Calculating Hermite-Gauss Modes in Python

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PauLisa.py was used to calculate Hermite-Gauss(HG) modes. This document presents intensity profiles for "pure" modes and phase and intensity slices for higher-order modes. Beam tilts and misalignments relative to the optical axis can be used to model tilt-to-length(TTL) coupling in LISA's long arm.

1. EQUATIONS FOR HG MODES

The Living Reviews article [1] gives the general expression for HG modes as

$$u_{nm}(x,y,z) = (2^{n+m-1}n!m!\pi)^{-1/2}\frac{1}{w(z)}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)H_m\Big(\frac{\sqrt{2}y}{w(z)}\Big)\exp\Big(\frac{-ik(x^2+y^2)}{2R_c(z)} - \frac{x^2+y^2}{w(z)^2}\Big)\ . \tag{1}$$

In terms of the Gaussian beam parameter,

$$q(z) = iz_R + z - z_0 = q_0 + z - z_0\ , \tag{2}$$

this may be expressed as

$$\begin{aligned} u_{nm}(x,y,z) =& u_n(x,z)u_m(y,z) \\ =& \Big(\frac{2}{\pi}\Big)^{1/4}\Big(\frac{1}{2^n n! w_0}\Big)^{1/2}\Big(\frac{q_0}{q(z)}\Big)^{1/2}\Big(\frac{q_0 q*(z)}{q_0 * q(z)}\Big)^{n/2}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)\exp\Big(\frac{-ik(x^2)}{2q_z}\Big)\times u_m(y,z)\ , \end{aligned} \tag{3}$$

where the first three Hermite polynomials are given by

$$H_n(x) = \begin{cases} 1 & (n=0) \\ 2x & (n=1) \\ 4x^2-2 & (n=2) \end{cases}$$

For $n = 0, 1, 2$ and $m = 0$ at the beam waist(letting $z_0 = 0$):

$$u_{00}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{1}{w_0}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big]\ , \tag{4}$$

$$u_{10}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{2x}{w_0^2}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big]\ , \tag{5}$$

$$u_{20}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big[\Big(\frac{2\sqrt{2}x^2}{w_0^3}\Big) - \frac{\sqrt{2}}{2}\Big]\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big]\ , \tag{6}$$

Expressing u_{10} and u_{20} in terms of u_{00} at the beam waist,

$$u_{10}(x,y,0) = \frac{2x}{w_0}u_{00}(x,y,0) \tag{7}$$

$$u_{20}(x,y,0) = \Big[2\sqrt{2}\Big(\frac{x^2}{w_0^2}\Big) - \frac{\sqrt{2}}{2}\Big]u_{00}(x,y,0) \tag{8}$$

In PauLisa.py, Eq. 1 is represented by the *calculate* function and Eq. 3 as *calculate_q*. The outputs of both functions agree exactly.

