2-Categories from a Gray Perspective

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Abstract

In this paper we present 2-category theory from the perspective of Gray-categories using the graphical calculus of separated surface diagrams. As an extended example we consider cones and limits of 2-functors. Then we use the canonical adjunction between 2-computads and 2-categories to interpret the comparison structure of lax functors and extend the surface diagram calculus with compositor sheets in order to represent and reason about them.

1 A Gloss on Natural Transformations

An elementary presentation of a *natural transformation* [Mac98] between functors between (ordinary) categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})$ consists of an assignment of a component morphism $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GA})$ to each object $\mathrm{A} : \mathbb{C}$ such that for each morphism $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ there is a commuting naturality square in $\mathbb{D}\,(\mathrm{FA} \to \mathrm{GB})$:

$$\begin{tikzcd} \mathrm{FA} \arrow[r, "\mathrm{F}f"] \arrow[d, "\alpha\mathrm{A}"'] & \mathrm{FB} \arrow[d, "\alpha\mathrm{B}"] \\ \mathrm{GA} \arrow[r, "\mathrm{G}f"'] & \mathrm{GB} \end{tikzcd} \tag{1.1}$$

Such diagrammatic representations of categorical structures and properties are useful, especially when the composition involved is strictly unital and associative, because we may freely assemble them into compound diagrams of structures and properties by pasting them together along compatible boundaries.

A logically equivalent and often more perspicuous perspective is achieved by considering the dual-graph representation of such a diagram, where each k-dimensional cell of an n-dimensional diagram is drawn as an $(n-k)$-dimensional region [JS91; Sel11], with points sometimes "fattened up" into beads to facilitate labeling. The dual diagram of the naturality square for α, conventionally called a *string diagram*, can be drawn as follows, with the apparent depth ordering at

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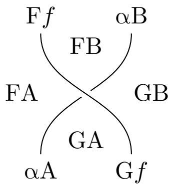
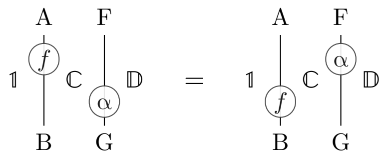
the intersection point to be explained shortly.



(1.2)

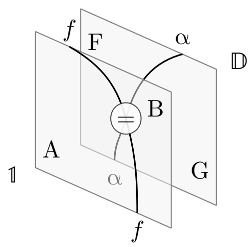
This diagram asserts the equality of the composites of the parallel 1-dimensional paths depicted as the top and bottom boundaries, $\mathrm{F}f \cdot \alpha\mathrm{B}$ and $\alpha\mathrm{A} \cdot \mathrm{G}f$, between the 0-dimensional objects depicted as the left and right boundaries, FA and GB. Note that the square interior in diagram (1.1), and its graph dual, the intersection point in diagram (1.2), represent a property (namely, morphism equality) rather than a structure in $\mathbb{D}$, as there are no 2-dimensional structures in a 1-dimensional category.

Rather than regarding the naturality of α as a property in the codomain category $\mathbb{D}$, we may instead regard it as a property in an ambient globular 2-dimensional category where objects are suitably small categories and arrows are functors between them. We do this by regarding an object of the domain category $\mathrm{A} : \mathbb{C}$ as a *global element* or constant functor from the singleton category $\mathrm{A} : \mathbb{1} \to \mathbb{C}$, and an arrow of the domain category $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ as a (necessary natural) 2-cell between such[1]. In this setting the naturality condition for α asserts that:



(1.3)

Here we have expressed this equation as a relation *between* diagrams, which necessarily have equal boundaries in all dimensions. We can also express it as a relation *within* a diagram by enlisting another dimension, in which the boundaries are the two diagrams related in equation (1.3):



(1.4)

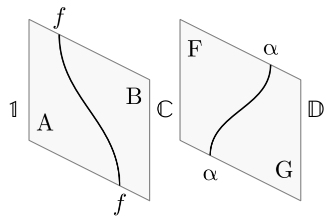
This 3-dimensional diagram, conventionally called a *surface diagram*[2], asserts the equality of the composites of the parallel 2-dimensional paths depicted as the top and bottom boundaries, $(f \cdot\cdot\, \mathrm{F}) \cdot (\mathrm{B} \cdot\cdot\, \alpha)$ and $(\mathrm{A} \cdot\cdot\, \alpha) \cdot (f \cdot\cdot\, \mathrm{G})$, between the

[1]The global elements are sometimes distinguished from the objects and morphisms of the category using a variant notation, but I prefer to disambiguate using either context or explicit boundary specifications.

[2]Personally, I find the convention of naming these diagrams for their codimension 1 spaces unintuitive, but this is not a curve that I'm willing to die on.

composites of the parallel 1-dimensional paths depicted as the back and front boundaries, $\mathrm{A}\cdot\mathrm{F}$ and $\mathrm{B}\cdot\mathrm{G}$, between the 0-dimensional objects depicted as the left and right boundaries, $\mathbb{1}$ and $\mathbb{D}$. Briefly, our notation for globular composition is ordered diagrammatically, with the number of infix dots indicating the number of dimensions below that of the highest-dimensional cell being composed at which the composition takes place.

For reasons that will soon be explained, we will use a shorthand variant of this diagram, called a *separated surface diagram*, in which the surface containing the natural transformation is separated from the surface preceding it in the composition order:

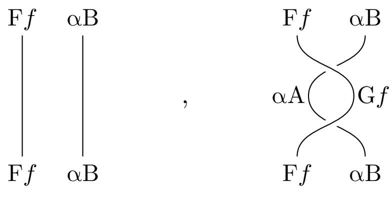


(1.5)

If we were to view either of these diagrams, which we can think of as embedded within a cube, from the perspective of the normal vector of its 0-dimensional domain face $\mathbb{1}$ we would see projected onto its 0-dimensional codomain face $\mathbb{D}$ precisely the naturality square depicted in diagram (1.2), with the intersection point representing the only 2-dimensional attribute available in a 1-dimensional category, namely, the equality relation between parallel arrows.

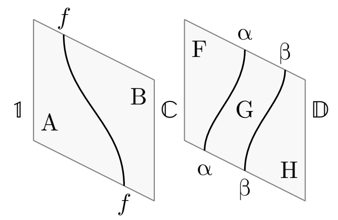
We can elaborate the separated surface diagram (1.5) back into (1.4) by inferring the presence of the implied intersection point from the intersection point in the projection string diagram (1.2). But whereas diagram (1.4) makes it clear that the lines for f and α intersect at exactly one point, diagram (1.5) is potentially ambiguous. Depending on the geometry it could represent any odd number of consecutive crossing points.

Later when we consider directed naturality of higher-dimensional structures we will need to ensure that we elaborate separated surface diagrams only into diagrams with properly-oriented explicit intersection points. But in the present case of natural transformations between functors we need only stipulate that the lines in the projection string diagrams intersect *transversely*; that is, each intersection point has a neighborhood in which it forms a crossing. Morphism equality is an equivalence relation so both of the following string diagrams represent the equality $\mathrm{F}f \cdot \alpha\mathrm{B} = \mathrm{F}f \cdot \alpha\mathrm{B}$, the former by reflexivity and the latter by diagram (1.2) together with symmetry and transitivity. Only the crossing parity matters, and this is determined by the diagram's boundary.



(1.6)

If we add another natural transformation $\beta :\ (\mathbb{C} \to \mathbb{D})\,(\mathrm{G} \to \mathrm{H})$ then the surface diagram corresponding to the naturality of the composition of α and β along the functor G, the so-called “vertical” composition, is shown below on the left. Its projection string diagram in $\mathbb{D}$ is shown in the middle, whose dual diagram, the conventional pasting diagram depicting the naturality of vertical composites of natural transformations, is shown on the right.



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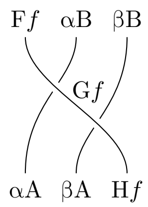
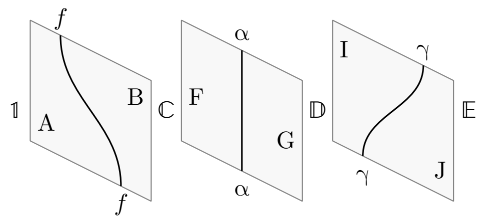


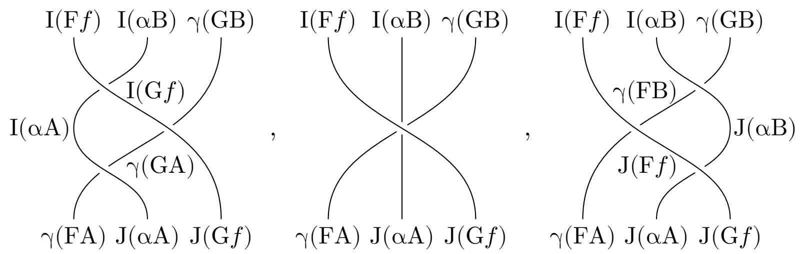
,

$$\begin{tikzcd}
\mathrm{FA} \arrow[r, "\mathrm{F}f"] \arrow[d, "\alpha\mathrm{A}"'] & \mathrm{FB} \arrow[d, "\alpha\mathrm{B}"] \\
\mathrm{GA} \arrow[r, "\mathrm{G}f"'] \arrow[d, "\beta\mathrm{A}"'] & \mathrm{GB} \arrow[d, "\beta\mathrm{B}"] \\
\mathrm{HA} \arrow[r, "\mathrm{H}f"'] & \mathrm{HB}
\end{tikzcd}$$

Alternatively, we can add another category $\mathbb{E}$, functors $\mathrm{I}, \mathrm{J} : \mathbb{D} \to \mathbb{E}$, and a natural transformation $\gamma : (\mathbb{D} \to \mathbb{E})\,(\mathrm{I} \to \mathrm{J})$. The surface diagram corresponding to the naturality of the composition of α and γ along the category $\mathbb{D}$, the so-called “horizontal” composition, is shown below.



Depending on the geometry, its projection string diagram in $\mathbb{E}$ looks, up to homotopy and modulo gratuitous consecutive crossings, like one of the following:



(1.7)

Leaving aside for the moment the diagram with the triple crossing point, each pairwise crossing in each of the other diagrams represents an equality of morphisms as an instance of diagram (1.2), two by the naturality of γ and the third as a functor image of the naturality of α. Because morphism equality is transitive and a congruence with respect to composition, each diagram witnesses the equality of its top and bottom boundary.

We might be tempted to wonder whether these different projections represent the *same* way of equating the composite morphisms $\mathrm{I}(\mathrm{F}f) \cdot \mathrm{I}(\alpha\mathrm{B}) \cdot \gamma(\mathrm{GB})$ and $\gamma(\mathrm{FA}) \cdot \mathrm{J}(\alpha\mathrm{A}) \cdot \mathrm{J}(\mathrm{G}f)$ between $\mathrm{I}(\mathrm{FA})$ and $\mathrm{J}(\mathrm{GB})$ in $\mathbb{E}$, but the question is not meaningful: the attribute of morphism equality in a 1-category is a property, not a structure. We can meaningfully ask whether a given property holds, but we can’t compare various ways of satisfying it. This motivates us to investigate naturality in higher-dimension categorical structures.

2 Picturing Gray-Categories

The globular 2-dimensional categorical structure formed by categories, functors and natural transformations is known as a 2-category. 2-categories, in turn, are related by morphisms of three different dimensions that come in several variations. The structure of 2-categories and their hierarchy of morphisms was studied by Gray in [Gra74]. This structure has come to be known as a Gray-category. We build it up in stages.

Definition 2.1 (sesquicategory)
A *sesquicategory* $\mathbb{C}$ consists of a category of 0-cells (objects) and 1-cells (arrows) together with the following additional structure:

2-cells: for each parallel pair of 1-cells $f, f' : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ a collection of 2-cells[3] or "disks" $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$,

nullary 2-cell composition: for each 1-cell $f : \mathrm{A} \to \mathrm{B}$ a 2-cell $\mathrm{id}\, f : f \to f$,

binary 2-cell composition: for consecutive 2-cells $\alpha : (\mathrm{A} \to \mathrm{B})\,(f \to f')$ and $\beta : (\mathrm{A} \to \mathrm{B})\,(f' \to f'')$ a 2-cell $\alpha \cdot \beta : (\mathrm{A} \to \mathrm{B})\,(f \to f'')$,

2-cell right-whiskering: for a 2-cell $\alpha : (\mathrm{A} \to \mathrm{B})\,(f \to f')$ and an adjacent 1-cell $g : \mathrm{B} \to \mathrm{C}$ a 2-cell $\alpha \cdot\cdot\, g : (\mathrm{A} \to \mathrm{C})\,(f \cdot g \to f' \cdot g)$,

2-cell left-whiskering: for a 1-cell $f : \mathrm{A} \to \mathrm{B}$ and an adjacent 2-cell $\beta : (\mathrm{B} \to \mathrm{C})\,(g \to g')$ a 2-cell $f \cdot\cdot\, \beta : (\mathrm{A} \to \mathrm{C})\,(f \cdot g \to f \cdot g')$.

In addition to the composition laws for the underlying category we have the following relations:

strict 2-cell composition: for 0-cells $\mathrm{A}, \mathrm{B} : \mathbb{C}$, the 1-cells and 2-cells of $\mathbb{C}$ in $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ form a category:

$$\mathrm{id}\, f \cdot \alpha = \alpha = \alpha \cdot \mathrm{id}\, f' \;,\; (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) \tag{2.1}$$

2-whiskering functoriality: the whiskering operations are functorial in their 2-cell argument:

$$\begin{gathered} \mathrm{id}\, f \cdot\cdot\, g = \mathrm{id}(f \cdot g) = f \cdot\cdot\, \mathrm{id}\, g \;, \\ (\alpha \cdot \beta) \cdot\cdot\, g = (\alpha \cdot\cdot\, g) \cdot (\beta \cdot\cdot\, g) \;,\; f \cdot\cdot\, (\gamma \cdot \delta) = (f \cdot\cdot\, \gamma) \cdot (f \cdot\cdot\, \delta) \end{gathered} \tag{2.2}$$

2-whiskering algebra laws: the whiskering operations are compatible with 1-cell composition:

$$\begin{gathered} \alpha \cdot\cdot\, \mathrm{id}\, \mathrm{B} = \alpha = \mathrm{id}\, \mathrm{A} \cdot\cdot\, \alpha \;, \\ \alpha \cdot\cdot\, (g \cdot h) = (\alpha \cdot\cdot\, g) \cdot\cdot\, h \;,\; (f \cdot g) \cdot\cdot\, \gamma = f \cdot\cdot\, (g \cdot\cdot\, \gamma) \end{gathered} \tag{2.3}$$

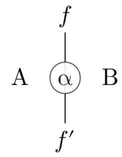
2-whiskering bialgebra law: the whiskering operations are compatible with each other:

$$(f \cdot\cdot\, \beta) \cdot\cdot\, h = f \cdot\cdot\, (\beta \cdot\cdot\, h) \tag{2.4}$$

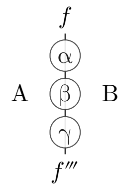
[3] For brevity we may omit any well-formed prefix of a boundary specification if it is either clear from context or irrelevant. This lets us write $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$ as either $(\mathrm{A} \to \mathrm{B})\,(f \to f')$ or $f \to f'$, dropping the brackets as well when we drop the entire prefix.

