homology of Δ with coefficients in $\mathbb{K}$.

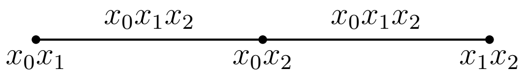
The shift in homological degree comes about as follows: the homological degree of a simplex in $\mathcal{C}(\Delta)$ is the number of vertices in the simplex, which is one more than the dimension of the simplex, so that $H_i(\Delta;\mathbb{K})$ is the $(i+1)$-st homology of $\mathcal{C}(\Delta;\mathbb{K})$.

If $H_i(\Delta;\mathbb{K}) = 0$ for all $i \geq -1$ (that is, for all i), then we say that Δ is $\mathbb{K}$-acyclic, and since S is a free module over $\mathbb{K}$, this is the same as saying that $H_i(\Delta;S) = 0$ for all $i \geq -1$.

The homology $H_i(\Delta;\mathbb{K})$ and the homology $H_i(\mathcal{C}(\Delta;S))$ are independent of the orientation of Δ - in fact they depend only on the homotopy type of the geometric realisation of Δ and the ring $\mathbb{K}$ or S. Thus we often ignore orientations.

Roughly speaking we may say that the complex $\mathcal{C}(\Delta;S)$, for an arbitrary labeling, is obtained by extending scalars from $\mathbb{K}$ to S and 'homogenizing' the formula for the differential of $\mathcal{C}(\Delta;\mathbb{K})$ with respect to the degrees of the generators of the F_i defined for the S-labeling of Δ.

Example 3.4. Suppose that Δ is the labeled simplicial complex



with orientation obtained by ordering the vertices from left to right. The complex $\mathcal{C}(\Delta)$ is

$$0 \to S(-3)^2 \xrightarrow{\begin{pmatrix} -x_2 & 0 \\ x_1 & -x_1 \\ 0 & x_0 \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} x_0x_1 & x_0x_2 & x_1x_2 \end{pmatrix}} S.$$

This complex is represented by the Betti diagram

	0	1	2
0	1	-	-
1	-	3	2

We will soon show that the complex $\mathcal{C}(\Delta)$ is a minimal free resolution of the S-module $S/(x_0x_1, x_0x_2, x_1x_2)$.

Definition 3.5. Let Δ be a labeled simplicial complex. If m is any monomial, we write Δ_m for the subcomplex of Δ consisting of those faces of Δ whose labels divide m.

For example, if m is not divisible by any of the vertex labels , then Δ_m is the empty simplicial complex, with no vertices and the single face $\emptyset$. On the other hand, if m is divisible by all the labels of Δ, then $\Delta_m = \Delta$.

Moreover, Δ_m is equal to $\Delta_{\mathrm{lcm}\{m_i|i\in\mathcal{N}'\}}$ for some subset $\mathcal{N}'$ of the vertex set of Δ. More precisely, $\mathcal{N}' = \{i \in \mathcal{N} : m_i|m\}$, so by the definition of least common multiple, $\mathrm{lcm}\{m_i|i \in \mathcal{N}'\}$ divides m, and so $\Delta_{\mathrm{lcm}\{m_i|i\in\mathcal{N}'\}} \subset \Delta_m$. Conversely, if $A \in \Delta_m$ is a face, then $m_A|m$ by definition of Δ_m, then as $m_A = \mathrm{lcm}\{m_i|i \in A\}$ for all $i \in A$ we have $m_i|m_A|m$ which means $i \in \mathcal{N}'$, so $A \subset \mathcal{N}'$ and thus $m_A|\mathrm{lcm}\{m_i|i \in \mathcal{N}'\}$ and therefore $A \in \Delta_{\mathrm{lcm}\{m_i|i\in\mathcal{N}'\}}$.

Definition 3.6. A full subcomplex of Δ is a subcomplex consisting of all the faces of Δ that involve a particular set of vertices. Note that we have shown that all the subcomplexes $\Delta_m \subset \Delta$ are full in Δ.

3.2 Syzygies of Monomial Ideals

Theorem 3.7 (Bayer, Peeva, and Sturmfels)**.** *Let Δ be a simplicial complex labeled by monomials $m_1, \ldots, m_t \in S$, and let $I = (m_1, \ldots, m_t) \subset S$ be the ideal in S generated by the vertex labels. The complex $\mathcal{C}(\Delta) = \mathcal{C}(\Delta; S)$ is a free resolution of S/I if and only if the reduced simplicial homology $H_i(\Delta_m; \mathbb{K})$ vanishes for every monomial m and every $i \geq 0$. Moreover, $\mathcal{C}(\Delta)$ is a minimal free resolution if and only if $m_A \neq m_{A'}$ for every proper subface A' of a face A for all faces $A \in \Delta$.*

By the above remarks, using this theorem we can determine whether $\mathcal{C}(\Delta)$ is a resolution just by checking the vanishing condition for monomials that are least common multiples of sets of vertex labels.

Proof. Let $\mathcal{C}(\Delta)$ be the complex $\mathcal{C}(\Delta) : \ldots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta} F_0$. Then $S/I = \mathrm{coker}(\delta_1 : F_1 \to F_0) = F_0/\mathrm{im}\delta_1 = S/\mathrm{im}\delta_1$ since $\mathrm{im}\delta_1 = I$. We will identify the homology of $\mathcal{C}(\Delta)$ at F_i with a direct sum of copies of vector spaces $H_i(\Delta_m; \mathbb{K})$ and then the complex $\mathcal{C}(\Delta)$ is a resolution if and only if it is exact if and only if its homology vanishes, i.e. $H_i(\mathcal{C}(\Delta; S)) = 0$, which will be if and only if all $H_i(\Delta_m; \mathbb{K}) = 0$. The identification is as follows:

For each $\alpha \in \mathbb{Z}^{r+1}$ we will compute the homology of the complex of vector spaces $\mathcal{C}(\Delta)_\alpha : \ldots \longrightarrow (F_i)_\alpha \xrightarrow{\delta} (F_{i-1})_\alpha \longrightarrow \ldots \xrightarrow{\delta} (F_0)_\alpha$, formed from the degree-α components of each free module F_i in $\mathcal{C}(\Delta)$. If any of the components of α are negative then $\mathcal{C}(\Delta)_\alpha = 0$ as a complex, so the homology vanishes in this degree. That is to say, we may suppose $\alpha \in \mathbb{N}^{r+1}$ as we are dealing with monomials and an $\mathbb{N}^{r+1}$-grading. Now fix some $m = x^\alpha = x_0^{\alpha_0} \ldots x_r^{\alpha_r} \in S$. For each face A of Δ, the complex $\mathcal{C}(\Delta)$ has a rank-one free summand $S.A$ which, as a vector space, has a $\mathbb{K}$-basis $\{n.A | n \in S \text{ is a monomial}\}$. The degree of $n.A$ is the exponent of $n.m_A$ where m_A is the label of the face A. Thus for the degree α part of $S.A$ we have

$$(S.A)_\alpha = \begin{cases} \mathbb{K}.\frac{x^\alpha}{m_A}.A & \text{if } m_A | m \\ 0 & \text{otherwise} \end{cases}$$

It follows that the complex $\mathcal{C}(\Delta)_\alpha$ has a $\mathbb{K}$-basis corresponding bijectively to the faces of Δ_m where $m = x^\alpha$.

