Module 1 Problem Set

Reviewing Proof Techniques

Group Members

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Problem 1. Prove that if $a, b \in \mathbb{R}$ are real numbers and $a < b < 0$, then $a^2 > b^2$.

Proof.
Suppose $a < b < 0$
From $a < b$, multiplying by $a < 0$ gives

$$ab > a^2,$$

Similarly, multiplying $a < b$ by $b < 0$ yields

$$b^2 < ab$$

Therefore we have

$$b^2 < ab < a^2$$

By the transitive property that equates to

$$b^2 < a^2$$

as required.

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Problem 2. Prove that $\sqrt{3}$ is irrational.

Proof.
Suppose to the contrary that $\sqrt{3}$ is rational, i.e. there exist $m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n}$$

Without loss of generality, suppose m and n do not share any factors (> 1), otherwise divide both by the common factors until this is true.
Next, square the equation

$$3 = \frac{m^2}{n^2}$$

and rearrange such that

$$m^2 = 3 \cdot n^2$$

Therefore we know 3 is a factor of m^2 and based on the property of *Unique Factorization* from the *Fundamental Theorem of Arithmetic*, we know that since 3 is a factor of m^2 and $m \in \mathbb{Z}$, m must also contain 3 in its prime factorization. With this we can rewrite m as

$$m = 3k \text{ for some integer } k$$

Then substitute $p = 3k$ for m,

$$(3k)^2 = 3 \cdot n^2$$
$$9 \cdot k^2 = 3 \cdot n^2$$

then divide both sides by 3

$$3 \times k^2 = n^2$$

Once again, given that 3 is a factor of n^2, by *Unique Factorization* we know 3 is also a factor of n. We found that

$$3|m \text{ and } 3|n$$

which contradicts the previous assumption of m and n not sharing a factor, thus $\sqrt{3}$ must be irrational.

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Problem 3. Prove that if $n \in \mathbb{N}$ and $n \geq 5$, then $2^n > n^2$.

Proof.
Argue by induction on $n \in \mathbb{N}$.
Base case
$n = 5$ and $2^n > n^2$. This is true as

$$2^5 = 32 > 25 = 5^2$$

Inductive Step
Consider any $n \geq 5$ and suppose by the induction hypothesis that there exists an integer k such that $2^k > k^2$. Then, take the inductive step from k to $k+1$ to get $2^{k+1} > (k+1)^2$. By the definition of exponents

$$2^{k+1} = 2 \cdot 2^k$$

Our Inductive Hypothesis of $2^k > k^2$ remains true when scaled by 2 (as 2 is positive)

$$2 \cdot 2^k > 2 \cdot k^2$$

From our previous step we know that $2^{k+1} = 2 \cdot 2^k$, so

$$2^{k+1} > 2 \cdot k^2$$

With this in mind we can then check the difference of $2k^2$ and $(k+1)^2$

$$\begin{aligned}
\Delta = && 2k^2 - (k+1)^2 && \\
\Delta = && 2k^2 - (k^2 + 2k + 1) && \text{expand the binomial} \\
\Delta = && 2k^2 - k^2 - 2k - 1 && \text{distribute the negative} \\
\Delta = && k^2 - 2k - 1 && \text{combine like terms} \\
\Delta = && (k-1)^2 && \text{rewrite as a perfect square}
\end{aligned}$$

Since we know the base case for k is 5,

$$(k-1)^2 \geq (5-1)^2 = 16$$

Meaning $\Delta = 16$, and thus $2k^2 > (k+1)^2$. From our previous steps we know that $2^{k+1} > 2k^2$, so $2^{k+1} > 2k^2 > (k+1)^2$, or simply put, $2^{k+1} > (k+1)^2$.
By induction, $2^n > n^2$ for all $n \geq 5$.

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