In light of the strict associativity of 2-cell composition we adopt an unbracketed notation for it, letting us write expressions like $\alpha \cdot \beta \cdot \gamma$ without ambiguity. Because whiskering a 2-cell with any number of 1-cells on either side is also unambiguous we adopt an unbracketed notation for this as well, letting us write the unique 2-cells appearing in the binary composition instances of the algebra laws and in the bialgebra law as $\alpha \cdot\cdot\, g \cdot\cdot\, h$, $f \cdot\cdot\, g \cdot\cdot\, \gamma$ and $f \cdot\cdot\, \beta \cdot\cdot\, h$.

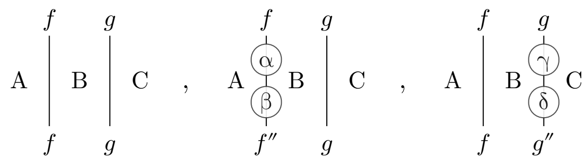
Because of the unit laws for 2-cell composition in (2.1) and for whiskering in (2.3) the following string diagram represents a unique 2-cell.



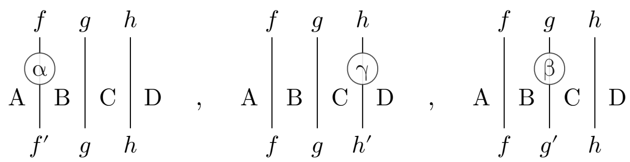
Because of the associative law for 2-cell composition in (2.1) the following string diagram represents a unique 2-cell.



Because of whiskering functoriality (2.2) each of the following string diagrams represents a unique 2-cell.



Because of the whiskering algebra laws for binary composites in (2.3) and the biaglebra law (2.4) each of the following string diagrams represents a unique 2-cell.



In a sesquicategory we can compose 2-cells vertically, but only whisker them by 1-cells horizontally. A string diagram is called *ordered* if it can be decomposed into horizontal layers such that each layer contains at most one 2-cell. Two ordered string diagrams are equal just in case they have equal ordered layer decompositions.

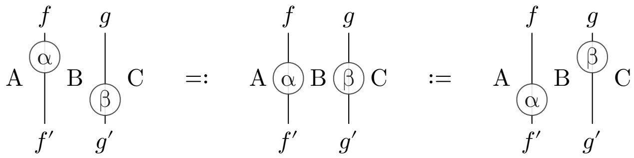
For our purposes it will be convenient to characterize a 2-category as a sesquicategory with an additional relation that loosens the ordering restriction.

Definition 2.2 (2-category)
A 2-*category* $\mathbb{C}$ is a sesquicategory satisfying the following additional relation:

interchange law: for adjacent 2-cells $\alpha : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$ and $\beta : \mathbb{C}\,(\mathrm{B} \to \mathrm{C})\,(g \to g')$ the relation:

$$(\alpha \cdot\cdot\, g) \cdot (f' \cdot\cdot\, \beta) \;=\; (f \cdot\cdot\, \beta) \cdot (\alpha \cdot\cdot\, g') \tag{2.5}$$

This relation allows us to coherently define a *horizontal composition* operation on 2-cells $\alpha \cdot\cdot\, \beta : (f \cdot g \to f' \cdot g')$ that is equal to each of its perturbations into a composition of whiskerings:

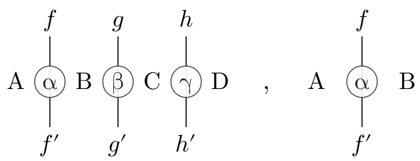


(2.6)

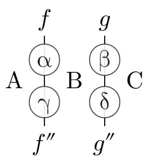
The horizontal composition of 2-cells is associative with units given by the vertical identity 2-cell on the identity 1-cell on each 0-cell, $\mathrm{id}^2\mathrm{A} := \mathrm{id}(\mathrm{id}\,\mathrm{A})$.

$$\begin{array}{rl|rl}
 & & & \mathrm{id}^2\mathrm{A} \cdot\cdot\, \alpha \\
 & (\alpha \cdot\cdot\, \beta) \cdot\cdot\, \gamma & = & (\mathrm{id}^2\mathrm{A} \cdot\cdot\, f) \cdot (\mathrm{id}\,\mathrm{A} \cdot\cdot\, \alpha) \\
= & ((\alpha \cdot\cdot\, \beta) \cdot\cdot\, h) \cdot ((f' \cdot g') \cdot\cdot\, \gamma) & = & \mathrm{id}(\mathrm{id}\,\mathrm{A} \cdot f) \cdot \alpha \\
= & (((\alpha \cdot\cdot\, g) \cdot (f' \cdot\cdot\, \beta)) \cdot\cdot\, h) \cdot (f' \cdot\cdot\, g' \cdot\cdot\, \gamma) & = & \mathrm{id}\, f \cdot \alpha \\
= & (\alpha \cdot\cdot\, g \cdot\cdot\, h) \cdot (f' \cdot\cdot\, \beta \cdot\cdot\, h) \cdot (f' \cdot\cdot\, g' \cdot\cdot\, \gamma) & = & \alpha \\
= & (\alpha \cdot\cdot\, g \cdot\cdot\, h) \cdot (f' \cdot\cdot\, ((\beta \cdot\cdot\, h) \cdot (g' \cdot\cdot\, \gamma))) & = & \alpha \cdot \mathrm{id}\, f' \\
= & (\alpha \cdot\cdot\, (g \cdot h)) \cdot (f' \cdot\cdot\, (\beta \cdot\cdot\, \gamma)) & = & \alpha \cdot \mathrm{id}(f' \cdot \mathrm{id}\,\mathrm{B}) \\
= & \alpha \cdot\cdot\, (\beta \cdot\cdot\, \gamma) & = & (\alpha \cdot\cdot\, \mathrm{id}\,\mathrm{B}) \cdot (f' \cdot\cdot\, \mathrm{id}^2\mathrm{B}) \\
 & & = & \alpha \cdot\cdot\, \mathrm{id}^2\mathrm{B}
\end{array} \tag{2.7}$$

The 0-cells and 2-cells of a 2-category themselves form a category so we adopt an unbracketed notation for horizontal composition of 2-cells, and each of the following string diagrams represents a unique 2-cell.



Moreover, the 2-cells satisfy the *middle-four exchange law* $(\alpha \cdot \gamma) \cdot\cdot\, (\beta \cdot \delta) = (\alpha \cdot\cdot\, \beta) \cdot (\gamma \cdot\cdot\, \delta)$, which is a 2-dimensional associative law asserting that the following string diagram represents a unique 2-cell.



Because of the horizontal composition operation string diagrams of 2-cells in 2-categories need not be ordered. Two such diagrams are equal just in case they are related by applications of the ternary interchange law (2.6).

Definition 2.3 (box-category)
A *box-category* $\mathbb{C}$ consists of a sesquicategory together with the following additional structure:

3-cells: for each parallel pair of 2-cells, $\alpha, \alpha' : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$ a collection of 3-cells or "globes" $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')\,(\alpha \to \alpha')$,

nullary 3-cell composition: for each 2-cell $\alpha : f \to f'$ a 3-cell $\mathrm{id}\,\alpha : \alpha \to \alpha$,

binary 3-cell composition: for consecutive 3-cells $\Gamma : (f \to f')\,(\alpha \to \alpha')$ and $\Gamma' : (f \to f')\,(\alpha' \to \alpha'')$ a 3-cell $\Gamma \cdot \Gamma' : (f \to f')\,(\alpha \to \alpha'')$,

3-cell horizontal composition: for adjacent 3-cells $\Gamma : (f \to f')\,(\alpha \to \alpha')$ and $\Delta : (f' \to f'')\,(\beta \to \beta')$ a 3-cell $\Gamma \cdot\cdot\, \Delta : (f \to f'')\,(\alpha \cdot \beta \to \alpha' \cdot \beta')$,

3-cell right-whiskering: for a 3-cell $\Gamma : (\mathrm{A} \to \mathrm{B})\,(f \to f')\,(\alpha \to \alpha')$ and a proximate 1-cell $g : \mathrm{B} \to \mathrm{C}$ a 3-cell $\Gamma \cdots g : (\mathrm{A} \to \mathrm{C})\,(f \cdot g \to f' \cdot g)\,(\alpha \cdot\cdot\, g \to \alpha' \cdot\cdot\, g)$,

3-cell left-whiskering: for a 1-cell $f : \mathrm{A} \to \mathrm{B}$ and a proximate 3-cell $\Phi : (\mathrm{B} \to \mathrm{C})\,(g \to g')\,(\gamma \to \gamma')$ a 3-cell $f \cdots \Phi : (\mathrm{A} \to \mathrm{C})\,(f \cdot g \to f \cdot g')\,(f \cdot\cdot\, \gamma \to f \cdot\cdot\, \gamma')$.

In addition to the composition laws for the underlying sesquicategory we have the following relations:

strict 3-cell composition: for parallel 1-cells $f, f' : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$, the 2-cells and 3-cells of $\mathbb{C}$ in $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$ form a category:

$$\mathrm{id}\,\alpha \cdot \Gamma = \Gamma = \Gamma \cdot \mathrm{id}\,\alpha'\ ,\ (\Gamma \cdot \Phi) \cdot \Omega = \Gamma \cdot (\Phi \cdot \Omega) \tag{2.8}$$

strict 3-cell horizontal composition: for 0-cells $\mathrm{A}, \mathrm{B} : \mathbb{C}$, the 1-cells and 3-cells of $\mathbb{C}$ in $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ form a category:

$$\mathrm{id}^2 f \cdot\cdot\, \Gamma = \Gamma = \Gamma \cdot\cdot\, \mathrm{id}^2 f'\ ,\ (\Gamma \cdot\cdot\, \Delta) \cdot\cdot\, \Lambda = \Gamma \cdot\cdot\, (\Delta \cdot\cdot\, \Lambda) \tag{2.9}$$

middle-four-exchange for 3-cell composition: there is a unique configuration of four 3-cells with the same object boundary arranged in a square:

$$(\Gamma \cdot \Phi) \cdot\cdot\, (\Delta \cdot \Psi) = (\Gamma \cdot\cdot\, \Delta) \cdot (\Phi \cdot\cdot\, \Psi) \tag{2.10}$$

3-whiskering functoriality: the whiskering operations are functorial in their 3-cell argument:

$$\begin{gathered}\mathrm{id}\,\alpha \cdots g = \mathrm{id}(\alpha \cdot\cdot\, g)\,,\ f \cdots \mathrm{id}\,\beta = \mathrm{id}(f \cdot\cdot\, \beta)\,,\\ (\Gamma \cdot \Gamma') \cdots g = (\Gamma \cdots g) \cdot (\Gamma' \cdots g)\,,\ f \cdots (\Phi \cdot \Phi') = (f \cdots \Phi) \cdot (f \cdots \Phi')\,,\\ (\Gamma \cdot\cdot\, \Delta) \cdots g = (\Gamma \cdots g) \cdot\cdot\, (\Delta \cdots g)\,,\ f \cdots (\Phi \cdot\cdot\, \Psi) = (f \cdots \Phi) \cdot\cdot\, (f \cdots \Psi)\end{gathered} \tag{2.11}$$

3-whiskering algebra laws: the whiskering operations are compatible with 1-cell composition:

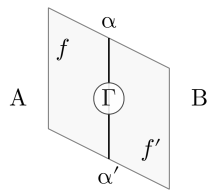
$$\begin{gathered}\Gamma \cdots \mathrm{id}\,\mathrm{B} = \Gamma = \mathrm{id}\,\mathrm{A} \cdots \Gamma\ ,\\ \Gamma \cdots (g \cdot h) = (\Gamma \cdots g) \cdots h\ ,\ (f \cdot g) \cdots \Lambda = f \cdots (g \cdots \Lambda)\end{gathered} \tag{2.12}$$

3-whiskering bialgebra law: the whiskering operations are compatible with each other:

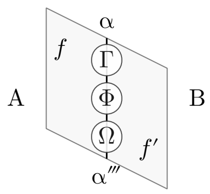
$$(f \cdots \Delta) \cdots h = f \cdots (\Delta \cdots h) \tag{2.13}$$

In light of the strict associativity of 3-cell composition along both 2-cells and 1-cells we adopt unbracketed notations for these operations. Because whiskering a 3-cell with any number of 1-cells on either side is also unambiguous we adopt an unbracketed notation for this as well, letting us write the unique 3-cells appearing in the binary composition instances of the algebra laws and in the bialgebra law as $\Gamma \cdots g \cdots h$, $f \cdots g \cdots \Lambda$ and $f \cdots \Delta \cdots h$.

Because of the unit laws for 3-cell composition in (2.8) and (2.9), and for whiskering in (2.12) the following surface diagram represents a unique 3-cell.



Because of the associative laws for 3-cell composition in (2.8) and (2.9) each of the following surface diagrams represents a unique 3-cell.



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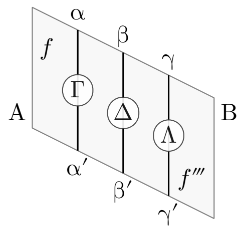
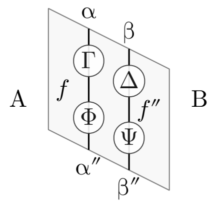
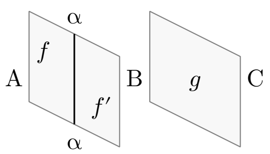


Because of the middle-four-exchange law (2.10) the following surface diagram represents a unique 3-cell.