Using this correspondence we identify the terms of the complex $\mathcal{C}(\Delta)_\alpha$ with the terms of the reduced chain complex of Δ_m having coefficients in $\mathbb{K}$ (up to a shift in homological degree as for the case where the vertex labels are all 1). Further the differentials of the complexes agree:

For $\mathcal{C}(\Delta)_\alpha : \ldots \longrightarrow (F_i)_\alpha \xrightarrow{\delta} (F_{i-1})_\alpha \longrightarrow \ldots$ we have

$$\begin{aligned} \delta\left(\sum_{A \in \Delta_m, \#A = i} \beta_A \frac{m}{m_A} A\right) &= \sum_{A \in \Delta_m, \#A = i} \beta_A \frac{m}{m_A} \delta A \\ &= \sum_{A \in \Delta_m, \#A = i} \beta_A \frac{m}{m_A} \sum_{n \in A} (-1)^{\mathrm{pos}(n,A)} \frac{m_A}{m_{A \setminus n}} A \setminus n \\ &= \sum_{A \in \Delta_m, \#A = i, n \in A} (-1)^{\mathrm{pos}(n,A)} \beta_A \frac{m}{m_{A \setminus n}} A \setminus n \end{aligned}$$

and correspondingly in $\mathcal{C}(\Delta_m; \mathbb{K})$ we have

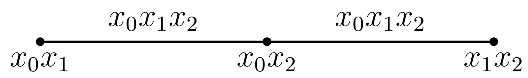
$$\begin{aligned} \delta\left(\sum_{A \in \Delta_m, \#A = i} \beta_A A\right) &= \sum_{A \in \Delta_m, \#A = i} \beta_A \delta A \\ &= \sum_{A \in \Delta_m, \#A = i} \beta_A \sum_{n \in A} (-1)^{\mathrm{pos}(n,A)} A \setminus n \\ &= \sum_{A \in \Delta_m, \#A = i, n \in A} (-1)^{\mathrm{pos}(n,A)} \beta_A A \setminus n \end{aligned}$$

Thus, having identified $\mathcal{C}(\Delta)_\alpha$, $\alpha \in \mathbb{N}^{r+1}$ with the reduced chain complex of Δ_m where $m = x^\alpha$, we see that the complex $\mathcal{C}(\Delta)$ is a resolution of S/I if and only if $H_i(\Delta_m; \mathbb{K}) = 0$ for all $i \geq 0$, as required for the first statement.

In summary, $\mathcal{C}(\Delta; S) : \ldots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \ldots$ is a resolution of S/I iff it is exact at each F_i, $\forall i \geq 1$ iff the homology $H_i(\mathcal{C}(\Delta; S)) = 0$, $\forall i \geq 0$ iff the homology of vector spaces $H_i(\mathcal{C}(\Delta)_\alpha) = 0$, $\forall i \geq 0$ for all graded components $\mathcal{C}(\Delta)_\alpha$, $\alpha \in \mathbb{N}^{r+1}$, since the degree of the differential is zero so preserves graded pieces and restricts to an exact sequence of graded pieces/vector spaces; and $H_i(\mathcal{C}(\Delta; S)_\alpha) \cong H_i(\Delta_m; \mathbb{K})$ where $m = x^\alpha$.

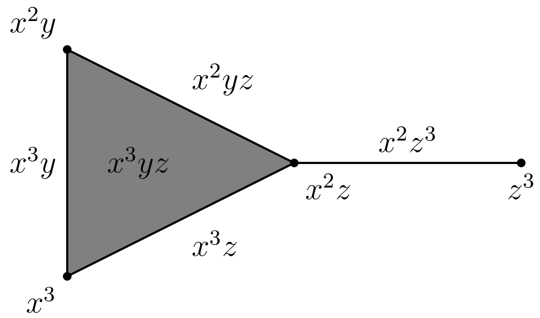
For minimality, note that if A is an $(i+1)$-face and A' is an i-face of Δ, then the component of the differential of $\mathcal{C}(\Delta)$ that maps $S.A$ to $S.A'$ is zero unless $A' \subset A$, in which case it is $\pm \frac{m_A}{m_{A'}}$ by definition. Thus $\mathcal{C}(\Delta)$ is minimal if and only if $m_A \neq m_{A'}$ for all $A' \subset A$, as required. $\square$

Example 3.8. We continue with the ideal (x_0x_1, x_0x_2, x_1x_2) as above in example 3.4. For the labeled simplicial complex Δ



the distinct subcomplexes Δ' of the form Δ_m are the empty complex Δ_1, the complexes $\Delta_{x_0x_1}, \Delta_{x_0x_2}, \Delta_{x_1x_2}$, each of which consists of a single point, and the whole complex Δ itself. As all the subcomplexes of Δ of the form Δ_m are contractible, they have no higher reduced homology (with any coefficients) than H_0, and thus we can conclude that the complex $\mathcal{C}(\Delta)$ is the minimal free resolution of $S/(x_0x_1, x_0x_2, x_1x_2)$.

Example 3.9. The ideal $I = (x^3, x^2y, x^2z, z^3) \subset S = \mathbb{K}[x, y, z]$ has minimal free resolution $\mathcal{C}(\Delta)$ where Δ is the labeled simplicial complex



The distinct subcomplexes of the form Δ_m are the empty complex Δ_1, the vertices $\Delta_{x^3}, \Delta_{x^2y}, \Delta_{x^2z}, \Delta_{z^3}$, the edges $\Delta_{x^3y}, \Delta_{x^3z}, \Delta_{x^2yz}, \Delta_{x^2z^3}$, two paths in Δ described by $\Delta_{x^3z^3}$ and $\Delta_{x^2yz^3}$, the 2-face Δ_{x^3yz}, and the whole complex $\Delta = \Delta_{x^3yz^3}$, all of which are contractible. The Betti diagram for S/I is

	0	1	2	3
0	1	-	-	-
1	-	-	-	-
2	-	4	3	1
3	-	-	1	-

Then by corollary 2.21 we have Hilbert function

$$H_{S/I}(d) = \binom{d+2}{2} - 4\binom{d-1}{2} + 3\binom{d-2}{2}$$

and Hilbert polynomial $P_{S/I}(d) \equiv 6$. By directly computing small values of the Hilbert function we find that in this case the bound given in corollary 2.8 is not sharp. That is, the Hilbert function and polynomial coincide for all $d \geq 2$ where $2 < 3 = \max_{i,j}\{a_{i,j}\} - r$.

Example 3.10. Noting that any full subcomplex of a simplex is itself a simplex, and the above examples, we come to an idea which gives a result first proved (in a different way) by Diana Taylor [Eis95].

3.3 Taylor and Koszul complexes

Definition 3.11. Let $I = (m_1, \ldots, m_n) \subset S$ be any monomial ideal, and let Δ be a simplex with n vertices, labeled $m_1, \ldots, m_n$. The complex $\mathcal{C}(\Delta)$, called the Taylor complex of $m_1, \ldots, m_n$ is a free resolution of S/I. This is because a simplex is contractible, so has no higher reduced homology.

