Calculating Hermite-Gauss Modes in Python

Paul Edwards

PauLisa.py was used to calculate Hermite-Gauss(HG) modes. This document presents intensity profiles for "pure" modes and phase and intensity slices for higher-order modes. Beam tilts and misalignments relative to the optical axis can be used to model tilt-to-length(TTL) coupling in LISA's long arm.

1. EQUATIONS FOR HG MODES

Hermite-Gauss (HG) modes represent a set of exact solutions of the paraxial wave equation

$$\nabla_t^2 u(x,y,z) - 2ik\partial z u(x,y,z) = 0 \ . \tag{1}$$

The Living Reviews article [1] gives the general expression for HG modes as

$$u_{nm}(x,y,z) = (2^{n+m-1}n!m!\pi)^{-1/2}\frac{1}{w(z)}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)H_m\Big(\frac{\sqrt{2}y}{w(z)}\Big)\exp\Big(\frac{-ik(x^2+y^2)}{2R_c(z)} - \frac{x^2+y^2}{w(z)^2}\Big) \ . \tag{2}$$

In terms of the Gaussian beam parameter,

$$q(z) = iz_R + z - z_0 = q_0 + z - z_0 \ , \tag{3}$$

HG modes may be expressed as

$$\begin{aligned} u_{nm}(x,y,z) =& u_n(x,z)u_m(y,z) \\ =& \Big(\frac{2}{\pi}\Big)^{1/4}\Big(\frac{1}{2^n n! w_0}\Big)^{1/2}\Big(\frac{q_0}{q(z)}\Big)^{1/2}\Big(\frac{q_0 q*(z)}{q_0 * q(z)}\Big)^{n/2} H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)\exp\Big(\frac{-ik(x^2)}{2q_z}\Big) \times u_m(y,z) \ , \end{aligned} \tag{4}$$

where the first three Hermite polynomials are given by

$$H_n(x) = \begin{cases} 1 & (n=0) \\ 2x & (n=1) \\ 4x^2 - 2 & (n=2) \end{cases}$$

The HG modes are orthonormal, such that

$$\int\int dxdy \ u_{nm}u^*_{n'm'} = \delta_{nn'}\delta_{mm'}. \tag{5}$$

For approximations used in later sections, it is useful to express higher-order modes in terms of the fundamental mode. For $n = 0, 1, 2$ and $m = 0$ at the beam waist(letting $z_0 = 0$):

$$u_{00}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{1}{w_0}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] \ , \tag{6}$$

$$u_{10}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{2x}{w_0^2}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] \ , \tag{7}$$

$$u_{20}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big[\Big(\frac{2\sqrt{2}x^2}{w_0^3}\Big) - \frac{\sqrt{2}}{2}\Big]\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] \ , \tag{8}$$

Expressing u_{10} and u_{20} in terms of u_{00} at the beam waist,

$$u_{10}(x,y,0) = \frac{2x}{w_0} u_{00}(x,y,0) \tag{9}$$

$$u_{20}(x,y,0) = \left[2\sqrt{2}\Big(\frac{x^2}{w_0^2}\Big) - \frac{\sqrt{2}}{2} \right] u_{00}(x,y,0) \tag{10}$$

In PauLisa.py, Eq. 2 is represented by the *calculate* function and Eq. 4 as *calculate_q*. The outputs of both functions agree exactly.

2. INTENSITY PROFILES

In the appendix, intensity profiles produced by PauLisa.py for u_{00} to u_{33} are shown in Fig. A.1.

