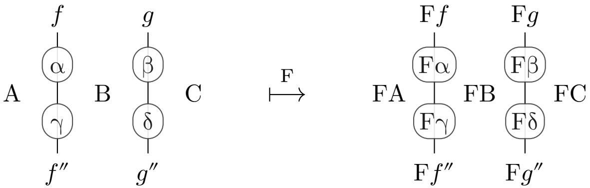
Functor Boxes, Compositor Sheets, and Lithographic Duality

TallCat Retreat

2022-04-12

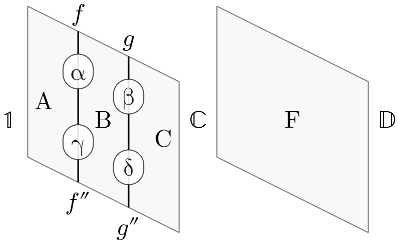
Strict Functors of 2-Categories

A *strict functor* between 2-categories $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ preserves all composition structure:



Strict Functors of 2-Categories

We can see this operation as arising by projection from a configuration in a 3-dimensional category of 2-categories, strict functors, transformations, and modifications:



Lax Functors of 2-Categories

A *lax functor* between 2-categories $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ preserves composition only up to a *comparison structure* $\theta(f_1 , \cdots , f_n) : \mathrm{F}f_1 \cdot \ldots \cdot \mathrm{F}f_n \to \mathrm{F}(f_1 \cdot \ldots \cdot f_n)$

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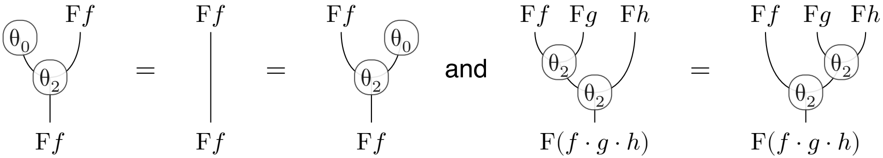

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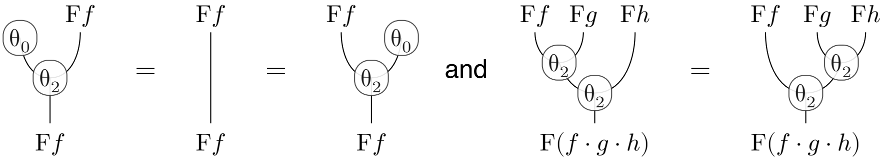
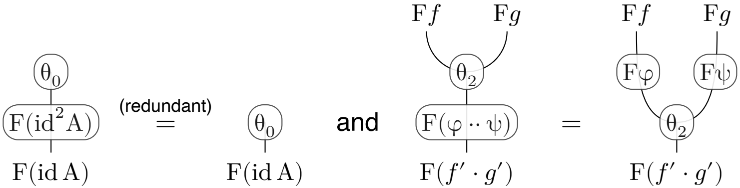


and natural:



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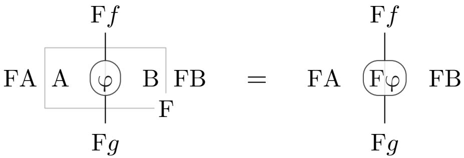

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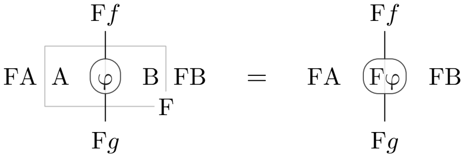
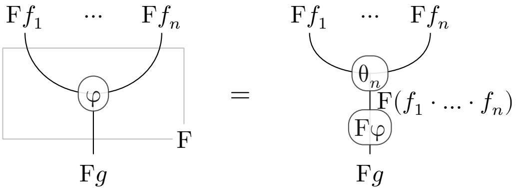
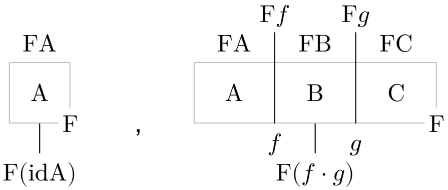


The side and bottom boundaries of an F-box represent the identity, and the top boundary represents the lax comparitor component θ_n.



