1. FUNDAMENTALS OF MATHEMATICS

1.2: Algebraic Expressions And Manipulations

- An algebraic expression is a combination of constants, variables and algebraic operations (+, -, ×, ÷). We can derive the algebraic expression for a given situation or condition by using these combinations.

1.2.1: Rules of Algebra

Rules of algebra

The rules of arithmetic that we met in the previous Programme for integers also apply to any type of number and we express this fact in the *rules of algebra* where we use variables rather than numerals as specific instances. The rules are:

Commutativity

Two numbers x and y can be added or multiplied in any order without affecting the result. That is:

$x + y = y + x$ and
$xy = yx$

Addition and multiplication are commutative operations

The order in which two numbers are subtracted or divided *does* affect the result. That is:

$x - y \neq y - x$ unless $x = y$ and

$x \div y \neq y \div x, \left(\frac{x}{y} \neq \frac{y}{x}\right)$ unless $x = y$ and neither equals 0

Subtraction and division are not commutative operations except in very special cases

Associativity

The way in which the numbers x, y and z are associated under addition or multiplication *does not* affect the result. That is:

$x + (y + z) = (x + y) + z = x + y + z$ and
$x(yz) = (xy)z = xyz$

Addition and multiplication are associative operations

Addition and multiplication are associative operations

The way in which the numbers are associated under subtraction or division *does* affect the result. That is:

$x - (y - z) \neq (x - y) - z$ unless $z = 0$ and

$x \div (y \div z) \neq (x \div y) \div z$ unless $z = 1$ and $y \neq 0$

Subtraction and division are not associative operations except in very special cases

Distributivity

Multiplication is distributed over addition and subtraction from both the left and the right. For example:

$x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

$x(y - z) = xy - xz$ and $(x - y)z = xz - yz$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$(x + y) \div z = (x \div z) + (y \div z)$ but

$x \div (y + z) \neq (x \div y) + (x \div z)$

that is:

$\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$ but $\frac{x}{y+z} \neq \frac{x}{y} + \frac{x}{z}$

Take care here because it is a common mistake to get this wrong

1.2.2 : Terms and Coefficients, Expanded Brackets and Nested Brackets

Terms and coefficients

An algebraic expression consists of alphabetic characters and numerals linked together with the arithmetic operators. For example:

$8x - 3xy$

is an algebraic expression in the two variables x and y. Each component of this expression is called a *term* of the expression. Here there are two terms, namely:

the x term and the xy term.

The numerals in each term are called the *coefficients* of the respective terms. So that:

8 is the coefficient of the x term and -3 is the coefficient of the xy term.

Collecting like terms

Terms which have the same variables are called *like* terms and like terms can be collected together by addition or subtraction. For example:

$4x + 3y - 2z + 5y - 3x + 4z$ can be rearranged as $4x - 3x + 3y + 5y - 2z + 4z$ and simplified to:

$x + 8y + 2z$

Similarly, $4uv - 7uz - 6wz + 2uv + 3wz$ can be simplified to …………

Check your answer with the next frame

$$\boxed{6uv - 7uz - 3wz}$$

Next frame

Similar terms

In the algebraic expression:

$ab + ac$

both terms contain the letter a and for this reason these terms, though not like terms, are called *similar* terms. Common symbols such as this letter a are referred to as *common factors* and by using brackets these common factors can be *factored out*. For example, the common factor a in this expression can be factored out to give:

$ab + ac = a(b + c)$ This process is known as *factorization*.

Expanding brackets

Sometimes it will be desired to reverse the process of factorizing an expression by *removing* the brackets. This is done by:

(a) multiplying or dividing each term inside the bracket by the term outside the bracket, but

(b) if the term outside the bracket is negative then each term inside the bracket changes sign.

For example, the brackets in the expression:

$3x(y - 2z)$ are removed to give $3xy - 6xz$

and the brackets in the expression:

$-2y(2x - 4z)$ are removed to give $-4yx + 8yz$.

As a further example, the expression:

$$\frac{y+x}{8x} - \frac{y-x}{4x}$$

is an alternative form of $(y + x) \div 8x - (y - x) \div 4x$ and the brackets can be removed as follows:

$$\begin{aligned}\frac{y+x}{8x} - \frac{y-x}{4x} &= \frac{y}{8x} + \frac{x}{8x} - \frac{y}{4x} + \frac{x}{4x} \\ &= \frac{y}{8x} + \frac{1}{8} - \frac{y}{4x} + \frac{1}{4} \\ &= \frac{3}{8} - \frac{y}{8x}\end{aligned}$$

which can be written as $\frac{1}{8}\left(3 - \frac{y}{x}\right)$ or as $\frac{1}{8x}(3x - y)$

Nested brackets **13**

Whenever an algebraic expression contains brackets nested within other brackets the innermost brackets are removed first. For example:

$$\begin{aligned}7(a - [4 - 5(b - 3a)]) &= 7(a - [4 - 5b + 15a]) \\ &= 7(a - 4 + 5b - 15a) \\ &= 7a - 28 + 35b - 105a \\ &= 35b - 98a - 28\end{aligned}$$

So that the algebraic expression $4(2x + 3[5 - 2(x - y)])$ becomes, after the removal of the brackets ....