3. SHIFTED BEAM

For a small shift of the input axis, $a{<<}w_0$, in the +x-direction relative to the cavity axis, the shifted u_{00} mode can be solved in terms of an added u_{10} mode up to a constant factor:

$$\begin{aligned} u_{00}(x-a,y,0) =& \Big(\frac{2}{\pi}\Big)^{-1/2} \Big(\frac{1}{w_0}\Big) \exp\Big(-\frac{(x-a)^2+y^2}{w_0^2}\Big) \\ =& \Big(\frac{2}{\pi}\Big)^{-1/2} \Big(\frac{1}{w_0}\Big) \exp\Big(-\frac{y^2}{w_0^2}\Big) \exp\Big(-\frac{(x-a)^2}{w_0^2}\Big) \\ =& u_{00}(x,y,0) \times exp\Big(\frac{2ax+a^2}{w_0^2}\Big) \\ =& u_{00}(x,y,0) \left[1 + \frac{2ax}{w_0^2} + \mathcal{O}\Big(\frac{a}{w}\Big)^2 \right] \\ \approx& u_{00} + \Big(\frac{2ax}{w_0^2}\Big) u_{00} \\ =& u_{00}(x,y,0) + \frac{a}{w_0} u_{10}(x,y,0) \ . \end{aligned} \tag{11}$$

The shift for mode coefficients C_{nm} is then

$$a \approx \frac{\Re(C_{10})}{\Re(C_{00})} w_0 \ . \tag{12}$$

Results for HG(0,0) and HG(1,0) addition are shown in Fig. 1.

(1,0) Scale	Pred. Shift[$\times 10^{-5} m$]	Act. Shift[$\times 10^{-5} m$]	%Error
0.04	4.0	3.9849	0.37
0.08	8.0	7.8998	1.25
0.16	16.0	15.2546	4.65

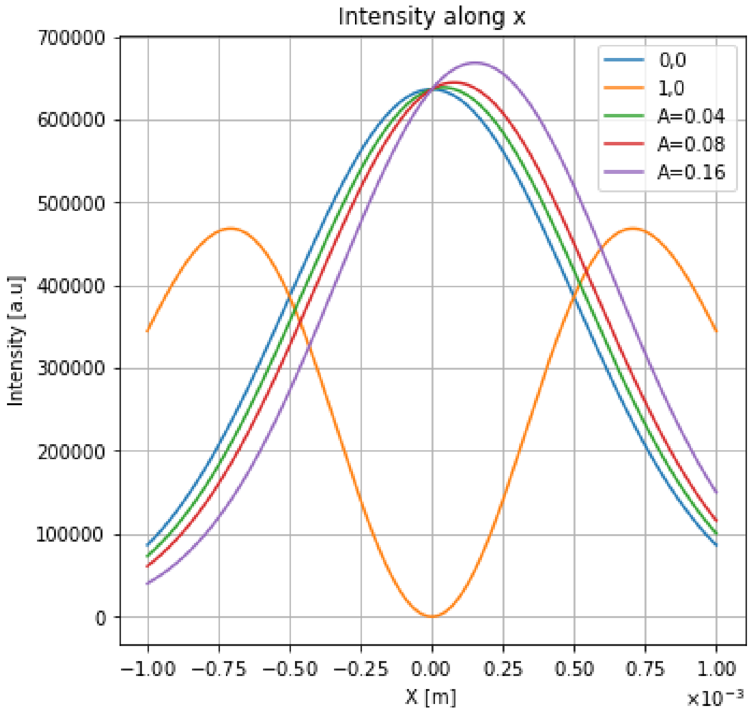


FIG. (1) Intensity at the beam waist and $y = 0$ for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

4. TILTED BEAM

A relative tilt between the input and cavity axis can be expressed as an addition of a first-order mode in quadrature phase,

$$u_{tilt}(x, y, 0) = u_{00}(x, y, 0) + iu_{10}(x, y, 0)\ . \tag{13}$$

This addition varies the phase of the wave in the x-direction

$$u_{tilt}(x, y, 0) = u_{00}(x, y, 0) \exp(i\phi)\ . \tag{14}$$

Expressing this in terms of a quadrature phase addition of u_{10}, with $\alpha << \frac{\lambda}{w_0 \pi}$:

$$\begin{aligned} u_{tilt(0,0)} =& u_{00} \exp(i\phi) \\ =& u_{00} \exp\left[ikx \sin(\alpha)\right] \\ \approx& u_{00} \exp\left[ikx\alpha\right] \\ =& u_{00} \exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ \approx& u_{00}\left[1 + i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ =& u_{00}(x, y, 0) + i\Big(\frac{\pi w_0 \alpha}{\lambda}\Big) u_{10}\ . \end{aligned} \tag{15}$$

