Two-Dimensional Stochastic Navier-Stokes Equations

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We consider the following incompressible stochastic Navier-Stokes Equation in two dimensions.

$$\begin{cases} \dot{Z}(t,\xi) - \Delta Z(t,\xi) + (Z(t,\xi)\cdot\nabla)Z(t,\xi) + \operatorname{grad} p(t,\xi) = f(t,\xi) + \dot{W}(t,\xi) \quad \text{on } [0,T]\times D \\ Z(t,\xi) = 0 \quad \text{on } [0,T]\times\partial D \\ \operatorname{div} Z(t,\xi) = 0 \quad \text{on } [0,T]\times D \\ Z(0,\xi) = z(\xi) \quad \text{on } D, \end{cases} \tag{1}$$

where $D \subset \mathbb{R}^2$ is bounded, open and connected with smooth boundary ∂D. In the physics context, the function $Z\colon [0,T]\times \bar{D} \to \mathbb{R}^2$ represents the unknown velocity of some fluid, and $p\colon [0,T]\times\bar{D}\to\mathbb{R}$ is the pressure, which is also unknown but can be determined by Z. Thus, it suffices to solve for Z and the pair (Z,p) of solutions then follows. For $z\colon \bar{D}\to\mathbb{R}^2$, its kinetic energy and enstrophy are defined by

$$E(z) := \sum_{i=1}^{2}\int_D z_i^2 \,\mathrm{d}\xi,$$
$$e(z) := \sum_{i,j=1}^{2}\int_D \left(\frac{\partial z_i}{\partial \xi_j}\right)^2 \mathrm{d}\xi,$$

corresponding to $\mathcal{L}^2$-norm and H^1-norm respectively.

1 Function Spaces and Operators

We need to specify in what space we are seeking a solution. This section begins with some steady (time-independent) function spaces, studies some operators defined on them, and moves on to (time-dependent) process spaces. We introduce function spaces as follows.

$$C^\infty_{0,\mathrm{div}=0} := \left\{ z\in C^\infty(D;\mathbb{R}^2) \,\middle|\, z \text{ is compactly supported}, \operatorname{div} z = 0\right\},$$
$$H := \text{ the closure of } C^\infty_{0,\mathrm{div}=0} \text{ in } \mathcal{L}^2(D;\mathbb{R}^2), \qquad \langle z,w\rangle_H := \sum_{i=1}^{2}\int_D z_i w_i \,\mathrm{d}\xi,$$
$$V := \text{ the closure of } C^\infty_{0,\mathrm{div}=0} \text{ in } H^1(D;\mathbb{R}^2), \qquad \langle z,w\rangle_V := \sum_{i,j=1}^{2}\int_D \frac{\partial z_i}{\partial \xi_j}\frac{\partial w_i}{\partial \xi_j}\,\mathrm{d}\xi.$$

Now, we want to drop the term $\operatorname{grad} p$ in (1) so that the equation would only concern about Z. This can be done by the Helmholtz-Leray decomposition.

Proposition 1 (Helmholtz-Leray decomposition). *The space $\mathcal{L}^2(D;\mathbb{R}^2)$ admits the decomposition $\mathcal{L}^2(D;\mathbb{R}^2) = H \oplus H^\perp$, and*

$$H = \left\{ z \in \mathcal{L}^2(D;\mathbb{R}^2) \middle| \operatorname{div} z = 0, z \cdot \boldsymbol{n}|_{\partial D} = 0 \right\},$$
$$H^\perp = \left\{ z \in \mathcal{L}^2(D;\mathbb{R}^2) \middle| z = \operatorname{grad} p, p \in H^1(D;\mathbb{R}) \right\}.$$

The intuition of this decomposition is that every vector field consists of a conservative part and a non-conservative part. The conservative part is of the gradient form, say $\operatorname{grad} p$, so we hope, for $z \in C^\infty(D;\mathbb{R}^2)$, there holds

$$z = \operatorname{grad} p + w \quad \text{and} \quad \langle \operatorname{grad} p, w \rangle_{\mathcal{L}^2} = 0.$$

From integration by part, we obtain $\operatorname{div} w = 0$, which is a characterization for the non-conservative part.

Proof. We first show $H^\perp = \{ z \in \mathcal{L}^2(D;\mathbb{R}^2) | \, z = \operatorname{grad} p, p \in H^1(D;\mathbb{R})\}$. For $z \in$"RHS", we compute, for any $w \in C^\infty_{0,\operatorname{div}=0}$,

$$\langle z, w \rangle_{\mathcal{L}^2} = \int_D \operatorname{grad} p \cdot w \, \mathrm{d}\xi = -\int_D p \operatorname{div} w = 0,$$

so $z \in H^\perp$. Conversely, if $z \in H^\perp$, we need to find some $p \in H^1(D;\mathbb{R})$ such that $z = \operatorname{grad} p$ in $\mathcal{L}^2(D;\mathbb{R}^2)$. The existence of p can be shown by de Rham's current theory, see Theorem 17' on page 95 in [6]. A constructive but elementary proof is given in [8], or see Appendix A for detailed account.

Next, we show $H = \{ z \in \mathcal{L}^2(D;\mathbb{R}^2) | \operatorname{div} z = 0, z \cdot \boldsymbol{n}|_{\partial D} = 0\}$. If $z \in H$, there exists a sequence $\{z_n\} \subset C^\infty_{0,\operatorname{div}=0}$ such that $z_n \to z$ in $\mathcal{L}^2(D;\mathbb{R}^2)$. For $p \in C_0^\infty(D;\mathbb{R})$, we have

$$(\operatorname{div} z)(p) = -\langle z, \operatorname{grad} p \rangle_{\mathcal{L}^2} = -\lim_{n\to\infty} \langle z_n, \operatorname{grad} p \rangle_{\mathcal{L}^2} = \lim_{n\to\infty} \langle \operatorname{div} z_n, p \rangle_{\mathcal{L}^2} = 0,$$

so $\operatorname{div} z = 0$ in the distributional sense. It remains to show $z \cdot \boldsymbol{n}|_{\partial D} = 0$. If $z \in \mathcal{L}^2(D;\mathbb{R}^2)$ and if $\operatorname{div} z \in \mathcal{L}^2(D;\mathbb{R})$, $z \cdot \boldsymbol{n}|_{\partial D}$ as a distribution acts on $\phi \in C_0^\infty(\partial D;\mathbb{R})$ by

$$(z \cdot \boldsymbol{n}|_{\partial D})(\phi) := \langle \operatorname{div} z, p \rangle_{\mathcal{L}^2} + \langle z, \operatorname{grad} p \rangle_{\mathcal{L}^2} \quad \text{for some } p \in H^1(\mathbb{D};\mathbb{R}) \text{ with } p|_{\partial D} = \phi.$$

See Theorem 1.2 on page 7 in [7] for more on definition of $z \cdot \boldsymbol{n}|_{\partial D}$ or see Appendix B. It follows

$$(z \cdot \boldsymbol{n}|_{\partial D})(\phi) = \lim_{n\to\infty} \left(\langle \operatorname{div} z_n, p \rangle_{\mathcal{L}^2} + \langle z_n, \operatorname{grad} p \rangle_{\mathcal{L}^2} \right) = 0$$

and $z \cdot \boldsymbol{n}|_{\partial D} = 0$. To show the converse, we suppose $z \in H^\perp \cap$"RHS" and hope $z = 0$. By the previous arguement, $z = \operatorname{grad} p$ for some $p \in H^1(D;\mathbb{R})$. Then we obtain a Poisson equation in p.

$$\begin{cases} \Delta p = \operatorname{div} z = 0 \\ \frac{\partial p}{\partial \boldsymbol{n}}|_{\partial D} = z \cdot \boldsymbol{n}|_{\partial D} = 0 \end{cases}$$

This gives $p = \text{const}$ and $z = 0$. ☐

Also, we have a similar result for the space V, but notice a different boundary feature as depicted below.

Proposition 2.

$$V = \left\{ z \in \mathbb{H}^1(D;\mathbb{R}^2) \middle| \operatorname{div} z = 0, z|_{\partial D} = 0 \right\}.$$

Why we have $z|_{\partial D} = 0$ in V but merely $z \cdot \boldsymbol{n}|_{\partial D} = 0$ in H? Intuitively, this is because we have the trace operator in H^1 extracting boundary values, while there is no such a counterpart in $\mathcal{L}^2$.

Proof. For $z \in V$, there exists a sequence $\{z_n\} \subset C^\infty_{0,\mathrm{div}=0}$ such that $z_n \to z$ in $H^1(D; \mathbb{R}^2)$. We have

$$\operatorname{div} z = \lim_{n\to\infty} \operatorname{div} z_n = 0 \quad \text{in } \mathcal{L}^2(D;\mathbb{R}),$$
$$z|_{\partial D} = \lim_{n\to\infty} z_n|_{\partial D} = 0 \quad \text{in } \mathcal{L}^2(\partial D, \mathbb{R}^2).$$

If $z \in V^\perp \cap$"RHS", it is then obvious $z \in H$, so we only need to show $z \in H^\perp$. For any $w \in C^\infty_{0,\mathrm{div}=0}$,

$$0 = \langle z, w\rangle_V = \sum_{i,j=1}^{2} \int_D \frac{\partial z_i}{\partial \xi_j}\frac{\partial w_i}{\partial \xi_j}\,\mathrm{d}\xi = -\sum_{i,j=1}^{2}\int_D z_i \frac{\partial^2 w_i}{\partial \xi_j^2}\,\mathrm{d}\xi = -\langle z, \Delta w\rangle_{\mathcal{L}^2}.$$

For any $\tilde{w} \in C^\infty_{0,\mathrm{div}=0}$, the Poisson equation

$$\begin{cases} \Delta w = \tilde{w} \\ w|_{\partial D} = 0 \end{cases}$$

has a unique solution $w \in C^\infty_{0,\mathrm{div}=0}$, so $\langle z, \tilde{w}\rangle_{\mathcal{L}^2} = 0$ and $z \in H^\perp$. □

Let P_H be the projection onto H in $\mathcal{L}^2(D;\mathbb{R}^2)$ and define

$$A := -P_H\Delta, \qquad \mathcal{D}(A) = H^2(D;\mathbb{R}^2) \cap V.$$

Then A maps $\mathcal{D}(A)$ into H, and actually, we can go further to see A is bijective from $\mathcal{D}(A)$ to H. Let $w \in H$ and we want to find some $z \in \mathcal{D}(A)$ such that $-P_H\Delta z = w$. That is to say there exist $z \in \mathcal{D}(A)$ and $p \in H^1(D;\mathbb{R})$ satisfying $-\Delta z + \operatorname{grad} p = w$. This reduces to a steady-state Stokes equation.

$$\begin{cases} -\Delta z + \operatorname{grad} p = w \\ \operatorname{div} z = 0 \\ z|_{\partial D} = 0 \end{cases}$$

By the Riesz representation theorem, it has a unique solution $z \in V$ such that

$$\langle z, v\rangle_V = \langle w, v\rangle_H \quad \text{for all } v \in V.$$

A regularity result gives $z \in H^2(D;\mathbb{R}^2)$ and this implies A is a bijection. See Section 1.2 of [7] for steady-state Stokes equations and a regularity discussion is presented in Appedix C. Moreover, A is positive and symmetric.

$$\langle Az, z\rangle_H = \langle w, z\rangle_H = \langle z, z\rangle_V \geq 0, \quad \text{"=" holds if and only if } \|z\|_V = 0,$$
$$\langle Az_1, z_2\rangle_H = \langle w_1, z_2\rangle_H = \langle z_1, z_2\rangle_V = \langle z_1, Az_2\rangle_H.$$

We observe A^{-1} maps H onto $\mathcal{D}(A)$ and $\mathcal{D}(A)$ can be compactly embedded back into H, so A^{-1} is a self-ajoint compact operator on H. There exists an orthonormal basis $\{u_k\} \subset H$ consisting of eigenvectors of A with

$$Au_k = \lambda_k u_k, \qquad \lambda_k > 0, \quad \lambda_k \uparrow +\infty.$$

For $s \in [0,1]$, we can define interpolation spaces between H and V by

$$V^{2s} := \mathcal{D}(A^s) = \left\{ z \in H \,\middle|\, \sum_{k=1}^{\infty} \lambda_k^{2s} \langle z, u_k \rangle_H^2 < \infty \right\}, \qquad \langle z, w \rangle_{V^{2s}} := \sum_k \lambda_k^{2s} \langle z, u_k \rangle_H \langle w, u_k \rangle_H .$$

In particular, when $s = \frac{1}{2}$,

$$\sum_k \lambda_k \langle z, u_k \rangle_H^2 = \sum_k \langle z, \lambda_k u_k \rangle_H \langle z, u_k \rangle_H = \sum_k \langle z, A u_k \rangle_H \langle z, u_k \rangle_H = \sum_k \langle Az, u_k \rangle_H \langle z, u_k \rangle_H = \langle Az, z \rangle_H = \|z\|_V^2 ,$$

so we have $V = \mathcal{D}(A^{1/2})$. The definition of V^{2s} extends to all $s \in \mathbb{R}$ and the following is an interpolation inequality for V^{2s}.

Proposition 3. *Suppose $s_1 < s < s_2$, $0 < \theta < 1$ and $s = \theta s_1 + (1-\theta) s_2$. Then*

$$\|z\|_{V^{2s}} \le \|z\|_{V^{2s_1}}^{\theta} \|z\|_{V^{2s_2}}^{1-\theta} .$$

This is similar to the interpolation inequality for Sobolev spaces H^s, $\|z\|_{H^s} \le C \|z\|_{H^{s_1}}^{\theta} \|z\|_{H^{s_2}}^{1-\theta}$. And actually, V^{2s} is continuously embedded into H^{2s}.

Proof.

$$\begin{aligned} \|z\|_{V^{2s}}^2 &= \sum_k \lambda_k^{2s} \langle z, u_k \rangle_H^2 \\ &= \sum_k \lambda_k^{2\theta s_1} \langle z, u_k \rangle_H^{2\theta} \lambda_k^{2(1-\theta)s_2} \langle z, u_k \rangle_H^{2(1-\theta)} \\ &\le \left(\sum_k \lambda_k^{2s_1} \langle z, u_k \rangle_H^2 \right)^{\theta} \left(\sum_k \lambda_k^{2s_2} \langle z, u_k \rangle_H^2 \right)^{1-\theta} && \left(\frac{1}{1/\theta} + \frac{1}{1/(1-\theta)} = 1 \right) \\ &= \|z\|_{V^{2s_1}}^{2\theta} \|z\|_{V^{2s_2}}^{2(1-\theta)} \end{aligned}$$

□

Now we introduce some process spaces that play a role in the sequel.

$$\begin{aligned} E &:= \mathcal{L}^4(0,T;\mathcal{L}^4(D;\mathbb{R}^2)), \\ H^1(0,T;V;V^{-1}) &:= \{ z \in \mathcal{L}^2(0,T;V) \,|\, \dot{z} \in \mathcal{L}^2(0,T;V^{-1}) \}, \\ &\qquad \langle z, w \rangle_{H^1(0,T;V;V^{-1})} := \langle z, w \rangle_{\mathcal{L}^2(0,T;V)} + \langle \dot{z}, \dot{w} \rangle_{\mathcal{L}^2(0,T;V^{-1})} , \\ C_{\mathrm{wk}}(0,T;H) &:= \{ z \colon [0,T] \to H \,|\, t \mapsto \langle z(t), w \rangle_H \text{ is continuous for any } w \in H \} , \\ C_{\mathrm{wk}}(0,T;V^{-1}) &:= \left\{ z \colon [0,T] \to V^{-1} \,\middle|\, t \mapsto \langle z(t), w \rangle_{V^{-1},V} \text{ is continuous for any } w \in V \right\} . \end{aligned}$$

The subscript "wk" means being continuous w.r.t. the weak topology or being weakly continuous.

