The Cubical Menagerie

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The State of the Cube

Several different cubical structures have been proposed
as a basis for models of higher-dimensional type theory.

We will briefly survey a few of their features
and propose a general framework to specify such constructions.

Cubes in the Wild

Grandis $\square(\partial\,,\varepsilon\,,\rho\,,\gamma)$ (dioids)

Awodey $\square(\partial\,,\varepsilon\,,\Delta)$ (cartesian cubes)

Coquand $\square(\partial\,,\varepsilon\,,\rho\,,\Delta\,,\gamma)$ (nominal de Morgan algebras)

Spitters $\square(\partial\,,\varepsilon\,,\rho\,,\Delta\,,\gamma)$ (nominal Kleene algebras)

$\vdots$

A Modest Proposal

We propose to describe cubical sets as:

- ordinary (contravariant) presheaves
- on finitely-presented symmetric monoidal categories
- with a single generating object (the abstract interval)
- and a decidable theory of morphism equality.

Why:

contravariance? Dimensions form a *context*, substitution is pre-composition.

finite presentation? Specify cubical structure based on intended geometry.

symmetric monoidal? Dimension contexts are unbiased and independent.

decidable theory? Computation!

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Abstract Cubes

In a cube category, $\square$,
for each $n \in \mathbb{N}$, we have an abstract n-dimensional cube, $[n]$.

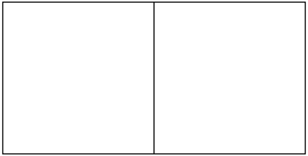
0-Dimensional Cube (point)

$[0]$

1-Dimensional Cube (interval)

$$\bullet \longrightarrow \bullet$$

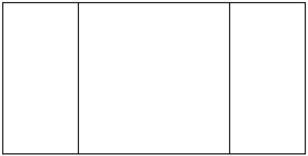
$$[1]$$



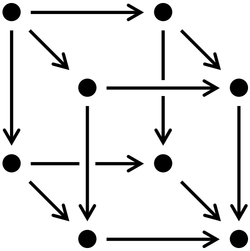
2-Dimensional Cube (square)

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

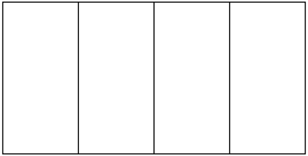
$[2]$



3-Dimensional Cube (cube)



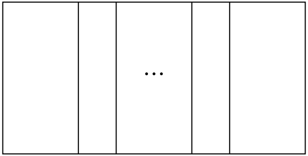
$[3]$



n-Dimensional Cube

???

$[n]$



Cubiness

We seek an equational presentation of cubes
so we can describe cubes of any dimension
and the relationships between them.

Cubical Sets

Cubical sets are presheaves on a cube category:

$a \quad b \quad c \quad d$	$\xrightarrow{f}$ $\xrightarrow{g}$ $\xrightarrow{h}$	$\boxed{\alpha}$ $\boxed{\beta}$	
$[0]$	$[1]$	$[2]$	$\cdots$

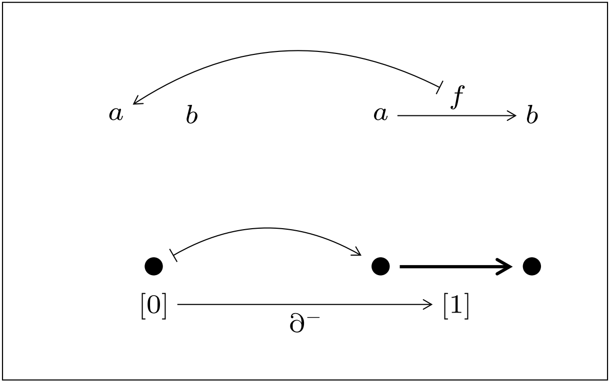
where the cubes of various dimensions are related by certain maps.

Face Maps

An **abstract interval** has two distinguishable boundary points.

This gives us a notion of a path.

$$\partial^{-}, \partial^{+} : \square([0] \to [1])$$



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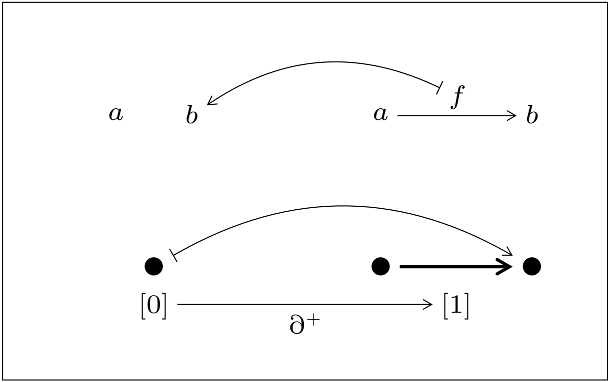
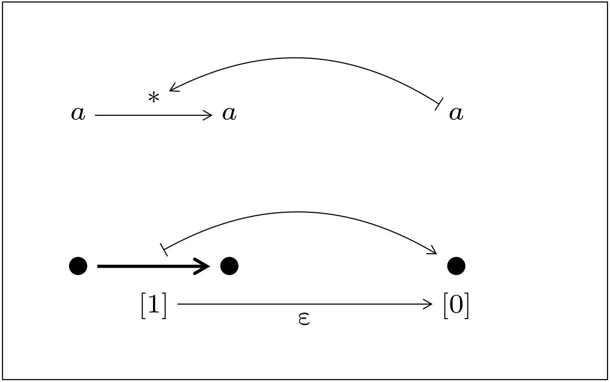