Next frame

$$\boxed{24y - 16x + 60}$$

Because

$$\begin{aligned} 4(2x + 3[5 - 2(x - y)]) &= 4(2x + 3[5 - 2x + 2y]) \\ &= 4(2x + 15 - 6x + 6y) \\ &= 8x + 60 - 24x + 24y \\ &= 24y - 16x + 60 \end{aligned}$$

<u>Examples</u>

1. Simplify each of the following by collecting like terms:

(a) $\begin{aligned}[t] 4xy + 3xz - 6zy - 5zx + yx &= 4xy + xy + 3xz - 5xz - 6yz \\ &= 5xy - 2xz - 6yz \end{aligned}$

Notice that the characters are written in alphabetic order.

(b) $\begin{aligned}[t] -2a + 4ab + a - 4ba &= -2a + a + 4ab - 4ab \\ &= -a \end{aligned}$

(c) $\begin{aligned}[t] 3rst - 10str + 8ts - 5rt + 2st &= 3rst - 10rst + 8st + 2st - 5rt \\ &= -7rst + 10st - 5rt \end{aligned}$

2. Expand the following and then refactorize where possible:

(a) $\begin{aligned}[t] 8x(y - z) + 2y(7x + z) &= 8xy - 8xz + 14xy + 2yz \\ &= 22xy - 8xz + 2yz \\ &= 2(x[11y - 4z] + yz) \end{aligned}$

(b) $\begin{aligned}[t] (3a - b)(b - 3a) + b^2 &= 3a(b - 3a) - b(b - 3a) + b^2 \\ &= 3ab - 9a^2 - b^2 + 3ab + b^2 \\ &= 6ab - 9a^2 \\ &= 3a(2b - 3a) \end{aligned}$

(c) $\begin{aligned}[t] -3(w - 7[x - 8(3 - z)]) &= -3(w - 7[x - 24 + 8z]) \\ &= -3(w - 7x + 168 - 56z) \\ &= -3w + 21x - 504 + 168z \end{aligned}$

(d) $\begin{aligned}[t] \frac{2a - 3}{4b} + \frac{3a + 2}{6b} &= \frac{2a}{4b} - \frac{3}{4b} + \frac{3a}{6b} + \frac{2}{6b} \\ &= \frac{a}{2b} - \frac{3}{4b} + \frac{a}{2b} + \frac{1}{3b} \\ &= \frac{a}{b} - \frac{5}{12b} \\ &= \frac{1}{12b}(12a - 5) \end{aligned}$

1.2.3: Algebraic Multiplication and Division

Algebraic multiplication and division

Multiplication

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Example 1

$$(x+2)(x+3) = x(x+3) + 2(x+3)$$
$$= x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

Now a slightly harder one

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Example 2

$(2x+5)(x^2+3x+4)$

Each term in the second expression is to be multiplied by $2x$ and then by 5 and the results added together, so we set it out thus:

$$\begin{array}{lr}
 & x^2 + 3x + 4 \\
 & 2x + 5 \\ \hline
\text{Multiply throughout by } 2x & 2x^3 + 6x^2 + 8x \\
\text{Multiply by } 5 & 5x^2 + 15x + 20 \\ \hline
\text{Add the two lines} & 2x^3 + 11x^2 + 23x + 20 \\ \hline
\end{array}$$

$$\text{So } (2x+5)(x^2+3x+4) = 2x^3 + 11x^2 + 23x + 20$$

Be sure to keep the same powers of the variable in the same column.

Next frame

Determine $(2x + 6)(4x^3 - 5x - 7)$

You will notice that the second expression is a cubic (highest power x^3), but that there is no term in x^2. In this case, we insert $0x^2$ in the working to keep the columns complete, that is:

$$\begin{array}{l} 4x^3 + 0x^2 - 5x - 7 \\ \underline{2x \ \ + 6 \qquad\qquad} \end{array}$$

which gives …………

Finish it

$$\boxed{8x^4 + 24x^3 - 10x^2 - 44x - 42}$$

Here it is set out:

$$\begin{array}{rrrrr} 4x^3 & +\,0x^2 & -\,5x & -\,7 & \\ 2x & +\,6 & & & \\ \hline 8x^4 + & 0x^3 & -\,10x^2 & -\,14x & \\ & 24x^3 + & 0x^2 & -\,30x & -\,42 \\ \hline 8x^4 & +\,24x^3 & -\,10x^2 & -\,44x & -\,42 \\ \hline \end{array}$$

They are all done in the same way, so here is one more for practice.

Example 4

Determine the product $(3x - 5)(2x^3 - 4x^2 + 8)$

You can do that without any trouble.