Proposition 4 **($\mathcal{L}^\infty(0,T;H) \cap \mathcal{L}^2(0,T;V) \subset E$).** *The space $\mathcal{L}^\infty(0,T;H) \cap \mathcal{L}^2(0,T;V)$ is continuously embedded into E.*

Proof. Let $z \in \mathcal{L}^\infty(0,T;H) \cap \mathcal{L}^2(0,T;V)$.

$$\begin{aligned} \|z\|_{\mathcal{L}^4} &\le C \|z\|_{H^{1/2}} && (H^{1/2} \text{ is embedded into } \mathcal{L}^4) \\ &\le C \|z\|_{\mathcal{L}^2}^{1/2} \|z\|_{H^1}^{1/2} && (\text{interpolation for } H^s) \end{aligned}$$

$$
\begin{aligned}
&\leq C \|z\|_H^{1/2} \|z\|_V^{1/2}. &(2)\\
\|z\|_E &= \left(\int_0^T \|z(t)\|_{\mathcal{L}^4}^4 \,\mathrm{d}t\right)^{1/4}\\
&\leq C \left(\int_0^T \|z(t)\|_H^2 \|z(t)\|_V^2 \,\mathrm{d}t\right)^{1/4}\\
&\leq C \|z\|_{\mathcal{L}^\infty(0,T;H)}^{1/2} \|z\|_{\mathcal{L}^2(0,T;V)}^{1/2}\\
&\leq C \left(\|z\|_{\mathcal{L}^\infty(0,T;H)} + \|z\|_{\mathcal{L}^2(0,T;V)}\right).
\end{aligned}
$$

□

Proposition 5 ($H^1(0,T;V;V^{-1}) \subset C(0,T;H) \cap \mathcal{L}^2(0,T;V)$)**.** *The space $H^1(0,T;V;V^{-1})$ is continuously embedded into $C(0,T;H) \cap \mathcal{L}^2(0,T;V)$.*

Proof. For $z \in H^1(0,T;V;V^{-1})$, it is obvious $z \in \mathcal{L}^2(0,T;V)$ by definition and it suffices to show $z \in C(0,T;H)$. Before reaching there, the first step is to show $z \in \mathcal{L}^\infty(0,T;H)$. By basic calculus, we have

$$
\|z(t)\|_H^2 = \|z(s)\|_H^2 + 2\int_s^t \langle \dot{z}(r), z(r)\rangle \,\mathrm{d}r \leq \|z(s)\|_H^2 + \|z\|_{H^1(0,T;V;V^{-1})}^2.
$$

Integrate s on $[0,T]$ and we obtain

$$
\sup_{t\in[0,T]} \|z(t)\|_H \leq C \|z\|_{H^1(0,T;V;V^{-1})}.
$$

This means $z \in \mathcal{L}^\infty(0,T;H)$ and it remains to show z is continuous in H.

Actually, we can show z is weakly continuous in H, i.e. $z \in C_{\mathrm{wk}}(0,T;H)$. As a primitive function of $\dot{z} \in \mathcal{L}^2(0,T;V^{-1})$, z is continuous in V^{-1}. In particular, for any $w \in V$, $t \mapsto \langle z(t), w\rangle_{V^{-1},V}$ is continuous, so $z \in C_{\mathrm{wk}}(0,T;V^{-1})$. A nontrivial result that $\mathcal{L}^\infty(0,T;H) \cap C_{\mathrm{wk}}(0,T;V^{-1}) = C_{\mathrm{wk}}(0,T;H)$ gives $z \in C_{\mathrm{wk}}(0,T;H)$. For a proof of this nontrivial result, see Lemma 1.4 on page 178 in [7] or Appendix D. An alternative proof of this proposition, but in a more general form, is given by Theorem 3.1 on page 19 in [3] and does not use $\mathcal{L}^\infty(0,T;H) \cap C_{\mathrm{wk}}(0,T;V^{-1}) = C_{\mathrm{wk}}(0,T;H)$.

Now that $z \in C_{\mathrm{wk}}(0,T;H)$, when $s \to t$, we have

$$
\|z(s) - z(t)\|_H^2 = \|z(s)\|_H^2 + \|z(t)\|_H^2 - 2\langle z(s), z(t)\rangle_H \to 0,
$$

by continuity of $s \mapsto \|z(s)\|_H^2$ and continuity of $s \mapsto \langle z(s), z(t)\rangle_H$. □

2 Existence and Uniqueness of Solutions by Fixed-Point Iteration

Apply P_H on both sides of the equation (1) and let $\operatorname{grad} p$ absorb the gradient part of f. We obtain

$$
\begin{cases}
\dot{Z}(t) + AZ(t) + P_H(Z(t)\cdot\nabla)Z(t) = f(t) + \dot{W}_Q(t), \quad t \in [0,T]\\
Z(0) = z \in H,
\end{cases}
$$

where W_Q is a Q-Weiner process in H with Q of trace class. The term $P_H(Z\cdot\nabla)Z$ is still arguable, and to make sense of it, we define a multi-linear form $b\colon V \times V \times V \to \mathbb{R}$ by

$$
b(u,v,w) := \langle (u\cdot\nabla)v, w\rangle_{\mathcal{L}^2} = \sum_{i,j=1}^2 \int_D u_i \frac{\partial v_j}{\partial \xi_i} w_j \,\mathrm{d}\xi.
$$

Then we have

$$b(u,v,w) = -b(u,w,v), \quad b(u,v,v) = 0 \qquad \text{(integration by part)}$$

$$|b(u,v,w)| \le C \|u\|_{\mathcal{L}^4} \|v\|_V \|w\|_{\mathcal{L}^4} \qquad \text{(Hölder's inequality).}$$

Denote V^{-1} the dual space of V, and for $z \in V$, the functional $w \mapsto b(z,z,w)$ is in V^{-1}. Hence, $P_H(z \cdot \nabla)z = (z \cdot \nabla)z$ in V^{-1} and this gives rise to

$$\begin{cases} \dot{Z}(t) + AZ(t) + (Z(t) \cdot \nabla)Z(t) = f(t) + \dot{W}_Q(t), \quad t \in [0,T] \\ Z(0) = z \in H. \end{cases} \tag{3}$$

The equality is interpreted in V^{-1} as follows.

Definition 6. *Given $z \in H$ and $f \in \mathcal{L}^2(0,T;V^{-1})$, a stochastic process $Z \in C(0,T;H) \cap \mathcal{L}^2(0,T;V)$ is called a pathwise weak solution of the equation (3) if*

$$\langle Z(t), w\rangle_H + \int_0^t \langle Z(s), w\rangle_V \,\mathrm{d}s + \int_0^t b(Z(s), Z(s), w)\,\mathrm{d}s = \langle z, w\rangle_H + \int_0^t \langle f(s), w\rangle_{V^{-1},V}\,\mathrm{d}s + \langle W_Q(t), w\rangle_H$$

for all $w \in V$ and all $t \in [0,T]$ a.s.

We will show existence and uniqueness in the space E, and to this end, first notice the map $B\colon V \to V^{-1}$, $z \mapsto b(z,z,\cdot)$, can extend continuously to $\mathcal{L}^4(D;\mathbb{R}^2) \to V^{-1}$, $z \mapsto -b(z,\cdot,z)$, and to $E \to \mathcal{L}^2(0,T;V^{-1})$, $z(t) \mapsto -b(z(t),\cdot,z(t))$, by involving time. Next, we shall reduce the equation (3) to a deterministic one whose coefficients, nonetheless, have randomness due to an Ornstein-Uhlenbeck process.

Define

$$S(t) := \mathrm{e}^{-tA}, \quad X(t) := \int_0^t S(t-s)\,\mathrm{d}W_Q(s), \quad Y(t) := Z(t) - X(t).$$

Then we can check $X \in E$ a.s. and obtain the following deterministic equation,

$$\begin{cases} \dot{Y}(t) = -AY(t) - B(Y(t) + X(t)) + f(t), \quad t \in [0,T] \\ Y(0) = z \in H. \end{cases} \tag{4}$$

Let $\{v_l\} \subset H$ be the orthonormal basis consisting of eigenvectors of Q with

$$Qv_l = \sigma_l^2 v_l, \qquad \sigma_l \ge 0, \quad \sum_{l=1}^{\infty} \sigma_l^2 < \infty,$$

and W_Q admits a representation

$$W_Q(t) = \sum_{l=1}^{\infty} \sigma_l \beta_l(t) v_l,$$

where $\{\beta_l\}$ is an independent sequence of $\mathbb{R}$-valued Weiner processes. We compute

$$\begin{aligned} X(t) &= \sum_l \int_0^t \mathrm{e}^{-(t-s)A} \sigma_l \,\mathrm{d}\beta_l(s) v_l \\ &= \sum_{k,l} \int_0^t \mathrm{e}^{-(t-s)\lambda_k} \sigma_l \,\mathrm{d}\beta_l(s) \langle v_l, u_k\rangle_H u_k, \\ \|X(t)\|_H^2 &= \sum_k \left(\sum_l \int_0^t \mathrm{e}^{-(t-s)\lambda_k} \sigma_l \langle v_l, u_k\rangle_H \,\mathrm{d}\beta_l(s) \right)^2, \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left\|X(t)\right\|_H^2 &= \sum_{k,l} \int_0^t \mathrm{e}^{-2(t-s)\lambda_k} \sigma^2 \left\langle v_l, u_k \right\rangle_H^2 \mathrm{d}s \\
&= \sum_{k,l} \frac{1-\mathrm{e}^{-2t\lambda_k}}{2\lambda_k} \sigma_l^2 \left\langle v_l, u_k \right\rangle_H^2 \\
&= \sum_{k} \frac{1-\mathrm{e}^{-2t\lambda_k}}{2\lambda_k} \left\langle Q u_k, u_k \right\rangle_H < \infty, \\
\mathbb{E} \sup_{t\in[0,T]} \left\|X(t)\right\|_H^2 &\leq 4\mathbb{E}\left\|X(T)\right\|_H^2 < \infty \qquad\qquad \text{(Doob's } \mathcal{L}^p \text{ inequality)}, \\
\left\|X(t)\right\|_V^2 &= \sum_k \lambda_k \left(\sum_l \int_0^t \mathrm{e}^{-(t-s)\lambda_k} \sigma_l \left\langle v_l, u_k \right\rangle_H \mathrm{d}\beta_l(s) \right)^2, \\
\mathbb{E}\left\|X(t)\right\|_V^2 &= \sum_{k,l} \lambda_k \int_0^t \mathrm{e}^{-2(t-s)\lambda_k} \sigma^2 \left\langle v_l, u_k \right\rangle_H^2 \mathrm{d}s \\
&= \sum_{k,l} \frac{1-\mathrm{e}^{-2t\lambda_k}}{2} \sigma_l^2 \left\langle v_l, u_k \right\rangle_H^2 \\
&\leq \sum_k \frac{1}{2} \left\langle Q u_k, u_k \right\rangle_H \\
\mathbb{E} \int_0^T \left\|X(t)\right\|_V^2 \mathrm{d}t &\leq T \sum_k \frac{1}{2} \left\langle Q u_k, u_k \right\rangle_H < \infty,
\end{aligned}
$$

so it gives $X \in \mathcal{L}^\infty(0,T;H) \cap \mathcal{L}^2(0,T;V) \subset E$ a.s.

Moreover, the integral form of (4) is

$$
Y(t) = S(t)z - \int_0^t S(t-s) \left[B(Y(s)+X(s)) - f(s) \right] \mathrm{d}s. \tag{5}
$$

To apply the fixed-point method to show existence, we need to play with $F \colon \mathcal{L}^2(0,T;V^{-1}) \to E$ defined by

$$
y(t) \mapsto \int_0^t S(t-s) y(s) \, \mathrm{d}s.
$$

To see F is well-defined, we need to check $z(t) = \int_0^t S(t-s)y(s)\,\mathrm{d}s \in E$.

$$
\begin{aligned}
\dot{z}(t) &= -Az(t) + y(t), \quad z(0) = 0 \\
\left\langle \dot{z}(t), z(t) \right\rangle_{V^{-1},V} &= -\left\langle Az(t), z(t) \right\rangle_{V^{-1},V} + \left\langle y(t), z(t) \right\rangle_{V^{-1},V} \qquad \text{(acting on } z\text{)} \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\|z(t)\right\|_H^2 &= -\left\|z(t)\right\|_V^2 + \left\langle y(t), z(t) \right\rangle_{V^{-1},V} \\
\frac{1}{2}\left\|z(t)\right\|_H^2 + \int_0^t \left\|z(s)\right\|_V^2 \mathrm{d}s &= \int_0^t \left\langle y(s), z(s) \right\rangle_{V^{-1},V} \mathrm{d}s \qquad \text{(operated by } \int_0^t \cdots \mathrm{d}s\text{)} \\
&\leq \int_0^t \left\|y(s)\right\|_{V^{-1}} \left\|z(s)\right\|_V \mathrm{d}s \\
&\leq \frac{1}{2} \int_0^t \left\|y(s)\right\|_{V^{-1}}^2 \mathrm{d}s + \frac{1}{2} \int_0^t \left\|z(s)\right\|_V^2 \mathrm{d}s \\
\left\|z(t)\right\|_H^2 + \int_0^t \left\|z(s)\right\|_V^2 \mathrm{d}s &\leq \int_0^t \left\|y(s)\right\|_{V^{-1}}^2 \mathrm{d}s. \qquad (6)
\end{aligned}
$$

This means $\left\|z\right\|_{\mathcal{L}^\infty(0,T;H)} + \left\|z\right\|_{\mathcal{L}^2(0,T;V)} \leq C \left\|y\right\|_{\mathcal{L}^2(0,T;V^{-1})}$ and $z \in E$. We now can examine the composition $\mathcal{F} := F \circ (B - f)$ from E into itself, $z(t) \mapsto \int_0^t S(t-s)[B(z(s)) - f(s)]\,\mathrm{d}s$. The following result is a basic tool when looking for a fixed point later.

Lemma 7. *There is a constant M independent of T such that*

$$\|\mathcal{F}z - \mathcal{F}w\|_E \le M(\|z\|_E + \|w\|_E)\,\|z - w\|_E\,.$$

Proof. For $z, w \in E$ and $v \in V$, we have

$$\begin{aligned}
(Bz - Bw)(v) &= -b(z, v, z) + b(w, v, w) \\
&= \sum_{i,j}\left(\int_D w_i w_j \frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi - \int_D z_i z_j \frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi\right) \\
&= \sum_{i,j}\int_D (w_i w_j - z_i z_j)\frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi \\
&= \sum_{i,j}\int_D (w_i w_j - w_i z_j + w_i z_j - z_i z_j)\frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi \\
&= \sum_{i,j}\int_D w_i (w_j - z_j)\frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi + \sum_{i,j}\int_D (w_i - z_i) z_j \frac{\partial v_j}{\partial \xi_i}\,\mathrm{d}\xi, \\
|(Bz - Bw)(v)| &\le C\left(\|w\|_{\mathcal{L}^4}\|z - w\|_{\mathcal{L}^4}\|v\|_V + \|z\|_{\mathcal{L}^4}\|z - w\|_{\mathcal{L}^4}\|v\|_V\right) && \text{(Hölder's inequality)} \\
&= C(\|z\|_{\mathcal{L}^4} + \|w\|_{\mathcal{L}^4})\,\|z - w\|_{\mathcal{L}^4}\,\|v\|_V\,, \\
\|Bz - Bw\|_{\mathcal{L}^2(0,T;V^{-1})} &\le C\left[\int_0^T (\|z\|_{\mathcal{L}^4} + \|w\|_{\mathcal{L}^4})^2\,\|z - w\|_{\mathcal{L}^4}^2\,\mathrm{d}t\right]^{1/2} \\
&\le C(\|z\|_E + \|w\|_E)\,\|z - w\|_E && \text{(Cauchy's inequality).}
\end{aligned}$$

For $x, y \in \mathcal{L}^2(0, T; V^{-1})$, we combine Proposition 4 and (6) to see

$$\|Fx - Fy\|_E \le C(\|F(x - y)\|_{\mathcal{L}^\infty(0,T;H)} + \|F(x - y)\|_{\mathcal{L}^2(0,T;V)}) \le C\,\|x - y\|_{\mathcal{L}^2(0,T;V^{-1})}\,.$$

It follows $\|\mathcal{F}z - \mathcal{F}w\|_E = \|(F \circ B)z - (F \circ B)w\|_E \le C\,\|Bz - Bw\|_{\mathcal{L}^2(0,T;V^{-1})} \le C(\|z\|_E + \|w\|_E)\,\|z - w\|_E$. ☐

Then consider the fixed-point problem:

$$y = Sz - \mathcal{F}(y + X).$$

Define

$$y_0 := Sz - \mathcal{F}(X), \qquad y_{n+1} = Sz - \mathcal{F}(y_n + X), \quad n \in \mathbb{N},$$

and we see

$$\|y_{n+1} - y_n\|_E = \|\mathcal{F}(y_n + X) - \mathcal{F}(y_{n-1} + X)\|_E \le M(\|y_n + X\|_E + \|y_{n-1} + X\|_E)\,\|y_n - y_{n-1}\|_E\,.$$

Thus, we hope $\|y_n + X\|_E + \|y_{n-1} + X\|_E \le \frac{1}{2M}$ to make the sequence $\{y_n\}$ converge.

