Linear Algebra for Data Science: Home assignment #1

Due on Novermber 04, 2023

Variant 43

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Problem 1

Find the singular value decomposition for the matrix

$$A = \begin{bmatrix} -114 & -64 & -100 & -54 \\ 78 & 52 & -98 & 0 \\ -21 & -38 & 94 & 108 \end{bmatrix}$$

Solution
Firstly, find matrix V − orthogonal

$$A^T \cdot A = \begin{bmatrix} -114 & 78 & -21 \\ -64 & 52 & -38 \\ -100 & -98 & 94 \\ -54 & 0 & 108 \end{bmatrix} \cdot \begin{bmatrix} -114 & -64 & -100 & -54 \\ 78 & 52 & -98 & 0 \\ -21 & -38 & 94 & 108 \end{bmatrix} = \begin{bmatrix} 19521 & 12150 & 1782 & 3888 \\ 12150 & 8244 & -2278 & -648 \\ 1782 & -2268 & 28440 & 15552 \\ 3888 & -648 & 15552 & 14580 \end{bmatrix}$$

Then, find the eigenvalues and eigenvectors: $|A^T \cdot A - \lambda I| = 0$

$$\begin{vmatrix} 19521-\lambda & 12150 & 1782 & 3888 \\ 12150 & 8244-\lambda & -2278 & -648 \\ 1782 & -2268 & 28440-\lambda & 15552 \\ 3888 & -648 & 15552 & 14580-\lambda \end{vmatrix} \xrightarrow{\textit{Applying the elementary row operations}} \lambda \cdot (\lambda - 39204) \cdot (\lambda - 27225) \cdot (\lambda - 4356) = 0$$

The roots of the equation are:

$$\begin{bmatrix} \lambda_1 = 39204 \\ \lambda_2 = 27225 \\ \lambda_3 = 4356 \\ \lambda_4 = 0 \end{bmatrix}$$

The eigenvalues are arranged in non-decreasing order (in such order will put the eigenvectors then).

We can obtain $\Sigma_{4\times3}$ calculating $\sigma_i = \sqrt{\lambda_i}$ and put the on the main diagonal of it:

$$\sigma_1 = 198,\ \sigma_2 = 165,\ \sigma_3 = 66$$

$$\text{And the } \Sigma = \begin{bmatrix} 198 & 0 & 0 & 0 \\ 0 & 165 & 0 & 0 \\ 0 & 0 & 66 & 0 \end{bmatrix}$$

Now, let find eigenvectors for each of the calculated eigenvalue. Substitute λ_i into the $|A^T \cdot A - \lambda_i I| = 0$.

$\boxed{\lambda_1 = 39204}$:

$$v_1 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{3}{2} & 1 \end{bmatrix}^T$$

$\boxed{\lambda_2 = 27225}$:

$$v_2 = \begin{bmatrix} -\frac{9}{2} & -3 & 1 & 0 \end{bmatrix}^T$$

$\boxed{\lambda_3 = 4356}$:

$$v_3 = \begin{bmatrix} 0 & -\frac{2}{9} & -\frac{2}{3} & 1 \end{bmatrix}^T$$

$\boxed{\lambda_4 = 0}$:

$$v_4 = \begin{bmatrix} -3 & \frac{9}{2} & 0 & 1 \end{bmatrix}^T$$

Now we can construct $V_{4\times4}$ matrix putting the normalized eigenvectors:

$$V = \begin{bmatrix} \frac{2}{11} & -\frac{9}{11} & 0 & -\frac{6}{11} \\ 0 & -\frac{6}{11} & -\frac{2}{11} & \frac{9}{11} \\ \frac{9}{11} & \frac{2}{11} & -\frac{6}{11} & 0 \\ \frac{6}{11} & 0 & \frac{9}{11} & \frac{2}{11} \end{bmatrix}$$

To build $U_{3\times3}$ find $u_i = \frac{1}{\sigma_i} \cdot Av_i$.