Therefore, the predicted angle scales proportionally with the coefficient of the u_{10} mode, C_{10},

$$\alpha \approx \frac{|\Im(C_{10})|}{\Re(C_{00})} \frac{\lambda}{\pi w_0} \approx \frac{|\Im(C_{10})|}{\Re(C_{00})} \Theta\ , \tag{16}$$

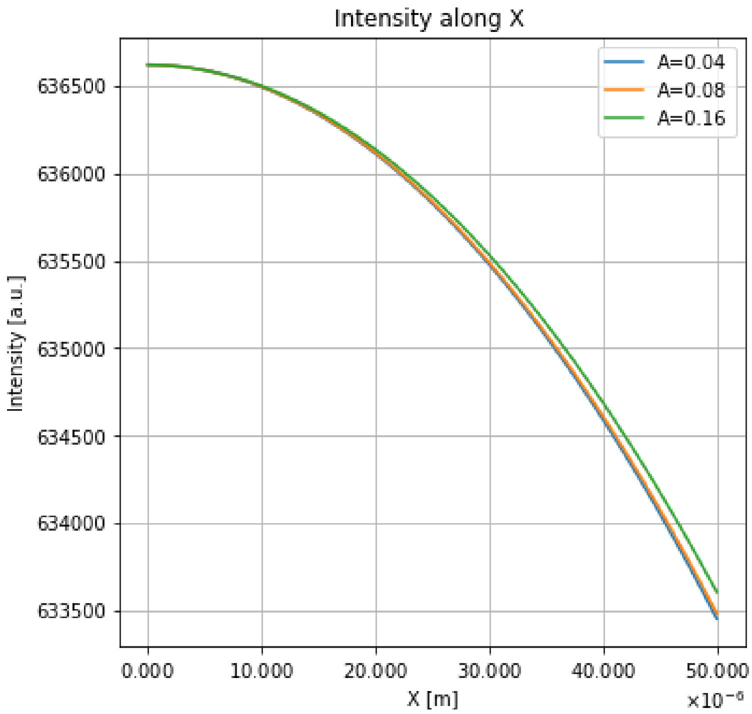


FIG. (2) Intensity at the beam waist and $y = 0$ for HG(0,0) and HG(1,0) addition in quadrature phase. The variable A represents the (imaginary) scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

where $\Theta = \frac{\pi w_0}{\lambda}$ is the diffraction angle. Results for HG_{10} quadrature-phase addition are shown in Table 2. Graphical results are shown in Fig. 2.

(1,0) Scale (Imag.)	Pred. Angle[$\times 10^{-5}$ rad.]	Act. Angle[$\times 10^{-5}$ rad.]	%Error
0.04	1.3547	1.3518	0.00212
0.08	2.7094	2.6866	0.00840
0.16	5.4189	5.2445	0.03217

By Eq. 15, phase should vary with x as

$$\frac{d\phi}{dx} \approx \frac{2\pi\alpha}{\lambda} \; . \tag{17}$$

Figure 3 shows agreement with this approximation.

5. TILTING A MISALIGNED BEAM

Building on the two previous sections, a tilt to an already shifted beam is

$$\sum_{n,m} u_{nm}(x,y,0) = u_{00(tilt)}(x,y,0) + \Big(\frac{a}{w_0}\Big) u_{10(tilt)}(x,y,0) \; . \tag{18}$$

Following the same approximations for a general tilted beam ($\alpha << \frac{\lambda}{w_0 \pi}$), the first term is the same as in Eq. 15,

$$u_{00(tilt)}(x,y,0) = u_{00}(x,y,0) + i\Big(\frac{\pi w_0 \alpha}{\lambda}\Big) u_{10} \; ,$$