Theorem 8 (local existence and uniqueness)**.** *For any initial condition $z \in H$, and for any $f \in \mathcal{L}^2(0, T; V^{-1})$, there exists a random variable $\tau \in (0, T]$ a.s. such that the equation (4) has a unique mild solution Y on the time interval $[0, \tau]$.*

Proof. Choose $T > 0$ small enough such that $\|y_0\|_E + \|X\|_E \le \frac{1}{8M}$. Then

$$\begin{aligned}
\|y_1 - y_0\|_E &\le M(\|y_0 + X\|_E + \|X\|_E)\,\|y_0\|_E \le \frac{1}{2}\,\|y_0\|_E\,, \\
\|y_1 + X\|_E &\le \|y_0 + X\|_E + \|y_1 - y_0\|_E \le \frac{1}{8M}\left(1 + \frac{1}{2}\right).
\end{aligned}$$

By induction, we have $\|y_n + X\|_E \le \frac{1}{8M}\sum_{k=0}^n 2^{-k} \le \frac{1}{4M}$ and $\|y_{n+1} - y_n\|_E \le \frac{1}{2}\,\|y_n - y_{n-1}\|_E$ for all n. The sequence $\{y_n\}$ converges to some $y \in E$ and $y = Sz - \mathcal{F}(y + X)$. This fixed point y is unique in the case of $\|y + X\|_E \le \frac{1}{4M}$. ☐

Global existence and uniqueness are a consequence of the following a priori estimates.

Proposition 9 (a priori estimates)**.** *Assume $y \in E$ is a mild solution of (4) with initial condition $z \in H$. Then we have*

$$\sup_{t\in[0,T]} \|y(t)\|_H^2 \le \mathrm{e}^{C\int_0^T \|X(r)\|_{\mathcal{L}^4}^4 \,\mathrm{d}r} \|z\|_H^2 + C \int_0^T \mathrm{e}^{C\int_s^T \|X(r)\|_{\mathcal{L}^4}^4 \,\mathrm{d}r} \left(\|X(s)\|_{\mathcal{L}^4}^4 + \|f(s)\|_{V^{-1}}^2\right) \mathrm{d}s, \tag{7}$$

$$\int_0^T \|y(t)\|_V^2 \,\mathrm{d}t \le \|z\|_H^2 + C \sup_{s\in[0,T]} \|y(s)\|_H^2 \int_0^T \|X(s)\|_{\mathcal{L}^4}^4 \,\mathrm{d}s + C\int_0^T \left(\|X(s)\|_{\mathcal{L}^4}^4 + \|f(s)\|_{V^{-1}}^2\right) \mathrm{d}s. \tag{8}$$

Assume $y_1, y_2 \in E$ are mild solutions of (4) corresponding to initial conditons $z_1, z_2 \in H$ respectively. Then

$$\sup_{t\in[0,T]} \|y_1(t) - y_2(t)\|_H^2 \le \mathrm{e}^{C\int_0^T (\|y_1(s)+y_2(s)\|_{\mathcal{L}^4}^4 + \|X(s)\|_{\mathcal{L}^4}^4)\,\mathrm{d}s} \|z_1 - z_2\|_H^2, \tag{9}$$

$$\int_0^T \|y_1(t) - y_2(t)\|_V^2 \,\mathrm{d}t \le \|z_1 - z_2\|_H^2 + C \sup_{s\in[0,T]} \|y_1(s) - y_2(s)\|_H^2 \int_0^T (\|y_1(s)+y_2(s)\|_{\mathcal{L}^4}^4 + \|X(s)\|_{\mathcal{L}^4}^4)\,\mathrm{d}s. \tag{10}$$

Proof. We begin with

$$\begin{aligned}
\dot{y} + Ay &= -B(y+X) + f \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle y, y\rangle_H + \langle y, y\rangle_V &= b(y+X, y, y+X) + \langle f, y\rangle_{V^{-1},V} && \text{(acting on } y) \\
&= b(y, y, X) + b(X, y, X) + \langle f, y\rangle_{V^{-1},V} && \text{(linearity and } b(u,v,v)=0) \\
&\le C\|y\|_{\mathcal{L}^4}\|y\|_V\|X\|_{\mathcal{L}^4} + C\|y\|_V\|X\|_{\mathcal{L}^4}^2 + \|f\|_{V^{-1}}\|y\|_V && \text{(Hölder's inequality)} \\
&\le C\|y\|_H^{1/2}\|y\|_V^{3/2}\|X\|_{\mathcal{L}^4} + C\|y\|_V\|X\|_{\mathcal{L}^4}^2 + \|f\|_{V^{-1}}\|y\|_V && \text{(invoke (2))} \\
&\le \frac{1}{6}\|y\|_V^2 + C\|y\|_H^2\|X\|_{\mathcal{L}^4}^4 + \frac{1}{6}\|y\|_V^2 + C\|X\|_{\mathcal{L}^4}^4 \\
&\quad + \frac{1}{6}\|y\|_V^2 + C\|f\|_{V^{-1}}^2 && \text{(Young's inequality)} \\
&= \frac{1}{2}\|y\|_V^2 + C\|y\|_H^2\|X\|_{\mathcal{L}^4}^4 + C\|X\|_{\mathcal{L}^4}^4 + C\|f\|_{V^{-1}}^2
\end{aligned}$$

Integrating both sides on the time interval $[0,t]$ gives

$$\|y(t)\|_H^2 + \int_0^t \|y(s)\|_V^2 \,\mathrm{d}s \le \|z\|_H^2 + C\int_0^t \left(\|y(s)\|_H^2\|X(s)\|_{\mathcal{L}^4}^4 + \|X(s)\|_{\mathcal{L}^4}^4 + \|f(s)\|_{V^{-1}}^2\right)\mathrm{d}s,$$

and the estimates of y follows from Gronwall's inequality. In a similar manner, the estimates of $y_1 - y_2$ come from

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_1 - y_2\|_H^2 + \|y_1 - y_2\|_V^2 \\
&= b(y_1 + X, y_1 - y_2, y_1 + X) - b(y_2 + X, y_1 - y_2, y_2 + X) \\
&= \frac{1}{2}b(y_1 - y_2, y_1 - y_2, y_1 + y_2) + b(y_1 - y_2, y_1 - y_2, X) \\
&\le C\|y_1 - y_2\|_{\mathcal{L}^4}\|y_1 - y_2\|_V\|y_1 + y_2\|_{\mathcal{L}^4} + C\|y_1 - y_2\|_{\mathcal{L}^4}\|y_1 - y_2\|_V\|X\|_{\mathcal{L}^4} \\
&\le C\|y_1 - y_2\|_H^{1/2}\|y_1 - y_2\|_V^{3/2}\|y_1 + y_2\|_{\mathcal{L}^4} + C\|y_1 - y_2\|_H^{1/2}\|y_1 - y_2\|_V^{3/2}\|X\|_{\mathcal{L}^4} \\
&\le \frac{1}{4}\|y_1 - y_2\|_V^2 + C\|y_1 - y_2\|_H^2\|y_1 + y_2\|_{\mathcal{L}^4}^4 + \frac{1}{4}\|y_1 - y_2\|_V^2 + C\|y_1 - y_2\|_H^2\|X\|_{\mathcal{L}^4}^4 \\
&= \frac{1}{2}\|y_1 - y_2\|_V^2 + C\|y_1 - y_2\|_H^2\|y_1 + y_2\|_{\mathcal{L}^4}^4 + C\|y_1 - y_2\|_H^2\|X\|_{\mathcal{L}^4}^4
\end{aligned}$$

namely,

$$\|y_1(t) - y_2(t)\|_H^2 + \int_0^t \|y_1(s) - y_2(s)\|_V^2 \,\mathrm{d}s \le \|z_1 - z_2\|_H^2 + C\int_0^t \|y_1(s) - y_2(s)\|_H^2 \left(\|y_1(s) + y_2(s)\|_{\mathcal{L}^4}^4 + \|X(s)\|_{\mathcal{L}^4}^4\right)\mathrm{d}s$$

□

Theorem 10 (global existence and uiqueness)**.** *For any initial condition $z \in H$, and for any $f \in \mathcal{L}^2(0,T;V^{-1})$, there exists a unique weak solution Z of the equation (3) a.s. and the solution depends continuously on z a.s.*

Proof. The estimates in Proposition 9 gives $Z \in \mathcal{L}^\infty(0,T;H) \cap \mathcal{L}^2(0,T;V)$ exists, is unique and depends continuously on z. Due to the integral equation (5) that Z satisfies, we have $Z \in C(0,T;H)$. $\square$

3 Existence and Uniqueness of Solutions by Galerkin's Approximation

In this section, we prove existence and uniqueness of solutions by Galerkin's approximation. Technically, it invloves approximation of solutions in finite but increasing dimensions and passing to the limit in an infinite-dimensional space. Instead of working in E, we will focus on the space $H^1(0,T;V;V^{-1})$ in this case.

As done before, we split the equation (3) into an Ornstein-Uhlenbeck process and a deterministic equation with random coefficients.

$$\begin{cases} \dot{X}(t) = -AX(t) + \dot{W}_Q(t), \quad t \in [0,T] \\ X(0) = 0. \end{cases}$$

$$\begin{cases} \dot{Y}(t) = -AY(t) - B(Y(t) + X(t)) + f(t), \quad t \in [0,T] \\ Y(0) = z \in H. \end{cases} \tag{4}$$

Now we approximate Y in finite-dimensional spaces. Recall $\{u_k\}$ is an orthonormal basis in H, consisting of eigenvectors of A. Suppose $Y_m(t) = \sum_{k=1}^m \gamma_{m,k}(t) u_k$ and consider the following equation.

$$\begin{cases} \dot{Y}_m(t) = -AY_m(t) - B(Y_m(t) + X(t)) + f(t), \quad t \in [0,T] \\ Y_m(0) = \sum_{k=1}^m \langle z, u_k \rangle_H u_k. \end{cases}$$

Both sides act on u_k, $k = 1, 2, \ldots, m$, and we obtain a system of ODEs.

$$\begin{cases} \dot{\gamma}_{m,k}(t) = -\lambda_k \gamma_{m,k}(t) - \sum_{i,j=1}^m b(u_i, u_j, u_k)\gamma_{m,i}(t)\gamma_{m,j}(t) - \sum_{i=1}^m b(u_i, X(t), u_k)\gamma_{m,i}(t) \\ \qquad\qquad - \sum_{j=1}^m b(X(t), u_j, u_k)\gamma_{m,j}(t) - b(X(t), X(t), u_k) + \langle f(t), u_k \rangle_{V^{-1},V}, \quad t \in [0,T],\ k = 1, 2, \ldots, m \\ \gamma_{m,k}(0) = \langle z, u_k \rangle_H, \quad k = 1, 2, \ldots, m \end{cases}$$

Then there is a random variable $\tau \in (0,T]$ a.s. such that $\gamma_{m,k}$, $k = 1, 2, \ldots, m$, exist on the time interval $[0,\tau]$. By establishing a priori estimates on Y_m in the same manner of (7)(8) in Propostion 9, we see each Y_m does not blow up and exists on the whole interval $[0,T]$. Moreover, the sequence $\{Y_m\}$ is bounded in $H^1(0,T;V;V^{-1})$. By passing to a weakly convergent subsequence, we may assume $\{Y_m\}$ weakly converges to Y in $H^1(0,T;V;V^{-1})$ and want to show the weak limit Y is a solution of (4).

Fix k, and for all $m \geq k$, we have

$$\int_0^t \left\langle \dot{Y}_m(s), u_k \right\rangle_{V^{-1},V} \mathrm{d}s = -\int_0^t \langle Y_m(s), u_k \rangle_V \mathrm{d}s - \int_0^t b(Y_m(s) + X(s), Y_m(s) + X(s), u_k) \,\mathrm{d}s + \int_0^t \langle f(s), u_k \rangle_{V^{-1},V} \mathrm{d}s \tag{11}$$

for all $t \in [0,T]$ a.s. From weak convergence of $\{Y_m\}$, when $m \to \infty$, it follows

$$\int_0^t \left\langle \dot{Y}_m(s), u_k \right\rangle_{V^{-1},V} \mathrm{d}s \to \int_0^t \left\langle \dot{Y}(s), u_k \right\rangle_{V^{-1},V} \mathrm{d}s,$$

$$\int_0^t \langle Y_m(s), u_k\rangle_V \,\mathrm{d}s \to \int_0^t \langle Y(s), u_k\rangle_V \,\mathrm{d}s.$$

But how about $\int_0^t b(Y_m(s)+X(s), Y_m(s)+X(s), u_k)\,\mathrm{d}s$ as $m\to\infty$? Actually, since $H^1(0,T;V;V^{-1})$ is continuously embedded into E by combining Proposition 4 and Proposition 5, and since the map $B\colon E\to \mathcal{L}^2(0,T;V^{-1})$, $z(t)\mapsto -b(z(t),\cdot,z(t))$, is continuous, we can pass to a further subsequence $\{B(Y_m+X)\}$ weakly converging to $\tilde{Y}$. Thanks to the bilinear structure behind B, the weak limit $\tilde{Y}$ is exactly $B(Y+X)$. On the one hand,

$$Y_m \to Y \text{ weakly in } \mathcal{L}^2(0,T;V) \quad\Longrightarrow\quad Y_m\to Y \text{ strongly in } \mathcal{L}^2(0,T;H)\subset \mathcal{L}^2([0,T]\times D;\mathbb{R}^2).$$

On the other hand,

$$Y_m \to Y \text{ weakly in } \mathcal{L}^2(0,T;V) \quad\Longrightarrow\quad \nabla Y_m\to \nabla Y \text{ weakly in } \mathcal{L}^2(0,T;\mathcal{L}^2(D;\mathbb{R}^2))\subset \mathcal{L}^2([0,T]\times D;\mathbb{R}^2).$$

It follows

$$(Y_m\cdot\nabla)Y_m \to (Y\cdot\nabla)Y \text{ weakly in } \mathcal{L}^1([0,T]\times D;\mathbb{R}^2).$$

By uniqueness of limit in the distribution space on $[0,T]\times D$, we obtain $\tilde{Y}=[(Y+X)\cdot\nabla](Y+X)=B(Y+X)$. That is to say

$$\int_0^t b(Y_m(s)+X(s), Y_m(s)+X(s), u_k)\,\mathrm{d}s \to \int_0^t b(Y(s)+X(s), Y(s)+X(s), u_k)\,\mathrm{d}s \quad \text{as } m\to\infty.$$

Passing to the limit on both sides of (11) gives

$$\begin{aligned}\langle Y(t), u_k\rangle_H - \langle z, u_k\rangle_H &= \int_0^t \left\langle \dot{Y}(s), u_k\right\rangle_{V^{-1},V} \mathrm{d}s\\ &= -\int_0^t \langle Y(s), u_k\rangle_V \,\mathrm{d}s - \int_0^t b(Y(s)+X(s), Y(s)+X(s), u_k)\,\mathrm{d}s + \int_0^t \langle f(s), u_k\rangle_{V^{-1},V}\,\mathrm{d}s\end{aligned}$$

for all $t\in[0,T]$ a.s. Note that the linear span of $\{u_k\}$ is dense in V, so Y is a pathwise weak solution.