$$u_1 = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^T;\ u_2 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}^T;\ u_3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^T$$

$$\text{Then, } U = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A = U \cdot \Sigma \cdot V^T = \begin{bmatrix} -0.667 & 0.667 & 0.333 \\ -0.333 & -0.667 & 0.667 \\ 0.667 & 0.333 & 0.667 \end{bmatrix} \cdot \begin{bmatrix} 198 & 0 & 0 & 0 \\ 0 & 165 & 0 & 0 \\ 0 & 0 & 66 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.182 & 0 & 0.818 & 0.545 \\ -0.818 & -0.545 & 0.182 & 0 \\ 0 & -0.182 & -0.545 & 0.818 \\ -0.545 & 0.818 & 0 & 0.182 \end{bmatrix}$$

Problem 2

Find a full rank decomposition and the pseudoinverse of the matrix

$$A = \begin{bmatrix} -1 & 5 & 10 \\ 3 & -11 & -6 \\ -2 & 9 & 14 \\ 4 & -15 & -10 \end{bmatrix}$$

Solution

Full-ranked decomposition is $A = F_{mr} \cdot G_{rn}$.

Applied the elementary row operations we get the echelon form of $A = \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $rank(A) = 2$.

So, the F matrix is two first columns of matrix A, G is two first rows of echelon form of A.

$$F = \begin{bmatrix} -1 & 5 \\ 3 & -11 \\ -2 & 9 \\ 4 & -15 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 6 \end{bmatrix}$$

Full-ranked decomposition:

$$\begin{bmatrix} -1 & 5 & 10 \\ 3 & -11 & -6 \\ -2 & 9 & 14 \\ 4 & -15 & -10 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 3 & -11 \\ -2 & 9 \\ 4 & -15 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 6 \end{bmatrix}$$

Now we can find pseudoinverse matrix to A:

$$A^+ = G^*(GG^*)^{-1}(F^*F)^{-1}F^*,\ A^* = A^T \text{ as } A \in M_{mn}(\mathbb{R}).$$

$$G^*(GG^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 20 & 6 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 20 & 6 \end{bmatrix} \right)^{-1} = \tfrac{1}{437} \cdot \begin{bmatrix} 37 & -120 \\ -120 & 401 \\ 20 & 6 \end{bmatrix}$$

$$(F^*F)^{-1}F^* = \left(\begin{bmatrix} -1 & 3 & -2 & 4 \\ 5 & -11 & 9 & -15 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & -11 \\ -2 & 9 \\ 4 & -15 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} -1 & 3 & -2 & 4 \\ 5 & -11 & 9 & -15 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 16 & 10 & \frac{35}{2} & \frac{17}{2} \\ \frac{17}{4} & \frac{9}{4} & \frac{19}{4} & \frac{7}{4} \end{bmatrix}$$

Multiplying both parts:

$$A^+ = \frac{1}{437} \cdot \begin{bmatrix} 37 & -120 \\ -120 & 401 \\ 20 & 6 \end{bmatrix} \cdot \frac{1}{13} \begin{bmatrix} 16 & 10 & \frac{35}{2} & \frac{17}{2} \\ \frac{17}{4} & \frac{9}{4} & \frac{19}{4} & \frac{7}{4} \end{bmatrix} = \frac{1}{22724} \cdot \begin{bmatrix} 82 & 100 & 145 & 109 \\ -863 & -1191 & -781 & 1273 \\ 691 & 427 & 757 & 361 \end{bmatrix}$$

Problem 3

Find the minimal length least squares solution of the system of linear equations

$$\begin{cases} -1 \cdot x + 6 \cdot y - 8 \cdot z + 11 \cdot t = 6 \\ 3 \cdot x - 10 \cdot y + 7 \cdot z + 5 \cdot t = 0 \\ -2 \cdot x + 10 \cdot y - 12 \cdot z + 12 \cdot t = 4 \\ 4 \cdot x - 14 \cdot y + 9 \cdot z + 0 \cdot t = 3 \end{cases}$$