Example 3.12. The Taylor complex is rarely minimal. For instance, taking $(m_1, m_2, m_3) = (x_0x_1, x_0x_2, x_1x_2)$ as before, we get a Taylor complex

$$0 \to S(-3) \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} S(-3)^3 \xrightarrow{\begin{pmatrix} 0 & -x_2 & -x_2 \\ -x_1 & 0 & x_1 \\ x_0 & x_0 & 0 \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} x_0x_1 & x_0x_2 & x_1x_2 \end{pmatrix}} S$$

This Taylor complex is a nonminimal resolution with Betti diagram:

	0	1	2	3
0	1	-	-	1
1	-	3	3	-

Further, via the change of basis $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ at $S(-3)^3$ we may decompose the above resolution into a direct sum of the resolution seen in example 3.4 and a trivial complex.

Definition 3.13. We may define the Koszul complex $\mathbf{K}(x_0, \ldots, x_r)$ of $x_0, \ldots, x_r$ to be the Taylor complex in the special case where the $m_i = x_i$ are variables. The smallest cases were exhibited in example 2.4. By theorem 3.7 the Koszul complex is a minimal free resolution of the residue class field $\mathbb{K} = S/(x_0, \ldots, x_r)$ since each face has unique label which is the product (as least common multiple) of its vertex labels.

Example. $\mathbf{K}(x_0, x_1, x_2)$:

$$0 \to S(-3) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}} S^3(-1) \xrightarrow{\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}} S$$

We can replace the variables $x_0, \ldots, x_r$ by any polynomials $f_0, \ldots, f_r$ to obtain a complex which we will write as $\mathbf{K}(f_0, \ldots, f_r)$, the Koszul complex of the sequence $f_0, \ldots, f_r$. In fact, since the differentials have only $\mathbb{Z}$-coefficients, we could even take the f_i to be elements of an arbitrary commutative ring. Under nice circumstances, for example when the f_i are homogeneous elements of positive degree in a graded ring, this complex is a resolution if and only if the f_i form a regular sequence. [Eis95, Theorem 17.6]

Example 3.14. By theorem 3.7, the Koszul complex $\mathbf{K}(x_0, \ldots, x_{n-1})$ of $x_0, \ldots, x_{n-1}$ in $S = k[x_0, \ldots, x_r]$, $n \leq r+1$ is a minimal free resolution of $S_X = S/(x_0, \ldots, x_{n-1})$ where the associated variety $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a complete intersection of hyperplanes. Each i-face of this simplex on n-vertices has (unique) label which is the product of its vertex labels, so is of degree i, and there are $\binom{n}{i}$ such i-faces in a simplex on n vertices. Therefore

$$F_i = S(-i)^{\binom{n}{i}}, \; \beta_{i,j} = \begin{cases} \binom{n}{i} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

and we have Betti diagram:

	0	...	i	...	n
0	1	...	$\binom{n}{i}$	...	1

and in the notation of corollary 2.21 we have

$$B_j = \sum_{i=0}^{\infty}(-1)^i \beta_{i,j} = (-1)^j \binom{n}{j},$$

thus we may deduce the combinatorial formula

$$H_{S_X}(d) = \sum_{j=0}^{\infty} B_j \binom{r+d-j}{r} = \sum_{j=0}^{n}(-1)^j \binom{n}{j}\binom{r+d-j}{r} = \binom{r+d-n}{r-n}$$

by identifying $S_X = S/(x_0, \ldots, x_{n-1}) \cong k[x_n, \ldots, x_r] \cong k[x_0, \ldots, x_{r-n}]$ which may otherwise follow inductively from the combinatorial relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Note that $H_{S_X}(d)$ is given by the associated polynomial for all $d \geq 0$.

3.4 Hilbert's Syzygy Theorem - a strengthening

We can use the Koszul complex and theorem 3.7 to prove a sharpening of Hilbert's Syzygy theorem (theorem 2.10), which is the vanishing statement in the following proposition. We also get an alternate way to compute the graded Betti numbers.

Proposition 3.15. *Let M be a finitely generated graded module over $S = k[x_0, \ldots, x_r]$.* The graded Betti number $\beta_{i,j}(M)$ is the dimension of the homology, at the term $M_{j-i} \bigotimes \bigwedge^i \mathbb{K}^{r+1}$, of the complex:

$$0 \to M_{j-(r+1)} \bigotimes \bigwedge^{r+1} \mathbb{K}^{r+1} \to \ldots$$
$$\to M_{j-i-1} \bigotimes \bigwedge^{i+1} \mathbb{K}^{r+1} \to \quad M_{j-i} \bigotimes \bigwedge^i \mathbb{K}^{r+1} \to \quad M_{j-i+1} \bigotimes \bigwedge^{i-1} \mathbb{K}^{r+1} \to$$
$$\ldots \to M_j \bigotimes \bigwedge^{0} \mathbb{K}^{r+1} \to 0$$

In particular, we have $\beta_{i,j}(M) \leq H_M(j-i)\binom{r+1}{i}$, so $\beta_{i,j}(M) = 0$ if $i > r+1$.

Proof. By proposition 2.17 and the symmetry of the Tor operator, $\beta_{i,j}(M) = \dim_{\mathbb{K}} \mathrm{Tor}_i(\mathbb{K}, M)_j = \dim_{\mathbb{K}} \mathrm{Tor}_i(M, \mathbb{K})_j$. Then as $\mathbf{K}(x_0, \ldots, x_r)$ is the minimal free resolution of $\mathbb{K} = S/(x_0, \ldots, x_r)$ we may compute $\mathrm{Tor}_i^S(M, \mathbb{K})_j$ as the degree-j part of the graded $\mathbb{K}$-vector space which is the i-th homology of the complex $M \bigotimes_S \mathbf{K}(x_0, \ldots, x_r)$. Now $(\mathbf{K}(x_0, \ldots, x_r))_i$ is the free S-module with basis consisting of faces with i elements of the simplex with $r+1$ vertices, labeled by $x_0, \ldots, x_r$. Faces (i.e. subsets) of $\{x_0, \ldots, x_r\}$ of size i can be identified with multi-indices on $\{0, \ldots, r\}$ of length i, and so with tuples $e_{k_1} \wedge \ldots \wedge e_{k_i}$, $0 \leq k_1 <$

$\ldots < k_i \leq r$. So the basis of $(\mathbf{K}(x_0, \ldots, x_r))_i$ by i-faces corresponds to a basis of $\bigwedge S^{r+1}(-i) = S(-i)^{\binom{r+1}{i}}$. Thus we need the homology at the term

$$\left(M \bigotimes_S \mathbf{K}(x_0, \ldots, x_r)\right)_i = M \bigotimes_S \bigwedge^i S^{r+1}(-i) = M \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}(-i).$$

Decomposing M into its homogeneous components $M = \bigoplus M_k$, we see that the degree-j part of $M \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}(-i)$ is

$$\left(M \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}(-i)\right)_j = M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}.$$

The differentials of $M \bigotimes_S \mathbf{K}(x_0, \ldots, x_r)$ preserves degrees, so the complex decomposes as a direct sum of complexes of $\mathbb{K}$-vector spaces of the form:

$$\ldots \to M_{j-i-1} \bigotimes_{\mathbb{K}} \bigwedge^{i+1} \mathbb{K}^{r+1} \to M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^{i} \mathbb{K}^{r+1} \to M_{j-i+1} \bigotimes_{\mathbb{K}} \bigwedge^{i-1} \mathbb{K}^{r+1} \to \ldots$$

Now homology and graded components commute since quotient and grading (i.e. direct sum) commute and the maps are degree preserving. Thus $\beta_{i,j}(M)$ is the dimension of the homology at the term $M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}$ of the above complex, thus proving the first statement.