Uniqueness in $H^1(0,T;V;V^{-1})$ follows from the estimates (9)(10).

4 Existence of Invariant Measures by Krylov-Bogoliubov Arguement

Let $(P_t, t\geq 0)$ be the Markov semigroup on H associated to the equation (3). We show existence of invariant measures for P_t in this section, namely there is some $\mu\in\mathcal{P}(H)$ such that $P_t^*\mu=\mu$ for all $t\geq 0$. Also, we derive several moment estimates for invariant measures along the way. First recall the Krylov-Bogoliubov theorem and see Chapter 3 of [4] for a proof.

Theorem 11 (Krylov-Bogoliubov)**.** *Assume $(P_t)_{t\geq 0}$ is a Feller semigroup. For $\nu\in\mathcal{P}(H)$, define*

$$\bar{\nu}_t(\Gamma) := \frac{1}{t}\int_0^t P_s^*\nu(\Gamma)\,\mathrm{d}s = \frac{1}{t}\int_0^t\int_H P_s(z,\Gamma)\nu(\mathrm{d}z)\,\mathrm{d}s.$$

If some sequence $\{\bar{\nu}_{t_n}\}$ converges weakly to μ, then μ is an invariant measure for P_t.

By this arguement, we may set $\nu=\delta_0$ and want to find a weakly convergent subsequence of $\bar{\nu}_t$, which can be achieved by tightness. That is to say, for any $\varepsilon>0$, we need to find a compact set $\Gamma_\varepsilon\subset H$ such that

$$\bar{\nu}_t(H\setminus\Gamma_\varepsilon)\leq\varepsilon \quad \text{for all } t\geq 0.$$

Notice V is compactly embedded in H, a natural way is to take Γ_ε as closed balls $B_V(0,R)$ in V.

Theorem 12 (existence of invariant measures)**.** *Let $f \in V^{-1}$ be constant. Then the equation (3) has at least one invariant measure.*

Proof. Let $\nu = \delta_0$ and we observe

$$\begin{aligned}
P_t^* \nu(\Gamma) &= \int_H P_t(z, \Gamma) \delta_0(\mathrm{d}z) = P_t(0, \Gamma), \\
\bar{\nu}_t(\Gamma) &= \frac{1}{t} \int_0^t P_s(0, \Gamma) \, \mathrm{d}s = \frac{1}{t} \int_0^t \mathbb{P}(Z(s) \in \Gamma) \, \mathrm{d}s, \\
\bar{\nu}_t(H \setminus \bar{B}_V(0, R)) &= \frac{1}{t} \int_0^t \mathbb{P}(Z(s) \notin \bar{B}_V(0, R)) \, \mathrm{d}s \\
&= \frac{1}{t} \int_0^t \mathbb{P}(\|Z(s)\|_V > R) \, \mathrm{d}s \\
&\leq \frac{1}{R^2 t} \mathbb{E} \int_0^t \|Z(s)\|_V^2 \, \mathrm{d}s. \qquad \text{(Chebyshev's inequality)}
\end{aligned}$$

It remains to compute the expectation value $\mathbb{E} \int_0^t \|Z(s)\|_V^2 \, \mathrm{d}s$. By Itô's formula, we have

$$\|Z(t)\|_H^2 = \|Z(0)\|_H^2 + 2 \int_0^t \langle -AZ(s) - BZ(s) + f, Z(s) \rangle_{V^{-1}, V} + t \sum_{l=1}^{\infty} \sigma_l^2 + 2 \sum_{l=1}^{\infty} \int_0^t \sigma_l \langle Z(s), v_l \rangle_H \, \mathrm{d}\beta_l(s),$$

$$\mathbb{E} \|Z(t)\|_H^2 + 2\mathbb{E} \int_0^t \|Z(s)\|_V^2 \, \mathrm{d}s = \mathbb{E} \|Z(0)\|_H^2 + 2\mathbb{E} \int_0^t \langle f, Z(s) \rangle_{V^{-1}, V} \, \mathrm{d}s + t \sum_{l=1}^{\infty} \sigma_l^2, \tag{12}$$

and it follows

$$\bar{\nu}_t(H \setminus \bar{B}_V(0, R)) \leq \frac{1}{R^2} \left(\sum_{l=1}^{\infty} \sigma_l^2 + \|f\|_{V^{-1}}^2 \right).$$

Thus, $(\bar{\nu}_t, t \geq 0)$ is tight and has a weakly convergent subsequence. □

If μ is an invariant measure and if $Z(0)$ obeys the law of μ, then $(Z(t), t \geq 0)$ is a stationary process. In particular, we have $\mathbb{E} \|Z(t)\|_H^2 = \mathbb{E} \|Z(0)\|_H^2$, $\mathbb{E} \|Z(t)\|_V^2 = \mathbb{E} \|Z(0)\|_V^2$ and $\mathbb{E} \int_0^t \langle f, Z(s) \rangle_{V^{-1}, V} \, \mathrm{d}s = t\mathbb{E} \langle f, Z(0) \rangle_{V^{-1}, V}$. The equality (12) becomes

$$\mathbb{E} \|Z(0)\|_V^2 = \mathbb{E} \langle f, Z(0) \rangle_{V^{-1}, V} + \frac{1}{2} \sum_{l=1}^{\infty} \sigma_l^2,$$

whenever $\mathbb{E} \|Z(0)\|_H^2 < \infty$ to allow subtraction from both sides. This gives a second moment relation of μ, but we first need to check μ itself has a finite second moment. Before doing so, here is a second moment estimate of solutions with initial conditions having finite second moments.

Lemma 13 (second moment estimates of solutions)**.** *Let $f \in V^{-1}$ be constant and suppose $\mathbb{E} \|Z(0)\|_H^2 < \infty$. Then*

$$\mathbb{E} \|Z(t)\|_H^2 \leq \mathrm{e}^{-\lambda_1 t} \mathbb{E} \|Z(0)\|_H^2 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2.$$

Proof. We combine (12) and Poincaré's inequality, $\|z\|_V^2 \geq \lambda_1 \|z\|_H^2$. It follows

$$\mathbb{E} \|Z(t)\|_H^2 + \lambda_1 \mathbb{E} \int_0^t \|Z(s)\|_H^2 \, \mathrm{d}s \leq \mathbb{E} \|Z(0)\|_H^2 + t \|f\|_{V^{-1}}^2 + t \sum_{l=1}^{\infty} \sigma_l^2.$$

Gronwall's inequality applies then. □

Theorem 14 (second moment relation of invariant measures)**.** *Let $f \in V^{-1}$ be constant. Then any invariant measure μ of the equation (3) satisfies*

$$\int_H \|z\|_V^2 \, \mu(\mathrm{d}z) = \int_H \langle f, z \rangle_{V^{-1}, V} \, \mu(\mathrm{d}z) + \frac{1}{2} \sum_{l=1}^{\infty} \sigma_l^2.$$

Proof. It suffices to show μ has a finite second moment, and the relation follows from Ito's formula by letting $Z(0)$ coinciding the law of μ in (12). Define $G_R\colon H \to \mathbb{R}$ by

$$G_R(z) := \begin{cases} \|z\|_H^2, & \|z\|_H^2 < R, \\ R, & \|z\|_H^2 \geq R, \end{cases}$$

and we compute

$$\begin{aligned}
\int_H G_R(z)\mu(\mathrm{d}z) &= \int_H \int_H P_t(z,\mathrm{d}w)G_R(w)\mu(\mathrm{d}z) && (P_t^*\mu = \mu) \\
&= \int_{\|z\|_H^2<\rho} \int_H P_t(z,\mathrm{d}w)G_R(w)\mu(\mathrm{d}z) + \int_{\|z\|_H^2\geq\rho} \int_H P_t(z,\mathrm{d}w)G_R(w)\mu(\mathrm{d}z) \\
&\leq \int_{\|z\|_H^2<\rho} \mathbb{E}\|Z(t)\|_H^2 \mu(\mathrm{d}z) + R\mu(H \setminus B_H(0,\rho)) && (Z(0)=z, \|z\|_H^2 < \rho) \\
&\leq \mathrm{e}^{-\lambda_1 t}\rho + C + R\mu(H \setminus B_H(0,\rho)) && (\text{Lemma 13}).
\end{aligned}$$

Let $t \to +\infty$, $\rho \to +\infty$, and it follows $\int_H G_R(z)\mu(\mathrm{d}z) \leq C$. Then Fatou's lemma applies. ☐

In the same manner, we can deduce a second exponential moment estimate for invariant measures if there is a moment decay for solutions with initial conditions having finite second exponential moments.

Lemma 15 (second exponential moment estimates of solutions)**.** *Let $f \in V^{-1}$ be constant and suppose* $\sup_l \kappa\sigma_l^2 \leq \frac{\lambda_1}{4}$. *If* $\mathbb{E}\exp(\kappa\|Z(0)\|_H^2) < \infty$, *then*

$$\mathbb{E}\exp\left(\kappa\|Z(t)\|_H^2\right) \leq \mathrm{e}^{-\kappa t}\mathbb{E}\exp\left(\kappa\|Z(0)\|_H^2\right) + C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^\infty \sigma_l^2\right)\exp\left[C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^\infty \sigma_l^2\right)\right].$$

Proof. By Itô's formula, we have

$$\begin{aligned}
&\mathbb{E}\exp\left(\kappa\|Z(t)\|_H^2\right) = \mathbb{E}\exp\left(\kappa\|Z(0)\|_H^2\right) \\
&\quad+\mathbb{E}\int_0^t \kappa\exp\left(\kappa\|Z(s)\|_H^2\right)\left(-2\|Z(s)\|_V^2 + 2\langle f, Z(s)\rangle_{V^{-1},V} + 2\kappa\sum_{l=1}^\infty \sigma_l^2\langle Z(s), v_l\rangle_H^2 + \sum_{l=1}^\infty \sigma_l^2\right)\mathrm{d}s,
\end{aligned}$$

and we do some estimates on the integrand. Poincaré's inequality gives

$$-2\|Z(s)\|_V^2 \leq -2\lambda_1\|Z(s)\|_H^2.$$

Cauchy's inequality and Young's inequality give

$$2\langle f, Z(s)\rangle_{V^{-1},V} \leq 2\|f\|_{V^{-1}}\|Z(s)\|_V \leq \|f\|_{V^{-1}}^2 + \|Z(s)\|_V^2.$$

The condition $\sup_l \kappa\sigma_l^2 \leq \frac{\lambda_1}{4}$ implies

$$2\kappa\sum_l \sigma_l^2\langle Z(s), v_l\rangle_H^2 \leq \frac{\lambda_1}{2}\|Z(s)\|_H^2.$$

It follows from combining above that

$$\begin{aligned}
&\kappa\exp\left(\kappa\|Z(s)\|_H^2\right)\left(-2\|Z(s)\|_V^2 + 2\langle f, Z(s)\rangle_{V^{-1},V} + 2\kappa\sum_{l=1}^\infty \sigma_l^2\langle Z(s), v_l\rangle_H^2 + \sum_{l=1}^\infty \sigma_l^2\right) \\
&\leq \kappa\exp\left(\kappa\|Z(s)\|_H^2\right)\left(-\frac{\lambda_1}{2}\|Z(s)\|_H^2 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^\infty \sigma_l^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\kappa \exp\left(\kappa \|Z(s)\|_H^2\right) \left(\frac{\lambda_1}{2} \|Z(s)\|_H^2 - \|f\|_{V^{-1}}^2 - \sum_{l=1}^{\infty} \sigma_l^2\right) \\
&\le \begin{cases} -\kappa \exp(\kappa \|Z(s)\|_H^2), & \text{if } \frac{\lambda_1}{2} \|Z(s)\|_H^2 - \|f\|_{V^{-1}}^2 - \sum_l \sigma_l^2 \ge 1 \\ \kappa \exp[\frac{2\kappa}{\lambda_1}(1 + \|f\|_{V^{-1}}^2 + \sum_l \sigma_l^2)](\|f\|_{V^{-1}}^2 + \sum_l \sigma_l^2), & \text{if } \frac{\lambda_1}{2} \|Z(s)\|_H^2 - \|f\|_{V^{-1}}^2 - \sum_l \sigma_l^2 < 1 \end{cases} \\
&\le -\kappa \exp\left(\kappa \|Z(s)\|_H^2\right) + \kappa \exp\left[\frac{2\kappa}{\lambda_1}\left(1 + \|f\|_{V^{-1}}^2 + \sum_l \sigma_l^2\right)\right]\left(1 + \|f\|_{V^{-1}}^2 + \sum_l \sigma_l^2\right).
\end{aligned}
$$

We invoke Gronwall's inequality to complete the proof. □

Theorem 16 (second exponential moment estimates of invariant measures)**.** *Let $f \in V^{-1}$ be constant and suppose $\sup_l \kappa\sigma_l^2 \le \frac{\lambda_1}{4}$. Then any invariant measure μ of the equation (3) satisfies*

$$
\int_H \exp(\kappa \|z\|_H^2)\mu(\mathrm{d}z) \le C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2\right) \exp\left[C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2\right)\right].
$$

Proof. At this time, we take $G_R\colon H \to \mathbb{R}$ as

$$
G_R(z) := \begin{cases} \exp(\kappa \|z\|_H^2), & \|z\|_H^2 < R, \\ \exp(\kappa R), & \|z\|_H^2 \ge R. \end{cases}
$$

Apply Lemma 15 and we see

$$
\begin{aligned}
\int_H G_R(z)\mu(\mathrm{d}z) &= \int_H \int_H P_t(z, \mathrm{d}w) G_R(w)\mu(\mathrm{d}z) \\
&\le \int_{\|z\|_H^2 < \rho} \mathbb{E} \exp\left(\kappa \|Z(t)\|_H^2\right) \mu(\mathrm{d}z) + \exp(\kappa R)\mu(H \setminus B_H(0,\rho)) \\
&\le \mathrm{e}^{-\kappa t + \kappa\rho} + C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2\right) \exp\left[C\left(1 + \|f\|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2\right)\right] + \exp(\kappa R)\mu(H \setminus B_H(0,\rho)).
\end{aligned}
$$

Let $t \to +\infty$, $\rho \to +\infty$, and apply Fatou's lemma. □

5 Uniqueness of Invariant Measures by Coupling Technique

In this section, we show uniqueness of invariant measures for P_t by coupling techniques, which utilizes a Markov semigroup $\tilde{P}_t$ on the product space $\tilde{H} = H \times H$. Specifically, we consider extensions $\tilde{P}_t$ of P_t, and this means $\tilde{P}_t(\tilde{z}, \Gamma \times H) = P_t(z_1, \Gamma)$, $\tilde{P}_t(\tilde{z}, H \times \Gamma) = P_t(z_2, \Gamma)$, for $\tilde{z} = (z_1, z_2)$. As a convention, we add "tilde" over a symbol to indicate it is in the product space.