Solution

By the theorem we know, that vector $\vec{u} = A^{+}\vec{b}$ is the unique pseudosolution with minimal length.
Firsly, find A^{+}. Let us do it by full ranked decomposition.Applied the elementary row operations we get the echelon form of:

$$A = \begin{bmatrix} -1 & 6 & -8 & 11 \\ 3 & -10 & 7 & 5 \\ -2 & 10 & -12 & 12 \\ 4 & -14 & 9 & 0 \end{bmatrix} \xrightarrow{\textit{Applying the elementary row operations}} \begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} .rank(A) = 3$$

$$F = \begin{bmatrix} -1 & 6 & -8 \\ 3 & -10 & 7 \\ -2 & 10 & -12 \\ 4 & -14 & 9 \end{bmatrix}; G = \begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$G^*(GG^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & 9 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & 9 & 2 \end{bmatrix} \right)^{-1} = \frac{1}{815} \cdot \begin{bmatrix} 86 & -243 & -54 \\ -243 & 734 & -18 \\ -54 & -18 & 811 \\ 27 & 9 & 2 \end{bmatrix}$$

$$(F^*F)^{-1}F^* = \left(\begin{bmatrix} -1 & 3 & -2 & 4 \\ 6 & -10 & 10 & 14 \\ -8 & 7 & -12 & 9 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 & -8 \\ 3 & -10 & 7 \\ -2 & 10 & -12 \\ 4 & -14 & 9 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} -1 & 3 & -2 & 4 \\ 6 & -10 & 10 & 14 \\ -8 & 7 & -12 & 9 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{17}{14} & \frac{55}{14} & -0.5 & -\frac{37}{14} \\ 0.425 & 1.525 & -0.275 & -1.175 \\ \frac{4}{35} & \frac{22}{35} & -\frac{1}{5} & -\frac{19}{35} \end{bmatrix}$$

We get: $A^{+} = \frac{1}{45640} \begin{bmatrix} -281 & -3733 & 1939 & 4903 \\ 830 & 8590 & -4298 & -11786 \\ 1090 & 15130 & -7294 & 15478 \\ 2063 & 6779 & -917 & -4649 \end{bmatrix}$

And, $\vec{u} = \frac{1}{45640} \cdot \begin{bmatrix} -281 & -3733 & 1939 & 4903 \\ 830 & 8590 & -4298 & -11786 \\ 1090 & 15130 & -7294 & 15478 \\ 2063 & 6779 & -917 & -4649 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \\ 4 \\ 3 \end{bmatrix} = \frac{1}{45640} \cdot \begin{bmatrix} 20779 \\ -47570 \\ -69070 \\ -5237 \end{bmatrix} = \begin{bmatrix} 0.455 \\ 1.042 \\ 1.732 \\ 0.115 \end{bmatrix}$

Problem 4

Find and plot an interpolation polynomial in the Lagrange form that passes through the four points whose coordinates form the columns of the matrix