Note that the hom objects of a box category are themselves 2-categories.

Because of whiskering functoriality (2.11) each of the following surface diagrams represents a unique 3-cell.



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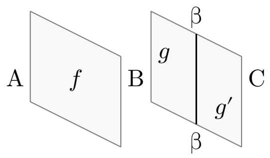

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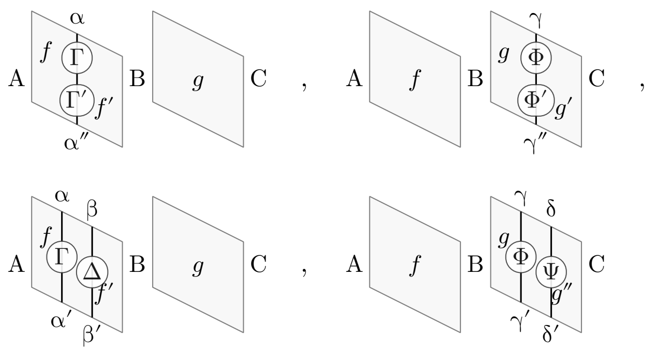
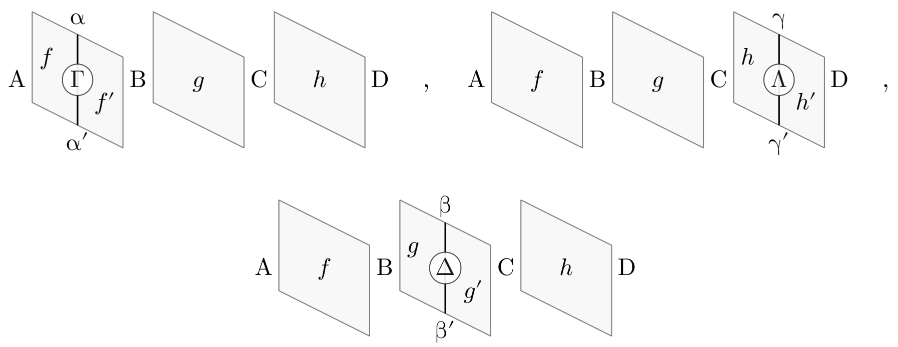


Because of the whiskering algebra laws for binary composites in (2.12) and the biaglebra law (2.13) each of the following surface diagrams represents a unique 3-cell.



The top and bottom boundaries of a surface diagram in a box-category are ordered string diagrams of its 0-, 1- and 2-cells, as are all horizontal slices that don't intersect a point representing a 3-cell. However, these surface diagrams are not themselves vertically ordered because the hom objects are 2-categories, which have horizontal composition and interchange.

We extended sesquicategories to 2-categories by adding the property of 2-cell interchange. Gray-categories extend box-categories in a similar way, but because box-categories have 3-cells there is now "room" for the interchange of 2-cells to be a structure rather than a property.

Definition 2.4 (oplax Gray-category)
An *oplax Gray-category* $\mathbb{C}$ consists of a box-category together with the following additional structure:

oplax interchangers: for adjacent 2-cells $\alpha : (\mathrm{A} \to \mathrm{B})\,(f \to f')$ and $\beta : (\mathrm{B} \to \mathrm{C})\,(g \to g')$ a 3-cell

$$\chi_{(\alpha,\beta)} : (\mathrm{A} \to \mathrm{C})\,(f \cdot g \to f' \cdot g')\,((\alpha \cdot\cdot\, g) \cdot (f' \cdot\cdot\, \beta) \to (f \cdot\cdot\, \beta) \cdot (\alpha \cdot\cdot\, g')). \tag{2.14}$$

In addition to the composition laws for the underlying box-category we have the following relations:

composite 2-cell interchangers: interchangers respect 2-cell composition in each index:

$$\begin{aligned} \chi_{(\mathrm{id} f,\beta)} &= \mathrm{id}(f \cdot\cdot \beta)\ ,\ \chi_{(\alpha\cdot\alpha',\beta)} = (\mathrm{id}(\alpha \cdot\cdot g) \cdot\cdot \chi_{(\alpha',\beta)}) \cdot (\chi_{(\alpha,\beta)} \cdot\cdot \mathrm{id}(\alpha' \cdot\cdot g'))\ , \\ \chi_{(\alpha,\mathrm{id} g)} &= \mathrm{id}(\alpha \cdot\cdot g)\ ,\ \chi_{(\alpha,\beta\cdot\beta')} = (\chi_{(\alpha,\beta)} \cdot\cdot \mathrm{id}(f' \cdot\cdot \beta')) \cdot (\mathrm{id}(f \cdot\cdot \beta) \cdot\cdot \chi_{(\alpha,\beta')}) \end{aligned} \tag{2.15}$$

interchanger extremal whiskering: interchangers are compatible with extremal whiskering in each index:

$$\chi_{(f\cdot\cdot\beta,\gamma)} = f \cdot\cdot\cdot \chi_{(\beta,\gamma)}\ ,\ \chi_{(\alpha,\beta\cdot\cdot h)} = \chi_{(\alpha,\beta)} \cdot\cdot\cdot h \tag{2.16}$$

interchanger medial whiskering: interchangers are compatible with medial whiskering:

$$\chi_{(\alpha\cdot\cdot g,\gamma)} = \chi_{(\alpha,g\cdot\cdot\gamma)} \tag{2.17}$$

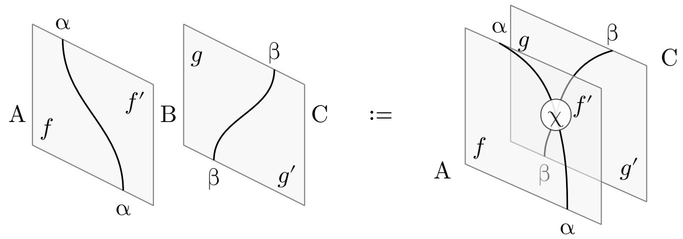
interchanger naturality: interchangers are natural in each index, in the sense that for 3-cells $\Gamma : \alpha \to \alpha'$ and $\Delta : \beta \to \beta'$,

$$\begin{aligned} ((\Gamma \cdot\cdot\cdot g) \cdot\cdot \mathrm{id}(f' \cdot\cdot \beta)) \cdot \chi_{(\alpha',\beta)} &= \chi_{(\alpha,\beta)} \cdot (\mathrm{id}(f \cdot\cdot \beta) \cdot\cdot (\Gamma \cdot\cdot\cdot g'))\ , \\ (\mathrm{id}(\alpha \cdot\cdot g) \cdot\cdot (f' \cdot\cdot\cdot \Delta)) \cdot \chi_{(\alpha,\beta')} &= \chi_{(\alpha,\beta)} \cdot ((f \cdot\cdot\cdot \Delta) \cdot\cdot \mathrm{id}(\alpha \cdot\cdot g')) \end{aligned} \tag{2.18}$$

In light of their extremal and medial whiskering laws we adopt an interchanger notation with arbitrarily many additional 1-cell indices, letting us write the unique 3-cells expressed by those laws as $\chi_{(f,\beta,\gamma)}$, $\chi_{(\alpha,\beta,h)}$, and $\chi_{(\alpha,g,\gamma)}$.

A *lax Gray-category* is one with interchangers oriented the other way. A *pseudo Gray-category* has invertible interchangers, making it both lax and oplax. A 3-*category* is a Gray-category whose interchangers are identity 3-cells.

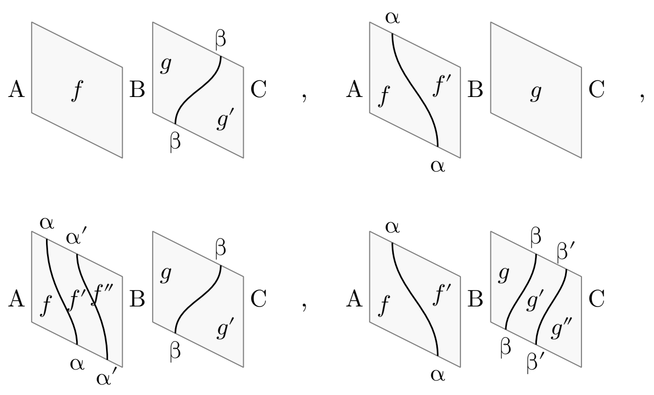
In diagrams interchanger 3-cells are drawn as lines on separated surfaces whose projections intersect at a crossing point:



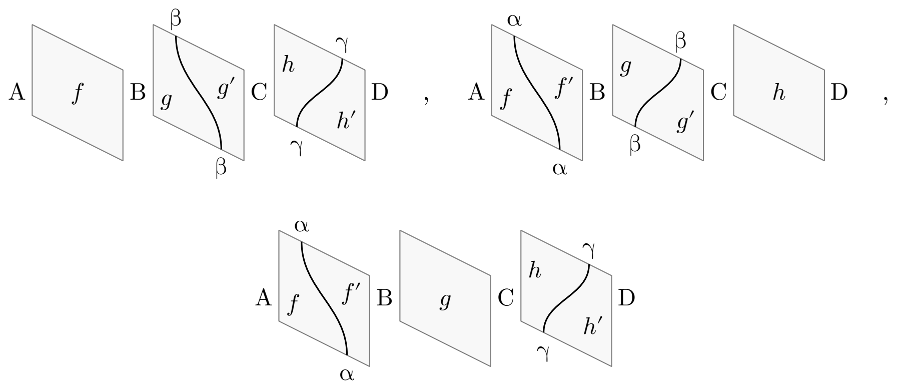
(2.19)

The topological properties of this representation turn out to be a good fit for the algebraic properties of interchangers. However, this representation introduces the same potential ambiguity that we encountered in section 1. We must still restrict the geometry so that the lines in the projection string diagram intersect transversely. We must also ensure that the line representing the second 2-cell index of an interchanger crosses the line representing its first 2-cell index from the upper right to the lower left only if the interchanger is oplax, and from the upper left to the lower right only if it is lax.

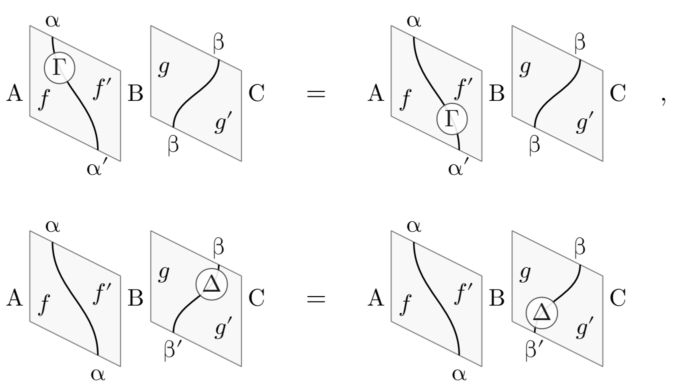
Because of the interchanger laws for composite 2-cells (2.15) each of the following surface diagrams represents a unique 3-cell.



By induction, interchangers for arbitrary 2-cell composites in each index are well-defined. Because of the interchanger extremal (2.16) and medial (2.17) whiskering laws each of the following surface diagrams represents a unique 3-cell.



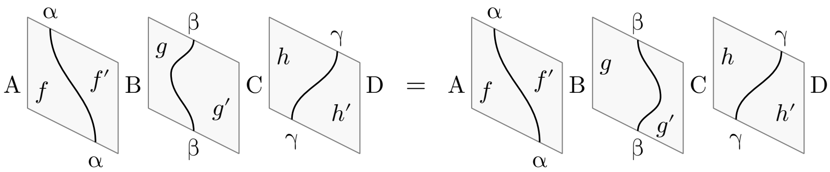
Because of the interchanger naturality laws (2.18) we have the following surface diagram equations which, thanks to the separation convention, are homotopies.



A notable consequence of the composite 2-cell interchanger laws and the interchanger naturality laws is that permuting the composition order of any number of whiskered 2-cells using interchangers is coherent [Gra76]. The special case of reversing the order of three 2-cells is known as the Yang-Baxter equation,

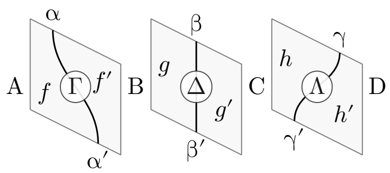
$$(\chi_{(\alpha,\beta,h)} \cdot\cdot \mathrm{id}(f' \cdot\cdot g' \cdot\cdot \gamma)) \cdot (\mathrm{id}(f \cdot\cdot \beta \cdot\cdot h) \cdot\cdot \chi_{(\alpha,g,\gamma)}) \cdot (\chi_{(f,\beta,\gamma)} \cdot\cdot \mathrm{id}(\alpha \cdot\cdot g' \cdot\cdot h'))$$
$$=$$
$$(\mathrm{id}(\alpha \cdot\cdot g \cdot\cdot h) \cdot\cdot \chi_{(f',\beta,\gamma)}) \cdot (\chi_{(\alpha,g,\gamma)} \cdot\cdot \mathrm{id}(f' \cdot\cdot \beta \cdot\cdot h')) \cdot (\mathrm{id}(f \cdot\cdot g \cdot\cdot \gamma) \cdot\cdot \chi_{(\alpha,\beta,h')})$$

which we can represent as the following equation between surface diagrams.



These have projection string diagrams with the shapes of the two in (1.7) having only pairwise crossings.

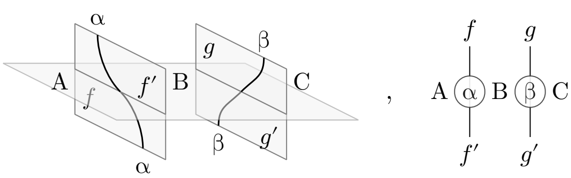
Thus we could coherently define n-ary 2-cell interchangers, such as $\chi_{(\alpha,\beta,\gamma)}$, and even n-ary 3-cell interchangers, such as $\chi_{(\Gamma,\Delta,\Lambda)}$. The latter would be equal to each of the 54 perturbations of the following surface diagram containing only binary 2-cell interchangers; namely, those whose projection string diagrams have a bead on one of the three segments of each of the three wires.



(2.20)

Even if we don't admit such *generalized interchangers* it is sometimes convenient to write an expression or draw a diagram involving them as a shorthand for any/all of their equal admissible perturbations.