Suppose Z_1 and Z_2 are weak solutions to the equaiton (3), we define the hitting time of $\tilde{Z} = (Z_1, Z_2)$ upon a closed subset $\tilde{B} \subset \tilde{H}$ to be

$$
\tilde{\tau}(\tilde{B}) := \inf\{t \ge 0, \tilde{Z}(t) \in \tilde{B}\}.
$$

We equip the space $L_b(H)$ of bounded Lipschitz functions on H with the following norm,

$$
\|\phi\|_L := \|\phi\|_{\mathcal{L}^\infty} + \sup_{\substack{z_1, z_2 \in H \\ z_1 \ne z_2}} \frac{|\phi(z_1) - \phi(z_2)|}{\|z_1 - z_2\|_H},
$$

and accordingly, we have the dual Lipschitz distance for the probability measures on H,

$$\|\mu_1-\mu_2\|_L^* := \sup_{\substack{\phi\in L_b(H)\\ \|\phi\|_L\le 1}} |\langle\phi,\mu_1\rangle-\langle\phi,\mu_2\rangle|\,.$$

This distance is an equivalent metric of weak convergence. Also, there is a stronger metric called total variation between two measures,

$$\|\mu_1-\mu_2\|_{\mathrm{var}} := \frac{1}{2}\sup_{\substack{\phi\in C_b(H)\\ \|\phi\|_{\mathcal{L}^\infty}\le 1}} |\langle\phi,\mu_1\rangle-\langle\phi,\mu_2\rangle| = \sup_{\Gamma \text{ measurable}} |\mu_1(\Gamma)-\mu_2(\Gamma)|\,.$$

Theorem 17. *Let $(P_t)_{t\ge 0}$ be a Markov semigroup and let $(\tilde{P}_t)_{t\ge 0}$ be an extenstion. If there exists a sequence $\{B_m\}_{m=1}^\infty$ of closed subsets in H satisfying the following conditions:*

$$\tilde{\mathbb{P}}_{\tilde z}(\tilde\tau(\tilde B_m)<\infty)=1 \text{ for all } \tilde z\in\tilde H \text{ and for all } m, \qquad \text{(recurrence)}$$

$$\sup_{\tilde z\in\tilde B_m}\sup_{t\ge 0}\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^* \to 0 \text{ as } m\to\infty, \qquad \text{(stability)}$$

then $\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^*\to 0$ as $t\to+\infty$ for all $\tilde z\in\tilde H$. Moreover, if μ is invariant, then it is unique and $\|P_t^*\nu-\mu\|_L^*\to 0$ for all $\nu\in\mathcal{P}(H)$.

Proof. We first show $\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^*\to 0$ as $t\to+\infty$ for all $\tilde z\in\tilde H$. Let $\tilde\tau(m,t)=\tilde\tau(\tilde B_m)\wedge t$ and observe

$$P_t(z,\Gamma)=\tilde P_t(\tilde z,\Gamma\times H)=\tilde{\mathbb{E}}_{\tilde z}\tilde P_{t-\tilde\tau(m,t)}(\tilde Z(\tilde\tau(m,t)),\Gamma\times H)=\tilde{\mathbb{E}}_{\tilde z}P_{t-\tilde\tau(m,t)}(Z(\tilde\tau(m,t)),\Gamma).$$

Then we have

$$\begin{aligned}\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^* &\le \tilde{\mathbb{E}}_{\tilde z}\left\|P_{t-\tilde\tau(m,t)}(Z_1(\tilde\tau(m,t)),\cdot)-P_{t-\tilde\tau(m,t)}(Z_2(\tilde\tau(m,t)),\cdot)\right\|_L^*\\ &\le \sup_{\tilde z\in\tilde B_m}\sup_{t\ge 0}\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^*+2\tilde{\mathbb{P}}_{\tilde z}(\tilde\tau(\tilde B_m)>t).\end{aligned}$$

It suffices to choose a large m and accordingly choose a large t.

The remaining part is to show, if μ is invariant, $\|P_t^*\nu-\mu\|_L^*\to 0$ for all $\nu\in\mathcal{P}(H)$. Let $\phi\in L_b$ with $\|\phi\|_L\le 1$.

$$\begin{aligned}|\langle\phi,P_t^*\nu-\mu\rangle| &= |\langle\phi,P_t^*\nu-P_t^*\mu\rangle|\\ &= |\langle P_t\phi,\nu-\mu\rangle|\\ &\le \iint_{\tilde H}|P_t\phi(z_1)-P_t\phi(z_2)|\,\nu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\\ &= \iint_{\tilde H}|\langle\phi,P_t(z_1,\cdot)-P_t(z_2,\cdot)\rangle|\,\nu(\mathrm{d}z_1)\mu(\mathrm{d}z_2),\end{aligned}$$

and Lebesgue's dominated convergence theorem applies. □

For our purpose in the Naver-Stokes context, we set $\tilde P_t$ to be an independent extension of P_t, namely

$$\tilde P_t(\tilde z,\Gamma_1\times\Gamma_2)=P_t(z_1,\Gamma_1)P_t(z_2,\Gamma_2),$$

and take B_m to be closed balls $\bar B_H(0,\frac{1}{m})$.

Lemma 18 (hitting time of V-balls)**.** *Let $f\in V^{-1}$ be constant and suppose $\kappa\le\frac{\lambda_1}{4}$. Then there exists some $R>0$ such that the hitting time $T_R:=\inf\{t\ge 0,\left\|\tilde Z(t)\right\|_{\tilde V}\le R\}$ has an exponential moment,*

$$\mathbb{E}_{\tilde z}\exp(\kappa T_R)\le C(1+\|\tilde z\|_{\tilde H}^2).$$

Proof. By Itô's formula, we have

$$\mathbb{E}_{\tilde{z}} e^{\kappa t} \left\| \tilde{Z}(t) \right\|_{\tilde{H}}^2 + \mathbb{E}_{\tilde{z}} \int_0^t 2 e^{\kappa s} \left\| \tilde{Z}(s) \right\|_{\tilde{V}}^2 \mathrm{d}s = \| \tilde{z} \|_{\tilde{H}}^2$$
$$+ \mathbb{E}_{\tilde{z}} \int_0^t \kappa e^{\kappa s} \left\| \tilde{Z}(s) \right\|_{\tilde{H}}^2 \mathrm{d}s + \mathbb{E}_{\tilde{z}} \int_0^t 2 e^{\kappa s} \left\langle \tilde{f}, \tilde{Z}(s) \right\rangle_{\tilde{V}^{-1}, \tilde{V}} \mathrm{d}s + \int_0^t 2 e^{\kappa s} \sum_{l=1}^{\infty} \sigma_l^2 \, \mathrm{d}s.$$

Apply Poincaré's and Young's inequalities,

$$\mathbb{E}_{\tilde{z}} e^{\kappa t} \left\| \tilde{Z}(t) \right\|_{\tilde{H}}^2 + \mathbb{E}_{\tilde{z}} \int_0^t e^{\kappa s} \left[(1 - \frac{\kappa}{\lambda_1}) \left\| \tilde{Z}(s) \right\|_{\tilde{V}}^2 - 2 \| f \|_{V^{-1}}^2 - 2 \sum_{l=1}^{\infty} \sigma_l^2 \right] \mathrm{d}s \le \| \tilde{z} \|_{\tilde{H}}^2 .$$

Notice $\kappa \le \frac{\lambda_1}{4}$ and choose $R > 0$ so large that

$$\| f \|_{V^{-1}}^2 + \sum_{l=1}^{\infty} \sigma_l^2 \le \frac{R^2}{8}.$$

It follows

$$\mathbb{E}_{\tilde{z}} \int_0^{T_R \wedge n} e^{\kappa s} \left(\frac{3}{4} R^2 - \frac{1}{4} R^2 \right) \mathrm{d}s \le \| \tilde{z} \|_{\tilde{H}}^2 ,$$
$$\frac{R^2}{2\kappa} \mathbb{E}_{\tilde{z}} \left(e^{\kappa (T_R \wedge n)} - 1 \right) \le \| \tilde{z} \|_{\tilde{H}}^2 ,$$
$$\mathbb{E}_{\tilde{z}} e^{\kappa (T_R \wedge n)} \le 1 + \frac{2\kappa}{R^2} \| \tilde{z} \|_{\tilde{H}}^2 .$$

Fatou's lemma gives the result. ☐

Lemma 19 (transition from V-balls to H-balls)**.** *Let $f \in V^{-1}$ be constant and suppose $\sigma_l^2 > 0$ for all l. Then for any $R > 0$, any ball B in H,*

$$\inf_{\| \tilde{z} \|_{\tilde{V}} \le R} \tilde{P}_1(\tilde{z}, \tilde{B}) > 0.$$

Proof. We may assume $B = B_H(w, \rho)$ for some $w \in H$ and $\rho > 0$, so we need to show, with a positive probability, $\| Z(1) - w \|_H \le \rho$ for Z being the solution with $Z(0) = z$ ranging over $\| z \|_V \le R$. We first establish a controllability property for our equations. For $z \in H$, $X \in E$, we denote $Y(z, X)(\cdot)$ by the solution of the following equation,

$$\begin{cases} \dot{Y} = -AY - B(X + Y) + f, \\ Y(0) = z. \end{cases}$$

For $\hat{w} \in V$, we can define a map $X_{\hat{w}} \colon H \to C(0,1; H) \cap \mathcal{L}^2(0,1; V)$ such that

$$X_{\hat{w}}(z)(1) + Y(z, X_{\hat{w}}(z))(1) = \hat{w}.$$

Let $\hat{Z}$ be the solution to the following,

$$\begin{cases} \dot{\hat{Z}} = -A\hat{Z} - B(\hat{Z}), \\ \hat{Z}(0) = z, \end{cases}$$

and let $Z(t) = (1 - t)\hat{Z}(t) + t\hat{w}$. Then we have $Z(0) = z$, $Z(1) = \hat{w}$. Define $X_{\hat{w}}(z)$ to be the solution of the following,

$$\begin{cases} \dot{X} = -AX + \dot{Z} + AZ + B(Z) - f, \\ X(0) = 0, \end{cases}$$

and we can verify $Z(t) = X_{\hat{w}}(z)(t) + Y(z, X_{\hat{w}}(z))(t)$. Moreover, by a priori estimes in Proposition 9, $X_{\hat{w}}$ is locally Lipschitz.

Next, we tansfer the comparison of the solution at time 1 with w to the one of stochastic terms X in the decompostion $Z = X + Y$. Since V is dense in H, choose $\hat{w} \in V$ with $\|w - \hat{w}\|_H \leq \frac{\rho}{2}$. We observe

$$\begin{aligned}
\|Z(1) - \hat{w}\|_H &= \|X(1) + Y(1) - X_{\hat{w}}(z)(1) - Y(z, X_{\hat{w}}(z))(1)\|_H \\
&\leq \|X(1) - X_{\hat{w}}(z)(1)\|_H + \|Y(1) - Y(z, X_{\hat{w}}(z))(1)\|_H \\
&\leq C \|X - X_{\hat{w}}(z)\|_{C(0,1;H)\cap\mathcal{L}^2(0,1;V)}, \quad \text{for } \|z\|_V \leq R.
\end{aligned}$$

It follows

$$\mathbb{P}_z\left(\|Z(1) - w\|_H \leq \rho\right) \geq \mathbb{P}_z\left(\|Z(1) - \hat{w}\|_H \leq \frac{\rho}{2}\right) \geq \mathbb{P}\left(\|X - X_{\hat{w}}(z)\|_{C(0,1;H)\cap\mathcal{L}^2(0,1;V)} \leq \frac{\rho}{2C}\right).$$

The condition $\sigma_l^2 \geq 0$ implies the law of X is supported in the whole space $C(0,1;H) \cap \mathcal{L}^2(0,1;V)$, so the RHS of the above is positive. Note $B_V(0,R)$ is compact in H, and we obtain

$$\inf_{\|z\|_V \leq R} \mathbb{P}_z\left(\|Z(1) - w\|_H \leq \rho\right) \geq \inf_{\|z\|_V \leq R} \mathbb{P}\left(\|X - X_{\hat{w}}(z)\|_{C(0,1;H)\cap\mathcal{L}^2(0,1;V)} < \frac{\rho}{2C}\right) > 0.$$

$\square$

Proposition 20 (recurrence of NS equations). *Let $f \in V^{-1}$ be constant, let $\tilde{\tau}_m = \tilde{\tau}(\tilde{B}_m)$, and suppose $\sigma_l^2 > 0$ for all l. For any m, there exists some $\kappa > 0$ such that*

$$\mathbb{E}_{\tilde{z}} \exp(\kappa\tilde{\tau}_m) \leq C(1 + \|\tilde{z}\|_{\tilde{H}}^2).$$

Proof. We define two sequences ,$\{T_n'\}$, $\{T_n\}$, of stopping times by

$$T_0' = 0, \quad T_n' = \inf\left\{t \geq T_{n-1}' + 1, \left\|\tilde{Z}(t)\right\|_{\tilde{V}} \leq R\right\}, \quad T_n = T_n' + 1.$$

By Lemma 18, we can choose $R > 0$ such that $T_n' < \infty$ a.s. Now observe

$$\mathbb{P}_{\tilde{z}}(\tilde{\tau}_m \geq M) = \mathbb{P}_{\tilde{z}}(\tilde{\tau}_m \geq M, T_n < M) + \mathbb{P}_{\tilde{z}}(\tilde{\tau}_m \geq M, T_n \geq M) \leq \mathbb{P}_{\tilde{z}}(\tilde{\tau}_m > T_n) + \mathbb{P}_{\tilde{z}}(T_n \geq M).$$

On the one hand,

$$\begin{aligned}
\mathbb{P}_{\tilde{z}}(\tilde{\tau}_m > T_n) &\leq \mathbb{P}_{\tilde{z}}\left(\bigcap_{j=0}^{n}\left\{\tilde{Z}(T_j) \notin \tilde{B}_m\right\}\right) \\
&\leq \mathbb{E}_{\tilde{z}}\left[P_1(\tilde{Z}(T_1'), \tilde{B}_m^{\complement}) P_1(\tilde{Z}(T_2'), \tilde{B}_m^{\complement}) \cdots P_1(\tilde{Z}(T_{n-1}'), \tilde{B}_m^{\complement})\right] \\
&\leq (1 - p_m^2)^{n-1},
\end{aligned}$$

where $p_m = \inf_{\|z\|_V \leq R} P_1(z, B_m) > 0$ as shown in Lemma 19. On the other hand,

$$\begin{aligned}
\mathbb{P}_{\tilde{z}}(T_n \geq M) &\leq \mathrm{e}^{-\kappa M}\mathbb{E}_{\tilde{z}}\mathrm{e}^{\kappa T_n} \qquad (\kappa > 0) \\
&= \mathrm{e}^{-\kappa(M-1)}\mathbb{E}_{\tilde{z}}\mathrm{e}^{\kappa T_n'} \\
&= \mathrm{e}^{-\kappa(M-1)}\mathbb{E}_{\tilde{z}}\left(\mathrm{e}^{\kappa T_{n-1}'}\mathbb{E}_{\tilde{Z}(T_{n-1}')}\mathrm{e}^{\kappa T_1'}\right) \\
&\leq \mathrm{e}^{-\kappa(M-1)}\mathbb{E}_{\tilde{z}}\mathrm{e}^{\kappa T_{n-1}'} \cdot \mathrm{e}^{\kappa} C(1 + R^2) \qquad \text{(Lemma 18)} \\
&\leq \cdots \leq C^{n-1}(1 + R^2)^{n-1}\mathrm{e}^{-\kappa(M-n)}\mathbb{E}_{\tilde{z}}\mathrm{e}^{\kappa T_1'}
\end{aligned}$$

$$\leq C^n(1+R^2)^{n-1}\mathrm{e}^{-\kappa(M-n-1)}(1+\|\tilde{z}\|_{\tilde{H}}^2).$$

We choose $n\sim\varepsilon M$ for $\varepsilon>0$ small enough, and then with smaller $\kappa>0$ than before,

$$\mathbb{P}_{\tilde{z}}(\tilde{\tau}_m\geq M)\leq C\mathrm{e}^{-\kappa M}(1+\|\tilde{z}\|_{\tilde{H}}^2).$$