$$P = \begin{bmatrix} -2 & -1 & 2 & 3 \\ 6 & 20 & 2 & -9 \end{bmatrix}$$

Solution

Lagrange interpolating polynomial: $f(x) = \sum_{i=0}^{n} \frac{(x-x_0)...(x-x_{i-1})((x-x_{i+1})...(x-x_n)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$. For our case:
$f(x) = 6 \cdot \frac{(x+1)(x-2)(x-3)}{(-2+1)(-2-2)(-2-3)} + 20 \cdot \frac{(x+2)(x-2)(x-3)}{(-1+2)(-1-2)(-1-3)} + 2 \cdot \frac{(x+2)(x+1)(x-3)}{(2+2)(2+1)(2-3)} - 9 \cdot \frac{(x+2)(x+1)(x-2)}{(3+2)(3+1)(3-2)} =$
$= 6 \cdot \frac{(x+1)(x-2)(x-3)}{-20} + 20 \cdot \frac{(x+2)(x-2)(x-3)}{12} + 2 \cdot \frac{(x+2)(x+1)(x-3)}{-12} - 9 \cdot \frac{(x+2)(x+1)(x-2)}{20} =$
$= -\frac{3}{10} \cdot (x^3 - 4x^2 + x + 6) + \frac{20}{12} \cdot (x^3 - 3x^2 - 4x + 12) - \frac{1}{6} \cdot (x^3 - 7x - 6) - \frac{9}{20} \cdot (x^3 + x^2 - 4x - 4) =$
$\frac{3}{4}x^3 - \frac{17}{4}x^2 - 4x + 21 = 0.75x^3 - 4.25x^2 - 4x + 21$

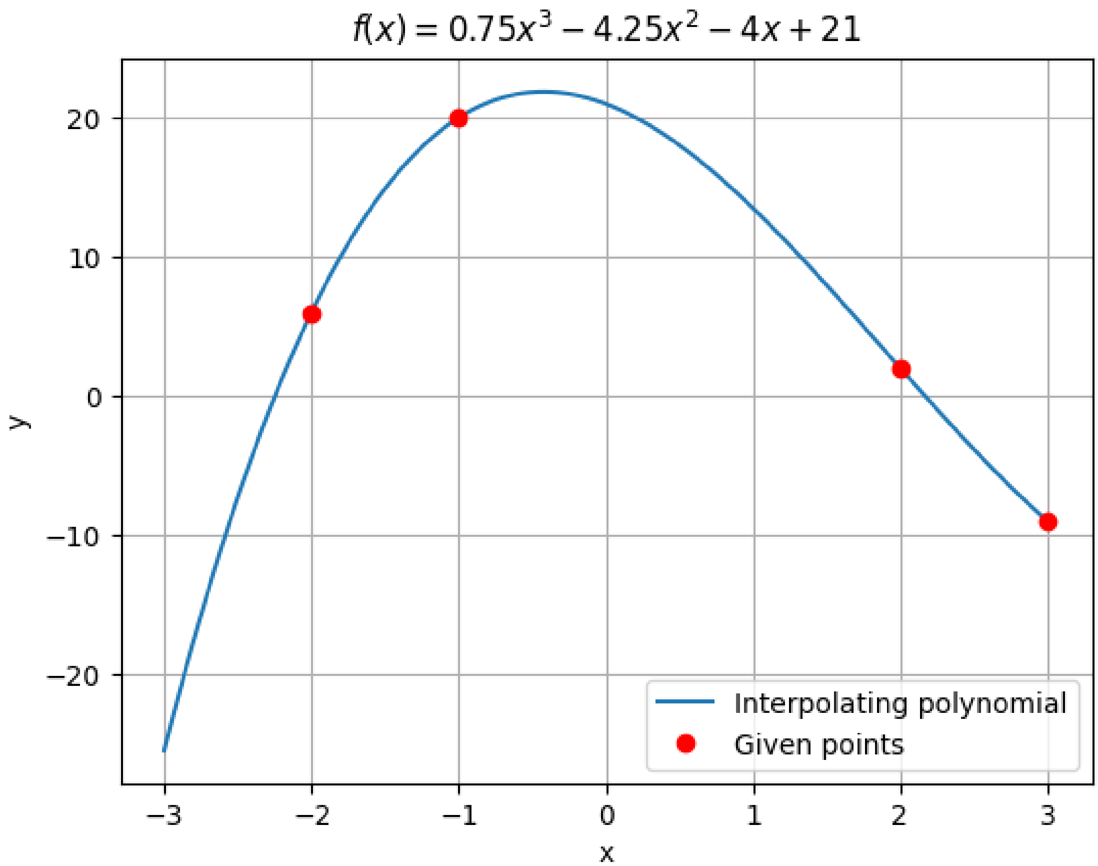

Figure 1: Lagrange Interpolating Polynomial.