The top and bottom boundaries of a surface diagram in a Gray-category are ordered string diagrams of its 0-, 1- and 2-cells, as in the case for a box-category. However, now there may be horizontal slices not intersecting a point representing a 3-cell of the underlying box-category (left) forming string diagrams that are not ordered (right). By the transverse intersection requirement, these correspond to line crossings in the projection string diagram, which we elaborate to interchanger 3-cells as in (2.19).



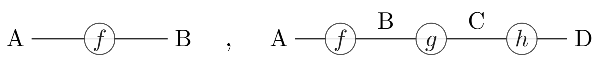
3 Diagram Semantics

Thus far we have used string diagrams for 2-dimensional categories and surface diagrams for 3-dimensional categories informally as convenient notations for sometimes unwieldy expressions. But in fact they have precise mathematical semantics. Such diagrams can be understood as presentations of free categorical structures. This was worked out in the case of string diagrams for monoidal categories in [JS91], and extended to string diagrams for 2-categories. The case of surface diagrams for Gray Categories appears in [Hum12].

In the following we will not address the geometric aspects of dual diagrams. While these are important, their development is rather complex, orthogonal to our present discussion, and can be found in the above references. For our purposes it will suffice to consider diagrams as algebraic objects, with equivalence given by an intuitive notion of configuration-preserving deformation.

A computad is an algebraic structure used to give a presentation for a free (generally higher-dimensional) category. The original construction was used by Street to give presentations for 2- and 3-categories [Str96]. Intuitively, an n-dimensional computad consists of an $(n-1)$-dimensional computad specifying its lower-dimensional structure together with a set of n-dimensional generator cells and functions specifying their boundaries. These boundaries are valued in the free $(n-1)$-dimensional category determined by the underlying $(n-1)$-dimensional computad, which is built up from its generator cells using the composition operations of the category and quotiented by its relations. We do not attempt to characterize the general construction here but rather describe the computads and free categories that we will need.

- A 0-*computad* G consists of a set of 0-generators G_0.
- The *free* 0-*category* presented by a 0-computad $\mathrm{diag}_0\mathrm{G}$ has as 0-cells the elements of the set G_0, with no further structure or relations.
- A 1-*computad* is a 0-computad G together with a set of 1-generators G_1 and boundary maps $\partial^-, \partial^+ : \mathrm{G}_1 \to \mathrm{diag}_0\mathrm{G}$, in other words, a directed graph.
- The *free* 1-*category* presented by a 1-computad $\mathrm{diag}_1\mathrm{G}$ has as 1-cells *dot diagrams* built from the generators of G using nullary and binary composition and satisfying the categorical unit and associative laws so that each of the following dot diagrams represents a unique 1-cell.



- A *sesquicomputad*, which is also a 2-*computad*, is a 1-computad G together with a set of 2-generators G_2 and boundary maps $\partial^-, \partial^+ : \mathrm{G}_2 \to \mathrm{diag}_1\mathrm{G}$. These are required to satisfy globularity relations asserting that the boundaries of 2-generators are parallel dot diagrams: $\partial^- \cdot \partial^- = \partial^+ \cdot \partial^-$ and $\partial^- \cdot \partial^+ = \partial^+ \cdot \partial^+$ in $\mathrm{G}_2 \to \mathrm{diag}_0\mathrm{G}$, where the first boundary operator of each composite is that of the computad G and the second is that of the category $\mathrm{diag}_1\mathrm{G}$.
- The *free sesquicategory* presented by a sesquicomputad $\mathrm{diag}_{\mathrm{ses}}\mathrm{G}$ has as 2-cells ordered string diagrams built from the generators of G. The *free* 2-*category* presented by a 2-computad $\mathrm{diag}_2\mathrm{G}$ is similar, except that the string diagrams need not be ordered and may contain horizontal compositions of 2-cells.

- A *Gray-computad* is a sesquicomputad G together with a set of 3-generators G_3 and boundary maps $\partial^-, \partial^+ : G_3 \to \mathrm{diag}_{\mathrm{ses}} G$. These are required to satisfy globularity relations asserting that the boundaries of 3-generators are parallel ordered string diagrams: $\partial^- \cdot \partial^- = \partial^+ \cdot \partial^-$ and $\partial^- \cdot \partial^+ = \partial^+ \cdot \partial^+$ in $G_3 \to \mathrm{diag}_1 G$.

- The *free Gray-category* presented by a Gray-computad $\mathrm{diag}_{\mathrm{Gray}} G$ has as 3-cells separated surface diagrams built from the generators of G. Interchanger 3-cells are inferred from line crossings in the projection string diagrams. We want the result to be a valid diagram in a Gray-category so we require that in the projection string diagram:

  - lines intersect transversely (so that line intersections represent interchangers),

  - such line crossings are oriented allowably (so that the interchangers are lax and/or oplax, as the case may be),

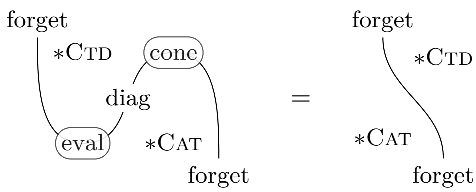
  - only two lines cross at a point (so that interchangers have only two 2-cell indices),

  - points do not intersect points or lines (so that compositions involving generator 3-cells along 0-cells are whiskerings).

  Such surface diagrams are sometimes called "photogenic". If we wish to allow n-ary 2-cell interchangers and 3-cell interchangers as in (2.20) then the last two conditions can be relaxed to requiring lines and line boundaries of points to obey the other conditions pairwise.

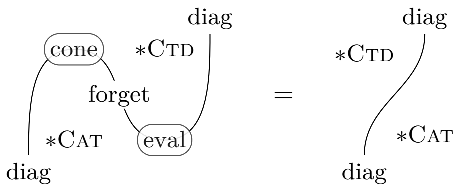
For $* \in \{0, 1, \mathrm{ses}, 2, \mathrm{Gray}\}$ the collection of $*$-categories itself forms a category $*$Cat, and the collection of $*$-computads forms a category $*$Ctd, whose morphisms are the boundary-respecting functions between generator sets. It is shown in [Hum12] that in each case the free $*$-category map is a functor diag : $*$Ctd $\to$ $*$Cat with a right adjoint forget : $*$Cat $\to$ $*$Ctd that regards each n-cell as an n-generator and forgets the composition structure. The adjunction units have components known as *cones*, which send n-generators to their singleton diagrams. The adjunction counits have components known as *evaluations*, which send diagrams of n-cells to their composites.

The right adjunction law says:



Given a $*$-category $\mathbb{C}$, the $*$-computad forget $\mathbb{C}$ has as n-generators n-cells of $\mathbb{C}$, while the $*$-computad forget(diag(forget $\mathbb{C}$)) has as n-generators diagrams of n-cells of $\mathbb{C}$. The relation says that for any n-cell φ of $\mathbb{C}$ if we form the singleton diagram comprising its cone and evaluate it then we get back φ.

The left adjunction law says:



Given a $*$-computad G, the $*$-category $\operatorname{diag} \mathrm{G}$ has as n-cells diagrams of n-generators of G, while the $*$-category $\operatorname{diag}(\operatorname{forget}(\operatorname{diag} \mathrm{G}))$ has as n-cells diagrams of diagrams of n-generators of G. The relation says that for any diagram δ of G's n-generators if we form the diagram of diagrams resulting from coning each n-generator in place and evaluate the resulting diagram of diagrams to a diagram then we get back δ.

4 Morphisms of 2-Categories

By combining Gray's classical results about the structure of the hierarchy of morphisms between 2-categories [Gra74] with Hummon's results about surface diagrams for Gray-categories [Hum12] we obtain an intuitive graphical calculus for 2-category theory. We can recover the conventional component-based presentation by using global elements and projection. This works because projection flattens a surface diagram in a Gray-category into a string diagram in the hom 2-category determined by its 0-dimensional boundary. By choosing the domain of this boundary to be the singleton 2-category $\mathbb{1}$ we obtain a string diagram in the codomain 2-category.

The simplest sort of 1-dimensional morphism between 2-categories is known as a 2-functor [KS74].

Definition 4.1 (2-functor between 2-categories)
A 2-*functor* between 2-categories $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ consists of the following structure:

object map: a function on objects $\mathrm{F}_0 : \mathbb{C}_0 \to \mathbb{D}_0$,

hom functors: for each pair of objects $\mathrm{A}, \mathrm{B} : \mathbb{C}$ a local functor ${\mathrm{F}_1}^{(\mathrm{A},\mathrm{B})} : \mathbb{C}\,(\mathrm{A} \to \mathrm{B}) \to \mathbb{D}\,(\mathrm{F}_0\mathrm{A} \to \mathrm{F}_0\mathrm{B})$.

The hom functors are required to strictly preserve the horizontal composition structure is the following sense[4].

arrow composition preservation: for each $\mathbb{C}$-object A and consecutive $\mathbb{C}$-arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ we have

$$\mathrm{F}(\mathrm{id}\mathrm{A}) = \mathrm{id}(\mathrm{F}\mathrm{A}) \quad \text{and} \quad \mathrm{F}(f \cdot g) = \mathrm{F}f \cdot \mathrm{F}g \tag{4.1}$$

disk horizontal composition preservation: for adjacent $\mathbb{C}$-disks $\varphi :\ (\mathrm{A} \to \mathrm{B})\,(f \to f')$ and $\psi :\ (\mathrm{B} \to \mathrm{C})\,(g \to g')$ we have

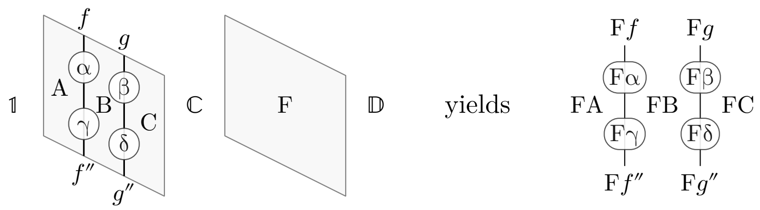
$$\mathrm{F}(\varphi \cdot\cdot \psi) = \mathrm{F}\varphi \cdot\cdot \mathrm{F}\psi \tag{4.2}$$

[4] Except for the sake of emphasis, we will follow the convention of omitting the super- and subscript annotations on the constituent maps.

Note that F necessarily preserves the vertical composition structure of disks because the F_1s are functors on the hom categories. Thus by (2.7) it preserves the horizontal composition units as well:

$$\mathrm{F}(\mathrm{id}^2\,\mathrm{A}) := \mathrm{F}(\mathrm{id}(\mathrm{id}\,\mathrm{A})) = \mathrm{id}(\mathrm{F}(\mathrm{id}\,\mathrm{A})) = \mathrm{id}(\mathrm{id}(\mathrm{FA})) =: \mathrm{id}^2(\mathrm{FA}).$$

In diagrams we represent a 2-functor as a surface separating the volumes representing its boundary 2-categories. This representation preserves all composition structure in the domain 2-category because given a string diagram δ in $\mathbb{C}$ the string diagram Fδ in $\mathbb{D}$ looks just like δ, but with "F" prefixed to each label. For example,



Like a natural transformation between 1-functors between 1-categories, a transformation between 2-functors between 2-categories [KS74; Lei98] has component arrows for objects. But because there is now "room" for 2-dimensional structure, we can categorify the property of being natural for arrows into the structure of component disks.

Definition 4.2 (oplax transformation between 2-functors)
An *oplax transformation* between 2-functors between 2-categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})$ consists of the following data:

object-component arrows: for each object of the domain 2-category $\mathrm{A} : \mathbb{C}$ an arrow of the codomain 2-category $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GA})$,

arrow-component disks: for each arrow of the domain 2-category $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ a disk of the codomain 2-category $\alpha f : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GB})\,(\mathrm{F}f \cdot \alpha\mathrm{B} \to \alpha\mathrm{A} \cdot \mathrm{G}f)$.

This data is required to satisfy the following relations:

compatibility with arrow composition: for each $\mathbb{C}$-object A and consecutive $\mathbb{C}$-arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ we have

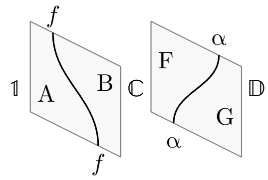
$$\alpha(\mathrm{id}\,\mathrm{A}) = \mathrm{id}(\alpha\mathrm{A}) \quad \text{and} \quad \alpha(f \cdot g) = (\mathrm{id}(\mathrm{F}f) \cdot\cdot\, \alpha g) \cdot (\alpha f \cdot\cdot\, \mathrm{id}(\mathrm{G}g)) \tag{4.3}$$

naturality for disks: for each $\mathbb{C}$-disk $\varphi : (\mathrm{A} \to \mathrm{B})\,(f \to f')$ we have

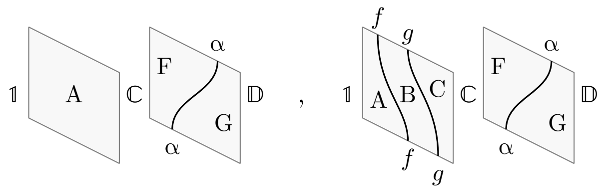
$$(\mathrm{F}\varphi \cdot\cdot\, \mathrm{id}(\alpha\mathrm{B})) \cdot \alpha f' = \alpha f \cdot (\mathrm{id}(\alpha\mathrm{A}) \cdot\cdot\, \mathrm{G}\varphi) \tag{4.4}$$

A *lax transformation* has its arrow-component disks oriented the other way and satisfies suitably dualized relations. A *pseudo transformation* has invertible arrow-component disks that make it lax one way and oplax the other. A pseudo transformation is *strict* if its arrow-component disks are identities.

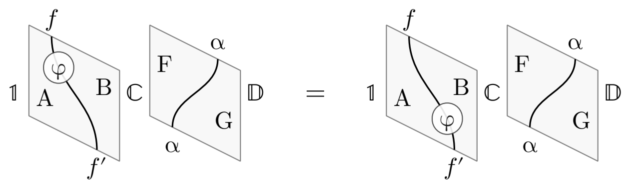
Transformations are 2-cells between parallel 1-cells, so we represent them in surface diagrams as lines separating co-bounded surfaces. Their arrow-component disks αf are the projections of Gray-interchangers $\chi_{(f,\alpha)}$.



Their compatibility with arrow composition (4.3) is the respect by interchangers for composition in the first index (2.15),



and their naturality for disks (4.4) is interchanger naturality in the first index (2.18).