Then

$$\begin{aligned}
\beta_{i,j}(M) &= \dim_{\mathbb{K}} \left(\frac{\ker\left(M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1} \to M_{j-i+1} \bigotimes_{\mathbb{K}} \bigwedge^{i-1} \mathbb{K}^{r+1}\right)}{\operatorname{Im}\left(M_{j-i-1} \bigotimes_{\mathbb{K}} \bigwedge^{i+1} \mathbb{K}^{r+1} \to M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^{i} \mathbb{K}^{r+1}\right)} \right) \\
&\leq \dim_{\mathbb{K}} M_{j-i} \bigotimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1} = \dim_{\mathbb{K}} M_{j-i} . \dim_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1} \\
&= H_M(j-i) \binom{r+1}{i}
\end{aligned}$$

and so $\beta_{i,j}(M) = 0$ if $i > r+1$ as then $\binom{r+1}{i} = 0$.

The (unique) minimal free resolution of M has length at most $r+1$ as there are no generators for F_i in any degree when $i > r+1$ in the minimal free resolution as $\beta_{i,j}(M) = 0$ for $i > r+1$. This proves Hilbert's syzygy theorem 2.10. $\square$

4 Castelnuovo-Mumford Regularity

4.1 Definition and first applications

The Castelnuovo-Mumford regularity of an ideal in S is an important measure of how complicated the ideal is. A first approximation is the maximum degree of a generator the ideal requires; the actual definition involves the syzygies (relations between generators) as well.
Definition 4.1. Let $S = k[x_0, \dots, x_r]$ and let

$$\mathbf{F} : \dots \longrightarrow F_i \xrightarrow{\varphi_i} F \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0.$$

be a graded complex of free S-modules, with $F_i = \sum_j S(-a_{i,j})$. We define the Castelnuovo-Mumford regularity, or simply regularity, of $\mathbf{F}$ to be $\mathrm{reg}\mathbf{F} = \sup_{i,j}(a_{i,j} - i)$. The regularity of a finitely generated graded S-module M is the regularity of a minimal graded free resolution of M, which we denote by $\mathrm{reg}M$. In case $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a projective variety and I_X is its ideal, then $\mathrm{reg}I$ is called the regularity of X, or $\mathrm{reg}X$.

Suppose M is free, so we have $M \cong F_0 = \bigoplus_j S(-a_{0,j})$, then the regularity of M is the supremum of the degrees of a set of homogeneous minimal generators of M, since $\mathrm{reg}M = \sup_j\{a_{0,j} - 0\} = \sup_j\{a_{0,j}\}$.

In general, the regularity of M is an upper bound for the largest degree of a minimal generator of M, which is the supremum of the numbers $a_{0,j} - 0$.

Assuming that M is generated by elements of degree 0, the regularity of M is the index of the last nonzero row in the Betti diagram of M:

Recall that the entry in the j-th row of the i-th column of the Betti diagram of M is $\beta_{i,i+j}$ and using alternative expressions

$$F_i = \bigoplus_{k=1}^{n_i} S(-a_{i,k}) = \bigoplus_{j=0}^{\infty} S(-j)^{\beta_{i,j}}$$

for the modules in the minimal free resolution we find that

$$\mathrm{reg}M = \sup_{i,k} a_{i,k} - i = \sup_{i,j:\beta_{i,j}>0} j - i = \sup_{i,j:\beta_{i,i+j}>0} j$$

The power of the notion of regularity comes from an alternate description in terms of cohomology, which might seem to have little to do with free resolutions. Historically, the cohomological interpretation came first. David Mumford defined the regularity of a coherent sheaf on projective space in order to generalise a classical argument of Castelnuovo. Mumford's definition is given in terms of sheaf cohomology. The definition for modules, which extends that for sheaves, and the equivalence with the condition on the resolution used as a definition above came from [Eisenbud and Goto, 1984]. In most cases the regularity of a sheaf, in the (cohomological) sense of Mumford, is equal to the regularity of the graded module of its twisted global sections.

Recall the Hilbert function of a finitely generated graded S-module M is $H_M(d) = \dim_{\mathbb{K}} M_d$, and it is equal to a polynomial function $P_M(d)$ for large d. The regularity of M provides a bound, which is sharp in the case of a Cohen-Macaulay module for when $H_M(d) = P_M(d)$.

Theorem 4.2. *Let M be a finitely generated graded module over the polynomial ring $S = k[x_0, \ldots, x_r]$. Then,*

(1) The Hilbert function $H_M(d)$ agrees with the Hilbert polynomial $P_M(d)$ for all $d \geq \mathrm{reg}M + 1$;

(2) More precisely, if M is a module of projective dimension δ, then $H_M(d) = P_M(d)$ for all $d \geq \mathrm{reg}M + \delta - r$;

(3) If $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a nonempty set of points and $M = S_X = S/I(X)$, then $H_M(d) = P_M(d)$ if and only if $d \geq \mathrm{reg}M$.

In general, if M is a Cohen-Macaulay module, then the bound in part (2) is sharp.

Proof. Part (1) follows from part (2), the Hilbert Syzygy theorem, and the result that the projective dimension δ is equal to the length of the minimal free resolution (Corollary 2.18), which is less than the number of variables in S. That is to say, $\delta \leq r + 1$, so $\mathrm{reg}M + \delta - r \leq \mathrm{reg}M + 1$.

Part (2): To prove that if M is a module of projective dimension δ, then $H_M(d) = P_M(d)$ for all $d \geq \mathrm{reg}M + \delta - r$.

Consider the minimal graded free resolution of M, which is of the form

$$\mathbf{F} : 0 \longrightarrow \bigoplus_{j=1}^{i_\delta} S(-a_{\delta,j}) \longrightarrow \ldots \longrightarrow \bigoplus_{j=1}^{i_0} S(-a_{0,j}) \to M \to 0$$

by corollary 2.18. In these terms $\mathrm{reg}M = \max_{i,j}(a_{i,j} - i)$.

We can compute the Hilbert function or polynomial of M by taking the alternating sum of the Hilbert functions or polynomials of free modules in the resolution of M. In this way we obtain expressions as in chapter 2:

$$
\begin{aligned}
H_M(d) &= \sum_{i=0}^{\delta}(-1)^i \sum_j \binom{r+d-a_{i,j}}{r} \\
P_M(d) &= \sum_{i=0}^{\delta}(-1)^i \sum_j \frac{(d-a_{i,j}+r)(d-a_{i,j}+r-1)\dots(d-a_{i,j}+1)}{r!}
\end{aligned}
$$

This expansion for P_M is the expression for H_M with each binomial coefficient replaced by the polynomial to which it is eventually equal.

