Linear Algebra for Data Science: Home assignment #2

Due on December 20, 2023

Variant 43

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All the calculations draft you may find there:
https://github.com/EdwardNee/hse-ds-masters/blob/main/la4ds/hw2/LA4DS2.ipynb

Problem 1

Find the best approximation matrix A_1 of rank 2 of the matrix A in the norm $||\cdot||_2$ and find $||A - A_1||_2$, where

$$A = \begin{bmatrix} -56 & -4 & 30 & 59 \\ -76 & -32 & -33 & -74 \\ 76 & 56 & -18 & -22 \end{bmatrix}$$

Solution

Let us compute a singular value decomposition for matrix $A = U \cdot \Sigma \cdot V^*$. I suppose, we do not need to show the steps here, as we did it in previous hometask.

Using numpy lib in Python we get:

$$\begin{bmatrix} -56 & -4 & 30 & 59 \\ -76 & -32 & -33 & -74 \\ 76 & 56 & -18 & -22 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 135 & 0 & 0 & 0 \\ 0 & 108 & 0 & 0 \\ 0 & 0 & 27 & 0 \end{bmatrix} \cdot \frac{1}{9} \cdot \begin{bmatrix} -8 & -4 & 0 & -1 \\ 1 & 0 & -4 & -8 \\ 4 & -8 & 1 & 0 \\ 0 & -1 & -8 & 4 \end{bmatrix}$$

Now, build Σ_1 by making lowest singular value $\sigma_3 = 0$: $\Sigma_1 = \begin{bmatrix} 135 & 0 & 0 & 0 \\ 0 & 108 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Computing matrix $A_1 = U \cdot \Sigma_1 \cdot V^* = \begin{bmatrix} -48 & -20 & 32 & 59 \\ -72 & -40 & -32 & -74 \\ 84 & 40 & -16 & -22 \end{bmatrix}$

And compute $||A - A_1||_2 = \left\| \begin{matrix} -8 & 16 & -2 & 0 \\ -4 & 8 & -1 & 0 \\ -8 & 16 & -2 & 0 \end{matrix} \right\|_2 = 27$

Problem 2

Solve the system $AX = b$ approximately, rounding the values to the closest whole numbers, and estimate the relative error of the solution in the norms $|\cdot|_1$ and $|\cdot|_2$ using the condition number of the matrix A, where

$$A = \begin{pmatrix} -3.05 & -0.06 \\ -0.13 & -8.05 \end{pmatrix}, \; b = \begin{pmatrix} -2.89 \\ -7.98 \end{pmatrix}$$

Solution

To solve the $AX = b$, rounding the matrices, $x = A^{-1}b = \frac{1}{24} \cdot \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

By the definition, $\mathcal{K}(A) = ||A|| \cdot ||A^{-1}||$.
$\mathcal{K}_1(\hat{A}) = ||\hat{A}||_1 \cdot ||\hat{A}^{-1}||_1 = max\{3,\ 8\} \cdot \frac{1}{24} \cdot max\{8,\ 3\} = 8 \cdot \frac{1}{24} \cdot 8 = \frac{8}{3} = 2.667$

To calculate $\mathcal{K}_2(\hat{A}) = \frac{\sigma_{max}(\hat{A}^T\hat{A})}{\sigma_{min}(\hat{A}^T\hat{A})} = \frac{8}{3} = 2.667$