This completes the proof. $\square$

Lemma 21 (energy growth estimate)**.** *Let $f\in V^{-1}$ be constant, let $\kappa=\frac{\lambda_1}{4\sup_l\sigma_l^2}$, and suppose $\mathbb{E}\|Z(0)\|_H^2<\infty$. Then for any $\rho>0$,*

$$\mathbb{P}\left[\sup_{t\geq 0}\left(\|Z(t)\|_H^2+\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s-t\sum_{l=1}^{\infty}\sigma_l^2-2t\|f\|_{V^{-1}}\right)\geq\|Z(0)\|_H^2+\rho\right]\leq\mathrm{e}^{-\kappa\rho}$$

Proof. It follows from Itô's formula that

$$\|Z(t)\|_H^2+\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s=\|Z(0)\|_H^2-\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s+2\int_0^t\langle f,Z(s)\rangle_{V^{-1},V}\,\mathrm{d}s+t\sum_{l=1}^{\infty}\sigma_l^2+M_t,$$

where M_t is a martingale given by

$$M_t=2\sum_{l=1}^{\infty}\sigma_l\int_0^t\langle Z(s),v_l\rangle_H\,\mathrm{d}\beta_l(s).$$

Then we give a bound to its quadratic variation,

$$\begin{aligned}\langle M\rangle_t&=4\sum_{l=1}^{\infty}\sigma_l^2\int_0^t\langle Z(s),v_l\rangle_H^2\,\mathrm{d}s\\&\leq\frac{4\sup_l\sigma_l^2}{\lambda_1}\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s\qquad\text{(Poincaré's inequality)}\\&=\frac{1}{\kappa}\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s.\end{aligned}$$

This implies

$$\begin{aligned}\|Z(t)\|_H^2+\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s-t\sum_{l=1}^{\infty}\sigma_l^2&=\|Z(0)\|_H^2+M_t-\frac{1}{2}\kappa\langle M\rangle_t+\left(\frac{1}{2}\kappa\langle M\rangle_t-\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s+2\int_0^t\langle f,Z(s)\rangle_{V^{-1},V}\,\mathrm{d}s\right),\\&\leq\|Z(0)\|_H^2+M_t-\frac{1}{2}\kappa\langle M\rangle_t+2t\|f\|_{V^{-1}}\qquad\text{(Young's inequality)},\end{aligned}$$

$$\|Z(t)\|_H^2+\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s-t\sum_{l=1}^{\infty}\sigma_l^2-2t\|f\|_{V^{-1}}\leq\|Z(0)\|_H^2+M_t-\frac{1}{2}\kappa\langle M\rangle_t.$$

Notice that $\exp(\kappa M_t-\frac{1}{2}\kappa^2\langle M\rangle_t)$ is a supermartingale, so Doob's inequality gives

$$\begin{aligned}&\mathbb{P}\left[\sup_{t\geq 0}\left(\|Z(t)\|_H^2+\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s-t\sum_{l=1}^{\infty}\sigma_l^2-2t\|f\|_{V^{-1}}\right)\geq\|Z(0)\|_H^2+\rho\right]\\&\qquad\leq\mathbb{P}\left[\sup_{t\geq 0}\exp\left(\kappa M_t-\frac{1}{2}\kappa^2\langle M\rangle_t\right)\geq\mathrm{e}^{\kappa\rho}\right]\leq\mathrm{e}^{-\kappa\rho}.\end{aligned}$$

$\square$

Lemma 22 (Foiaş-Prodi estimate)**.** *Let $f\in\mathcal{L}^2_{\mathrm{loc}}(0,\infty;V^{-1})$, let $P_k\colon H\to H_k=\mathrm{span}\{u_1,\ldots,u_k\}$ be the projection, and suppose $Z\in C(0,\infty;H)\cap\mathcal{L}^2_{\mathrm{loc}}(0,\infty;V)$ is the solution of a deterministic equaiton, such that*

$$\int_0^t\|Z(s)\|_V^2\,\mathrm{d}s\leq\rho+Mt\quad\textit{for all } t\geq 0.$$

Then there exist k, λ such that the solution $W \in C(0,\infty;H)\cap \mathcal{L}^2_{\text{loc}}(0,\infty;V)$ of the following equation

$$\begin{cases} \dot{W} = -AW - B(W) - \lambda P_k(W-Z) + f \\ W(0) = w \in H \end{cases}$$

satisfies

$$\|W(t)-Z(t)\|_H \le \mathrm{e}^{-t+C\rho}\|w-z\|_H \quad \textit{for all } t\ge 0.$$

Proof. We compute

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W-Z\|_H^2 + \|W-Z\|_V^2 + \lambda\|P_k(W-Z)\|_H^2 &= b(Z,Z,W-Z) - b(W,W,W-Z) \\ &= -b(W-Z,Z,W-Z) \\ &\le \|W-Z\|_{\mathcal{L}^4}\|Z\|_V\|W-Z\|_{\mathcal{L}^4} \\ &\le \|W-Z\|_H\|W-Z\|_V\|Z\|_V \qquad \text{(interpolation)} \\ &\le \frac{1}{2}\|W-Z\|_V^2 + C\|W-Z\|_H^2\|Z\|_V^2 \qquad \text{(Young's inequality)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\|W-Z\|_H^2 + \|W-Z\|_V^2 + 2\lambda\|P_k(W-Z)\|_H^2 &\le C\|W-Z\|_H^2\|Z\|_V^2. \end{aligned}$$

Note that $\|z\|_V^2 \ge \lambda_{k+1}^2\|(I-P_k)z\|_H^2$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W-Z\|_H^2 + \left(\min\{2\lambda,\lambda_{k+1}^2\} - C\|Z\|_V^2\right)\|W-Z\|_H^2 \le 0.$$

Choose λ and k sufficiently large so that $\min\{2\lambda,\lambda_{k+1}^2\} - CM \ge 2$, and it follows

$$\|W(t)-Z(t)\|_H \le \|w-z\|_H\exp(-t+C\rho).$$

□

Theorem 23 (stability of NS equations)**.** *Let $f\in V^{-1}$ be constant, and suppose $\sigma_l^2>0$ for all l. As $m\to\infty$,*

$$\sup_{\tilde{z}\in\tilde{B}_m}\sup_{t\ge 0}\|P_t(z_1,\cdot)-P_t(z_2,\cdot)\|_L^* \to 0.$$

Proof. Let Z_1, Z_2 be the solutions starting from $z_1\in B_m, z_2=0$ respectively, and we may assume the Stokes operator A and the covariance operator Q have identical eigenvectors. By Lemma 21, for any $\varepsilon>0$, there are some $M>0$ and some $\rho_\varepsilon>0$ such that

$$\mathbb{P}\left(\|Z_2(t)\|_H^2 + \int_0^t\|Z_2(s)\|_V^2 \le \rho_\varepsilon + Mt \quad \text{for all } t\ge 0\right)\ge 1-\varepsilon.$$

We denote this event by Γ_ε, and let Z_3 be the solution of the following equation.

$$\begin{cases} \dot{Z}_3 = -AZ_3 - B(Z_3) - \lambda P_k(Z_3-Z_2) + f + \dot{W}_Q, \\ Z_3(0) = z_1. \end{cases}$$

By Lemma 22, there exist some $\lambda>0$ and some $k\in\mathbb{Z}_+$ such that

$$\|Z_3(t)-Z_2(t)\|_H \le \mathrm{e}^{-t+C\rho_\varepsilon}\|z_1\|_H, \qquad \sup_{t\ge 0}\|Z_3(t)-Z_2(t)\|_H \le \mathrm{e}^{C\rho_\varepsilon}\|z_1\|_H \quad \text{on } \Gamma_\varepsilon.$$

If $\phi\colon C(0,\infty;H)\to\mathbb{R}$ is bounded Lipschitz with $\|\phi\|_L\le 1$, then

$$|\mathbb{E}(\phi(Z_3)-\phi(Z_2))|\le 2\mathbb{P}(\Gamma_\varepsilon^\complement)+\mathrm{e}^{C\rho_\varepsilon}\|z_1\|_H\le 2\varepsilon+\mathrm{e}^{C\rho_\varepsilon}\frac{1}{m}.$$

Choose $\varepsilon=m^{-\lambda_1/2C\sup_l\sigma_l^2}$ and the right hand side converges to zero. This means

$$\sup_{z_1\in B_m}\|\mathrm{Law}(Z_3)-\mathrm{Law}(Z_2)\|_L^*\to 0\quad\text{as } m\to\infty,$$

and it remains to show $\sup_{z_1\in B_m}\|\mathrm{Law}(Z_3)-\mathrm{Law}(Z_1)\|_L^*\to 0$.

We may assume $\Omega=C(0,\infty;H)$, and define a Girsanov transform Φ by

$$\Phi(\omega):=\omega(t)-\int_0^t a(s)\,\mathrm{d}s,\quad a(t):=\lambda P_k(Z_3(t)-Z_2(t)).$$

However, this process a may not satisfy Novikov's condition, so we modify it to be

$$\hat{a}(t):=a(t)\mathbb{I}\{\|Z_3(s)-Z_2(s)\|_H\le \mathrm{e}^{-s+C\rho_\varepsilon}\|z_1\|_H\ \text{ for all } s\in[0,t]\}.$$

Then $\hat{a}=a$ on the event Γ_ε, and $\hat{\Phi}\colon\omega(t)\mapsto\omega(t)-\int_0^t\hat{a}(s)\,\mathrm{d}s$ is admissible to Girsanov's theorem. Define an exponential martingale as below,

$$M_t:=\exp\left(\sum_{j=1}^k\int_0^t\sigma_j^{-1}\langle\hat{a}(s),u_j\rangle_H\,\mathrm{d}\beta_j(s)-\frac{1}{2}\sum_{j=1}^k\int_0^t\sigma_j^{-2}\langle\hat{a}(s),u_j\rangle_H^2\,\mathrm{d}s\right),$$

and it converges to some M_∞ in the $\mathcal{L}^1$-space. Girsanov's theorem then gives

$$\begin{aligned}\left\|\mathbb{P}-\hat{\Phi}_*(\mathbb{P})\right\|_{\mathrm{var}}&=\|\mathbb{P}-M_\infty\mathbb{P}\|_{\mathrm{var}}\\&\le\frac{1}{2}\mathbb{E}\,|M_\infty-1|\\&\le\frac{1}{2}(\mathbb{E}M_\infty^2-1)^{1/2}\\&\le\frac{1}{2}[\exp(C\mathrm{e}^{C\rho_\varepsilon}\|z_1\|_H^2)-1]^{1/2}.\end{aligned}$$

Now observe

$$\begin{aligned}\|\mathrm{Law}(Z_3)-\mathrm{Law}(Z_2)\|_{\mathrm{var}}&=\|(Z_3)_*(\mathbb{P})-(Z_2)_*(\mathbb{P})\|_{\mathrm{var}}\\&=\|(Z_2\circ\Phi)_*(\mathbb{P})-(Z_2)_*(\mathbb{P})\|_{\mathrm{var}}\\&\le\|\Phi_*(\mathbb{P})-\mathbb{P}\|_{\mathrm{var}}\\&\le\left\|\Phi_*(\mathbb{P})-\hat{\Phi}_*(\mathbb{P})\right\|_{\mathrm{var}}+\left\|\hat{\Phi}_*(\mathbb{P})-\mathbb{P}\right\|_{\mathrm{var}}\\&\le 2\mathbb{P}(\Gamma_\varepsilon)+\frac{1}{2}[\exp(C\mathrm{e}^{C\rho_\varepsilon}\frac{1}{m^2})-1]^{1/2}\\&\lesssim 2\varepsilon+C\mathrm{e}^{C\rho_\varepsilon}\frac{1}{m}.\qquad(\text{choose }\varepsilon=m^{-\lambda_1/2C\sup_l\sigma_l^2})\end{aligned}$$

The last term converges to zero as $m\to\infty$, and this completes the proof. □

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A If $z \in H^{\perp}$, How to Find p Such That $z = \operatorname{grad} p$?

In this section, we will define a distribution p acting on $q \in C_0^\infty(D;\mathbb{R})$ and check $z = \operatorname{grad} p$ in distribution. It follows then $p \in H^1(D;\mathbb{R})$ from $z \in \mathcal{L}^2(D;\mathbb{R}^2)$.

At the moment, we only have z in our hands, so we can define p by

$$p(q) := \langle z, \mathcal{F}(q)\rangle_{\mathcal{L}^2}, \quad q \in C_0^\infty(D;\mathbb{R}),$$

where $\mathcal{F}\colon C_0^\infty(D;\mathbb{R}) \to C_0^\infty(D;\mathbb{R}^2)$ is a map to be determined later. Assume $z = \operatorname{grad} p$ in distribution and, for any $w \in C_0^\infty(D;\mathbb{R}^2)$, we have

$$\langle z, w\rangle_{\mathcal{L}^2} = (\operatorname{grad} p)(w) = -p(\operatorname{div} w) = -\langle z, \mathcal{F}(\operatorname{div} w)\rangle_{\mathcal{L}^2}.$$

This means $\mathcal{F}(\operatorname{div} w) = -w$ and is a partial definition of $\mathcal{F}$. Namely, for $q \in C_0^\infty(D;\mathbb{R})$, if $q = \operatorname{div} w$ for some $w \in C_0^\infty(D;\mathbb{R}^2)$ then $\mathcal{F}(q) = -w$. But what about other functions that are NOT a divergence of some vector field? It is natural to decompose $q \in C_0^\infty(D;\mathbb{R})$ as the following

$$q = \operatorname{div} w + f,$$

and define $\mathcal{F}(q) = -w$. To obtain a characterization of f, we further require the two components, $\operatorname{div} w$ and f, are orthogonal. This results in $\nabla f = 0$ and f is a constant. Notice

$$\int_D q \,\mathrm{d}\xi = \int_D \operatorname{div} w \,\mathrm{d}\xi + \operatorname{Area}(D) f = \operatorname{Area}(D) f,$$

so we have $f = \int_D q\,\mathrm{d}\xi / \operatorname{Area}(D)$. However, this $f \notin C_0^\infty(D;\mathbb{R})$ in general and a modification is needed here.

Let $r \in C_0^\infty(D;\mathbb{R})$ with $\int_D r\,\mathrm{d}\xi = 1$ replacing the factor $\frac{1}{\operatorname{Area}(D)}$. That is, we modify f to be $f = r\int_D q\,\mathrm{d}\xi$ and hope every $q \in C_0^\infty(D;\mathbb{R})$ admits the following decomposition.

$$q = \operatorname{div} w + r\int_D q\,\mathrm{d}\xi.$$

Although $f = r\int_D q\,\mathrm{d}\xi$ is NOT orthogonal to $\operatorname{div} w$, at least, the integrals of both sides are equal.