In fact the binomial coefficient $\binom{d-a+r}{r}$ has the same value as the degree r polynomial $\frac{(d-a+r)(d-a+r-1)\dots(d-a+1)}{r!}$ for all $d \geq a-r$. Thus from $d \geq \mathrm{reg}M + \delta - r$ we get $d \geq a_{i,j} - i + \delta - r \geq a_{i,j} - r$ for each $a_{i,j}$ with $i \leq \delta$. For such d, each term in the expression of the Hilbert function is equal to the corresponding term in the expression of the Hilbert polynomial, and this proves part (2).

The proof of part (3) and the final remark can be found in [Eis05, Theorem 4.2]. $\square$

4.2 The Regularity of a Cohen-Macaulay Module

Definition 4.3. The (Krull) dimension of a ring R, written $\dim R$, is the supremum of lengths of chains of primes ideals of R. That is $P_0 \subsetneqq P_1 \subsetneqq \dots \subsetneqq P_n$ has length n. If P is a prime ideal, the codimension of P, written codim$P$, is the maximum of the lengths of chains of prime ideals $P \supsetneqq \dots \supsetneqq P_0$ descending from P. If I is any ideal, the codimension of I is the minimum of the codimension of primes containing I. See [Eis95, Chapter 8] for a discussion linking these rather algebraic notions with geometry.

The dimension $\dim M$ of an R-module M is defined $\dim M = \dim R/\mathrm{ann}M$.

Definition 4.4. A sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R is called a regular sequence if $x_1, \dots, x_n$ generate a proper ideal of R, and for each i the element x_i is a nonzerodivisor modulo $(x_1, \dots, x_{i-1})$.

Similarly, if M is an R-module, then $\mathbf{x}$ is a regular sequence on M (or M-sequence) if $(x_1, \dots, x_n)M \neq M$ and for each i the element x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$.

An ideal that can be generated by a regular sequence (or, in the geometric case, the variety it defines) is called a complete intersection.

Definition 4.5. If I is an ideal of R and M is a finitely generated module such that $IM \neq M$, then the depth of I on M, written $\mathrm{depth}(I, M)$, is the maximal length of a regular sequence on M contained in I. If $IM = M$ we set $\mathrm{depth}(I, M) = \infty$.

In the case where R is a local or homogeneous algebra and I is the maximal homogeneous ideal we write $\mathrm{depth}M$ in place of $\mathrm{depth}(I, M)$.

We define the grade of I to be $\mathrm{grade}I = \mathrm{depth}(I, R)$.

In particular, for $S = k[x_0, \ldots, x_r]$ and a finitely generated S-module M we have $\mathrm{depth}M = \mathrm{depth}(\mathfrak{m}, M)$ is the maximal length of a regular sequence on M contained in $\mathfrak{m} = (x_0, \ldots, x_r)$. It follows that $\mathrm{depth}S = r + 1$ from the maximal regular sequence $x_0, \ldots, x_r$.

Definition 4.6. Recall that the projective dimension $\mathrm{pd}M$ of an R-module M is the minimum length of a projective resolution of M (or ∞ if there is no finite projective resolution), and that in corollary 2.18 we proved that when M is a finitely generated S-module its projective dimension is equal to the length of the minimal free resolution of M.

Definition 4.7. A local ring R is Cohen-Macaulay if $\mathrm{depth}R = \dim R$. More generally, an R-module M is Cohen-Macaulay if $\mathrm{depth}M = \dim M$. If $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a projective variety, we say that X is arithmetically Cohen-Macaulay if the homogeneous coordinate ring $S_X = \mathbb{K}[x_0, \ldots, x_r]/I(X)$ is Cohen-Macaulay.

Fact. *$\mathbb{K}[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_k^{a_k})$ is a Cohen-Macaulay ring for any positive integers $k \leq n$ and $a_1, \ldots, a_k$.*

Lemma 4.8. *Suppose that M is a finitely generated graded S-module. If x is a linear form in S (i.e. a degree 1 element) that is a nonzerodivisor on M (that is, $xm = 0 \implies m = 0 \in M$), then $\mathrm{reg}M = \mathrm{reg}M/xM$.*

Proof. Let $\mathbf{F}$ be the minimal free resolution of M. From the free resolution $\mathbf{K}(x)$: $0 \to S(-1) \xrightarrow{x} S$ of $S/(x)$ we compute

$$\begin{aligned} \mathrm{Tor}_*(M, S/(x)) : \quad & 0 \to M \bigotimes_S S(-1) \xrightarrow{\mathrm{id}_M \otimes x} M \bigotimes_S S \\ \cong \quad & 0 \to M(-1) \xrightarrow{x} M \end{aligned}$$

As x is a nonzerodivisor on M, the sequence is exact at $M(-1)$ and so $\mathrm{Tor}_0(M, S/(x)) = M/xM$ and $\mathrm{Tor}_i(M, S/(x)) = 0$ for $i > 0$.

Consider the free double complex $\mathbf{F} \bigotimes \mathbf{K}(x)$,

$$\begin{array}{ccccccc}
\dots \longrightarrow S(-1)\bigotimes_S F_i & \xrightarrow{\mathrm{id}\otimes\delta_i} & S(-1)\bigotimes_S F_{i-1} & \xrightarrow{\mathrm{id}\otimes\delta_{i-1}} & S(-1)\bigotimes_S F_{i-2} \longrightarrow \dots \\
\downarrow{\scriptstyle x\otimes id} & & \downarrow{\scriptstyle x\otimes id} & & \downarrow{\scriptstyle x\otimes id} \\
\dots \longrightarrow S\bigotimes_S F_i & \xrightarrow{\mathrm{id}\otimes\delta_i} & S\bigotimes_S F_{i-1} & \xrightarrow{\mathrm{id}\otimes\delta_i} & S\bigotimes_S F_{i-2} \longrightarrow \dots
\end{array}$$

which (as above) is isomorphic to

$$\begin{array}{ccccccc}
\dots \longrightarrow F_i(-1) & \xrightarrow{\delta_i} & F_{i-1}(-1) & \xrightarrow{\delta_{i-1}} & F_{i-2}(-1) \longrightarrow \dots \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\
\dots \longrightarrow F_i & \xrightarrow{\delta_i} & F_{i-1} & \xrightarrow{\delta_i} & F_{i-2} \longrightarrow \dots
\end{array}$$

We can also compute Tor as the total homology of $\mathbf{F}\bigotimes\mathbf{K}(x)$, [Weibel 1995]

$$\dots \to F_i \bigoplus F_{i-1}(-1) \xrightarrow{\begin{pmatrix} \delta_i & x \\ 0 & \delta_{i-1} \end{pmatrix}} F_{i-1} \bigoplus F_{i-2}(-1) \to \dots$$

and we see that $\mathbf{F}\bigotimes\mathbf{K}(x)$ is the minimal free resolution of M/xM. The i-th free module in $\mathbf{F}\bigotimes\mathbf{K}(x)$ is $F_i\bigoplus F_{i-1}(-1)$ so $\mathrm{reg}M/xM = \mathrm{reg}M$ by the definition of regularity. □