Problem 5

Find and plot a (parametric) equation defining the Bezier curve defined by the four points whose coordinates form the columns of the matrix

$$P = \begin{bmatrix} 2 & 3 & 8 & 9 \\ 5 & 3 & 8 & 6 \end{bmatrix}$$

Solution

Bezier curve explicit formula: $B(t) = \sum_{i=0}^{n} P_i \cdot b_{n,i}(t)$, where $b_{n,i} = C_n^i (1-t)^{n-i} t^i$

$$B(t) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} C_3^0 (1-t)^3 t^0 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} C_3^1 (1-t)^2 t + \begin{bmatrix} 8 \\ 8 \end{bmatrix} C_3^2 (1-t) t^2 + \begin{bmatrix} 9 \\ 6 \end{bmatrix} C_3^3 (1-t)^0 t^3 = \begin{bmatrix} 2 - 6t + 6t^2 - 2t^3 \\ 5 - 15t + 15t^2 - 5t^3 \end{bmatrix} +$$
$$\begin{bmatrix} 9t - 18t^2 + 9t^3 \\ 9t - 18t^2 + 9t^3 \end{bmatrix} + \begin{bmatrix} 24t^2 - 24t^3 \\ 24t^2 - 24t^3 \end{bmatrix} + \begin{bmatrix} 9t^3 \\ 6t^3 \end{bmatrix} = \begin{bmatrix} 2 + 3t + 12t^2 - 8t^3 \\ 5 - 6t + 21t^2 - 14t^3 \end{bmatrix}$$