Functor Box Laws

The nullary and binary comparitor components become:



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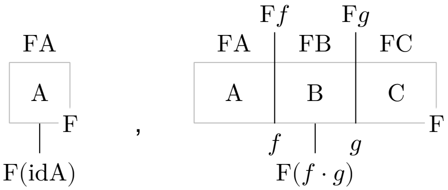

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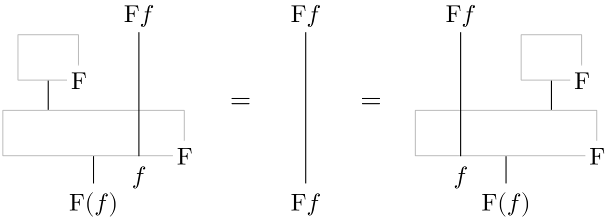

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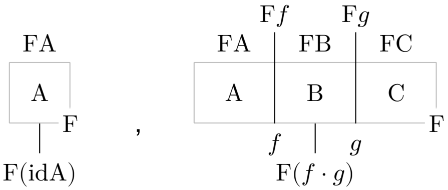

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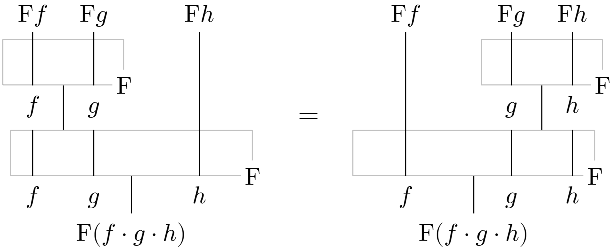

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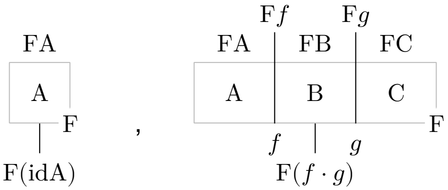

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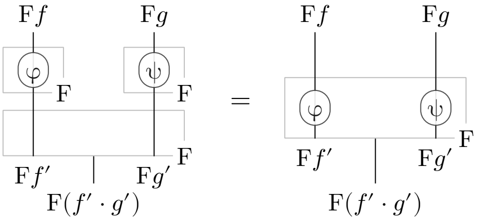

Compositor Sheets

For lax functor F, in surface diagrams we need to be able to distinguish

$$\mathrm{F}(\varphi \cdot\cdot\, \psi) : \mathbb{D}\left(\mathrm{F}(f \cdot g) \to \mathrm{F}(f' \cdot g')\right) \qquad \text{and} \qquad \mathrm{F}\varphi \cdot\cdot\, \mathrm{F}\psi : \mathbb{D}\left(\mathrm{F}f \cdot \mathrm{F}g \to \mathrm{F}f' \cdot \mathrm{F}g'\right)$$

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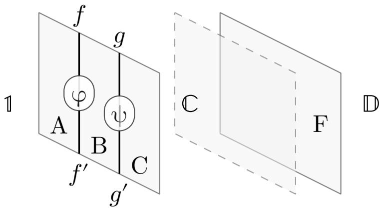
We can do this using *compositor sheets*:



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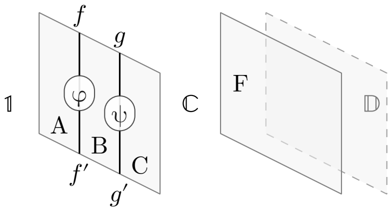

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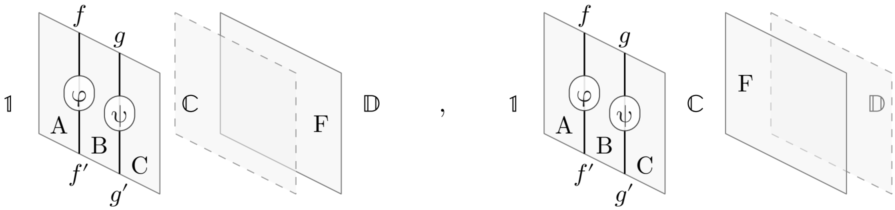

These arise from the free/forgetful adjunction between 2-categories and 2-computads.

They are idempotent, as you would expect.

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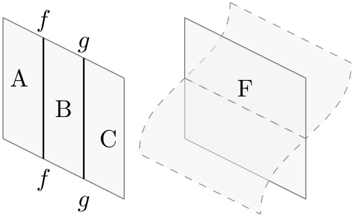

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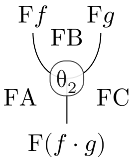

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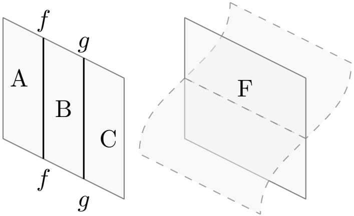

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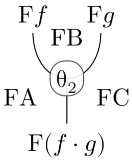

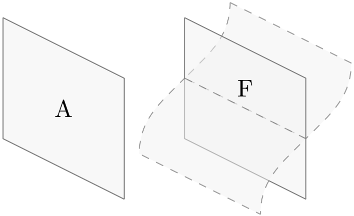

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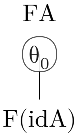

Comparison Naturality

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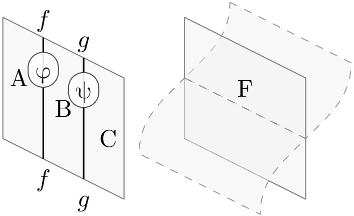