Now, we can find relative errors: $\delta A = \frac{||\Delta A||}{||A||}$, $\delta b = \frac{||\Delta b||}{||b||}$, $\delta x = \frac{||\Delta x||}{||x||}$

$$\Delta A = \begin{bmatrix} -0.05 & -0.06 \\ -0.13 & -0.05 \end{bmatrix}, \Delta b = \begin{bmatrix} 0.11 \\ 0.02 \end{bmatrix}$$
$$\delta_1 A = \frac{||\Delta A||_1}{||A||_1} = \frac{0.18}{8.11} = 0.022194$$
$$\delta_1 b = \frac{||\Delta b||_1}{||b||_1} = \frac{0.13}{10.87} = 0.01196$$

$$\delta_2 A = \frac{||\Delta A||_2}{||A||_2} = \frac{\sigma(\Delta A^T \Delta A)}{\sigma(A^T A)} = \frac{0.156}{8.0519} = 0.0194$$
$$\delta_2 b = \frac{||\Delta b||_2}{||b||_2} = \frac{0.1118}{8.544} = 0.0131$$

And we should find upper bound: $\delta x \leq \frac{\mathcal{K}(A)}{1 - \mathcal{K}(A)\delta A}(\delta A + \delta b)$.

For the $|\cdot|_1$ $\delta_1 x \leq 2.667 \cdot (0.022194 + 0.01196) \leq 0.09109$;
For the $|\cdot|_2$ $\delta_2 x \leq 2.667 \cdot (0.0194 + 0.0131) \leq 0.08668$.

Problem 3

Solve the system approximately and estimate the relative error of the solution in the norms$|\cdot|_2$ and $|\cdot|_\infty$:

$$\begin{cases} -(-4+\varepsilon_1)x + 2(-1+\varepsilon_2)y = 3+\varepsilon_3 \\ -4x + (-1+\varepsilon_1)y = 4+\varepsilon_4, \end{cases}$$

where the unknown numbers ε_j satisfy the conditions $|\varepsilon_j| < 0.05$ for all $j = 1, ..., 4$

Solution

We have: $A = \begin{bmatrix} 4 & -2 \\ -4 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Longrightarrow x = A^{-1}b = \dfrac{1}{12}\begin{bmatrix} 1 & -2 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} -5 \\ -28 \end{bmatrix}$

From the previous task we can compute $\mathcal{K}$:

$\mathcal{K}_2(A) = \dfrac{\sigma_{max}(A^TA)}{\sigma_{min}(A^TA)} = \dfrac{5.708}{2.102} = 2.716$
$\mathcal{K}_\infty(A) = max\{6,\ 5\} \cdot \dfrac{1}{12} \cdot max\{3,\ 8\} = 4$

Now, we should write ΔA, Δb to move further. $\Delta A = \begin{bmatrix} \varepsilon_1 & 2\varepsilon_2 \\ 0 & \varepsilon_1 \end{bmatrix}$, $\Delta b = \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$

$$\delta_2 A = \frac{||\Delta A||_2}{||A||_2} = \frac{\sigma(\Delta A^T \Delta A)}{\sigma(A^TA)} < \frac{0.1207}{5.7079} = 0.0211$$
$$\delta_2 b = \frac{||\Delta b||_2}{||b||_2} < \frac{0.0707}{5} = 0.01414$$

$$\delta_\infty A = \frac{||\Delta A||_\infty}{||A||_\infty} = \frac{max\{|\varepsilon_1| + 2|\varepsilon_2|,\ \varepsilon_1\}}{max\{6,\ 5\}} < \frac{0.15}{6} = 0.025$$
$$\delta_\infty b = \frac{||\Delta b||_\infty}{||b||_\infty} = \frac{\varepsilon_3, \varepsilon_4}{max\{3,\ 4\}} < \frac{0.05}{4} = 0.0125$$

And the upper bound for the relative error for the solutions:

For the $|\cdot|_2$ $\delta_2 x \leq 2.716 \cdot (0.0211 + 0.01414) \leq 0.09571184$;
For the $|\cdot|_\infty$ $\delta_\infty x \leq 4 \cdot (0.025 + 0.0125) \leq 0.15$.