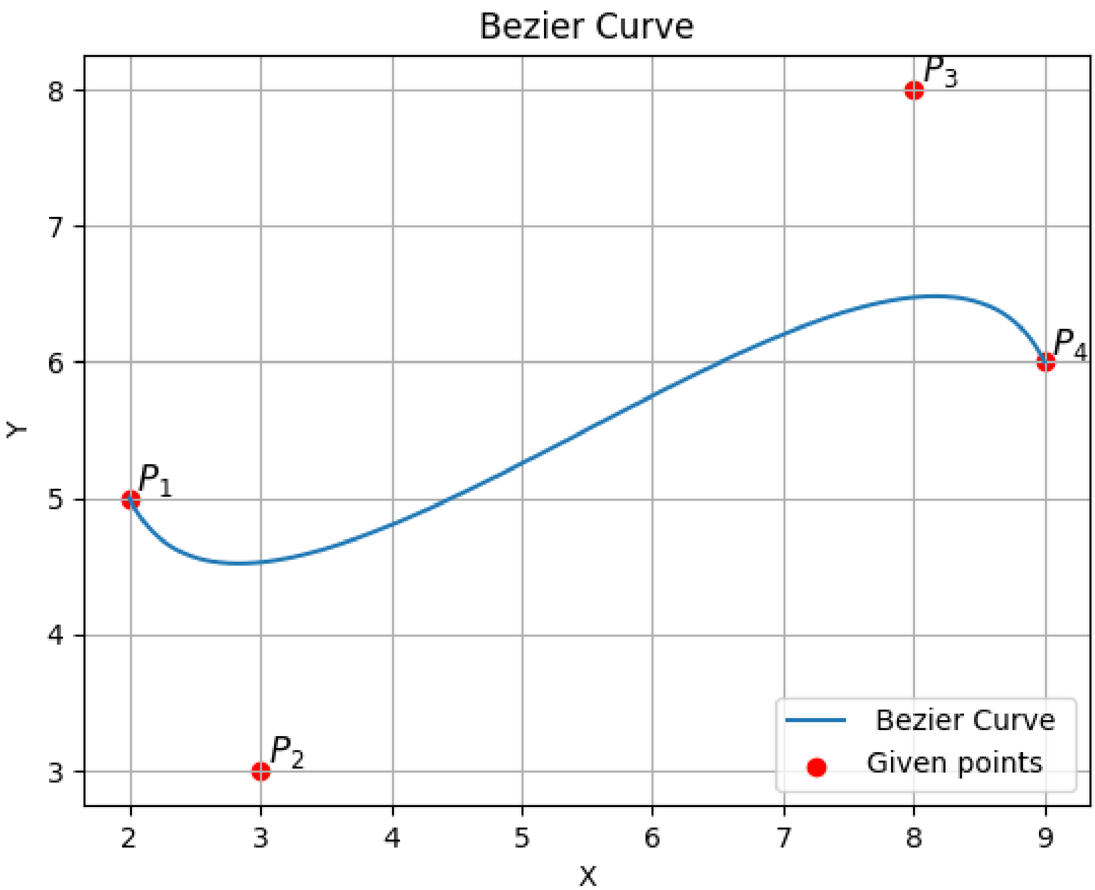

Figure 2: Bezier Curve.

Problem 6

For the polynomial $x^3 - 4x^2 - 4x - 5$ find the best approximation with respect to the max-norm $\int_0^5 |f(x)|dx$ by a polynomial of degree 2 on a line segment $[0, 5]$

Solution

By the Chebyshev polynomials of the second kind we can stay, that area under the curve on the segment $[-1, 1]$ $\tilde{U}_n(x) = \frac{1}{2^n}U_n(x)$. For our case, to transform to $[-1, 1]$ $\tilde{U}_n(x) = \frac{(b-a)^n}{2^n}U_n\Big(\frac{2x-(b+a)}{b-a}\Big)$. So, $\tilde{U}_3(x) = \frac{(5-0)^3}{2^{2\cdot 3}}U_3\Big(\frac{2x-(5+0)}{5-0}\Big) = \frac{125}{64}U_3\Big(\frac{2x-5}{5}\Big)$. We know, $U_3(x) = 8x^3 - 4x$

$\frac{125}{64}\left[8\cdot\left(\frac{2x-5}{5}\right)^3 - 4\cdot\left(\frac{2x-5}{5}\right)\right] = \frac{125}{8}\cdot\left(\frac{2x-5}{5}\right)^3 - \frac{125}{16}\cdot\left(\frac{2x-5}{5}\right) = \frac{1}{8}\cdot(2x-5)^3 - \frac{25}{8}x + \frac{125}{16} = x^3 - \frac{15}{2}x^2 + \frac{75}{4}x - \frac{125}{8} - \frac{25}{8}x + \frac{125}{16} = x^3 - \frac{15}{2}x^2 + \frac{125}{8}x - \frac{125}{16}$.

So, $\tilde{U}_3(x) = [x^3 - 4x^2 - 4x - 5] - g(x) \Rightarrow g(x) = [x^3 - 4x^2 - 4x - 5] - \tilde{U}_3(x) = [x^3 - 4x^2 - 4x - 5] - x^3 + \frac{15}{2}x^2 - \frac{125}{8}x + \frac{125}{16} = 3.5x^2 + 19.625x + 2.8125$

Problem 7

Find a polynomial of degree ≤ 3 that approximates the function $f(x) = \sqrt{2x+6}$ on the segment $[1, 3]$ in the norm $|h|_T = \sqrt{\int_1^3 \frac{h(x)^2}{\sqrt{1-(2x-4)^2/4}}dx}$

Solution

As we know, the best approximation of a function f by polynomial of degree $\leq n$:

$$\tilde{f}(x) = \sum_{i=0}^{n} \frac{\langle T_i, f\rangle}{\langle T_i, T_i\rangle} T_i(x)$$

with a scalar product of continuous functions on the segment $[-1, 1]$:

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}dx$$

and the orthogonality relations:

$$\langle T_m, T_n\rangle = \begin{cases} 0, & m \neq n, \\ \pi/2, & m = n \neq 0, \\ \pi, & m = n = 0. \end{cases}$$

