Calculating Hermite-Gauss Modes in Python

Paul Edwards

PauLisa.py was used to calculate Hermite-Gauss(HG) modes. This document presents intensity profiles for "pure" modes and phase and intensity slices for higher-order modes. Beam tilts and misalignments relative to the optical axis can be used to model tilt-to-length(TTL) coupling in LISA's long arm.

1. EQUATIONS FOR HG MODES

The Living Reviews article [1] gives the general expression for HG modes as

$$u_{nm}(x,y,z) = (2^{n+m-1}n!m!\pi)^{-1/2}\frac{1}{w(z)}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)H_m\Big(\frac{\sqrt{2}y}{w(z)}\Big)\exp\Big(\frac{-ik(x^2+y^2)}{2R_c(z)}-\frac{x^2+y^2}{w(z)^2}\Big). \tag{1}$$

In terms of the Gaussian beam parameter,

$$q(z) = iz_R + z - z_0 = q_0 + z - z_0 , \tag{2}$$

this may be expressed as

$$
\begin{aligned}
u_{nm}(x,y,z) =& u_n(x,z)u_m(y,z)\\
=& \Big(\frac{2}{\pi}\Big)^{1/4}\Big(\frac{1}{2^n n! w_0}\Big)^{1/2}\Big(\frac{q_0}{q(z)}\Big)^{1/2}\Big(\frac{q_0 q*(z)}{q_0*q(z)}\Big)^{n/2}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)\exp\Big(\frac{-ik(x^2)}{2q_z}\Big)\times u_m(y,z) ,
\end{aligned}
\tag{3}
$$

where the first three Hermite polynomials are given by

$$H_n(x) = \begin{cases} 1 & (n=0)\\ 2x & (n=1)\\ 4x^2-2 & (n=2) \end{cases}$$

For $n = 0,1,2$ and $m = 0$ at the beam waist(letting $z_0 = 0$):

$$u_{00}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{1}{w_0}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{4}$$

$$u_{10}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{2x}{w_0^2}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{5}$$

$$u_{20}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big[\Big(\frac{2\sqrt{2}x^2}{w_0^3}\Big)-\frac{\sqrt{2}}{2}\Big]\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{6}$$

Expressing u_{10} and u_{20} in terms of u_{00} at the beam waist,

$$u_{10}(x,y,0) = \frac{2x}{w_0}u_{00}(x,y,0) \tag{7}$$

$$u_{20}(x,y,0) = \Big[2\sqrt{2}\Big(\frac{x^2}{w_0^2}\Big)-\frac{\sqrt{2}}{2}\Big]u_{00}(x,y,0) \tag{8}$$

In PauLisa.py, Eq. 1 is represented by the *calculate* function and Eq. 3 as *calculate_q*. The outputs of both functions agree exactly.
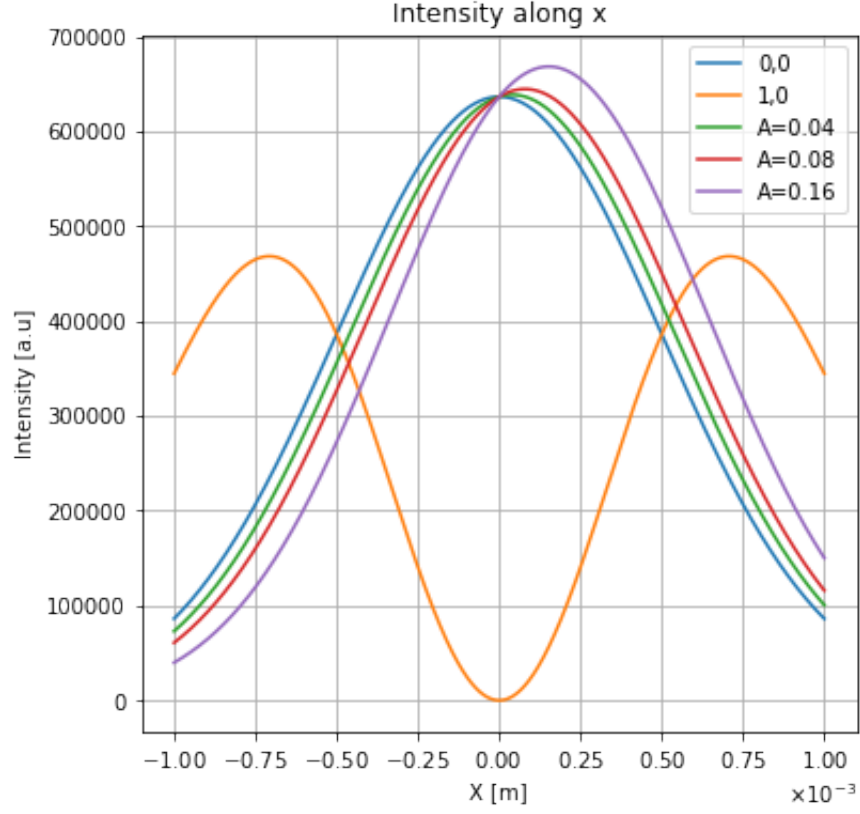
FIG. (1) Intensity at the beam waist and x=0 for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

2. INTENSITY PROFILES

In the appendix, intensity profiles produced by PauLisa.py for u_{00} to u_{33} are shown in Fig. A.1.