The *naturality* of transformations between 2-functors corresponds to the *parametricity* of their component structure in the sense that it is independent of the domain 2-category. This is what makes the separated surface notation so useful: with no connection between the surfaces there is no obstruction to moving a disk in the domain 2-category past the crossing in the projection string diagram representing an arrow-component disk, so naturality is just a homotopy.

In a setting where objects have 2-dimensional structure, so too should the homs between them. Indeed, we wouldn't be able to represent disks as global elements if this were not the case. This motivates the definition of the following 3-dimensional morphisms [KS74; JY20].

Definition 4.3 (modification between oplax transformations)
A *modification* between oplax transformations between 2-functors between 2-categories $\mu : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})\,(\alpha \to \beta)$ consists of the following data:

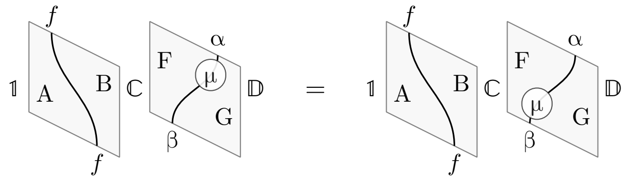
object-component disks: for each object of the domain 2-category $\mathrm{A} : \mathbb{C}$ a disk of the codomain 2-category $\mu\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GA})\,(\alpha\mathrm{A} \to \beta\mathrm{A})$.

These object-component disks are required to satisfy the following relation:

naturality for arrows: for each $\mathbb{C}$-arrow $f : \mathrm{A} \to \mathrm{B}$ we have

$$(\mathrm{id}(\mathrm{F}f) \cdot\cdot\, \mu\mathrm{B}) \cdot \beta f = \alpha f \cdot (\mu\mathrm{A} \cdot\cdot\, \mathrm{id}(\mathrm{G}f)) \tag{4.5}$$

Modifications are 3-cells between parallel 2-cells, so we represent them in surface diagrams as points separating co-bounded lines. Their naturality for arrows (4.5) is interchanger naturality in the second index (2.18).



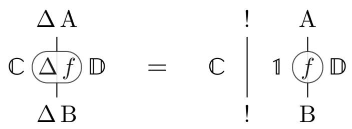
The composition structure of transformations and modifications is that of 2-cells and 3-cells in a Gray-category and, as we have seen in section 2, is apparent from their diagrammatic representations.

We see that in the construction of the Gray-category of 2-categories, 2-functors, (lax/oplax/pseudo) transformations and modification by explicit definitions, the structure of interchangers is given as part of the structure of transformations, namely, as their arrow-component disks, and the naturality of interchangers in their first index as a property of this structure; whereas the naturality of interchangers in their second index is given as a property of modifications. In contrast, in a general Gray-category interchangers are structures in their own right, and their naturality in each index is a property of this structure itself. Arguably, the nature of the Gray-category of 2-categories appears more clearly when viewed abstractly, where the interchanger 3-cells have an independent existence and properties uniform in their indices.

5 Cones and Limits of 2-Functors

As an extended example of using the graphical calculus of surface diagrams to represent and reason about 2-categorical constructions, in this section we consider 2-dimensional cones and limits. We begin by recapitulating the constructions of their 1-dimensional analogues in a form that will prove convenient.

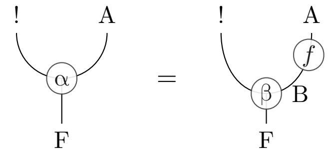
Given categories $\mathbb{C}$ and $\mathbb{D}$, for each object $\mathrm{A} : \mathbb{D}$ there is a constant functor $\Delta\,\mathrm{A} : \mathbb{C} \to \mathbb{D}$ sending every arrow to $\mathrm{id}\,\mathrm{A}$. Likewise, for each arrow $f : \mathbb{D}\,(\mathrm{A} \to \mathrm{B})$ there is a constant natural transformation $\Delta\, f : \Delta\,\mathrm{A} \to \Delta\,\mathrm{B}$, all of whose object-component arrows are f. Together these determine a *concentration functor* $\Delta : \mathbb{D} \to (\mathbb{C} \to \mathbb{D})$, or equivalently $(\mathbb{1} \to \mathbb{D}) \to (\mathbb{C} \to \mathbb{D})$, which can be defined as precomposition by the unique functor to the singleton category $! : \mathbb{C} \to \mathbb{1}$.



A *cone* over a functor[5] $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ with vertex $\mathrm{A} : \mathbb{D}$ is a natural transformation $\alpha : \Delta\,\mathrm{A} \to \mathrm{F}$. A *morphism* between cones α and β is an arrow between their

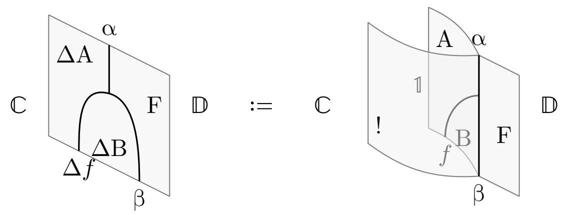
[5] not to be confused with the unit of the diag ⊣ forget adjunction from section 3

respective vertex objects $f : \mathbb{D}\,(\mathrm{A} \to \mathrm{B})$ that factors the domain cone through the codomain cone as $\alpha = \Delta f \cdot \beta$.



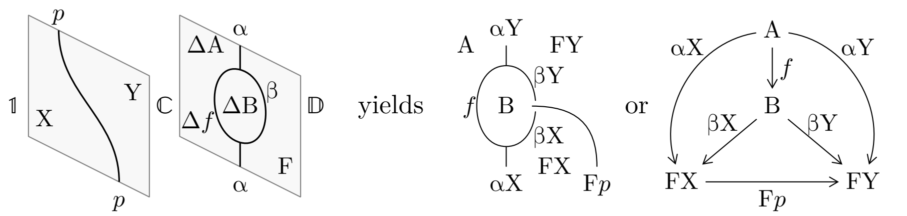
(5.1)

Rather than expressing this property as a relation between diagrams we can instead express it as a relation within a digram by enlisting another dimension, in which the boundaries are the two diagrams related in equation (5.1).

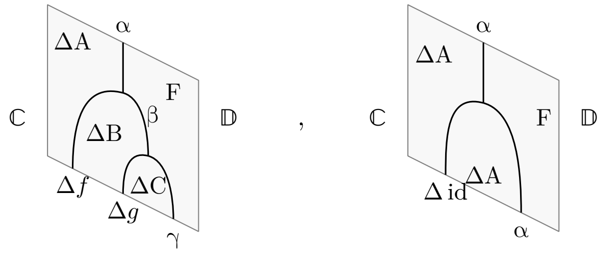


(5.2)

By composing the invertibility of this equation with the global element corresponding to a generic arrow $p : \mathbb{C}\,(\mathrm{X} \to \mathrm{Y})$ we recover the conventional presentation of cone component factorization by a cone morphism.



Consecutive cone morphisms compose by substitution: if $\alpha = \Delta\, f \cdot \beta$ and $\beta = \Delta\, g \cdot \gamma$ then $\alpha = \Delta\, f \cdot \Delta\, g \cdot \gamma$, which is $\Delta(f \cdot g) \cdot \gamma$ because Δ is a functor; and each cone has an identity morphism determined by the identity arrow on its vertex, $\alpha = \Delta(\mathrm{id}\,\mathrm{A}) \cdot \alpha$, which is just α, again by the functoriality of Δ.



For a given functor F, its cones their morphisms form a category $\textsc{Cone}\,\mathrm{F}$.

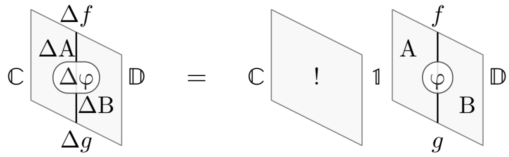
A *limit* for a functor $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ is a terminal object $\omega : \Delta\,\mathrm{L} \to \mathrm{F}$ of $\textsc{Cone}\,\mathrm{F}$. Its universal property determines a natural bijection between the objects of the category $\textsc{Cone}\,\mathrm{F}$ and the objects of the slice category $\textsc{Cone}\,\mathrm{F}/\omega$ (left).

Expanding the definitions, this says that for each $\mathbb{D}$-object A there is a bijection, natural in A, between the hom set of natural transformations $(\mathbb{C} \to \mathbb{D})(\Delta\,A \to F)$ and the hom set of arrows $\mathbb{D}(A \to L)$ that factor them by Δ through ω (right).

$$\frac{\alpha : \textsc{Cone}\,F}{\overline{f : \textsc{Cone}\,F\,(\alpha \to \omega)}} \qquad \text{or} \qquad \frac{\alpha : (\mathbb{C} \to \mathbb{D})(\Delta\,A \to F)}{\overline{f : \mathbb{D}(A \to L)\ .\ \alpha = \Delta\,f \cdot \omega}} \tag{5.3}$$

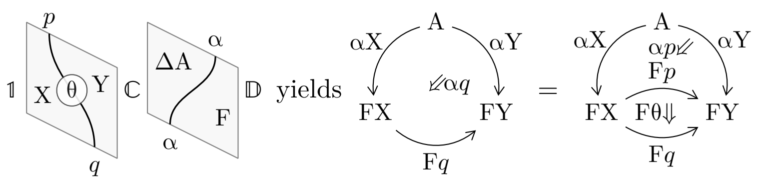
In other words, the functions $\Delta - \cdot\, \omega : \mathbb{D}(A \to L) \to (\mathbb{C} \to \mathbb{D})(\Delta\,A \to F)$ are bijections of sets natural in A.

Now let $\mathbb{C}$ and $\mathbb{D}$ be 2-categories. For each object $A : \mathbb{D}$ there is a constant 2-functor $\Delta\,A : \mathbb{C} \to \mathbb{D}$ sending every disk to $\mathrm{id}^2 A$. Likewise, for each arrow $f : \mathbb{D}(A \to B)$ there is a constant strict transformation $\Delta\,f : \Delta\,A \to \Delta\,B$, all of whose object-component arrows are f. And for each disk $\varphi : \mathbb{D}(A \to B)(f \to g)$ there is a constant modification $\Delta\,\varphi : (\mathbb{C} \to \mathbb{D})(\Delta\,A \to \Delta\,B)(\Delta\,f \to \Delta\,g)$, all of whose object-component disks are φ. Together these determine a *concentration 2-functor* $\Delta : \mathbb{D} \to (\mathbb{C} \to \mathbb{D})$, or equivalently $(\mathbb{1} \to \mathbb{D}) \to (\mathbb{C} \to \mathbb{D})$, which can be defined as precomposition by the unique 2-functor to the singleton 2-category $! : \mathbb{C} \to \mathbb{1}$.



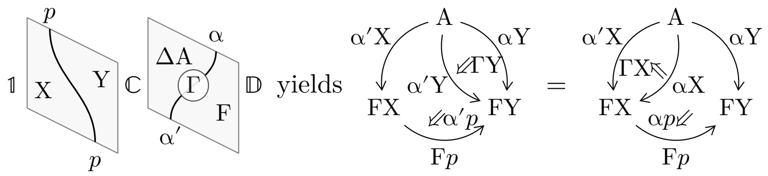
An *oplax cone* over a 2-functor $F : \mathbb{C} \to \mathbb{D}$ with vertex $A : \mathbb{D}$ is an oplax transformation $\alpha : \Delta\,A \to F$. So for a generic arrow $p : \mathbb{C}(X \to Y)$ the arrow-component disk αp is a directed triangle in $\mathbb{D}(A \to FY)(\alpha Y \to \alpha X \cdot Fp)$. We can define lax-, pseudo-, and strict cones analogously as the respective sort of transformation.

Composing an oplax cone $\alpha : \Delta\,A \to F$ with the global element corresponding to a generic disk $\theta : \mathbb{C}(X \to Y)(p \to q)$ as shown on the left determines the relation between component diagrams shown on the right by the naturality of interchangers in their first index (2.18).



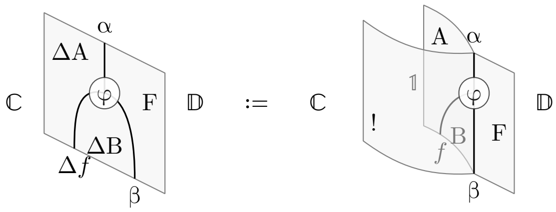
(5.4)

Composing a modification between parallel oplax cones $\Gamma : (\Delta\,A \to F)(\alpha \to \alpha')$ with the global element corresponding to a generic arrow $p : \mathbb{C}(X \to Y)$ as shown on the left determines the relation between component diagrams shown on the right by the naturality of interchangers in their second index (2.18).



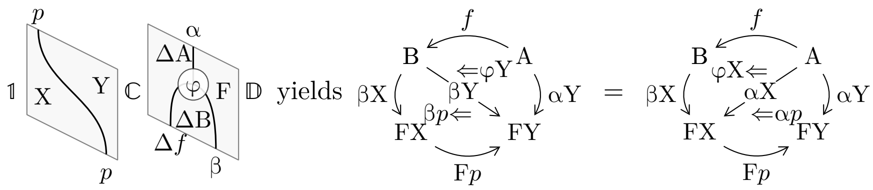
(5.5)

A *morphism* between oplax cones α and β is an arrow between their respective vertex objects $f : \mathbb{D}\,(\mathrm{A} \to \mathrm{B})$ together with a modification $\varphi : \alpha \to \Delta\, f \cdot \beta$.

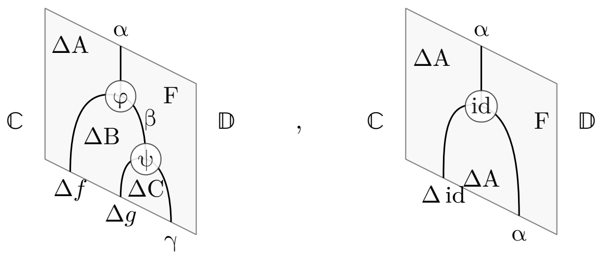


(5.6)

We can see this as a categorification of a cone morphism for a 1-functor, depicted in diagram (5.2), in the sense that there is "room" in a 2-category to turn the property of arrow equality into the structure of a disk[6]. Composing an oplax cone morphism $(f\,, \varphi)$ with the global element corresponding to a generic arrow $p : \mathbb{C}\,(\mathrm{X} \to \mathrm{Y})$ as shown on the left determines the relation between component diagrams shown on the right.