We need to transform our segment to corresponding to Chebyshev polynomials $[-1, 1]$. $y = \frac{a+b}{2} + \frac{b-a}{2}x = 2 + x$

Let us pass it to the given norm $|h|_t$: $|h|_T = \sqrt{\int_1^3 \frac{h(2+x)^2}{\sqrt{1-x^2}}dx}$

Now, just apply $\tilde{f}(x) = \sum_{i=0}^{3} \frac{\langle T_i, f(2+x)\rangle}{\langle T_i, T_i\rangle} T_i(x-2) = \frac{\langle T_0, f(2+x)\rangle}{\langle T_0, T_0\rangle} T_0(x-2) + \frac{\langle T_1, f(2+x)\rangle}{\langle T_1, T_1\rangle} T_1(x-2) + \frac{\langle T_2, f(2+x)\rangle}{\langle T_2, T_2\rangle} T_2(x-2) + \frac{\langle T_3, f(2+x)\rangle}{\langle T_3, T_3\rangle} T_3(x-2) =$

$= \frac{\langle 1, \sqrt{2x+10}\rangle}{\langle T_0, T_0\rangle} \cdot (x-2) + \frac{\langle x, \sqrt{2x+10}\rangle}{\langle T_1, T_1\rangle} \cdot (x-2) + \frac{\langle 2x^2-1, \sqrt{2x+10}\rangle}{\langle T_2, T_2\rangle} \cdot (2x^2-8x+7) + \frac{\langle 4x^3-3x, \sqrt{2x+10}\rangle}{\langle T_3, T_3\rangle} \cdot (4x^3-24x^2+45x-26) =$

$= \frac{9.9095}{\pi} + \frac{0.9972}{\pi} \cdot (x-2) + \frac{-0.0126}{\pi} \cdot (2x^2-8x+7) + \frac{0.0006}{\pi} \cdot (4x^3-24x^2+45x-26) =$

$= 0.000763944x^3 - 0.0126051x^2 + 0.358099x + 2.48641$

Problem 8

Find all the values of q such that the equation $x^2 \cdot (2q+2) + xy(2q+4) + y^2 \cdot (1-4q) = 1$ defines a unit circle with respect to some norm. Find the norm of the vector $(1,1)$ as a function of q

Solution

The equation is unit circle if rewriting by the unit ball definition $B = \{(x,y,z) | x^2 \cdot (2q+2) + xy(2q+4) + y^2 \cdot (1-4q) \leq 1\}$

Properties of ball are satisfied:
1) B is bounded;
2) B is closed;
3) $U_{\mathcal{E}} \in B$;
4) symmetricity also satisfied; 5) Let's check whether it convex

Write a Hessian matrix $H = \begin{bmatrix} 2q+2 & q+2 \\ q+2 & 2q+2 \end{bmatrix}$ it should be positive-definite. Check the minors:

$$2q+2 > 0 \Rightarrow \ when \ q > -1,$$
$$\begin{vmatrix} 2q+2 & q+2 \\ q+2 & 2q+2 \end{vmatrix} = 3q^2 + 4q > 0 \Rightarrow q_1 = -\frac{4}{3}; \ q_2 = 0 \Rightarrow \ when \ q \in (-\infty; -\frac{4}{3}) \cup (0; +\infty)$$

Let's find norm $\mu(x) = ||x||$. To find the intersection of balls we need $||\vec{v}|| = ||1 \cdot \alpha, 1 \cdot \alpha|| = 1$

$\alpha^2 \cdot (2q+2) + \alpha^2 \cdot (2q+4) + \alpha^2 \cdot (1-4q) = 1$
$7\alpha^2 = 1 \Rightarrow \alpha_1 = 1, \alpha_2 = -1$
$||w|| = \frac{\vec{v}}{\alpha} = 1$