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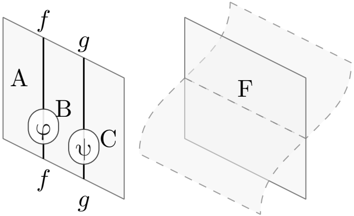

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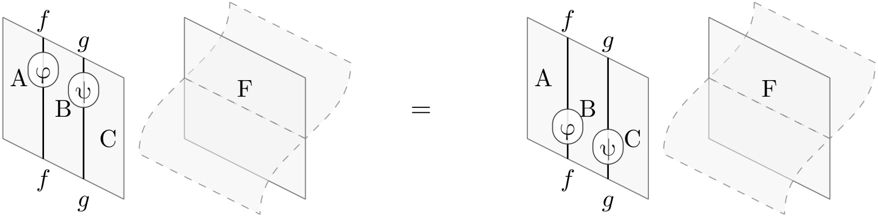

Because θ_1 is the identity and identity disks a strict unit for vertical composition, independent disks can move freely past the comparison structure, so long as the boundary is preserved.

Diagrammatic Calculation

We can calculate
using algebra:

$$
\begin{aligned}
& [(\theta(f\,, g) \cdot \mathrm{F}\varphi) \cdot\cdot\, (\theta(h\,, i) \cdot \mathrm{F}\psi)] \cdot [\theta(f' \cdot g'\,, h' \cdot i') \cdot \mathrm{F}(f' \cdot\cdot\, \sigma \cdot\cdot\, i')] \\
= \;& (\theta(f\,, g) \cdot\cdot\, \theta(h\,, i)) \cdot (\mathrm{F}\varphi \cdot\cdot\, \mathrm{F}\psi) \cdot \theta(f' \cdot g'\,, h' \cdot i') \cdot \mathrm{F}(f' \cdot\cdot\, \sigma \cdot\cdot\, i') \\
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= \;& \theta(f\,, g\,, h\,, i) \cdot \mathrm{F}((\varphi \cdot\cdot\, \psi) \cdot (f' \cdot\cdot\, \sigma \cdot\cdot\, i')).
\end{aligned}
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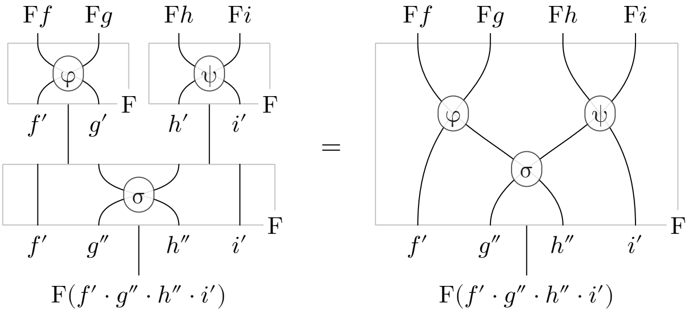

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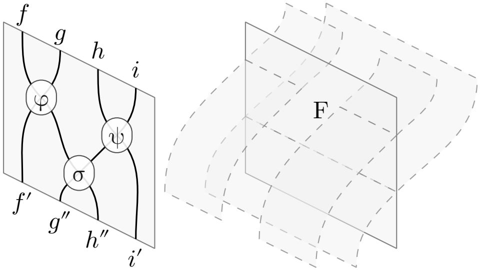

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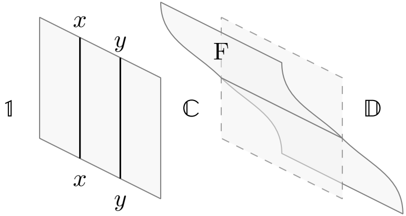

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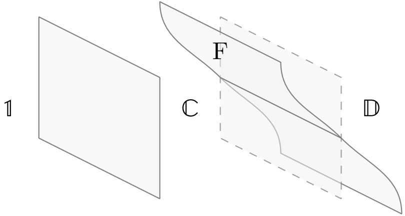

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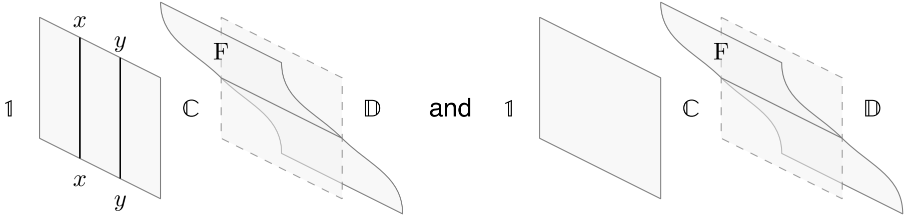

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