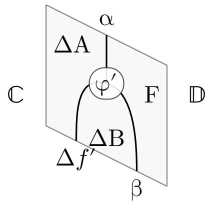
Consecutive oplax cone morphisms $(f\ ,\ \varphi)\ :\ (\mathrm{A}\,, \alpha)\ \to\ (\mathrm{B}\,, \beta)$ and $(g\ ,\ \psi)\ :$ $(\mathrm{B}\,, \beta)\ \to\ (\mathrm{C}\,, \gamma)$ compose as $(f \cdot g\ ,\ \varphi \cdot (\Delta\, f \cdot\cdot\, \psi))$, and each oplax cone $\alpha\ :$ $\Delta\,\mathrm{A} \to \mathrm{F}$ has an identity morphism $(\mathrm{id}\,\mathrm{A}\ ,\ \mathrm{id}\,\alpha)$. Similar to the 1-dimensional case, these are well-bounded by the (in this case, 2-) functoriality of Δ.



So for a given 2-functor we have a 1-category of its oplax cones and their morphisms. But we can do better. For parallel oplax cone morphisms $(f,\varphi), (f',\varphi')$:

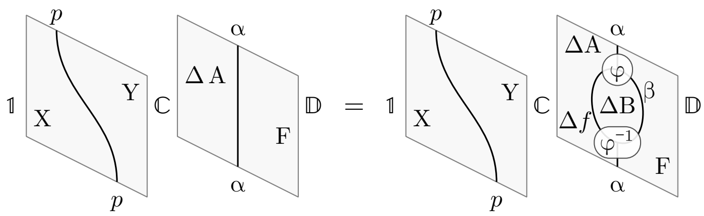
[6] As with the 2-dimensional cones themselves, we have options regarding the variance of this structure.

$(A, \alpha) \to (B, \beta)$ a transformation between them is a disk between their respective arrows $\mu : \mathbb{D}(A \to B)(f \to f')$ that factors the modification of the latter through that of the former as $\varphi' = \varphi \cdot (\Delta \mu \cdot\cdot \beta)$.



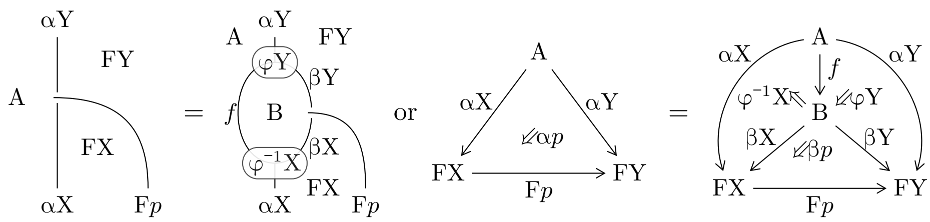
These transformations compose vertically by substitution and horizontally by the horizontal composition of their underlying disks. Thus for a given 2-functor F, we have a 2-category 2CONE F of its oplax cones, their morphisms, and the transformations of these.

We will call a cone morphism *factoring* if its modification is invertible. Each 2-category 2CONE F has a sub-2-category 2CONE^f F whose arrows are the factoring cone morphisms. For a factoring cone morphism $(f, \varphi) : (A, \alpha) \to (B, \beta)$ we have the following relation for each arrow $p : \mathbb{C}(X \to Y)$.



(5.7)

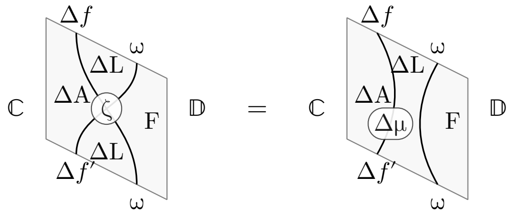
By projection this gives the following factorization of cone component diagrams.



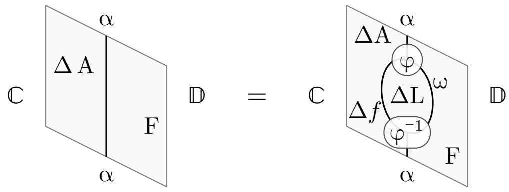
The singular concept of limit of a functor ramifies into a family of related concepts in the 2-dimensional case [Kel89; CM20]. We make no attempt to survey these here. Rather, we investigate a single conical, directed notion of limit of a 2-functor using the graphical calculus to elucidate its attributes.

For our present purposes at least, an *oplax limit* of a 2-functor $F : \mathbb{C} \to \mathbb{D}$ is an oplax cone $\omega : \Delta L \to F$ such that the functors $\Delta - \cdot \omega : \mathbb{D}(A \to L) \to (\mathbb{C} \to \mathbb{D})(\Delta A \to F)$ are equivalences of categories natural in A. This can be seen as a categorification of the notion of limit of a 1-functor in (5.3) in the sense that an equivalence of hom categories is a categorification of a bijection of hom sets. We obtain a better algebraic understanding of this universal property using the characterization of an equivalence of categories as a functor that is full, faithful, and essentially surjective on objects.

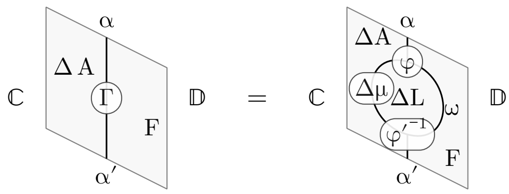
For a functor $\Delta - \cdot \omega$ to be *full and faithful* means that for each arrow in its codomain between objects in its image $\zeta : (\mathbb{C} \to \mathbb{D})\,(\Delta\,A \to F)\,(\Delta\,f \cdot \omega \to \Delta\,f' \cdot \omega)$ there is a unique arrow in its domain $\mu : \mathbb{D}\,(A \to L)\,(f \to f')$ that factors it through ω:



For a functor $\Delta - \cdot \omega$ to be *essentially surjective on objects* means that for each $\alpha : (\mathbb{C} \to \mathbb{D})\,(\Delta\,A \to F)$ there is some $f : \mathbb{D}\,(A \to L)$ such that α is isomorphic to $\Delta\,f \cdot \omega$. This is to say that there is an invertible modification $\varphi : (\alpha \to \Delta\,f \cdot \omega)$; that is, there is a factoring cone morphism $(f\,,\,\varphi) : (A\,,\,\alpha) \to (L\,,\,\omega)$.



Let $\Gamma : (\mathbb{C} \to \mathbb{D})\,(\Delta\,A \to F)\,(\alpha \to \alpha')$ be an arbitrary modification between parallel oplax cones. By essential surjectivity on objects we have invertible modifications $\varphi : (\alpha \to \Delta\,f \cdot \omega)$ and $\varphi' : (\alpha' \to \Delta\,f' \cdot \omega)$. Thus we have a modification $\varphi^{-1} \cdot \Gamma \cdot \varphi' : (\Delta\,A \to F)\,(\Delta\,f \cdot \omega \to \Delta\,f' \cdot \omega)$. By full-and-faithfulness this is equal to $\Delta\,\mu \cdot\cdot\, \omega$ for a unique $\mu : \mathbb{D}\,(A \to L)\,(f \to f')$. By composing with the respective inverses we obtain $\Gamma = \varphi \cdot (\Delta\,\mu \cdot\cdot\, \omega) \cdot {\varphi'}^{-1}$.



Taking the component at a generic object $X : \mathbb{C}$ yields the following two-dimensional property of this limit.

$$\begin{tikzcd}
A \arrow[rr, bend left, "\alpha X"] \arrow[rr, bend right, "\alpha' X"'] \arrow[rr, phantom, "\Downarrow \Gamma X"] & & FX & = & A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \mu"] \arrow[rr, bend left=60, "\alpha X"] \arrow[rr, bend right=60, "\alpha' X"'] \arrow[rr, phantom, bend left=35, "\Downarrow \varphi X"] \arrow[rr, phantom, bend right=35, "\Downarrow {\varphi'}^{-1} X"] & L \arrow[r, "\omega X"] & FX
\end{tikzcd}$$

Inspired by the 1-dimensional case, we might expect that an oplax limit cone for a 2-functor F is one that is terminal in $2\textsc{Cone}^{f}\,F$ in the sense that for each oplax cone (A, α) the hom 2-category $(\mathrm{A}, \alpha) \to (\mathrm{L}, \omega)$ is equivalent to the singleton 2-category $\mathbb{1}$. However, this seems to not be the case. Clingman and Moser have investigated the relationships between various notions of limit of a 2-functor and of terminal cone obtained by varying their strictness and variance, and shown them to be in general unrelated [CM20].

6 Lax Functoriality

In this section we consider the setting where we require the morphisms between 2-categories to preserve their composition only up to a specified directed comparison structure [Bén67; Lei98]. Such morphisms are called lax or oplax functors, depending on the variance of their comparison structure.

Definition 6.1 (lax functor between 2-categories)
A *lax functor* between 2-categories $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ has an object map and hom functors, just like a 2-functor. But instead of strictly preserving the composition of arrows and the horizontal composition of disks, a lax functor has a *lax comparison structure* θ, which for each path of consecutive arrows $f_1, \cdots, f_n$ in $\mathbb{C}$ gives a comparison disk $\theta(f_1, \cdots, f_n) : \mathrm{F}f_1 \cdot \ldots \cdot \mathrm{F}f_n \to \mathrm{F}(f_1 \cdot \ldots \cdot f_n)$ in $\mathbb{D}$. These comparison disks are required to be natural and coherent in the sense described next.

A presentation for a comparison structure for a lax functor can be given as follows.

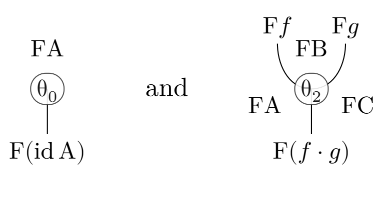
nullary comparitors: for each object $\mathrm{A} : \mathbb{C}$ a natural transformation $\theta_0^{\mathrm{A}} : \mathrm{id}_{\mathbb{D}}(\mathrm{F}_0\mathrm{A}) \to \mathrm{F}_1(\mathrm{id}_{\mathbb{C}}\mathrm{A})$,

$$\begin{tikzcd}
\mathbb{1} \arrow[r, equal] \arrow[d, "\mathrm{id}_{\mathbb{C}}\mathrm{A}"'] \arrow[dr, phantom, "\Leftarrow \theta_0^{\mathrm{A}}"] & \mathbb{1} \arrow[d, "\mathrm{id}_{\mathbb{D}}(\mathrm{FA})"] \\
\mathbb{C}\,(\mathrm{A} \to \mathrm{A}) \arrow[r, "\mathrm{F}"'] & \mathbb{D}\,(\mathrm{FA} \to \mathrm{FA})
\end{tikzcd}$$

binary comparitors: for objects $\mathrm{A}, \mathrm{B}, \mathrm{C} : \mathbb{C}$ a natural transformation $\theta_2^{\mathrm{A,B,C}} : (\mathrm{F}_1-) \cdot_{\mathbb{D}} (\mathrm{F}_1-) \to \mathrm{F}_1(- \cdot_{\mathbb{C}} -)$.

$$\begin{tikzcd}
\mathbb{C}\,(\mathrm{A} \to \mathrm{B}) \times \mathbb{C}\,(\mathrm{B} \to \mathrm{C}) \arrow[r, "\mathrm{F} \times \mathrm{F}"] \arrow[d, "- \cdot_{\mathbb{C}} -"'] \arrow[dr, phantom, "\Leftarrow \theta_2^{\mathrm{A,B,C}}"] & \mathbb{D}\,(\mathrm{FA} \to \mathrm{FB}) \times \mathbb{D}\,(\mathrm{FB} \to \mathrm{FC}) \arrow[d, "- \cdot_{\mathbb{D}} -"] \\
\mathbb{C}\,(\mathrm{A} \to \mathrm{C}) \arrow[r, "\mathrm{F}"'] & \mathbb{D}\,(\mathrm{FA} \to \mathrm{FC})
\end{tikzcd}$$

These, respectively, have the following component disks in $\mathbb{D}$:
$\theta(\mathrm{A}) \coloneqq \theta_0^{\mathrm{A}}(\star) : \mathrm{id}(\mathrm{FA}) \to \mathrm{F}(\mathrm{id}\,\mathrm{A})$ and $\theta(f, g) \coloneqq \theta_2^{\mathrm{A,B,C}}(f, g) : \mathrm{F}f \cdot \mathrm{F}g \to \mathrm{F}(f \cdot g)$

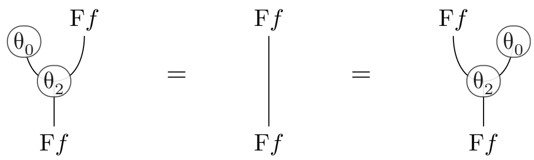


(6.1)

These are required to satisfy the following coherence laws.

comparitor unit laws: for $\mathbb{C}$-arrow $f : \mathrm{A} \to \mathrm{B}$, the relations

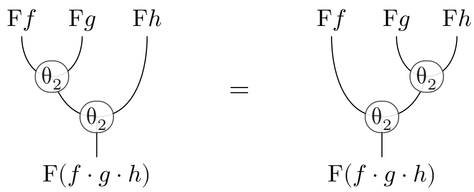
$$(\theta(\mathrm{A}) \cdot\cdot\, \mathrm{F}f) \cdot \theta(\mathrm{id}\,\mathrm{A}\,,\, f) \quad = \quad \mathrm{id}(\mathrm{F}f) \quad = \quad (\mathrm{F}f \cdot\cdot\, \theta(\mathrm{B})) \cdot \theta(f\,,\, \mathrm{id}\,\mathrm{B})$$



(6.2)

comparitor associative law: for consecutive $\mathbb{C}$-arrows $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$, and $h : \mathrm{C} \to \mathrm{D}$, the relation

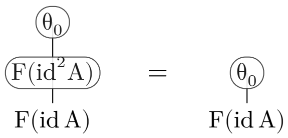
$$(\theta(f\,,\, g) \cdot\cdot\, \mathrm{F}h) \cdot \theta(f \cdot g\,,\, h) \quad = \quad (\mathrm{F}f \cdot\cdot\, \theta(g\,,\, h)) \cdot \theta(f\,,\, g \cdot h)$$



(6.3)

The naturality of the lax comparitors amount to the following relations on their components.