3. SHIFTED BEAM

For a small shift of the input axis, $a \ll w_0$, in the +x-direction relative to the cavity axis, the shifted u_{00} mode can be solved in terms of an added u_{10} mode up to a constant factor:

$$
\begin{aligned}
u_{00}(x-a,y,0) =& \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{(x-a)^2+y^2}{w_0^2}\right)\\
=& \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{y^2}{w_0^2}\right)\exp\left(-\frac{(x-a)^2}{w_0^2}\right)\\
=& u_{00}(x,y,0)\times exp\left(\frac{2ax+a^2}{w_0^2}\right)\\
=& u_{00}(x,y,0)\left[1+\frac{2ax}{w_0^2}+\mathcal{O}\left(\frac{a}{w}\right)^2\right]\\
\approx& u_{00}+\left(\frac{2ax}{w_0^2}\right)u_{00}\\
=& u_{00}(x,y,0)+\frac{a}{w_0}u_{10}(x,y,0)\ .
\end{aligned}
\tag{9}
$$

Results for HG(0,0) and HG(1,0) addition are shown in Fig. 1.

(1,0) Scale	Pred. Shift$[\times 10^{-5}m]$	Act. Shift$[\times 10^{-5}m]$	%Error
0.04	4.0	3.9849	0.37
0.08	8.0	7.8998	1.25
0.16	16.0	15.2546	4.65

4.   TILTED BEAM

A relative tilt between the input and cavity axis can be expressed as an addition of a first-order mode in quadrature phase,

$$u_{tilt}(x,y,0) = u_{00}(x,y,0) + \frac{ia\pi w_0}{\lambda}u_{10}(x,y,0) \ . \tag{10}$$

This addition varies the phase of the wave in the x-direction

$$u_{tilt}(x,y,0) = u_{00}(x,y,0)\exp(i\phi) \ . \tag{11}$$

Expressing this in terms of a quadrature phase addition of u_{10}, with $\alpha << \frac{\lambda}{w_0\pi}$:

$$
\begin{aligned}
u_{tilt(0,0)} &= u_{00}\exp(i\phi) \\
&= u_{00}\exp\left[ikx\sin(\alpha)\right] \\
&\approx u_{00}\exp\left[ikx\alpha\right] \\
&= u_{00}\exp\left[i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] \\
&\approx u_{00}\left[1 + i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] \\
&= u_{00}(x,y,0) + i\left(\frac{\pi w_0\alpha}{\lambda}\right)u_{10} \ .
\end{aligned} \tag{12}
$$

5.   TILTING A MISALIGNED BEAM

Building on the two previous sections, a tilt to an already shifted beam is

$$u(x,y,0) = u_{tilt(0,0)} + \left(\frac{a}{w_0}\right)u_{tilt(1,0)} \ . \tag{13}$$

Following the same approximations for a general tilted beam $(\alpha << \frac{\lambda}{w_0\pi})$, the first term is the same as in Eq. 9, while the second term is:

$$
\begin{aligned}
\left(\frac{a}{w_0}\right)u_{tilt(1,0)}(x,y,0) &= \left(\frac{a}{w_0}\right)u_{10}\exp(i\phi) \\
&= \left(\frac{a}{w_0}\right)u_{10}\exp\left[ikx\sin(\alpha)\right] \\
&\approx \left(\frac{a}{w_0}\right)u_{10}\exp\left[ikx\alpha\right] \\
&= \left(\frac{a}{w_0}\right)u_{10}\exp\left[i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] \\
&\approx \left(\frac{a}{w_0}\right)u_{10}\left[1 + i\left(\frac{2\pi x\alpha}{\lambda}\right)\right]
\end{aligned} \tag{14}
$$

4

In terms of u_{00}, the second term is:

$$i\left(\frac{2\pi x\alpha}{\lambda}\right)u_{10} = i\left(\frac{4\pi x^2\alpha a}{w_0^2\lambda}\right)u_{00} \tag{15}$$

Rewriting in terms of u_{20}:

$$i\left(\frac{4\pi x^2 a\alpha}{w_0^2\lambda}\right)u_{00} = i\left(\frac{2\alpha a}{\sqrt{2}\lambda}\right)\left[u_{20} + \frac{\sqrt{2}}{2}u_{00}\right] \tag{16}$$