Degeneracies

Represent the idea of a trivial path:

$$\varepsilon : \square([1] \to [0])$$



Face-Degeneracy Laws

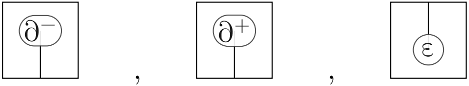
$$\partial^i \cdot \varepsilon = \mathrm{id}([0])$$

$$\begin{tikzcd}
 & & * \arrow[lld, mapsto, bend right] & & \\
a & a \arrow[rr] & & a & a \arrow[llu, mapsto, bend right]
\end{tikzcd}$$

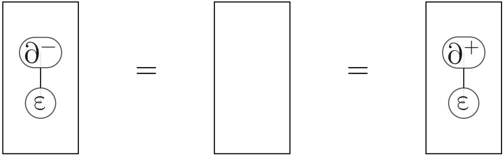
$$\begin{tikzcd}
{[0]} \arrow[r, "\partial^i"] \arrow[rr, bend right, "\mathrm{id}"'] & {[1]} \arrow[r, "\varepsilon"] & {[0]}
\end{tikzcd}$$

$\square(\partial\,,\,\varepsilon)$

generators



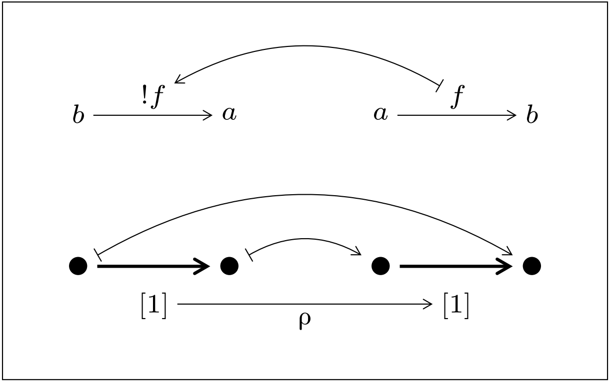
relations



Reversals

Represent the idea of following a path *backwards*:

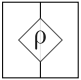
$$\rho : \Box\,([1] \to [1])$$



$\square(\partial\ ,\varepsilon\ ,\rho)$

The theory $\square(\partial\ ,\varepsilon)$ plus:

generator



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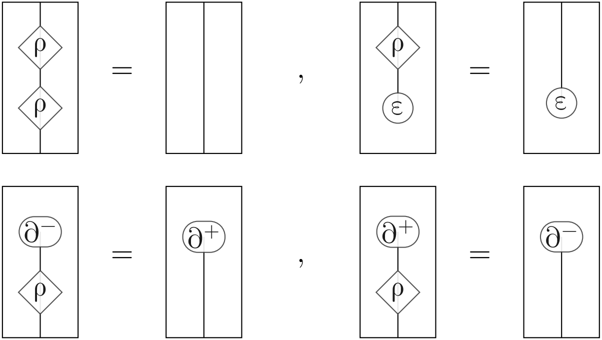
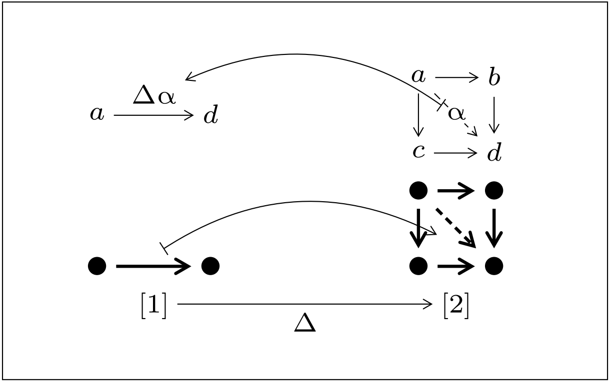


Diagonal Maps

Represent the idea of a path cutting through the middle of a square:

$$\Delta : \square\,([1] \to [2])$$



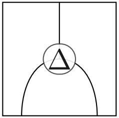
$\square(\partial\ ,\Delta)$

So far, the diagonal is under-specified:

we don't say *how* to cut through the middle of a square.

