Pure SU(2) Gauge Theory Applied on the Lattice

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Introduction

The strong and weak nuclear forces, describing the interaction of fundamental particles, require a special mathematical framework. Which in turn is near impossible to solve analytically.

The utilisation of lattice gauge theory allows for simulations of these systems. Where numerical values can be obtained from observables.
The aim of this project was to apply lattice gauge theory to obtain a simulation of the SU(2) Yang-Mills gauge theory on a 2D lattice using a Markov Chain Monte Carlos method.

Discretisation

For the computation of a continuous theory, it is a requirement that the theory is discretised. So, for each space-time point is described by a lattice point, separated by a lattice constant ‘a’**.**

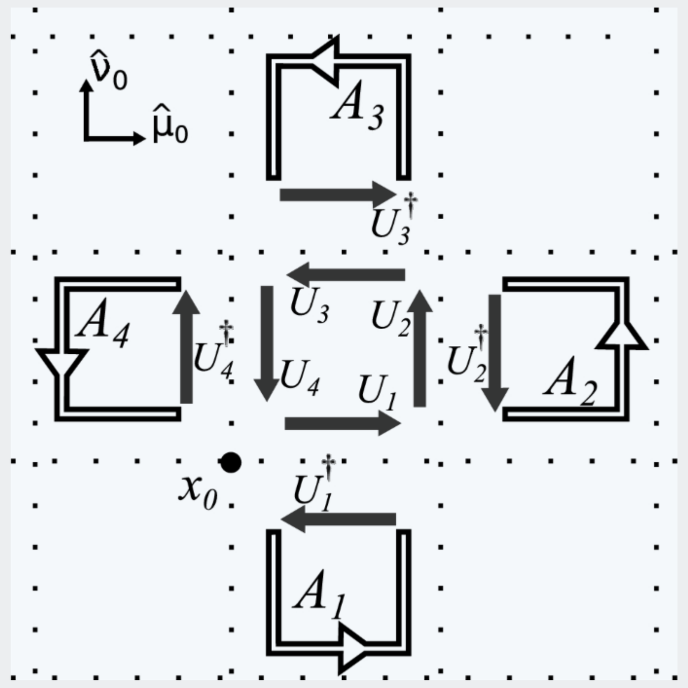
$$\Lambda_2 : \boldsymbol{x} \Rightarrow a\mathbf{n}, \quad n_i = 0,1,\dots,N-1 \quad for \quad i = 1,2$$

This discretisation transforms the initial action that describes the continuous SU(2) gauge theory. The Yang-Mills action,

$$S_{YM} = \frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}\,{}^2.$$

Noting this action is used because the system must adhere to non-Abelian symmetries, due to self-interaction of the gauge fields in question. The discretisation results in the Wilson action,

$$S_W[U] = \frac{\beta}{N}\sum_{\Box} \mathbb{R}Tr[\,\mathbb{I} - U_{\Box}\,(n)]\,{}^1.$$