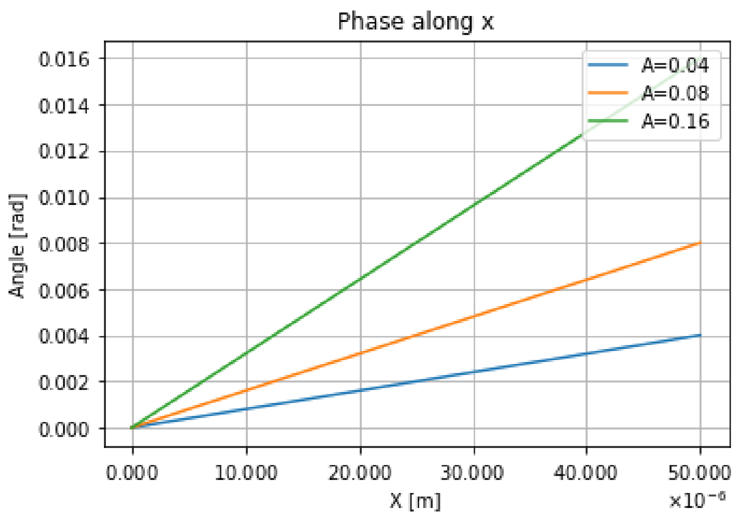


FIG. (3) Phase at the beam waist and $y = 0$ for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the (imaginary) scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient. Phase varies with x as predicted by Eq. 17

while the u_{10} term in Eq. 18 is:

$$\begin{aligned}\Big(\frac{a}{w_0}\Big)u_{10(tilt))}(x,y,0) &= \Big(\frac{a}{w_0}\Big)u_{10}\exp(i\phi)\\ &= \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\sin(\alpha)\right]\\ &\approx \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\alpha\right]\\ &= \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right]\\ &\approx \Big(\frac{a}{w_0}\Big)u_{10}\left[1+i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \ .\end{aligned} \tag{19}$$

In terms of u_{00}, the imaginary term in Eq. 19 is:

$$i\Big(\frac{2\pi x\alpha}{\lambda}\Big)u_{10} = i\Big(\frac{4\pi x^2\alpha a}{w_0^2\lambda}\Big)u_{00} \ . \tag{20}$$

Rewriting in terms of u_{00} and u_{20}:

$$i\Big(\frac{4\pi x^2 a\alpha}{w_0^2\lambda}\Big)u_{00} = i\Big(\frac{2\pi\alpha a}{\sqrt{2}\lambda}\Big)\left[u_{20}+\frac{\sqrt{2}}{2}u_{00}\right] \ . \tag{21}$$

Therefore, Eq. 18 for a shifted then tilted beam of fundamental mode at the waist is

$$\sum_{n,m} u_{nm}(x,y,0) = \left[1+i\Big(\frac{\pi a\alpha}{\lambda}\Big)\right]u_{00} + \left[\frac{a}{w_0}+i\Big(\frac{\pi w_0\alpha}{\lambda}\Big)\right]u_{10} + i\Big(\frac{\sqrt{2}\pi a\alpha}{\lambda}\Big)u_{20} \ . \tag{22}$$

6. TILT-TO-LENGTH COUPLING AND JITTER

LISA uses PD's to measure interference of a received beam with a reference beam, resulting in beat notes which are used to determine phase difference ($\Delta\phi = \phi_1 - \phi_2$) of the two beams. Fields of the reference and received beam defined with a basis (w_0, z_0) are, respectively:

$$E_{ref} = E_{0\ (ref)} e^{i((\omega_{ref} t)+\phi_1)} u_{00}(w_{0\ ref}, z_{0\ ref}) \; . \tag{23}$$

$$E_{rec} = E_{0\ (rec)} e^{i((\omega_{rec} t)+\phi_2)} \sum_{n,m=0} u_{nm}(w_{0\ rec}, z_{0\ rec}) \; , \tag{24}$$

These fields produce a power at a single photodetector

$$P_{pd} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E^* E \; dxdy, \tag{25}$$

where the electric field is the sum of the received and reference fields. Let the spatial component of the received beam be given by 22. By orthonormality of HG modes, only the imaginary part of the received beam's u_{00} mode contributes to an overall phase shift which is proportional to misalignment angles

$$\Delta\phi \propto \arctan\left(\frac{a\alpha\pi}{\lambda}\right) \approx \left(\frac{a\alpha\pi}{\lambda}\right) . \tag{26}$$