The product is …………

$$\boxed{6x^4 - 22x^3 + 20x^2 + 24x - 40}$$

All very straightforward:

$$\begin{array}{r}
2x^3 - 4x^2 + 0x + 8 \phantom{{}-40} \\
3x - 5 \phantom{{}+ 0x^2 + 24x - 40} \\
\hline
6x^4 - 12x^3 + 0x^2 + 24x \phantom{{}-40} \\
-10x^3 + 20x^2 + 0x - 40 \\
\hline
6x^4 - 22x^3 + 20x^2 + 24x - 40 \\
\hline
\end{array}$$

Division

Let us consider $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4)$. The result of this division is called the *quotient* of the two expressions and we find the quotient by setting out the division in the same way as we do for the long division of numbers:

$$3x + 4 \overline{\smash{\big)}\, 12x^3 - 2x^2 - 3x + 28}$$

To make $12x^3$, $3x$ must be multiplied by $4x^2$, so we insert this as the first term in the quotient, multiply the divisor $(3x+4)$ by $4x^2$, and subtract this from the first two terms:

$$\begin{array}{r|l} & 4x^2 \\ \cline{2-2} 3x+4 & 12x^3 - 2x^2 - 3x + 28 \\ & \underline{12x^3 + 16x^2} \\ & \quad -18x^2 - 3x \end{array}$$

Bring down the next term $(-3x)$ and repeat the process

To make $-18x^2$, $3x$ must be multiplied by $-6x$, so do this and subtract as before, not forgetting to enter the $-6x$ in the quotient.

Do this and we get ....

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$$\begin{array}{r|l} & 4x^2 - 6x \\ \cline{2-2} 3x+4 & 12x^3 - 2x^2 - 3x + 28 \\ & \underline{12x^3 + 16x^2} \\ & \quad -18x^2 - 3x \\ & \quad \underline{-18x^2 - 24x} \\ & \qquad\qquad 21x \end{array}$$

Now bring down the next term and continue in the same way and finish it off.

So $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4) =$ ....

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$\boxed{4x^2 - 6x + 7}$

As before, if an expression has a power missing, insert the power with zero coefficient. Now you can determine $(4x^3 + 13x + 33) \div (2x + 3)$

Here it is:

$$\begin{array}{r|l} & 2x^2 - 3x + 11 \\ \cline{2-2} 2x+3 & 4x^3 - 0x^2 + 13x + 33 \\ & \underline{4x^3 + 6x^2} \\ & \quad -6x^2 + 13x \\ & \quad \underline{-6x^2 - 9x} \\ & \qquad\quad 22x + 33 \\ & \qquad\quad \underline{22x + 33} \\ & \qquad\quad \underline{\;\bullet \quad \bullet\;} \end{array}$$

So $(4x^3 + 13x + 33) \div (2x + 3) = 2x^2 - 3x + 11$

And one more.

Determine $(6x^3 - 7x^2 + 1) \div (3x + 1)$

1 Perform the following multiplications and simplify your results:

(a) $(8x-4)(4x^2-3x+2)$

(b) $(2x+3)(5x^3+3x-4)$

2 Perform the following divisions:

(a) $(x^2+5x-6)\div(x-1)$

(b) $(x^2-x-2)\div(x+1)$

(c) $(12x^3-11x^2-25)\div(3x-5)$

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1 (a)
$$\begin{aligned}(8x-4)(4x^2-3x+2) &= 8x(4x^2-3x+2)-4(4x^2-3x+2)\\ &= 32x^3-24x^2+16x-16x^2+12x-8\\ &= 32x^3-40x^2+28x-8\end{aligned}$$

(b)
$$\begin{aligned}(2x+3)(5x^3+3x-4) &= 2x(5x^3+3x-4)+3(5x^3+3x-4)\\ &= 10x^4+6x^2-8x+15x^3+9x-12\\ &= 10x^4+15x^3+6x^2+x-12\end{aligned}$$

2 (a)
$$(x^2+5x-6)\div(x-1) = \begin{array}{r|r} & x+6 \\ \hline x-1 & x^2+5x-6 \\ & \underline{x^2-x\quad\;\;} \\ & 6x-6 \\ & \underline{6x-6} \\ & \underline{\bullet\quad\bullet} \end{array}$$

(b)
$$(x^2-x-2)\div(x+1) = \begin{array}{r|r} & x-2 \\ \hline x+1 & x^2-x-2 \\ & \underline{x^2+x\quad\;\;} \\ & -2x-2 \\ & \underline{-2x-2} \\ & \underline{\bullet\quad\bullet} \end{array}$$

(c)
$$(12x^3-11x^2-25)\div(3x-5) = \begin{array}{r|r} & 4x^2+3x+5 \\ \hline 3x-5 & 12x^3-11x^2+0x-25 \\ & \underline{12x^3-20x^2\qquad\qquad} \\ & 9x^2+0x\qquad \\ & \underline{9x^2-15x\qquad} \\ & 15x-25 \\ & \underline{15x-25} \\ & \underline{\bullet\quad\bullet} \end{array}$$

❖ LINEAR EQUATIONS