Problem 4

Find the approximate inverse matrix to the matrix A and evaluate the approximation error with respect to the uniform norm $||\cdot||_1$ if the elements of the matrix A are known with an absolute error of 0.01:

$$A \approx \begin{pmatrix} 5 & -1 \\ -2 & -3 \end{pmatrix}$$

Solution

So, the task is to find some $A = \begin{pmatrix} 5 & -1 \\ -2 & -3 \end{pmatrix} + \varepsilon = \begin{pmatrix} 5 & -1 \\ -2 & -3 \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$

$\hat{A}^{-1} = \frac{1}{17} \cdot \begin{pmatrix} 3 & -1 \\ -2 & -5 \end{pmatrix}$

Now, find the conditional number $\mathcal{K}_1(A) = max\{7,\ 4\} \cdot \frac{1}{17} \cdot max\{5, 6\} = \frac{42}{17}$ as we suppose $\mathcal{K}_1(A) \approx \mathcal{K}_1(\hat{A})$

And the approximation error is $\delta A^{-1} \leq \frac{\mathcal{K}(\hat{A}) \cdot \delta\varepsilon}{1 - \mathcal{K}(\hat{A}) \cdot \delta\varepsilon}$, $\delta\varepsilon = \frac{||\varepsilon||}{||\hat{A}||}$.

We het $\delta A^{-1} \leq \frac{\mathcal{K}_1(\hat{A}) \cdot \delta\varepsilon}{1 - \mathcal{K}_1(\hat{A}) \cdot \delta\varepsilon} = \frac{\frac{42}{17} \cdot \frac{max\{0.02,\ 0.02\}}{max\{6,\ 5\}}}{1 - \frac{42}{17} \cdot \frac{max\{0.02,\ 0.02\}}{max\{6,\ 5\}}} = \frac{0.0082}{1 - 0.0082} = 0.0083$

Problem 5

Use simple iteration method for finding the solution of the given linear system

$$\begin{cases} 25x + 6y + 4z = 5, \\ 2x + 23y + 2z = 9 \\ 6x + 1y + 25z = 1 \end{cases}$$

Determine the iteration number after which the approximation error for each coordinate does not exceed 0.01 and find the corresponding approximate solution. Start with $x_0 = [0, 0, 0]^T$.

Solution

So, the linear equation of the iteration $x_{n+1} = Px_n + b$. Let me use LDR-decomposition for the interation method to find the solution with Jacobi method $DX_{k+1} = -(L + R)x_k + b$. L − lower-diagonal matrix, D − diagonal matrix, R − upper-diagonal matrix. All the calculations processed in notebook.

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -6 & -1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 23 & 0 \\ 0 & 0 & 25 \end{bmatrix}, \ R = \begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Algorithm ended on 4th iteration with the result vector $x_4 = \begin{bmatrix} 0.10747 & 0.3812 & -0.00296 \end{bmatrix}^T$

Problem 6

Find the most influential vertex in the graph using the PageRank algorithm with damping factor $= \alpha = 1 - \beta = 0.85$, where the graph adjacency matrix is defined as follows

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution

Firstly, normalising and transposing A we get transitions stochastic matrix $P = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0.25 & 0.5 & 0 & 0 & \frac{1}{3} \\ 0.25 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0.25 & 0.5 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

Let's make matrix $PR = (1-\beta)P + \beta Q$, where each element of the Q is $\dfrac{1}{N}$. $Q = \dfrac{1}{N}$

So, $PR = \begin{bmatrix} 0.2425 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.0 & 0.03 & 0.3133 & 0.88 & 0.03 \\ 0.2425 & 0.455 & 0.03 & 0.03 & 0.3133 \\ 0.2425 & 0.03 & 0.3133 & 0.03 & 0.3133 \\ 0.2425 & 0.455 & 0.3133 & 0.03 & 0.3133 \end{bmatrix}$

To find most influential vertex let's find for eigenvalue $= 1$ most close eigenvalue. Using Python we get $v = \begin{bmatrix} -0.0779 & -0.5178 & -0.4683 & -0.3809 & -0.601 \end{bmatrix}^T$. As the result, most influential vertex is A.

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