The sum of the electric fields of the received and reference beam of the same basis is, effectively,

$$E = E_{0(ref)} e^{i((\omega_{ref} t)+\phi_1)} u_{00} + E_{0(rec)} e^{i((\omega_{rec} t)+\phi_2)} u_{00} \left[1 + i\Big(\frac{\pi a\alpha}{\lambda}\Big)\right] \; , \tag{27}$$

and, with $t = 0$,

$$E^* E = u_{00}^* u_{00} |E_0|^2 \{2 + (\frac{\pi a\alpha}{\lambda})^2 + e^{i\Delta\phi}\left[1 - \frac{\pi a\alpha}{\lambda}\right] + e^{-i\Delta\phi}\left[1 + \frac{\pi a\alpha}{\lambda}\right]\} \; . \tag{28}$$

Compared to a single misalignment of a beam (Eq. 15), which only has a phase shift contribution via a u_{10} term, the shifted and tilted beam has phase shift via a u_{00} term in quadrature phase to the reference beam. Moreover, the S/C rotating about the COM of the TM means the beam with a waist centered on the TM also appears to rotate about the TM CoM, resulting in no TTL coupling as the wavefronts at the receiving S/C appear spherically centered on the TM CoM. The result of a misaligned then tilted beam, however, is tilt-to-length coupling, where jitter has coupled into length.

Section C of the LISA Payload Description Document (PDD) outlines expected TTL couplings. Spacecraft jitter is assigned ~ 10 nrad/$\sqrt{\text{Hz}}$. Lateral alignment offset with optical bench (OB) is $\sim 20\ \mu m$. These combine for ~ 20 pm/$\sqrt{\text{Hz}}$.

7. MODULATING TTL OVER LISA'S LONG ARM

Jitter of a tilted beam, as outlined in the previous section, can be used to model TTL coupling in LISA's long arm. Sinusoidal modulation of the terms in Eq. 22 which came as an effect of jitter, scaled in m, yields:

$$u(x,y,0) = \left[u_{00} + \frac{a}{w_0} u_{10}\right] + i\frac{\pi\alpha}{\lambda}\left[a\Big(u_{00} + \sqrt{2}u_{20}\Big) + w_0 u_{10}\right] m \sin(\omega t - \phi) \; . \tag{29}$$

Assume an arm of 2.5 Gm, $w_0 = 15$ cm, and PD radius = 15 cm.

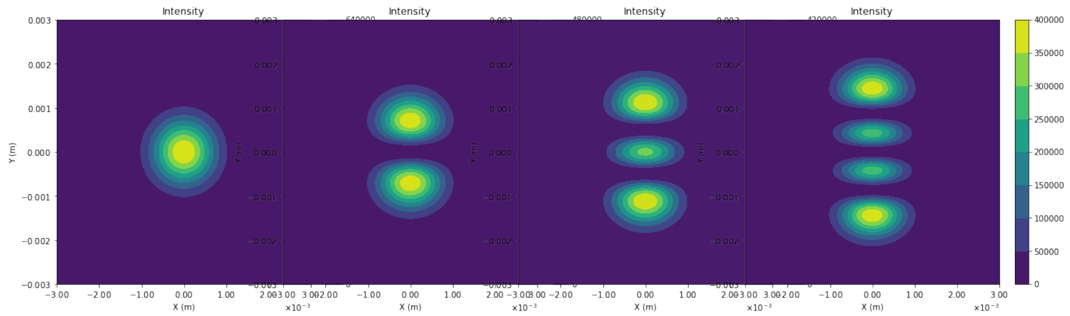
[1] C. Bond, D. Brown, A. Freise, and K. A. Strain, Living Reviews in Relativity **19**, 3 (2017), ISSN 1433-8351, URL https://doi.org/10.1007/s41114-016-0002-8.