But there is still something that we know for certain: its boundary.

generator



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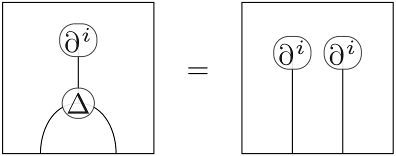


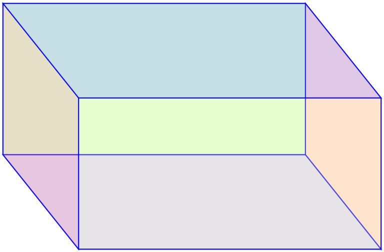
Symmetrical Diagonals

If the diagonal cuts through the square in "a straight line"
then we get more laws:

diagonal-diagonal law

$$\Delta \cdot (\Delta \otimes [1]) = \Delta \cdot ([1] \otimes \Delta)$$

represents cutting through the middle of a 3-cube.



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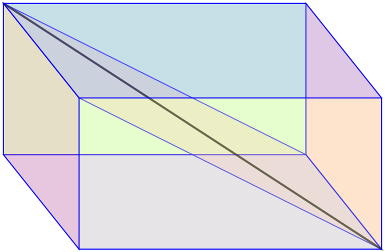

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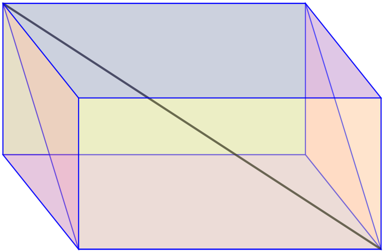


Symmetrical Diagonals

Also, putting the interval in the diagonal of the square and then squishing the square back into the interval along either dimension is identity:

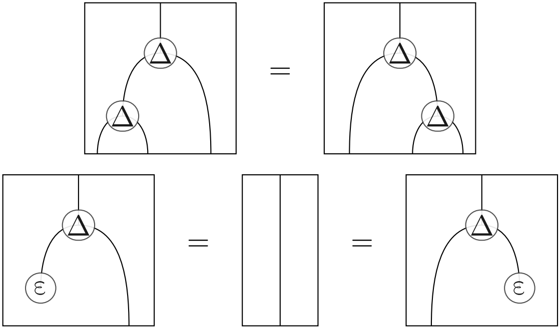
diagonal-degeneracy laws

$$\Delta \cdot (\varepsilon \otimes [1]) = \mathrm{id}([1]) = \Delta \cdot ([1] \otimes \varepsilon)$$

$\square(\varepsilon\ ,\ \Delta)$

You may recognize these as the *comonoid* laws:

relations

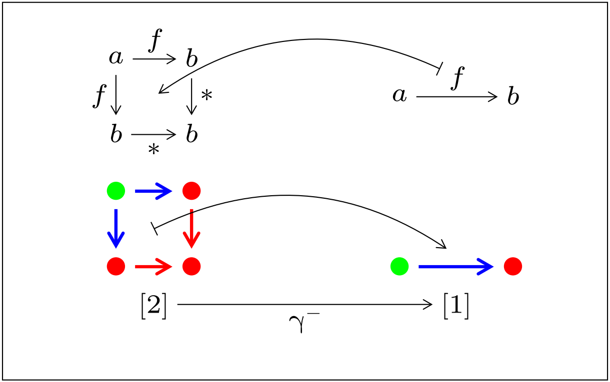


If we extend this comonoid structure *naturally* to all $[n]$,
then the monoidal structure becomes *cartesian*.

Connection Maps

Represent a collapsing a square to an interval, like a folding fan:

$$\gamma^-, \gamma^+ : \square([2] \to [1])$$



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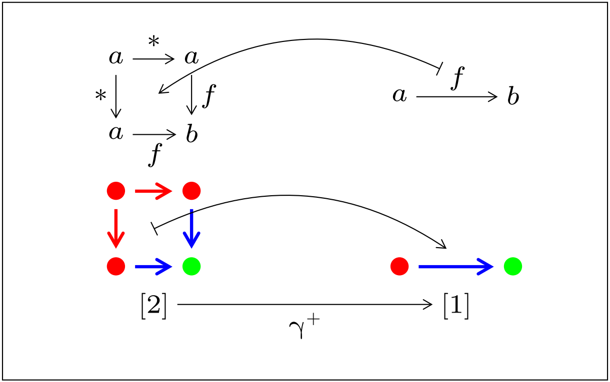

Connection Relations

Several possibilities:

- Grandis dioid
- de Morgan algebra
- Kleene algebra
- …

These all have decidable equational theories,
so are potential candidates for cubical models for type theory.

Next steps: investigate

- computational behavior,
- uniform Kan condition,
- relationship between Kan composition and Kan filling.

Thanks!