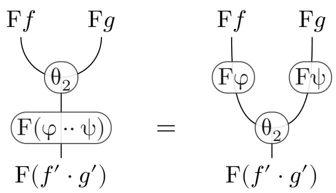
comparitor unit naturality: for object $\mathrm{A} : \mathbb{C}$, we have $\theta(\mathrm{A}) \cdot \mathrm{F}(\mathrm{id}^2\mathrm{A}) = \theta(\mathrm{A})$:



(6.4)

But because $\mathrm{F}(\mathrm{id}^2\mathrm{A}) \coloneqq \mathrm{F}(\mathrm{id}(\mathrm{id}\,\mathrm{A})) = \mathrm{id}(\mathrm{F}(\mathrm{id}\,\mathrm{A}))$ by the local functoriality of F, this condition is redundant.

comparitor composition naturality: for adjacent disks $\varphi : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to f')$ and $\psi : \mathbb{C}\,(\mathrm{B} \to \mathrm{C})\,(g \to g')$, we have $\theta(f\,,\, g) \cdot \mathrm{F}(\varphi \cdot\cdot\, \psi) = (\mathrm{F}\varphi \cdot\cdot\, \mathrm{F}\psi) \cdot \theta(f'\,,\, g')$:

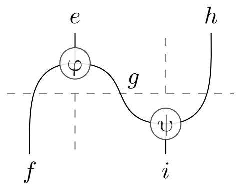


(6.5)

An *oplax functor* is one where the comparitors are oriented the other way and satisfy dual conditions. A *pseudo functor* is both lax and oplax with invertible comparitors, and a 2-functor is a pseudo functor with identity comparitors.

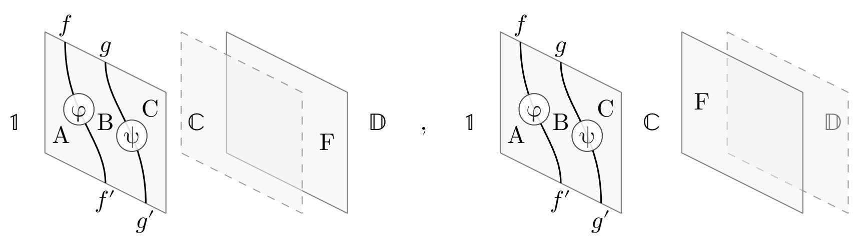
We would like to incorporate lax functors into our graphical calculus, but in trying to do so we encounter a problem. For a 2-functor $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ it doesn't matter whether we compose a diagram of cells in $\mathbb{C}$ and then send the composite cell to $\mathbb{D}$ by F or whether we send the cells over individually by F and then compose them in $\mathbb{D}$, the result is the same. But if F is only a lax functor then *where* we do the composition makes a difference: the composite in $\mathbb{D}$ of F-images is generally not the same as the F-image of the composite in $\mathbb{C}$ because a lax functor preserves the composition structure of its domain 2-category only up to its comparitor's components.

In fact, if we apply a lax functor to the cells of a diagram that is composable in its domain 2-category, the resulting collection of cells may not even be composable in its codomain 2-category. For example, for the diagram shown below, the vertical codomain of the F-image composite of its top half $\mathrm{F}\varphi \cdot\cdot \mathrm{F}h$ and the vertical domain of the F-image composite of its bottom half $\mathrm{F}f \cdot\cdot \mathrm{F}\psi$ coincide only if $\mathrm{F}(f \cdot g) \cdot \mathrm{F}h = \mathrm{F}f \cdot \mathrm{F}(g \cdot h)$.



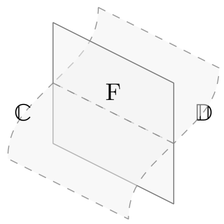
The lax functor images of the cells of a composable diagram themselves form a composable diagram if the diagram can be decomposed into a horizontal composition of vertical compositions of cells. We will call such diagrams *stratified*.

In order to accommodate lax functors we augment our surface diagrams with annotations to indicate where compositions take place. A *compositor* sheet is a surface drawn within a volume representing a 2-category. It is intended to represent the composition of the portion of the diagram preceding it into a single cell. A compositor sheet is significant only when it occurs before or after a surface representing a lax or oplax functor that is not a strict 2-functor. If the compositor sheet precedes the functor surface in the composition order then the diagram is first composed in the functor's domain and then the functor is applied to the resulting composite cell. If the compositor sheet follows the functor surface in the composition order then the functor is first applied to the constituent cells of a diagram in the functor's domain and then the resulting diagram is composed in the functor's codomain. The latter configuration requires the side-condition that the diagram's cells' functor images be composable, and is satisfied whenever the diagram is stratified. For example, the left diagram below represents $\mathrm{F}(\varphi \cdot\cdot \psi) : \mathbb{D}\,(\mathrm{F}(f \cdot g) \to \mathrm{F}(f' \cdot g'))$, whereas the right diagram represents $\mathrm{F}\varphi \cdot\cdot \mathrm{F}\psi : \mathbb{D}\,(\mathrm{F}f \cdot \mathrm{F}g \to \mathrm{F}f' \cdot \mathrm{F}g')$.

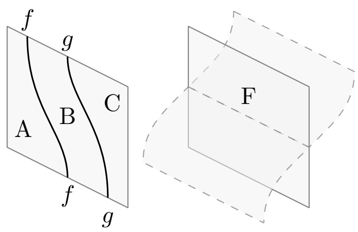


The global elements $A, B, C : \mathbb{1} \to \mathbb{C}$ should themselves be strict 2-functors, otherwise their lax comparison structures would induce nontrivial monads on the corresponding $\mathbb{C}$-objects, as was observed by Bénabou [Bén67].

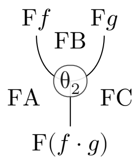
The comparison structure θ for a lax functor $F : \mathbb{C} \to \mathbb{D}$ provides a map from the composite in $\mathbb{D}$ of F-images to the F-image of the composite in $\mathbb{C}$. We can represent this in surface diagrams as the passing of a compositor sheet backward in the composition order through the lax functor surface.



If we compose this with a diagram in $\mathbb{C}$ consisting of a consecutive pair arrows then the projection gives the corresponding binary comparitor component.

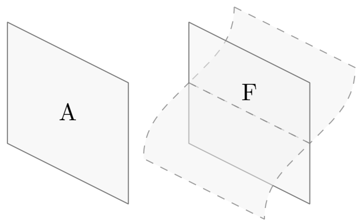


yields

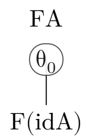


(6.6)

If we compose it with a diagram consisting of no arrows then the projection gives the corresponding nullary comparitor component.



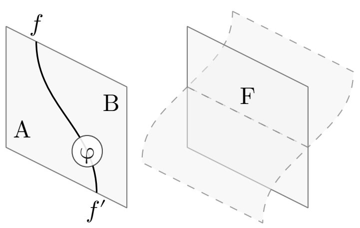
yields



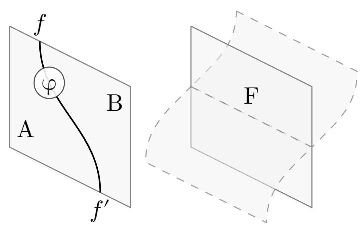
(6.7)

The comparitor unit (6.2) and associative (6.3) laws together imply that there is a unique $\theta_n^{A_0,\cdots,A_n}$ for each $n \in \mathbb{N}$, with $\theta(f) := \theta_1^{A,B}(f) : Ff \to Ff = \mathrm{id}(Ff)$.

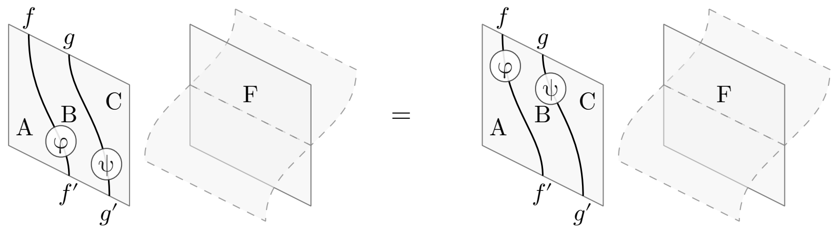
For a diagram consisting of a single disk $\varphi : \mathbb{C}\,(A \to B)\,(f \to f')$, the image of the diagram composite and the composite of the diagram comprising the disk's image are both just the image of the disk. So we have $\theta(f)\cdot F\varphi = F\varphi = F\varphi\cdot\theta(f')$.



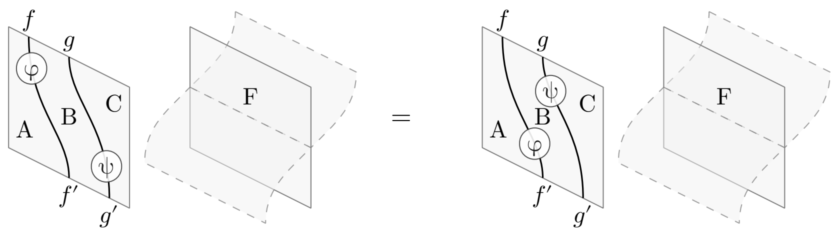
$=$



Thus vertical composites of disks can freely slide past unary comparitors. The same is true for horizontal composites of disks and binary comparitors due to the comparitor naturality condition (6.5).

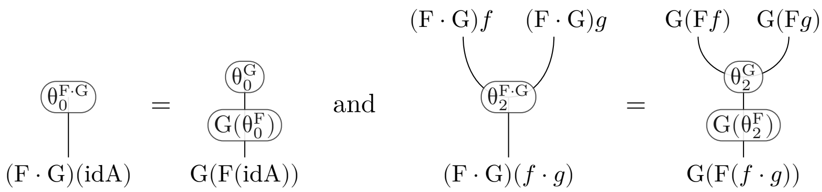


Moreover, because identity disks are strict units of vertical composition and their lax functor images are again identity disks, we may slide any disk of a horizontal composition past a comparitor independently of the others. So the next two diagrams are also equal to the previous two.



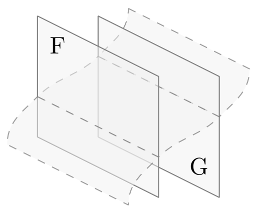
That is to say, disks in different strata can move past a comparitor independently. Thus any stratified diagram in $\mathbb{C}$ may be positioned arbitrarily with respect to the comparitor for a lax functor F, and all of the resulting projections into $\mathbb{D}$ will be equal.

Consecutive lax functors $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ and $\mathrm{G} : \mathbb{D} \to \mathbb{E}$ compose strictly, with composite comparitor components given by



(6.8)

In surface diagrams we represent these as the passage of a compositor sheet through the two lax functor surfaces sequentially.



This composition of lax functors is associative with identity 2-functors acting as composition units.

We can define transformations between lax functors as well [KS74; Lei98].

Definition 6.2 (oplax transformation between lax functors)
An *oplax transformation* between lax functors between 2-categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})$ has object-component arrows and arrow-component disks that are natural for disks in the domain 2-category (4.4) just like an oplax transformation between 2-functors.

However, now the arrow composition compatibility conditions (4.3) do not make sense because for object $\mathrm{A} : \mathbb{C}$ we have $\alpha(\mathrm{id\,A}) : \mathrm{F}(\mathrm{id\,A}) \cdot \alpha\mathrm{A} \to \alpha\mathrm{A} \cdot \mathrm{G}(\mathrm{id\,A})$ whereas $\mathrm{id}(\alpha\mathrm{A}) : \alpha\mathrm{A} \to \alpha\mathrm{A}$, and for consecutive arrows $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ and $g : \mathbb{C}\,(\mathrm{B} \to \mathrm{C})$ we have $\alpha(f \cdot g) : \mathrm{F}(f \cdot g) \cdot \alpha\mathrm{C} \to \alpha\mathrm{A} \cdot \mathrm{G}(f \cdot g)$ whereas $(\mathrm{F}f \cdot\cdot\, \alpha g) \cdot (\alpha f \cdot\cdot\, \mathrm{G}g) : \mathrm{F}f \cdot \mathrm{F}g \cdot \alpha\mathrm{C} \to \alpha\mathrm{A} \cdot \mathrm{G}f \cdot \mathrm{G}g$, but the lax functors F and G preserve arrow composition structure only up to their respective comparitors.

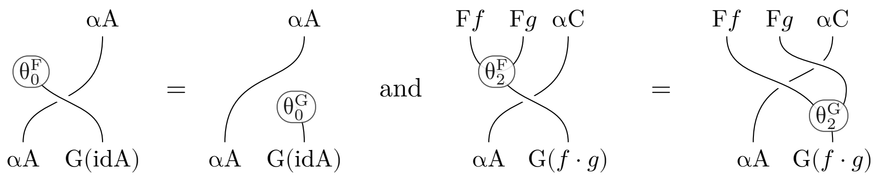
So instead we require the arrow-component disks to be compatible with these comparitors in the sense that

$$(\theta^{\mathrm{F}}(\mathrm{A}) \cdot\cdot\, \alpha\mathrm{A}) \cdot \alpha(\mathrm{id\,A}) \;=\; \alpha\mathrm{A} \cdot\cdot\, \theta^{\mathrm{G}}(\mathrm{A}) \tag{6.9}$$

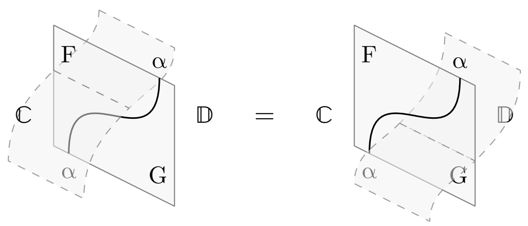
and

$$(\theta^{\mathrm{F}}(f\,,g) \cdot\cdot\, \alpha\mathrm{C}) \cdot \alpha(f \cdot g) \;=\; (\mathrm{F}f \cdot\cdot\, \alpha g) \cdot (\alpha f \cdot\cdot\, \mathrm{G}g) \cdot (\alpha\mathrm{A} \cdot\cdot\, \theta^{\mathrm{G}}(f\,,g)) \tag{6.10}$$

We can represent these lax comparitor compatibility conditions in string diagrams as



In surface diagrams this corresponds to the movement of the line representing the oplax transformation past the intersection of a compositor sheet with a lax functor surface.

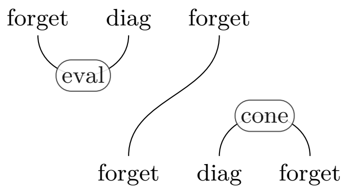


A *modification* between oplax transformations between lax functors is defined just like a modification between oplax transformations between 2-functors, with no additional compatibility relations involving the lax comparison structure [Lei98; JY20].