Lemma 24. *For any $r \in C_0^\infty(D;\mathbb{R})$ with $\int_D r\,\mathrm{d}\xi = 1$, and for any $q \in C_0^\infty(D;\mathbb{R})$, there exists $w \in C_0^\infty(D;\mathbb{R}^2)$ such that*

$$q = \operatorname{div} w + r\int_D q\,\mathrm{d}\xi.$$

Proof. We first consider a baby case where $D = (a_1, b_1) \times (a_2, b_2)$ is a rectangle. Let $g = q - r\int_D q\,\mathrm{d}\xi$ and we want to find some $w \in C_0^\infty(D;\mathbb{R}^2)$ such that $\operatorname{div} w = g$. Explicit formulas are given by

$$\begin{aligned} w_1(\xi_1,\xi_2) &= \int_{a_1}^{\xi_1} \left[g(t,\xi_2) - \int_{a_1}^{b_1} g(s_1,\xi_2)\,\mathrm{d}s_1 \int_{a_2}^{b_2} r(t,s_2)\,\mathrm{d}s_2 \right] \mathrm{d}t, \\ w_2(\xi_1,\xi_2) &= \int_{a_2}^{\xi_2} \int_{a_1}^{b_1} g(s_1,t)\,\mathrm{d}s_1 \int_{a_2}^{b_2} r(\xi_1,s_2)\,\mathrm{d}s_2\,\mathrm{d}t. \end{aligned}$$

Actually, we can compute

$$\begin{aligned} \frac{\partial w_1}{\partial \xi_1} &= g(\xi_1,\xi_2) - \int_{a_1}^{b_1} g(s_1,\xi_2)\,\mathrm{d}s_1 \int_{a_2}^{b_2} r(\xi_1,s_2)\,\mathrm{d}s_2, \\ \frac{\partial w_2}{\partial \xi_2} &= \int_{a_1}^{b_1} g(s_1,\xi_2)\,\mathrm{d}s_1 \int_{a_2}^{b_2} r(\xi_1,s_2)\,\mathrm{d}s_2. \end{aligned}$$

The next step is to glue two overlapping domains. If the decomposition holds in two domains, say D_1 and D_2, and if $D_1 \cap D_2 \neq \emptyset$, then we show the decomposition holds in $D_1 \cup D_2$. Notice $\operatorname{supp}(q) \cup \operatorname{supp}(r) \subset D_1 \cup D_2$ and we let $\{\phi_k \in C_0^\infty(D_k;\mathbb{R})\}_{k=1}^2$ be a partition of unity on $\operatorname{supp}(q) \cup \operatorname{supp}(r)$ subordinate to $\{D_k\}_{k=1}^2$. Also, choose a $\psi \in C_0^\infty(D_1 \cap D_2;\mathbb{R})$ with $\int_D \psi \,\mathrm{d}\xi = 1$. Then there are $w_1, w_2, w_3, w_4 \in C_0^\infty(D;\mathbb{R}^2)$ such that

$$
\begin{aligned}
q\phi_1 &= \operatorname{div} w_1 + \psi \int_D q\phi_1 \,\mathrm{d}\xi, & q\phi_2 &= \operatorname{div} w_2 + \psi \int_D q\phi_2 \,\mathrm{d}\xi, \\
r\phi_1 &= \operatorname{div} w_3 + \psi \int_D r\phi_1 \,\mathrm{d}\xi, & r\phi_2 &= \operatorname{div} w_4 + \psi \int_D r\phi_2 \,\mathrm{d}\xi.
\end{aligned}
$$

Adding each row up gives $q = \operatorname{div}(w_1 + w_2) + \psi \int_D q \,\mathrm{d}\xi$, $r = \operatorname{div}(w_3 + w_4) + \psi$, and hence

$$
q = \operatorname{div}\left[w_1 + w_2 - (w_3 + w_4)\int_D q \,\mathrm{d}\xi\right] + r \int_D q \,\mathrm{d}\xi.
$$

The general case follows from the fact that $\operatorname{supp}(q) \cup \operatorname{supp}(r)$ admits a finite open cover $\{R_k\}_{k=1}^n$ of rectangles in D. $\square$

The function $w \in C_0^\infty(D;\mathbb{R}^2)$ in the decomposition of $q \in C_0^\infty(D;\mathbb{R})$ is NOT unique, so $\mathcal{F}(q) = -w$ is NOT well-defined. But we can still define p as a distribution by first fixing some $r \in C_0^\infty(D;\mathbb{R})$ with $\int_D r \,\mathrm{d}\xi = 1$ and by

$$
p(q) := -\langle z, w\rangle_{\mathcal{L}^2}, \qquad q \in C_0^\infty(D;\mathbb{R}), q = \operatorname{div} w + r \int_D q \,\mathrm{d}\xi, w \in C_0^\infty(D;\mathbb{R}^2).
$$

We can check p is well-defined. Suppose $q = \operatorname{div} w_1 + r \int_D q \,\mathrm{d}\xi = \operatorname{div} w_2 + r \int_D q \,\mathrm{d}\xi$ with $w_1, w_2 \in C_0^\infty(D;\mathbb{R}^2)$ and this means $w_1 - w_2 \in C_{0,\mathrm{div}=0}^\infty$. It follows $\langle z, w_1 - w_2\rangle_{\mathcal{L}^2} = 0$ from the condition $z \in H^\perp$, so p is well-defined. Now take $w \in C_0^\infty(D;\mathbb{R}^2)$ and we see

$$
(\operatorname{grad} p)(w) = -p(\operatorname{div} w) = \langle z, w\rangle_{\mathcal{L}^2}.
$$

Thus, $z = \operatorname{grad} p$ in distribution.

B If $z \in \mathcal{L}^2(D;\mathbb{R}^2)$ and $\operatorname{div} z \in \mathcal{L}^2(D;\mathbb{R})$, How to Define $z \cdot \boldsymbol{n}|_{\partial D}$?

Denote $H^1_{\text{div}} := \{z \in \mathcal{L}^2(D;\mathbb{R}^2)\,|\operatorname{div} z \in \mathcal{L}^2(D;\mathbb{R})\,\}$, and H^1_{div} is a Hilbert space with an inner product defined by

$$\langle z, w\rangle_{H^1_{\text{div}}} := \langle z, w\rangle_{\mathcal{L}^2} + \langle \operatorname{div} z, \operatorname{div} w\rangle_{\mathcal{L}^2}.$$

In this section, we will give a meaning to $z \cdot \boldsymbol{n}|_{\partial D}$ for any $z \in H^1_{\text{div}}$.

Before the rigorous statement, we consider a special case. For $z \in C^\infty(\bar{D};\mathbb{R}^2)$, the term $z \cdot \boldsymbol{n}|_{\partial D}$ is well-defined and it gives the normal component of z on the boundary. Furthermore, Stokes' formula applies to $z \in C^\infty(\bar{D};\mathbb{R}^2)$, namely

$$\int_{\partial D} (z \cdot \boldsymbol{n}) p \,\mathrm{d}\sigma(\xi) = \int_D \operatorname{div}(pz)\,\mathrm{d}\xi = \int_D z(\operatorname{grad} p)\,\mathrm{d}\xi + \int_D (\operatorname{div} z) p\,\mathrm{d}\xi \quad \text{for all } p \in C^\infty(\bar{D};\mathbb{R}).$$

So a new definition of $z \cdot \boldsymbol{n}|_{\partial D}$ should be consistent with the smooth case in the above sense. Let's recall a similar situation arises when we try to assign a meaning to $p|_{\partial D}$ for $p \in H^1(D;\mathbb{R})$, and a solution is given by a trace operator $T_{\text{bdd}} : H^1(D;\mathbb{R}) \to H^{1/2}(\partial D;\mathbb{R})$ which coincides with the smooth case and which preserves Stokes' formula.

$$T_{\text{bdd}}(p) = p|_{\partial D} \quad \text{for all } p \in C^\infty(\bar{D};\mathbb{R}),$$
$$\langle T_{\text{bdd}}(p), z \cdot \boldsymbol{n}|_{\partial D}\rangle_{\mathcal{L}^2} = \langle p, \operatorname{div} z\rangle_{\mathcal{L}^2} + \langle \operatorname{grad} p, z\rangle_{\mathcal{L}^2} \quad \text{for all } z \in C^\infty(\bar{D};\mathbb{R}^2).$$

Here, the subscript "bdd" is an abbreviation of "boundary" to indicate the operator T_{bdd} extracts boundary information. Also, T_{bdd} has a continuous right inverse $l_{\text{bdd}} : H^{1/2}(\partial D;\mathbb{R}) \to H^1(D;\mathbb{R})$, where the alphabet "$l$" suggests the map is a lifting from boundary to the whole domain. For more on trace operators and more general trace theorems, see Section 1.8 of [3]. Likewise, we can define an operator T_{nor} from H^1_{div} to some space and this operator extracts the normal component of each $z \in H^1_{\text{div}}$ on the boundary. The subscript "nor" surely stands for "normal component".

Theorem 25. *There exists a linear continuous map $T_{\text{nor}} : H^1_{\text{div}} \to H^{-1/2}(\partial D;\mathbb{R})$ satisfying*

$$T_{\text{nor}}(z) = z \cdot \boldsymbol{n}|_{\partial D} \quad \textit{for all } z \in C^\infty(\bar{D};\mathbb{R}^2),$$
$$\langle T_{\text{nor}}(z), T_{\text{bdd}}(p)\rangle = \langle z, \operatorname{grad} p\rangle_{\mathcal{L}^2} + \langle \operatorname{div} z, p\rangle_{\mathcal{L}^2} \quad \textit{for all } p \in H^1(D;\mathbb{R}).$$

Proof. Take $z \in H^1_{\text{div}}$, directly define $T_{\text{nor}}(z) \in H^{-1/2}(\partial D;\mathbb{R})$ by

$$T_{\text{nor}}(z) \colon H^{1/2}(\partial D;\mathbb{R}) \ni \phi \mapsto \langle z, \operatorname{grad} p\rangle_{\mathcal{L}^2} + \langle \operatorname{div} z, p\rangle_{\mathcal{L}^2} \quad \text{for some } p \in H^1(D;\mathbb{R}) \text{ with } p|_{\partial D} = \phi.$$

We need to verify this definition is independent of p. Let $p_1, p_2 \in H^1(D;\mathbb{R})$ and $p_1|_{\partial D} = p_2|_{\partial D} = \phi$. Then we have $p_1 - p_2 \in H^1_0(D;\mathbb{R})$ and

$$\langle z, \operatorname{grad}(p_1 - p_2)\rangle_{\mathcal{L}^2} + \langle \operatorname{div} z, p_1 - p_2\rangle_{\mathcal{L}^2} = 0,$$

so $T_{\text{nor}}(z)$ is well-defined. Linearity is obvious and continuity follows from Cauchy's inequality and from continuity of the lifting $l_{\text{bdd}} : H^{1/2}(\partial D;\mathbb{R}) \to H^1(D;\mathbb{R})$ of T_{bdd}.

$$|T_{\text{nor}}(z)(\phi)| \le \|z\|_{H^1_{\text{div}}} \|p\|_{H^1} \le C\,\|z\|_{H^1_{\text{div}}} \|\phi\|_{H^{1/2}} \quad \Longrightarrow \quad \|T_{\text{nor}}(z)\|_{H^{-1/2}} \le C\,\|z\|_{H^1_{\text{div}}}.$$

To show $T_{\text{nor}}(z) = z \cdot \boldsymbol{n}|_{\partial D}$ for all $z \in C^\infty(\bar{D};\mathbb{R}^2)$, we just observe, for $p \in C^\infty(\bar{D};\mathbb{R})$,

$$T_{\text{nor}}(z)(p|_{\partial D}) = \int_D \operatorname{div}(pz)\,\mathrm{d}\xi = \int_{\partial D} (z \cdot \boldsymbol{n}) p\,\mathrm{d}\sigma(\xi),$$

and note the image of $C^\infty(\bar{D};\mathbb{R})$ under T_{bdd} is dense in $H^{1/2}(\partial D;\mathbb{R})$. $\square$

Even we may ask whether T_{nor} is surjective, and whether there is a continuous right inverse, say l_{nor}, of T_{nor}. Take an arbitrary $\phi \in H^{-1/2}(\partial D;\mathbb{R})$, and we want to find some $z \in H^1_{\text{div}}$ such that $z \cdot \boldsymbol{n}|_{\partial D} = \phi$. Roughly speaking, we can solve the following Poisson equation,

$$\begin{cases} \Delta p = 0 \\ \frac{\partial p}{\partial \boldsymbol{n}} = \phi \end{cases}$$

and letting "$z = \operatorname{grad} p$" serves our purpose. However, the above Poisson equation has a solution if and only if $\langle \phi, 1\rangle = 0$, which means a general ϕ does NOT guarantee a solution p. To tackle this issue, we modify the boundary value condition to be

$$\frac{\partial p}{\partial \boldsymbol{n}} = \phi - \frac{\langle \phi, 1\rangle}{\text{Length}(\partial D)},$$

and solve for $p \in H^1(D;\mathbb{R})$. Choose some $w \in C^\infty(\bar{D};\mathbb{R}^2)$ such that $w \cdot \boldsymbol{n}|_{\partial D} = 1$ and let

$$z = \operatorname{grad} p + \frac{\langle \phi, 1\rangle}{\text{Length}(\partial D)} w.$$

We can check $z \in H^1_{\text{div}}$ and $z \cdot \boldsymbol{n}|_{\partial D} = \phi$. Moreover, z continuously depends on ϕ and a right inverse l_{nor} exists.

C Regularity of Steady-State Stokes Equations

This part discusses the regularity problem for steady-state Stokes equations. We first establish local regularity and then up to the boundary.

Now, discard the boundary condtion temporarily and consider the following equation with a general divergence condition.

$$\begin{cases} -\Delta z + \operatorname{grad} p = f \quad \text{in } D, \\ \operatorname{div} z = g \quad \text{in } D. \end{cases} \tag{13}$$

We can derive local regularity for z and p from more regular f, g.

Theorem 26 (local regularity)**.** *Let $f \in \mathcal{L}^2(D;\mathbb{R}^2)$ and $g \in H^1(D;\mathbb{R})$. If $(z,p) \in H^1(D;\mathbb{R}^2) \times \mathcal{L}^2(D;\mathbb{R})$ is a solution of (13), then $(z,p) \in H^2_{\text{loc}}(D;\mathbb{R}^2) \times H^1_{\text{loc}}(D;\mathbb{R})$. Moreover, for any open set D' with $\overline{D'} \subset D$, there exists a constant $C(D')$ such that*

$$\|z\|_{H^2(D')} + \|p\|_{H^1(D')} \leq C \left(\|f\|_{\mathcal{L}^2(D)} + \|g\|_{H^1(D)} + \|z\|_{H^1(D)} + \|p\|_{\mathcal{L}^2(D)} \right).$$

Proof. Notice that $(z,p) \in H^2_{\text{loc}} \times H^1_{\text{loc}}$ if and only if $(rz, rp) \in H^2 \times H^1$ for any $r \in C_0^\infty(D;R)$. Let $\tilde{z} = rz$ and $\tilde{p} = rp$. Then we compute

$$\begin{cases} -\Delta \tilde{z} + \operatorname{grad} \tilde{p} = rf - (\Delta r) z - 2(\operatorname{grad} r) \cdot \operatorname{grad} z + p(\operatorname{grad} r) =: \tilde{f} \quad \text{in } D, \\ \operatorname{div} \tilde{z} = rg + (\operatorname{grad} r) \cdot z =: \tilde{g} \quad \text{in } D, \end{cases}$$

and obtain $\tilde{f} \in \mathcal{L}^2(D)$, $\tilde{g} \in H^1(D)$. More precisely, we have

$$\begin{aligned} \left\|\tilde{f}\right\|_{\mathcal{L}^2(D)} &\leq C(r) \left(\|f\|_{\mathcal{L}^2(D)} + \|z\|_{H^1(D)} + \|p\|_{\mathcal{L}^2(D)} \right), \\ \|\tilde{g}\|_{H^1(D)} &\leq C(r) \left(\|g\|_{H^1(D)} + \|z\|_{H^1(D)} \right). \end{aligned}$$

For $\eta \in \mathbb{R}^2$, define a difference operator δ_η by

$$(\delta_\eta z)(\xi) := z(\xi + \eta) - z(\xi),$$

and it acts on both sides of the equation of $(\tilde{z}, \tilde{p})$ for sufficiently small η.

$$\begin{cases} -\Delta \delta_\eta \tilde{z} + \operatorname{grad} \delta_\eta \tilde{p} = \delta_\eta \tilde{f} \quad \text{in } D, \\ \operatorname{div} \delta_\eta \tilde{z} = \delta_\eta \tilde{g} \quad \text{in } D, \end{cases}$$