The action is calculated by summing plaquettes,

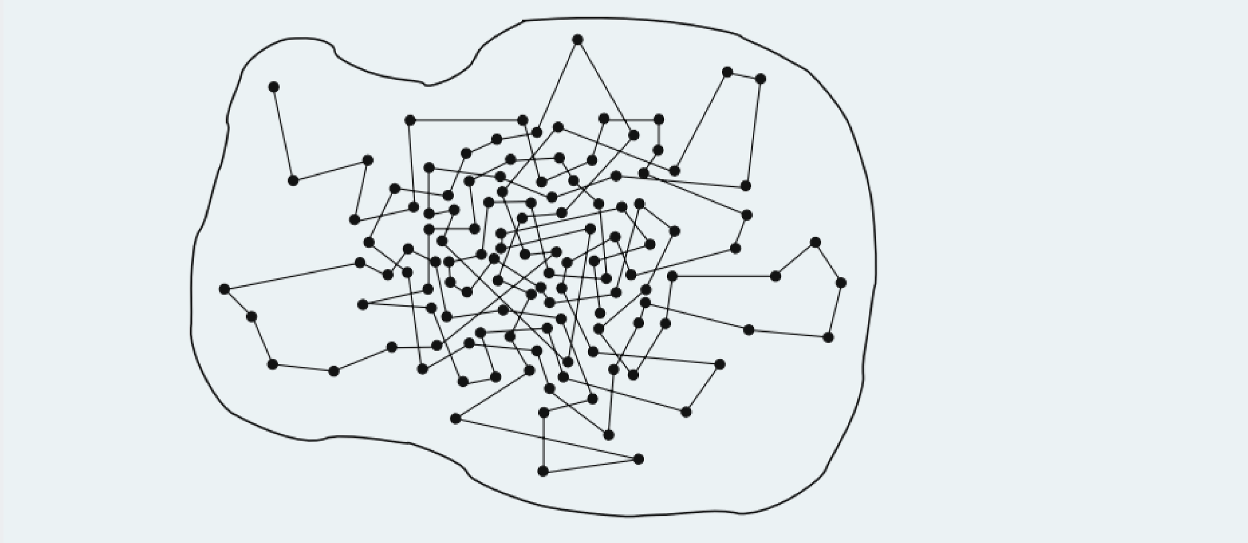
$$U_{\Box} = U_\mu(x)U_\nu(x+\hat{\mu})U^\dagger_\mu(x+\hat{\nu})U^\dagger_\nu(x)$$

where U is a link variable described by an SU(2) matrix.

The Wilson action is used to find an expression for finding expectation values of the lattice through the Euclidean path integral method.
Assuming the continuum limit is recovered of the Yang-Mills action, as $a \to 0$.

Markov Chain Monte Carlos Method

Expectation values of observables are obtained from the Euclidean integral method. However, these integrals are infinitely dimensional, integrated over all possible configurations and weighted by the Boltzmann factor.



Schematic sketch of a Markov chain in the space of all configurations

Generally speaking, the expectation of some function f(x) respect to a probability distribution is given by,

$$\langle f \rangle_\rho = \frac{\int_a^b dx\rho(x)f(x)}{\int_a^b dx\rho(x)}\,{}^1.$$

Markov chains are a stochastic sequence that follows an equilibrium distribution, where the future state depends on the current state. The update to the new field configuration is called a Monte Carlos step.
The Markov Chain Monte Carlo is a method that allows a system to explore the field configurations, which will have a Markov Chain with the required equilibrium distribution.
Leading to,

$$\langle f \rangle_\rho = lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)\,{}^1.$$

Where $x_n(a,b)$ is drawn randomly from a distribution with a normalised probability density.

$$dP(x) = \frac{\rho(x)dx}{\int_a^b dx\rho(x)}\,{}^1.$$

This is applied to an SU(2) field configuration, such that the link variables will follow the distribution, $P[U] \propto \exp(-\beta S[U])$
.
The Metropolis algorithm functions by iteratively proposing new states based on the proposal distribution. By updating a single link variable. It then accepts or rejects these proposals. The simulation is performed by allowing this Metropolis algorithm to update link variables many times, such that the probability distribution is adhered to.
After this, observables can be implemented and measured on the lattice.

Results, Discussion & Conclusion

This simulation was run on a lattice size of 10x10 at a β value of 1.2, for 1000 measurements.
As seen in [figure:2], one can see the action is fluctuating. This is intrinsic to this lattice simulation, arising from how the link variables are explored via the Markov Chain Monet Carlo sampling process.