APPENDIX A: ACRONYMS

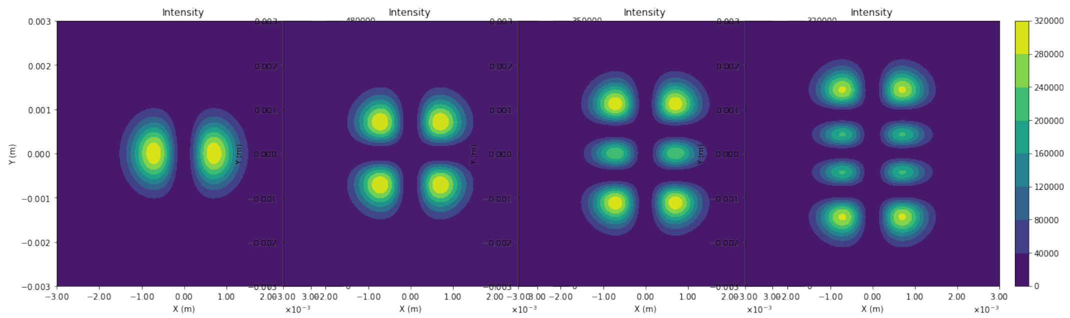
ACRONYMS

CoM	*center of mass*
HG	*Hermite – Gauss*
PDD	*Payload Definition Document*
S/C	*spacecraft*
TM	*test mass*
TTL	*tilt to length*

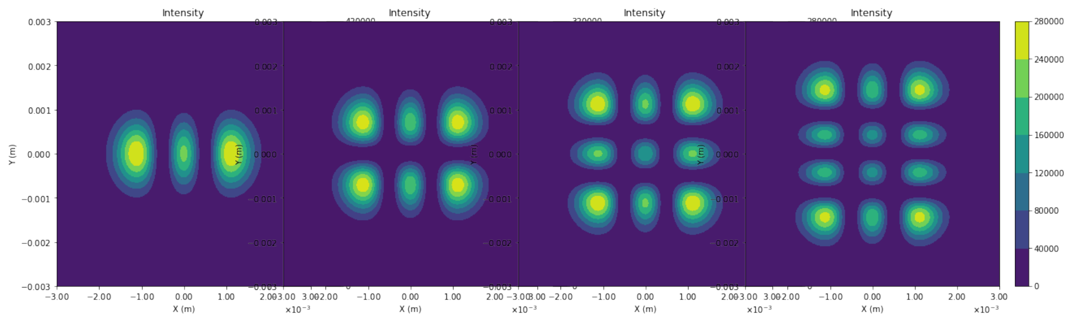
APPENDIX B: INTENSITY PLOTS



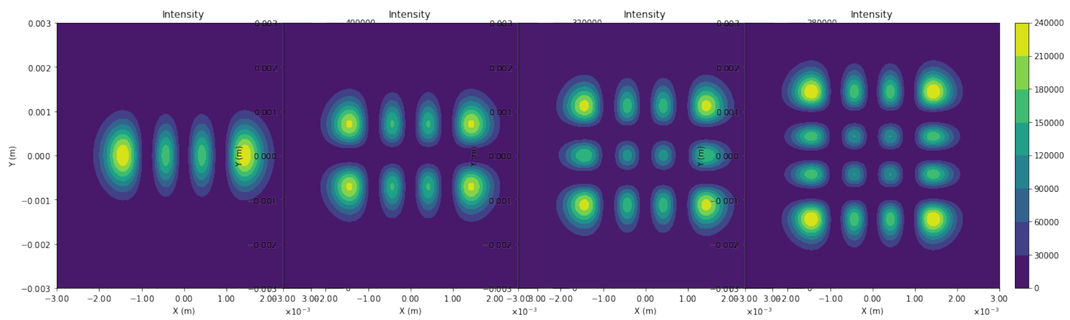
(a) HG(0,0) (b) HG(0,1) (c) HG(0,2) (d) HG(0,3)



(e) HG(1,0) (f) HG(1,1) (g) HG(1,2) (h) HG(1,3)



(i) HG(2,0) (j) HG(2,1) (k) HG(2,2) (l) HG(2,3)



(m) HG(3,0) (n) HG(3,1) (o) HG(3,2) (p) HG(3,3)

FIG. (A.1) Intensity profiles for HG modes u_{00} to u_{33} at the beam waist.