Therefore, Eq. 13 is

$$u(x,y,0) = \left[1 + i\left(\frac{a\alpha}{\lambda}\right)\right]u_{00} + \left(\frac{a}{w_0}\right)u_{10} + i\left(\frac{2a\alpha}{\sqrt{2}\lambda}\right)u_{20} \tag{17}$$

6. MODELING TTL IN LISA'S LONG ARM

Jitter of a tilted beam, as outlined in the previous section, can be used to model TTL coupling in LISA's long arm

[1] C. Bond, D. Brown, A. Freise, and K. A. Strain, Living Reviews in Relativity **19**, 3 (2017), ISSN 1433-8351, URL https://doi.org/10.1007/s41114-016-0002-8.
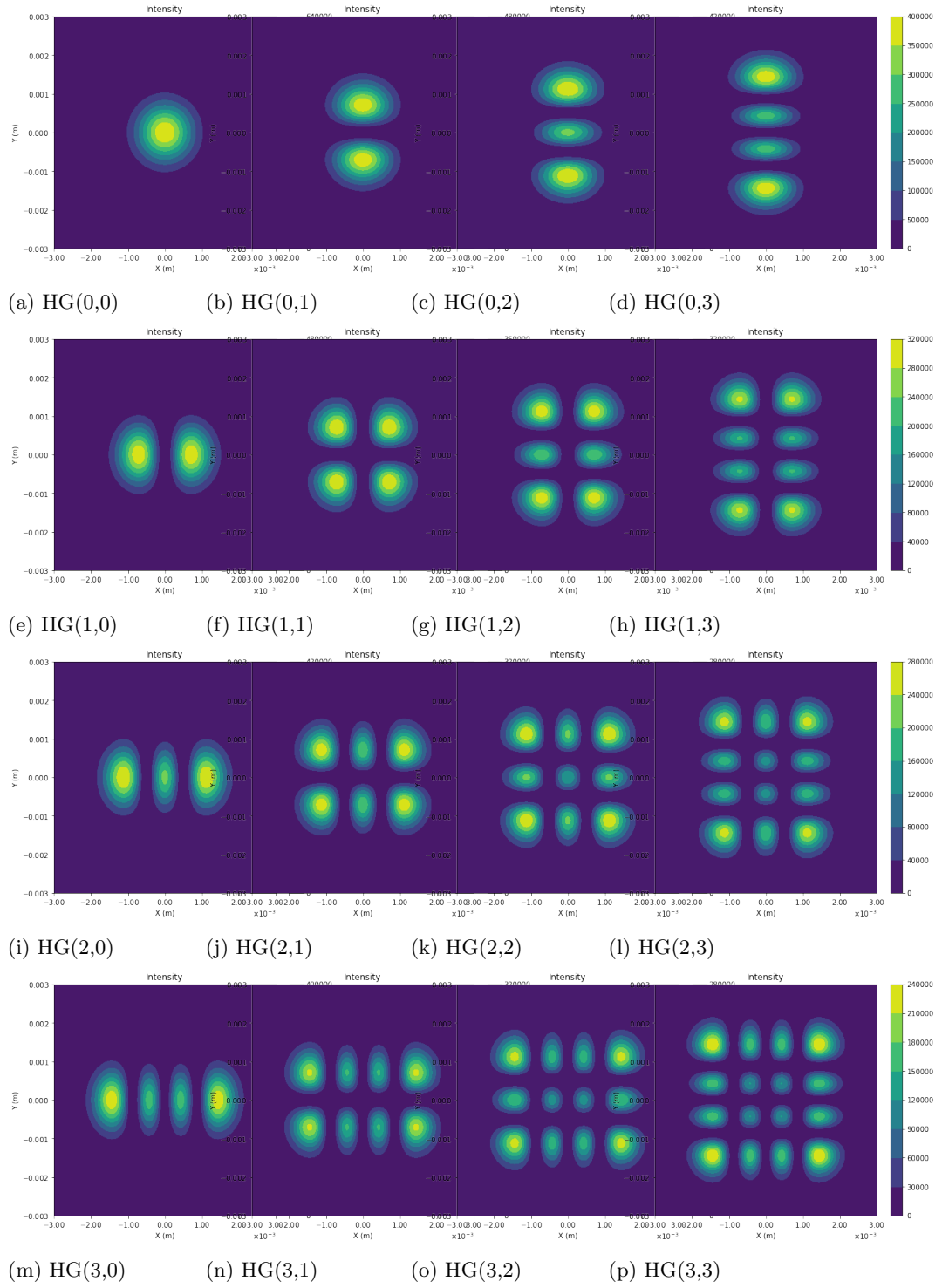
6

FIG. (A.1) Intensity profiles for HG modes u_{00} to u_{33} at the beam waist.