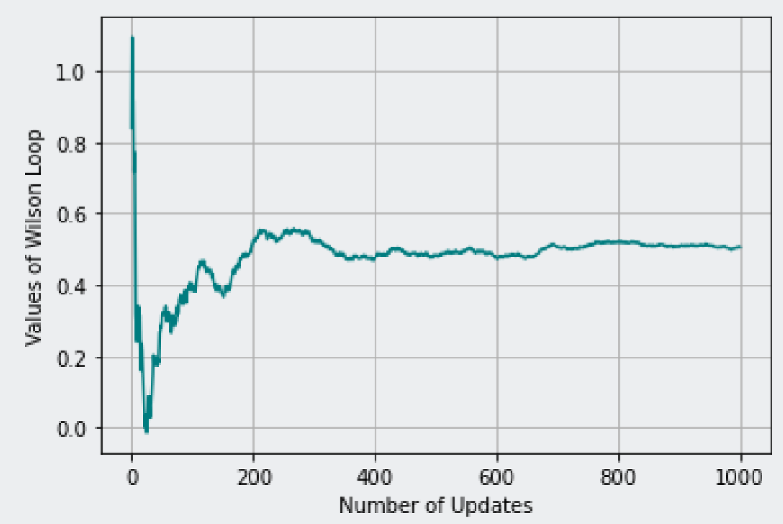


Figure 1: Expectation value of the Wilson loop observable.

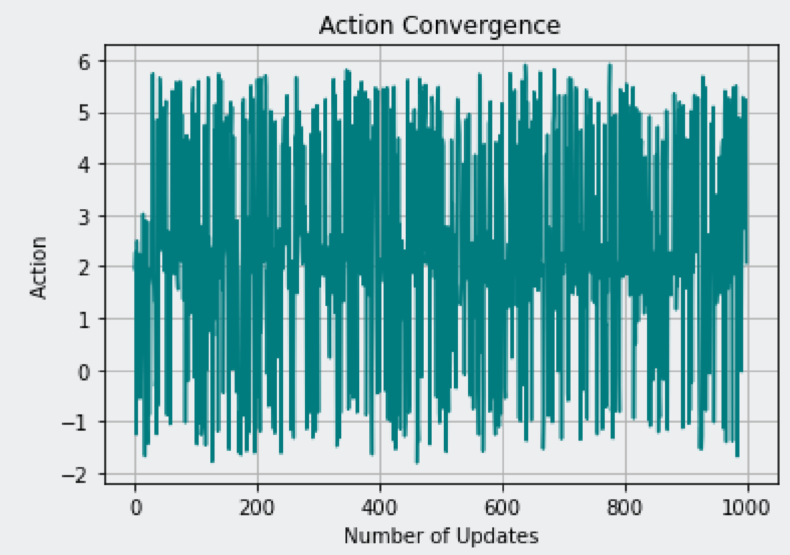


Figure 2: The action of the lattice

The expectation value of the observable converges at around 0.5. Although this doesn’t directly hint towards a phase transition, where changes in the ground state would need to be observed. It does however the direct convergence of the observable does imply potential physical properties of the lattice.
Improvements to the simulation could be simulating in 4D to obtain actual physical results applicable to the world around us. Also, implementing SU(3) to simulate fermions.

References

[1]. Gattringer and C. Lang. Quantum Chromodynamics on the Lattice: An Introductory Presentation. Lecture Notes in Physics. Springer Berlin Heidelberg, 2009. ISBN: 9783642018497. URL: https : / / books . google . co uk/books?id=l2hZKnlYDxoC.

[2]. R.Leme,O.Oliveira, and G. Krein. “Approximate dual representation for Yang–Mills SU(2) gauge theory”. In: The European Physical Journal C 78.8 (2018), p. 656. DOI: 10 . 1140 / epjc / s10052 - 018 - 6101 - 9. URL: https : //doi.org/10.1140/epjc/s10052-018-6101-9.

Acknowledgements

I would like to thank Prof Tony Kennedy for supervising me through this project. Also to the School of Physics and Astronomy Career Development Summer Scholarship at the University of Edinburgh for allowing this project to be financially possible.