To show $\tilde{z} \in H^2$, it suffices to show $\frac{1}{|\eta|} \operatorname{grad} \delta_\eta \tilde{z}$ is bounded in $\mathcal{L}^2$. Actually, it follows from the above equation that

$$\begin{aligned} \|\operatorname{grad} \delta_\eta \tilde{z}\|^2_{\mathcal{L}^2(D)} &= \langle \delta_\eta \tilde{p}, \delta_\eta \tilde{g} \rangle_{\mathcal{L}^2(D)} + \left\langle \delta_\eta \tilde{f}, \delta_\eta \tilde{z} \right\rangle_{H^{-1}(D), H^1(D)} \\ &\leq \|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)} \|\delta_\eta \tilde{g}\|_{\mathcal{L}^2(D)} + \left\|\delta_\eta \tilde{f}\right\|_{H^{-1}(D)} \|\delta_\eta \tilde{z}\|_{H^1(D)} \\ &= \|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)} \|\delta_\eta \tilde{g}\|_{\mathcal{L}^2(D)} + \left\|\delta_\eta \tilde{f}\right\|_{H^{-1}(D)} \left(\|\delta_\eta \tilde{z}\|_{\mathcal{L}^2(D)} + \|\operatorname{grad} \delta_\eta \tilde{z}\|_{\mathcal{L}^2(D)} \right). \end{aligned}$$

Since $\|\delta_\eta z\|_{H^{k-1}} \leq |\eta| \, \|z\|_{H^k}$ for all $z \in H^k$ and for all $k \geq 0$ (see Appendix C.1), this gives

$$\|\operatorname{grad} \delta_\eta \tilde{z}\|^2_{\mathcal{L}^2(D)} \leq \|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)} |\eta| \, \|\tilde{g}\|_{H^1(D)} + |\eta|^2 \left\|\tilde{f}\right\|_{\mathcal{L}^2(D)} \|\tilde{z}\|_{H^1(D)} + |\eta| \left\|\tilde{f}\right\|_{\mathcal{L}^2(D)} \|\operatorname{grad} \delta_\eta \tilde{z}\|_{\mathcal{L}^2(D)}.$$

It remains to give an upper bound for $\|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)}$. By Nečas inequality (see Appendix C.2),

$$\begin{aligned}
\|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)} &\leq C \left(\|\delta_\eta \tilde{p}\|_{H^{-1}(D)} + \|\operatorname{grad} \delta_\eta \tilde{p}\|_{H^{-1}(D)} \right) \\
&\leq C \left(\|\delta_\eta \tilde{p}\|_{H^{-1}(D)} + \left\| \delta_\eta \tilde{f} \right\|_{H^{-1}(D)} + \|\Delta \delta_\eta \tilde{z}\|_{H^{-1}(D)} \right) \\
&\leq C \left(|\eta| \, \|\tilde{p}\|_{\mathcal{L}^2(D)} + |\eta| \left\| \tilde{f} \right\|_{\mathcal{L}^2(D)} + \|\operatorname{grad} \delta_\eta \tilde{z}\|_{\mathcal{L}^2(D)} \right).
\end{aligned}$$

Apply Young's inequality and we derive

$$\begin{aligned}
\|\operatorname{grad} \delta_\eta \tilde{z}\|_{\mathcal{L}^2(D)} &\leq C |\eta| \left(\left\| \tilde{f} \right\|_{\mathcal{L}^2(D)} + \|\tilde{g}\|_{H^1(D)} + \|\tilde{z}\|_{H^1(D)} + \|\tilde{p}\|_{\mathcal{L}^2(D)} \right), \\
\|\delta_\eta \tilde{p}\|_{\mathcal{L}^2(D)} &\leq C |\eta| \left(\left\| \tilde{f} \right\|_{\mathcal{L}^2(D)} + \|\tilde{g}\|_{H^1(D)} + \|\tilde{z}\|_{H^1(D)} + \|\tilde{p}\|_{\mathcal{L}^2(D)} \right).
\end{aligned}$$

This completes the proof if we take $r \equiv 1$ on D'. □

C.1 Approximate weak derivatives by difference quotients

In calculus, if difference quotients of a function have a limit, then this limit is the derivative of this function. In Sobolev spaces, likewise, existence of the weak derivative of a function can be deduced from bounded difference quotients. This provides an approach to regularity promotion by estimating difference quotients.

Proposition 27. *Suppose $z \in H^{k-1}(\mathbb{R}^2)$, $k \geq 0$. Then*

$$\sup_{0<|\eta|<1} \frac{1}{|\eta|} \|\delta_\eta z\|_{H^{k-1}} < \infty \quad \iff \quad z \in H^k.$$

Proof. If $\frac{1}{|\eta|}\delta_\eta z$ is bounded in H^{k-1} for small η, then there exists a sequence $\{h_n\} \subset (0,1)$ decreasing to 0 such that

$$\frac{1}{h_n}\delta_{h_n e_1} z \to z_1, \quad \frac{1}{h_n}\delta_{h_n e_2} z \to z_2, \quad \text{both weakly in } H^{k-1}.$$

Indeed, z_1 and z_2 are weak derivatives of z, since $\frac{1}{h_n}\delta_{h_n e_i} z \to \partial_{\xi_i} z$, $i = 1, 2$, in distribution.

If $z \in H^k$, then there is a sequence $\{z_\varepsilon\} \subset C_0^\infty$ such that $z_\varepsilon \to z$ in H^k as $\varepsilon \to 0$. By passing to limit, we may assume z itself is in C_0^∞.

$$\begin{aligned}
\delta_\eta z(\xi) &= z(\xi + \eta) - z(\xi) \\
&= \int_0^1 \eta \cdot \operatorname{grad} z(\xi + h\eta) \, \mathrm{d}h \\
\|\delta_\eta z\|_{H^{k-1}} &\leq \int_0^1 |\eta| \, \|\operatorname{grad} z(\cdot + h\eta)\|_{H^{k-1}} \, \mathrm{d}h \\
&= |\eta| \, \|\operatorname{grad} z\|_{H^{k-1}} \\
&\leq |\eta| \, \|z\|_{H^k}.
\end{aligned}$$

□

C.2 Nečas inequality

Basically, Nečas inequality promotes regularity from H^{-1} to $\mathcal{L}^2$, and we prove it in this subsection.

Let $\chi(D) := \{z \in H^{-1}(D), \operatorname{grad} z \in H^{-1}(D)\}$ and $\chi(D)$ has a norm defined by

$$\|z\|_{\chi(D)} = \|z\|_{H^{-1}(D)} + \|\operatorname{grad} z\|_{H^{-1}(D)}.$$

We show $\chi(D)$ is actually $\mathcal{L}^2(D)$ with a norm bound.

Proposition 28 (Nečas inequality)**.** *There is a constant $C > 0$ such that*

$$\|z\|_{\mathcal{L}^2(D)} \leq C \|z\|_{\chi(D)} \quad \textit{for all } z \in \mathcal{L}^2(D).$$

Proof. Step 1. $D = \mathbb{R}^2$ is the whole plane. We can use Fourier transform to see

$$\begin{aligned}
\|z\|_{\mathcal{L}^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\hat{z}(\zeta)|^2 \,\mathrm{d}\zeta \\
&= \int_{\mathbb{R}^2} \frac{1}{1+|\zeta|^2} |\hat{z}(\zeta)|^2 \,\mathrm{d}\zeta + \int_{\mathbb{R}^2} \frac{|\zeta|^2}{1+|\zeta|^2} |\hat{z}(\zeta)|^2 \,\mathrm{d}\zeta \\
&= \|z\|_{H^{-1}(D)} + \|\operatorname{grad} z\|_{H^{-1}(D)}.
\end{aligned}$$

Step 2. $D = \mathbb{R}^2_+$ is the upper half-plane. To apply the result in Step 1, we need an extension operator from $\mathbb{R}^2_+$ to $\mathbb{R}^2$ and, conversely, a restriction operator from $\mathbb{R}^2$ to $\mathbb{R}^2_+$. Define $P\colon \mathcal{L}^2(\mathbb{R}^2_+) \to \mathcal{L}^2(\mathbb{R}^2)$ by zero extension,

$$(z(\xi), \xi \in \mathbb{R}^2_+) \quad \mapsto \quad \begin{cases} z(\xi), & \xi \in \mathbb{R}^2_+, \\ 0, & \xi \notin \mathbb{R}^2_+. \end{cases}$$

Then P is linear bounded and also maps $H^1_0(\mathbb{R}^2_+)$ into $H^1(\mathbb{R}^2)$. Its dual map $P^*\colon \mathcal{L}^2(\mathbb{R}^2) \to \mathcal{L}^2(\mathbb{R}^2_+)$, or $P^*\colon H^{-1}(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2_+)$, is a restriction, $z \mapsto z|_{\mathbb{R}^2_+}$. To construct an extension from $\chi(\mathbb{R}^2_+)$ to $\chi(\mathbb{R}^2)$, we begin with the other direction. Define $Q\colon H^1(\mathbb{R}^2) \to H^1_0(\mathbb{R}^2_+)$ by

$$z(\xi_1, \xi_2) \quad \mapsto \quad z(\xi_1, \xi_2) - \lambda_1 z(\xi_1, -\xi_2) - \lambda_2 z(\xi_1, -2\xi_2),$$

with $1 - \lambda_1 - \lambda_2 = 0$. Then Q is linear bounded and $Q^*\colon H^{-1}(\mathbb{R}^2_+) \to H^{-1}(\mathbb{R}^2)$. It follows $QP = \mathrm{Id}_{H^1_0(\mathbb{R}^2_+)}$ and, hence, $P^*Q^* = \mathrm{Id}_{H^{-1}(\mathbb{R}^2_+)}$, $P^*Q^*|_{\chi(\mathbb{R}^2_+)} = \mathrm{Id}_{\chi(\mathbb{R}^2_+)}$. We hope the values of λ_1, λ_2 should guarantee $Q^*|_{\chi(\mathbb{R}^2_+)}$ maps into $\chi(\mathbb{R}^2)$, and a specific constraint is to be determined later. In other words, we need to find proper λ_1, λ_2 such that $\operatorname{grad}(Q^*z) \in H^{-1}(\mathbb{R}^2)$ if $z \in \chi(\mathbb{R}^2_+)$. For $z \in \chi(\mathbb{R}^2_+)$ and for $w \in H^1(\mathbb{R}^2)$,

$$\begin{aligned}
\langle \partial_{\xi_1} Q^* z, w \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)} &= - \langle Q^* z, \partial_{\xi_1} w \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)} \\
&= - \langle z, Q \partial_{\xi_1} w \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)} \\
&= - \langle z, \partial_{\xi_1} Q w \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)} \\
&= \langle \partial_{\xi_1} z, Q w \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)}, \\
\langle \partial_{\xi_2} Q^* z, w \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)} &= - \langle Q^* z, \partial_{\xi_2} w \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)} \\
&= - \langle z, Q \partial_{\xi_2} w \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)} \\
&= - \langle z, \partial_{\xi_2} R w \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)}
\end{aligned}$$

$$R\colon w(\xi_1, \xi_2) \mapsto w(\xi_1, \xi_2) + \lambda_1 w(\xi_1, -\xi_2) + \frac{1}{2}\lambda_2 w(\xi_1, -2\xi_2)$$

$$= \langle \partial_{\xi_2} z, Rw \rangle_{H^{-1}(\mathbb{R}^2_+), H^1_0(\mathbb{R}^2_+)}$$

$$1 + \lambda_1 + \frac{1}{2}\lambda_2 = 0.$$

Thus, we obtain $\lambda_1 = -3, \lambda_2 = 4$ and $Q^*|_{\chi(\mathbb{R}^2_+)} \colon \chi(\mathbb{R}^2_+) \to \chi(\mathbb{R}^2)$ is linear bounded. By Step 1, we have $Q^*z \in \mathcal{L}^2(\mathbb{R}^2)$ and $\|Q^*z\|_{\mathcal{L}^2(\mathbb{R}^2)} \le C \|Q^*z\|_{\chi(\mathbb{R}^2)} \le C \|z\|_{\chi(\mathbb{R}^2)}$. Notice $z = P^*Q^*z$, so it gives

$$\|z\|_{\mathcal{L}^2(\mathbb{R}^2_+)} = \|P^*Q^*z\|_{\mathcal{L}^2(\mathbb{R}^2_+)} \le \|Q^*z\|_{\mathcal{L}^2(\mathbb{R}^2)} \le C \|z\|_{\chi(\mathbb{R}^2)} .$$

Step 3. In a general case, D admits a finite covering of open balls. Note each ball is contained in some half-plane, and we resort to a partiton of unity subordinate to this covering. $\square$

D A Proof of $\mathcal{L}^\infty(0,T;H)\cap C_{\mathrm{wk}}(0,T;V^{-1}) = C_{\mathrm{wk}}(0,T;H)$

The proof here is for completeness of the proof of Propostion 5.

Suppose $z \in \mathcal{L}^\infty(0,T;H)\cap C_{\mathrm{wk}}(0,T;V^{-1})$. Then $t \mapsto \|z(t)\|_H$ is essentially bounded, that is, $z(t) \in H$ for almost all $t \in [0,T]$. We first need to show $z(t) \in H$ for all $t \in [0,T]$. Extend z by reflection to a map defined on $\mathbb{R}$, which facilitates analysis of its convolutions with approximate identity. Let $\{\rho_n\}$ be a sequence of approximate identity and we have

$$\|(\rho_n \star z)(t)\|_H \le \int_{\mathbb{R}} \rho_n(t-s)\,\|z(s)\|_H\,\mathrm{d}s \le C\,\|z\|_{\mathcal{L}^\infty(0,T;H)}\,.$$

Thus, fix $t \in [0,T]$ and there exists a subsequence $\{(\rho_{n_k}\star z)(t)\}$ weakly converging in H to some $\tilde{z}(t) \in H$. At the same time, $z \in C_{\mathrm{wk}}(0,T;V^{-1})$ implies the sequence $\{(\rho_{n_k}\star z)(t)\}$ weakly converges in V^{-1} to $z(t)$. More precisely, for any $w \in V$, since $t \mapsto \langle z(t), w\rangle_{V^{-1},V}$ is continuous,

$$\langle (\rho_{n_k}\star z)(t), w\rangle_{V^{-1},V} = \int_{\mathbb{R}} \rho_{n_k}(t-s)\,\langle z(s), w\rangle_{V^{-1},V}\,\mathrm{d}s \to \langle z(t), w\rangle_{V^{-1},V} \quad \text{as } k\to\infty.$$

It follows from uniqueness of weak limit in V^{-1} that $z(t) = \tilde{z}(t) \in H$ and z is H-valued everywhere. Then we need to show z is weakly continuous in H. Let $\{t_m\} \subset [0,T]$ be a sequence convergent to t. Then $\{z(t_m)\}$ is bounded in H and a subsequence $\{z(t_{m_j})\}$ weakly convergent in H to some $\tilde{z} \in H$. Actually, $\{z(t_{m_j})\}$ weakly converges in V^{-1} to $z(t)$, so $\tilde{z} = z(t)$ is the unique weakly accumulation point of $\{z(t_m)\}$ in H.

Let $z \in C_{\mathrm{wk}}(0,T;H)$ and this z is just a map from $[0,T]$ to H without any measurability information. We need to show z is measurable w.r.t. the σ-algebra generated by the strong topology of H. Recall a closed ball $\bar{B}_H(x,r)$ in H is weakly closed and this is due to the following equality,

$$\bar{B}_H(x,r) = \bigcap_{\|y\|_H=1} \left\{ w \in H \,\middle|\, |\langle w-x, y\rangle_H| \le r \right\}.$$

Since z is continuous w.r.t. the weak topology of H, it follows $z^{-1}(\bar{B}_H(x,r))$ is a closed subset of $[0,T]$. We can write an open ball $B_H(x,r)$ in H, by separability, as a countable union of closed balls. Hence, the pre-image of $B_H(x,r)$ under z is measurable. The statement $z \in C_{\mathrm{wk}}(0,T;V^{-1})$ is trivial and it remains to show $z \in \mathcal{L}^\infty(0,T;H)$. The uniform boundedness principle (Banach-Steinhaus theorem) directly applies. To be concrete, for any $w \in H$, we see the map $t \mapsto \langle z(t), w\rangle_H$ is bounded, which means $\{\langle z(t), \cdot\rangle_H\}_{t\in[0,T]}$ is pointwise bounded. Then we pass boundedness to $\{z(t)\}_{t\in[0,T]}$, so $\|z(t)\|_H \le C$ for all $t \in [0,T]$.