Corollary 4.9. *Let M be a finitely generated Cohen-Macaulay graded S-module, and let $y_1, \dots, y_t$ be a maximal M-regular sequence of linear forms. The regularity of M is the largest d such that $(M/(y_1, \dots, y_t)M)_d \neq 0$.*

Proof. As M is Cohen-Macaulay we have $\dim M/(y_1,\dots,y_t)M = 0$, and by lemma 4.8 above we have $\mathrm{reg}M = \mathrm{reg}M/(y_1,\dots,y_t)M$, so we have reduced to the zero dimensional case. In the case $\dim M = 0$ we complete the proof by applying the following lemma. □

Lemma 4.10. *If M is a graded S-module of finite length, then*

$$\mathrm{reg}M = \max\{d \mid M_d \neq 0\}.$$

Proof. This is proven using cohomological techniques. See [Eis05, Theorem 4.3]. □

Theorem 4.11. *Suppose that $X \subset \mathbb{P}^r$ is a projective variety not contained in any hyperplane. If S_X is Cohen-Macaulay, then $\mathrm{reg}S_X \leq \deg X - \mathrm{codim}X$.*

Proof. Let $t = \dim X$, so that the dimension of S_X as a module is $t+1$. Without loss of generality we may extend the ground field in order to assume that it is algebraically closed, and in particular infinite. Thus we may assume that a sufficiently general sequence of linear forms $y_0, \ldots, y_t$ is a regular sequence in any order on S_X. Set $\bar{S}_X = S_X/(y_0, \ldots, y_t)$. As X is not contained in any hyperplane, then the corresponding homogeneous ideal I_X does not contain any linear form, so we have $\dim_{\mathbb{K}}(S_X)_1 = r+1$. Thus $\dim_{\mathbb{K}}(\bar{S}_X)_1 = (r+1)-(t+1) = r-t = \mathrm{codim}X$. If the regularity of S_X is d, then by corollary 4.9 we have $(\bar{S}_X)_d \neq 0$, so $H_{\bar{S}_X}(d) \neq 0$. Since $\bar{S}_X$ is generated as an S-module by 1 in degree 0, this implies that $H_{\bar{S}_X}(e) \neq 0$ for all $0 \leq e \leq d$. On the other hand, $\deg X$ is the number of points in which X meets the linear space L defined by $y_1 = \ldots = y_t = 0$ by definition, since this linear space has complementary dimension $\dim L = r-t = \mathrm{codim}X$. By induction on d, and using the graded exact sequence

$$0 \to S_X/(y_1, \ldots, y_t)(-1) \xrightarrow{y_0} S_X/(y_1, \ldots, y_t) \to \bar{S}_X \to 0$$

we see that $H_{S_X/(y_1,\ldots,y_t)}(d) = \sum_{e=0}^{d} H_{\bar{S}_X}(e)$. That is, passing to the exact sequence of the d-th graded components and applying rank-nullity theorem we find that $H_{S_X/(y_1,\ldots,y_t)}(d) = H_{\bar{S}_X}(d) + H_{S_X/(y_1,\ldots,y_t)}(d-1)$. For large d the polynomials of degree d induce all possible functions on the set of points $X \cap L$, so $\deg X = H_{S_X/(y_1,\ldots,y_t)}(d)$. [Eis05, Theorem 4.1]

It follows that for large d

$$\begin{aligned}
\deg X &= H_{S_X/(y_1,\ldots,y_t)}(d) = \sum_{e=0}^{d} H_{\bar{S}_X}(e) = H_{\bar{S}_X}(0) + H_{\bar{S}_X}(1) + \sum_{e=2}^{d} H_{\bar{S}_X}(e) \\
&= 1 + \mathrm{codim}X + \sum_{e=2}^{d} H_{\bar{S}_X}(e) \geq 1 + \mathrm{codim}X + (\mathrm{reg}X - 1)
\end{aligned}$$

as there are at least $\mathrm{reg}X - 1$ positive values of $H_{\bar{S}_X}(e)$ for $e = 2, \ldots, d$, since $H_{\bar{S}_X}(e) \neq 0$ for all $0 \leq e \leq d$. This gives $\mathrm{reg}X \leq \deg X - \mathrm{codim}X$ as required. $\square$

In the most general case, the regularity can be very large. Consider the case of a module of the form $M = S/I$. Groebner basis methods give a general bound for the regularity of M in terms of the degrees of generators of I and the number of variables, but these bounds are doubly exponential in the number of variables. [Eis05, Corollary 4.15]

5 Toric Ideals

In this chapter we introduce a class of polynomial ideals called toric ideals which allow interactions between algebra, geometry, and combinatorics. They are the defining ideals of toric varieties which are a rich but fairly accessible class of varieties in algebraic geometry.

Definition 5.1. Fix a subset $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that the matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix} \in \mathbb{Z}^{d\times n}$ has rank d. We have a semigroup homomorphism:

$$\pi : \mathbb{N}^n \to \mathbb{Z}^d,\ \mathbf{u} = (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} \mathbf{a}_i u_i = A\mathbf{u}$$

i.e. The map π distributes over addition of vectors. We define the monoid (semigroup) generated by $\mathcal{A}$ to be $\mathbb{N}\mathcal{A} := \pi(\mathbb{N}^n) = \{A\mathbf{u} : \mathbf{u} \in \mathbb{N}^n\}$.

The semigroup ring of $\mathbb{N}^n$ is $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ and the semigroup ring of $\mathbb{Z}^d$ is the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. The map π lifts to the ring homomorphism:

$$\hat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}],\ x_j \mapsto \mathbf{t}^{\mathbf{a}_j} := t_1^{a_{1j}} t_2^{a_{2j}} \ldots t_d^{a_{dj}}$$

from the matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix} = (a_{ij})_{d\times n}$.

The toric ideal of $\mathcal{A}$, denoted as $I_{\mathcal{A}}$, is the kernel of the map $\hat{\pi}$. That is $I_{\mathcal{A}} = \ker \hat{\pi} \lhd k[\mathbf{x}]$.

In this setting it is natural to grade the polynomial ring $k[\mathbf{x}]$ by setting $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$. Then the set of all degrees of polynomials in $k[\mathbf{x}]$ is $\mathbb{N}\mathcal{A}$. We say a polynomial in $k[\mathbf{x}]$ is $\mathcal{A}$-homogeneous if it is homogeneous under this multigrading.

Example 5.2. The Veronese and Segre embeddings each give rise to toric ideals. For example, the standard twisted cubic curve in $\mathbb{P}^3_{\mathbb{C}}$ is determined by the matrix $A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. More generally, the degree-m Veronese embedding of $\mathbb{P}^n$ yields a toric variety determined by an $N \times (n+1)$-matrix where $N = \binom{n+m}{n}$ whose columns consist of all the distinct $(n+1)$-tuples $i_0, \ldots, i_n$ such that $\sum_{j=0}^{n} i_j = m$, $0 \le i_j \le m$. That is, each column in the matrix A determining the variety $v_m(\mathbb{P}^n)$ consists of nonnegative integers whose sum is m.