Compositor sheets give us pretty pictures, but we would like to understand them in terms of the diagram semantics of section 3. A compositor is intended to take a string diagram of disks of a 2-category and compose it to a single disk of that 2-category. For a 2-category $\mathbb{C}$, a string diagram of $\mathbb{C}$-disks is a 2-generator of the 2-computad forget(diag(forget $\mathbb{C}$)). And in general, the operation that forms diagrams of cells of a $*$-category is the functor forget $\cdot$ diag $\cdot$ forget : $*$CAT $\to$ $*$CTD. For brevity we write this functor as "fdf".

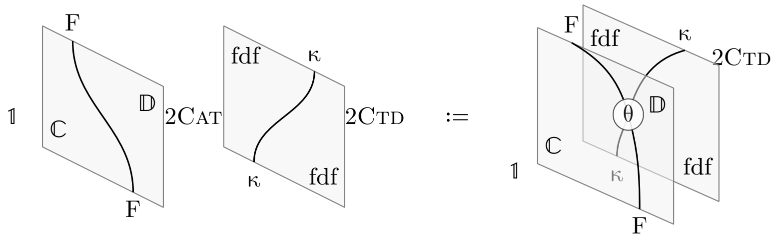
Recall that the adjunction diag $\dashv$ forget has as counit eval : forget $\cdot$ diag $\to$ id($*$CAT) whose components evaluate diagrams of cells to their composites, and as unit cone : id($*$CTD) $\to$ diag $\cdot$ forget whose components create singleton diagrams. From these we can form the following natural transformation:

κ :=

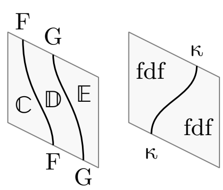


Given a $*$-category $\mathbb{C}$, $\kappa\mathbb{C}$ transforms a diagram of $\mathbb{C}$-cells into the singleton diagram of its composite $\mathbb{C}$-cell. This is just how we want compositors to behave. Moreover, κ is idempotent by the right adjunction law. This matches our expectation that if we compose an already-composed diagram then we leave it unchanged.

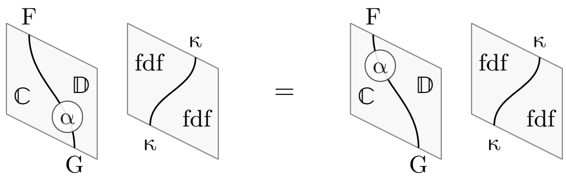
The comparison structure of a lax functor $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ allows us to pass a compositor sheet backward through the functor's surface. This gives us a morphism θ : (F $\cdot\cdot$ fdf) $\cdot$ ($\mathbb{D}$ $\cdot\cdot$ κ) $\to$ ($\mathbb{C}$ $\cdot\cdot$ κ) $\cdot$ (F $\cdot\cdot$ fdf), which we can think of as an *oplax* interchanger of F and κ:



The composition structure of lax functors (6.8) says that these interchangers respect composition in their first index (2.15):



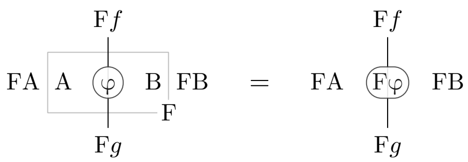
The comparitor compatibility requirement of an oplax transformation between lax functors (6.9) and (6.10) say that these interchangers are natural in their first index (2.18):



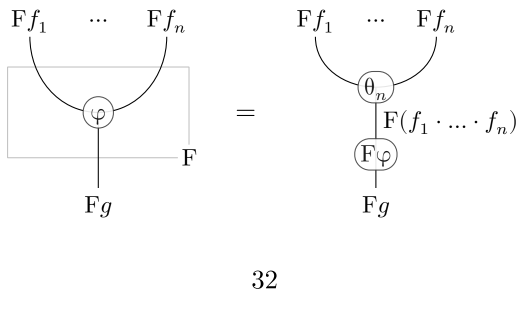
The graphical calculus of compositor sheets for surface diagrams can be seen as a "de-projection" of the graphical calculus of *functor boxes* for string diagrams [CS99; Mel06]. In the latter, parts of a string diagram may be annotated with boxes labeled by a lax or oplax functor of 2-categories $F : \mathbb{C} \to \mathbb{D}$. The part of a diagram outside of an F-box is a diagram with a "hole" in it in F's codomain 2-category $\mathbb{D}$. The part inside is a diagram in F's domain 2-category $\mathbb{C}$ whose F-image is a $\mathbb{D}$-disk that fits in the hole to complete the diagram in $\mathbb{D}$.

In accordance with globularity, a side boundary of an F-box may bound only a single region corresponding to a $\mathbb{C}$-object. Outside of the box this region is labeled by the F-image of that $\mathbb{C}$-object, while inside it is labeled by that $\mathbb{C}$-object itself. Because F is applied to everything inside the box these both represent the same $\mathbb{D}$-object and this part of the box boundary acts as an identity.

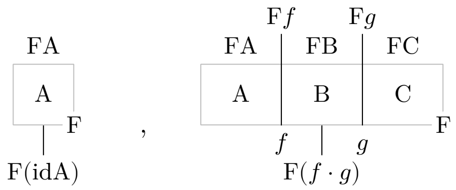
Regardless of whether F is a lax or oplax functor a single wire corresponding to a $\mathbb{C}$-arrow may pass through the top or bottom boundary of an F-box. Outside of the box this wire is labeled by the F-image of that $\mathbb{C}$-arrow, while inside it is labeled by that $\mathbb{C}$-arrow itself. Because F is applied to everything inside the box these both represent the same $\mathbb{D}$-arrow and this part of the box boundary also acts as an identity. For example, drawing a $\mathbb{C}$-disk $\varphi : f \to g$ within an F-box represents the $\mathbb{D}$-disk $F\varphi : Ff \to Fg$.



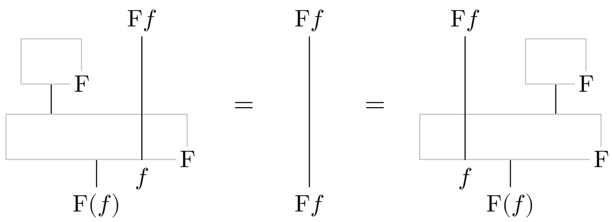
If F is a lax functor then the passage of any number of wires through the top boundary of an F-box represents the corresponding lax comparison disk $\theta(f_1, \cdots, f_n) : Ff_1 \cdot \ldots \cdot Ff_n \to F(f_1 \cdot \ldots \cdot f_n)$. Note that when $n = 1$ we have $\theta(f) = \mathrm{id}(Ff)$, agreeing with the case above. A $\mathbb{C}$-diagram drawn within an F-box represents the F-image of the composite of that diagram in $\mathbb{D}$. For example, drawing a $\mathbb{C}$-disk $\varphi : f_1 \cdot \ldots \cdot f_n \to g$ within an F-box represents the $\mathbb{D}$-diagram $\theta(f_1, \cdots, f_n) \cdot F\varphi$.



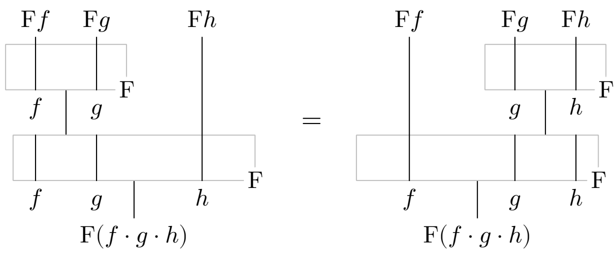
Unless F is oplax as well as lax, we do not generally have a morphism $\mathrm{F}(f_1 \cdot \ldots \cdot f_n) \to \mathrm{F}f_1 \cdot \ldots \cdot \mathrm{F}f_n$. In this case the bottom boundary of an F-box represents the identity on $\mathrm{F}(f_1 \cdot \ldots \cdot f_n)$. Graphically, this has n $\mathbb{C}$-wires coming in from above and a single $\mathbb{D}$-wire labeled by F of their composite leaving from below. For example, the nullary and binary lax comparitor components (6.1) are represented as follows.



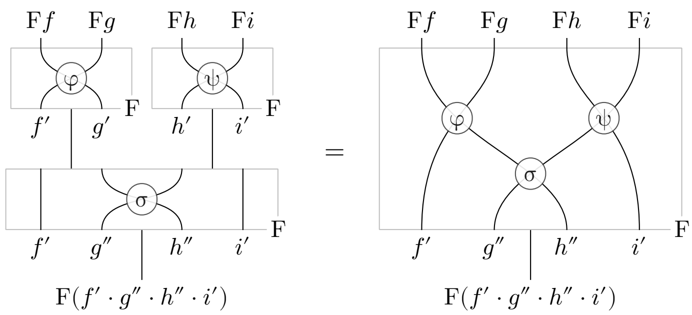
The lax comparitor unit laws (6.2) are drawn as follows.



The lax comparitor associative law (6.3) is drawn like this.



Together with comparitor composition naturality (6.5), these imply that boxes for lax functors support tree composition. For example, given $\mathbb{C}$-disks $\varphi : f \cdot g \to f' \cdot g'$, $\psi : h \cdot i \to h' \cdot i'$, and $\sigma : g' \cdot h' \to g'' \cdot h''$ we have the equation of string diagrams,



(6.11)

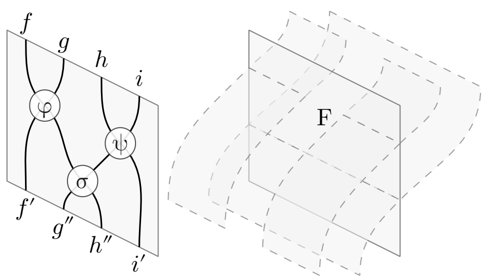
corresponding to the calculation

$$\begin{aligned}
& [(\theta(f\,,g)\cdot \mathrm{F}\varphi) \cdot\cdot (\theta(h\,,i)\cdot \mathrm{F}\psi)]\cdot[\theta(f'\cdot g'\,,h'\cdot i')\cdot \mathrm{F}(f' \cdot\cdot \sigma \cdot\cdot i')] \\
= \;& (\theta(f\,,g)\cdot\cdot\theta(h\,,i))\cdot(\mathrm{F}\varphi\cdot\cdot\mathrm{F}\psi)\cdot\theta(f'\cdot g'\,,h'\cdot i')\cdot\mathrm{F}(f'\cdot\cdot\sigma\cdot\cdot i') \\
= \;& (\theta(f\,,g)\cdot\cdot\theta(h\,,i))\cdot\theta(f\cdot g\,,h\cdot i)\cdot\mathrm{F}(\varphi\cdot\cdot\psi)\cdot\mathrm{F}(f'\cdot\cdot\sigma\cdot\cdot i') \\
= \;& \theta(f\,,g\,,h\,,i)\cdot\mathrm{F}((\varphi\cdot\cdot\psi)\cdot(f'\cdot\cdot\sigma\cdot\cdot i')).
\end{aligned}$$

The string diagrammatics for oplax functors is simply the vertical reflection of that for lax functors. In general we can tell whether the top or bottom boundary of a functor box represents a comparison disk by the number of wires connected to it on the outside.

The string diagram calculus for functor boxes can be seen as a projection of the surface diagram calculus for compositor sheets where string diagrams are projected onto the compositors as a movie is projected onto a cinema screen. The top boundary of a box for a lax functor F represents its lax comparitor component. This corresponds to the emergence of the compositor sheet from behind the surface representing F. Unless F is oplax as well, the compositor sheet remains in front of F's surface for the duration of the surface diagram, while in the string diagram the corresponding F-box seems to end at its bottom boundary. But this is just a cinematic illusion: the bottom boundary of an F-box represents the identity on the F-image of the composite of the arrows connected to it from the inside. Indeed, some authors prefer to depict lax functors in string diagrams using "tubes" rather than "boxes". These tubes simply extend the bottom boundaries of functor boxes down to top boundary of the next functor box.

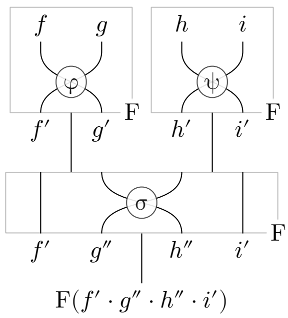
The equation between string diagrams depicted in (6.11) arises by projection from the following surface diagram, where in the string diagram on the left the disks φ and ψ are positioned vertically between the comparitor components and in the one on the right they are positioned below the lower comparitor component.



(6.12)

Note that in (6.12) we can't move φ and ψ up above the upper comparitor components without changing the top boundary of the diagram to $\mathrm{F}(f\cdot g)\cdot \mathrm{F}(h\cdot i)$. Doing so would correspond to a string diagram in which the upper

F-boxes extend beyond the top of the diagram as follows.



7 Conclusion

The preceding has essentially been an exercise in connecting the work of Gray [Gra74] with that of Hummon [Hum12] in order to present a surface diagram calculus for 2-category theory. By using the canonical adjunction between Street's 2-computads [Str96] and 2-categories we are able to incorporate lax functors between 2-categories as well as strict ones. The result is a framework to represent and reason about 2-categorical constructions in a manner that is intended to be natural, in both the homotopical and psychological sense.

There is of course a great deal of literature on 2-, 3-, n-, and ∞-dimensional categorical structures. Reaching arbitrarily high dimensions generally requires some sort of inductive or coinductive approach, and each stage of categorification requires making choices about what structures to keep strict or make weak or directed.

2-categories inhabit a particularly interesting point on this spectrum because they arise from the structure of category theory itself. Similarly, Gray-categories are what 2-categories and 2-functors naturally assemble into. But these structures are also interesting for the reason that they have coherence theorems providing equivalences with their fully-weak counterparts, bicategories [MP85] and tricategories [GPS95; Gur13], as well as compositional graphical calculi that provide non-experts a means to help tame the daunting bureaucracy involved in trying to do algebra in 2 or 3 dimensions. Tools such as the proof assistant *Globular* [BKV18] can be a great aid in this regard, helping users discover the cells they seek by constructing diagrams incrementally while verifying automatically that the constituent parts fit together properly along their boundaries.

Acknowledgement: I'd like to express my gratitude to the members of the Compositional Systems and Methods Group at TalTech for many enjoyable and enlightening discussions about diagrams and their semantics, and especially to Mario Román, who encouraged me to clarify the relationship between compositors and boxes for lax functors.

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