Proposition 5.3. *(1) The toric ideal $I_{\mathcal{A}}$ is a prime ideal in $k[\mathbf{x}]$.*

(2) The ring $k[\mathbf{x}]/I_{\mathcal{A}}$ has Krull dimension d.

(3) The ideal $I_{\mathcal{A}}$ is generated as a $\mathbf{k}$-vector space by the infinitely-many binomials $\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \pi(\mathbf{u}) = \pi(\mathbf{v}),\ u, v \in \mathbb{N}^n\}$, and hence $I_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \pi(\mathbf{u}) = \pi(\mathbf{v}) \rangle$.

(4) For every term order $\prec$ the reduced Groebner basis of $I_\mathcal{A}$ with respect to $\prec$ consists of a finite set of binomials of the form $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in I_\mathcal{A}$.

Proof. (1) By the first isomorphism theorem we have $k[\mathbf{x}]/I_\mathcal{A} \cong \pi\,(\hat{k}[\mathbf{x}]) = k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$, which is an integral domain, hence $I_\mathcal{A}$ is a prime ideal.

(2) This follows from our requirement that the matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix} \in \mathbb{Z}^{d \times n}$ has rank d. See [Eis95, Chapter 8].

(3), (4) See [Maclagan et al., Prop. 3.2.2]. $\square$

Example 5.4. Let $T = \mathbb{K}[z_0, \ldots, z_r]$ be a graded polynomial ring whose variables have degrees $\deg z_i = \alpha_i \in \mathbb{N}$. Consider the graded exact sequence of T-modules

$$0 \to T(-\alpha_r) \xrightarrow{z_r} T \to T/(z_r) \to 0.$$

We may pass to the d-th graded components to obtain an exact sequence of $\mathbb{K}$-vector spaces. Then by the rank-nullity theorem we have that $H_{T/(z_r)}(d) = H_T(d) - H_{T(-\alpha_r)}(d) = H_T(d) - H_T(d - \alpha_r)$. As $T/(z_r) \cong \mathbb{K}[z_0, \ldots, z_{r-1}]$ we may apply induction on r to deduce that the Hilbert function H_T of T is, for large d, equal to a polynomial with periodic coefficients: that is,

$$H_T(d) = h_0(d)d^r + h_1(d)d^{r-1} + \ldots$$

for some periodic functions $h_i(d)$ with values in $\mathbb{Q}$, whose periods divide the least common multiple of the α_i.

Example 5.5. Consider the toric variety given by $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}$. Then

$$\hat{\pi} : S = k[x_1, x_2, x_3, x_4] \to k[t_1, t_2],\; x_1 \mapsto t_1,\; x_2 \mapsto t_1 t_2,\; x_3 \mapsto t_1 t_2^2,\; x_4 \mapsto t_1 t_2^4$$

describes a projection to $\mathbb{P}^3$ of a rational normal quartic in $\mathbb{P}^4$. We have

$$\ker A = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle,\; I_A = (x_1x_3 - x_2^2, x_1x_4 - x_3^2).$$

We note this to be of the form as we have seen in example 2.4 so we have minimal free resolution

$$0 \to S \xrightarrow{\begin{pmatrix} x_1x_4 - x_3^2 \\ x_2^2 - x_1x_3 \end{pmatrix}} S(-2)^2 \xrightarrow{\begin{pmatrix} x_1x_3 - x_2^2 & x_1x_4 - x_3^2 \end{pmatrix}} S.$$

6 Eisenbud-Goto Conjecture

Recall that the regularity of M is an upper bound for the largest degree of a minimal generator of M. The following theorem of Gruson, Lazarsfeld, and Peskine gives an optimal upper bound for the regularity of a projective curve in terms of its degree, even if S_X is not Cohen-Macaulay (Compare with theorem 4.11):
Theorem 6.1 (Gruson-Lazarsfeld-Peskine)**.** *Suppose* $\mathbb{K}$ *is an algebraically closed field. If* $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a reduced and irreducible curve, not contained in a hyperplane, then $\mathrm{reg} S_X \leq \deg X - \mathrm{codim} X = \deg X - r + 1$, and thus $\mathrm{reg} I_X \leq \deg X - r + 2$.

In particular, this implies that the degrees of the polynomials needed to generate I_X are bounded by $\deg X - r + 2$. If the field $\mathbb{K}$ is the complex numbers, the degree of X may be thought of as the homology class of X in $\mathrm{H}_2(\mathbb{P}^r; \mathbb{K}) = \mathbb{Z}$, so the bound given depends only on the topology of the embedding of X. [Eis05, Theorem 5.1]

6.1 A General Regularity Conjecture

We say that a variety in a projective space is nondegenerate if it is not contained in any hyperplane. Correspondingly, we say that a homogeneous ideal is nondegenerate if it does not contain a linear form. In theorem 4.11 we proved that if $X \subset \mathbb{P}^r_{\mathbb{K}}$ is arithmetically Cohen-Macaulay and nondegenerate, then $\mathrm{reg} S_X \leq \deg X - \mathrm{codim} X$, which gives $\deg X - r + 1$ in the case of curves.

This result does not hold for schemes which are not arithmetically Cohen-Macaulay, even in the case of curves; the simplest example is where X is the union of two disjoint lines in $\mathbb{P}^3$ (see exercise 5.2), and the result can also fail when X is not reduced or the ground field is not algebraically closed (see exercises 5.3, 5.4). Further it is not enough to assume that the scheme is reduced and connected, since the cone over a disconnected set is connected and has the same codimension and regularity.

A possible way around these examples is to insist that X be reduced, and connected in codimension 1, meaning that X is pure-dimensional and cannot be disconnected by removing any algebraic subset of codimension 2.
Conjecture 6.2 (Eisenbud and Goto, 1984)**.** *If* $\mathbb{K}$ *is an algebraically closed field and* $X \subset \mathbb{P}^r_{\mathbb{K}}$ is a nondegenerate algebraic set that is connected in codimension 1, then $\mathrm{reg} S_X \leq \deg X - \mathrm{codim} X$.

The conjecture is known to hold in the Cohen-Macaulay and dimension-one cases, and for smooth surfaces in characteristic 0, arithmetically Buchsbaum surfaces, and toric varieties of low codimension. For a discussion of the literature, see [Eis05, Conjecture 5.2].

For the conjecture to have a chance, the number $\deg X - \mathrm{codim} X$ must at least be nonnegative. The following proposition establishes this inequality. Further, the hypotheses can be shown to be necessary [Eis05, Exercises 5.2-5.4].

Proposition 6.3. *Suppose that $\mathbb{K}$ is an algebraically closed field. If X is a nondegenerate algebraically closed set in $\mathbb{P}^r = \mathbb{P}^r_{\mathbb{K}}$ and X is connected in codimension 1, then $\deg X \geq 1 + \mathrm{codim} X$.*

To understand the bound, set $c = \mathrm{codim} X$ and let $p_0, \ldots, p_c$ be general points on X. As X is nondegenerate, these points can be chosen to span a plane L of dimension c. The degree of X is the number of points in which X meets a general c-plane and, by construction, L meets X in at least $c+1$ points. The problem with this argument is that L might, a priori, meet X in a set of positive dimension (i.e. L is not a sufficiently general c-plane), and this can indeed happen without some extra hypothesis, such as "connected in codimension 1". For the proof, see [Eis05, Proposition 5.3].

Remark. One can show that most questions about free resolutions of ideals can be reduced to the nondegenerate case, just as can most questions about varieties in projective space. This is discussed in more detail in [Eis05, Exercise 4.4].

6.2 Examples using Macaulay 2

In this section we verify conjecture 6.2 for some randomly generated toric ideals using the Macaulay2 software system as the corresponding toric varieties satisfy the hypothesis for the conjecture. We use the following formula to check whether our randomly generated toric ideal is Cohen-Macaulay.

Theorem 6.4 (Auslander-Buchsbaum formula)**.** *If R is a local ring and M is a finitely generated R-module such that $\mathrm{pd} M$ is finite, then $\mathrm{depth} M = \mathrm{depth} R - \mathrm{pd} M$.*

If M is a finitely generated S-module this allows us to compute $\mathrm{depth} M$ when we know the minimal free resolution of M, using corollary 2.18 and that the polynomial ring S is Cohen-Macaulay with $\mathrm{depth} S = r + 1$. It follows that the module M is Cohen-Macaulay if and only if $\mathrm{pd} M = \mathrm{codim}_S M$. [Eis95, Chapter 19].

We use the following method for each example:

1. Choose $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d \backslash \{\mathbf{0}\}$ such that the matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix} \in \mathbb{Z}^{d \times n}$ has rank d (so $d \leq n$). Further to ensure the toric ideal we obtain is

nondegenerate, we should take care that the columns of the matrix do not satisfy any linear relation.

```
A = transpose(matrix {{a_1},...,{a_n}});
```

2. Initialize a ring $S = k[\mathbf{x}] = k[x_1, \dots, x_n]$, which may be graded by setting $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \dots, n$.

```
S = QQ[x_1..x_n];
S = QQ[x_1..x_n,Degrees=>{{a_1},...,{a_n}}];
```

3. Compute $\ker A$ in order to determine I_A. Initialize the toric ideal I_A and the homogeneous coordinate ring $M = S/I_A$.

```
ker A
I = ideal(f_1,...,f_{n-d});
M = S^1/I;
M = coker gens I;
```

4. Compute the minimal free resolution of M to determine its projective dimension and hence check whether the ring S/I_A is Cohen Macaulay using the Auslander-Buchsbaum formula, or use the projective dimension function directly.

```
res M
pdim M
```

5. Compute the quantities appearing in conjecture 6.2 and check whether $\mathrm{reg} S_X \leq \deg X - \mathrm{codim} X$.

```
regularity M
degree M
codim M
```

Example 6.5. The matrix $A_1 = \begin{pmatrix} 7 & -5 & 10 & 6 & 4 & 4 \\ 9 & 1 & 0 & 10 & 5 & -10 \\ -8 & 10 & 6 & 3 & 5 & -5 \end{pmatrix}$ gives rise to the toric ideal $I_{A_1} = (x_5^6x_6^2 - x_1x_2x_3^3, x_3^{12}x_4^{31} - x_1^{10}x_5^{54}x_6^5, x_1^{10}x_3^{25} - x_4^{20}x_5^{26}x_6^{24})$.

Example 6.6. The matrix $A_2 = \begin{pmatrix} 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 2 & 1 \\ 3 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$ gives rise to the toric ideal $I_{A_2} = (x_5 - x_2, x_4x_6 - x_1, x_6^4 - x_1x_2x_3x_4^4)$. Remark: degenerate.

Example 6.7. The matrix $A_3 = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$ gives rise to the toric ideal $I_{A_3} = (x_5 - x_2, x_4^7x_5^2 - x_1x_2^4x_3^2, x_4^4x_5 - x_2^2x_3x_6)$. Remark: degenerate.

The ideal I_A is not homogeneous with the standard grading (with variables in degree 1) and so the module S/I_A does not admit a traditional graded structure, but if we choose the grading by setting $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$, then I_A becomes a homogeneous ideal. We will use a dash to denote when we use the implicit toric grading. We give the outputs from Macaulay2 when computing in these settings:

	pdim$M$	dim$M$	codim$M$	degree$M$	regularity$M$	isHomogeneous I_A
A_1	4	4	2	87	Failed	false
A_1'	4	4	2	Failed	187552	true
A_2	4	3	3	11	Failed	false
A_2'	3	3	3	840	38	true
A_3	3	4	2	1	Failed	false
A_3'	3	4	2	1	11	true

Remark. I am not confident in how much faith can be put in the above results due to their many seeming inconsistencies. One might deduce that the consequences of conjecture 6.2 do not hold when considering A_3 with its associated grading; however proposition 6.3 also fails in this case because $\deg X - \mathrm{codim} X$ is negative, which is also true in the standard grading. From case A_2 we see that we get differing outputs for both projective dimension and degree depending on our choice of grading. Further, it's suspected that the degree computation in case A_1' timed out due to an expected large "degree". Finally, as previously mentioned, if we use the standard grading on $k[x_1, \ldots, x_n]$ naively when analysing these toric ideals in Macaulay2, then the ideals are recognised as not homogeneous, so the resolution function does not return a minimal resolution and the regularity can not be computed since the expected input is a homogeneous (graded) S-module.

A Additional Macaulay2 Code

Some explicit examples of Macaulay2 code used is provided for error-checking purposes:

```
S = QQ[x_1..x_6]

A = transpose(matrix{{7,9,-8},{-5,1,10},{10,0,6},
{6,10,3},{4,5,5},{4,-10,-5}})

B = matrix {{-1,-10,10},{-1,0,0},{-3,12,25},
{0,31,-20},{6,-54,-26},{2,-5,-24}}
```

```
S = QQ[x_1..x_6,Degrees=>{{7,9,-8},{-5,1,10},{10,0,6},
{6,10,3},{4,5,5},{4,-10,-5}}]

I = ideal(x_5^6*x_6^2 - x_1*x_2*x_3^3,
x_3^12*x_4^31-x_1^10*x_5^54*x_6^5,
x_1^10*x_3^25 - x_4^20*x_5^26*x_6^24)

M = coker gens I

A = transpose(matrix{{1,1,3},{1,2,0},{2,1,1},
{0,0,1},{1,2,0},{1,1,2}})

S = QQ[x_1..x_6,Degrees=>{{1,1,3},{1,2,0},{2,1,1},
{0,0,1},{1,2,0},{1,1,2}}]

I = ideal(x_5-x_2, x_4*x_6 - x_1,
x_6^4 - x_1*x_2*x_3*x_4^4)

A = transpose(matrix{{1,1,3},{1,2,1},{2,1,1},
{1,1,1},{1,2,1},{1,1,2}})
S = QQ[x_1..x_6,Degrees=>{{1,1,3},{1,2,1},{2,1,1},
{1,1,1},{1,2,1},{1,1,2}}]

I = ideal(x_5-x_2, x_4^7*x_5^2 - x_1*x_2^4*x_3^2,
x_4^4*x_5 - x_2^2*x_3*x_6)
```

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