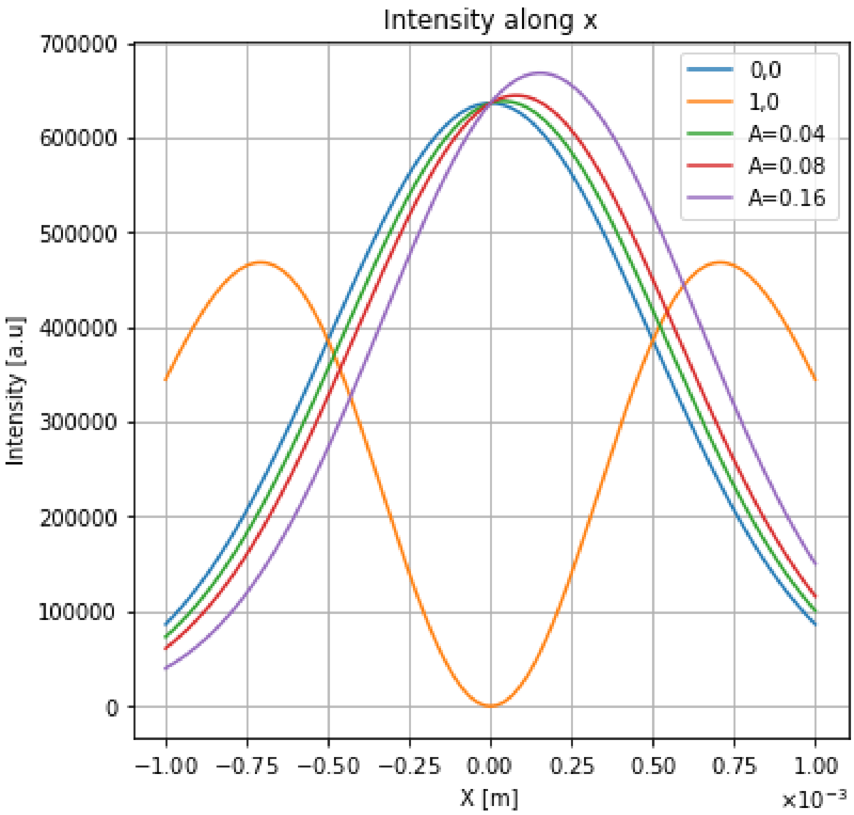


FIG. (1) Intensity at the beam waist and x=0 for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

2. INTENSITY PROFILES

In the appendix, intensity profiles produced by PauLisa.py for u_{00} to u_{33} are shown in Fig. A.1.

3. SHIFTED BEAM

For a small shift of the input axis, $a << w_0$, in the +x-direction relative to the cavity axis, the shifted u_{00} mode can be solved in terms of an added u_{10} mode up to a constant factor:

$$\begin{aligned}
u_{00}(x-a, y, 0) =& \Big(\frac{2}{\pi}\Big)^{-1/2} \Big(\frac{1}{w_0}\Big) \exp\Big(-\frac{(x-a)^2+y^2}{w_0^2}\Big) \\
=& \Big(\frac{2}{\pi}\Big)^{-1/2} \Big(\frac{1}{w_0}\Big) \exp\Big(-\frac{y^2}{w_0^2}\Big) \exp\Big(-\frac{(x-a)^2}{w_0^2}\Big) \\
=& u_{00}(x,y,0) \times exp\Big(\frac{2ax+a^2}{w_0^2}\Big) \\
=& u_{00}(x,y,0) \left[1 + \frac{2ax}{w_0^2} + \mathcal{O}\Big(\frac{a}{w}\Big)^2\right] \\
\approx& u_{00} + \Big(\frac{2ax}{w_0^2}\Big) u_{00} \\
=& u_{00}(x,y,0) + \frac{a}{w_0} u_{10}(x,y,0)\ .
\end{aligned} \tag{9}$$

Results for HG(0,0) and HG(1,0) addition are shown in Fig. 1.

(1,0) Scale	Pred. Shift[$\times 10^{-5}m$]	Act. Shift[$\times 10^{-5}m$]	%Error
0.04	4.0	3.9849	0.37
0.08	8.0	7.8998	1.25
0.16	16.0	15.2546	4.65

4. TILTED BEAM

A relative tilt between the input and cavity axis can be expressed as an addition of a first-order mode in quadrature phase,

$$u_{tilt}(x,y,0) = u_{00}(x,y,0) + \frac{ia\pi w_0}{\lambda} u_{10}(x,y,0)\ . \tag{10}$$

This addition varies the phase of the wave in the x-direction

$$u_{tilt}(x,y,0) = u_{00}(x,y,0)\exp(i\phi)\ . \tag{11}$$

Expressing this in terms of a quadrature phase addition of u_{10}, with $\alpha << \frac{\lambda}{w_0 \pi}$:

$$\begin{aligned} u_{tilt(0,0)} =& u_{00}\exp(i\phi) \\ =& u_{00}\exp\left[ikx\sin(\alpha)\right] \\ \approx& u_{00}\exp\left[ikx\alpha\right] \\ =& u_{00}\exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ \approx& u_{00}\left[1 + i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ =& u_{00}(x,y,0) + i\Big(\frac{\pi w_0 \alpha}{\lambda}\Big)u_{10}\ . \end{aligned} \tag{12}$$

5. TILTING A MISALIGNED BEAM

Building on the two previous sections, a tilt to an already shifted beam is

$$u(x,y,0) = u_{tilt(0,0)} + \Big(\frac{a}{w_0}\Big)u_{tilt(1,0)}\ . \tag{13}$$

Following the same approximations for a general tilted beam ($\alpha << \frac{\lambda}{w_0 \pi}$), the first term is the same as in Eq. 12, while the second term is:

$$\begin{aligned} \Big(\frac{a}{w_0}\Big)u_{tilt(1,0)}(x,y,0) =& \Big(\frac{a}{w_0}\Big)u_{10}\exp(i\phi) \\ =& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\sin(\alpha)\right] \\ \approx& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\alpha\right] \\ =& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ \approx& \Big(\frac{a}{w_0}\Big)u_{10}\left[1 + i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \end{aligned} \tag{14}$$

In terms of u_{00}, the second term is:

$$i\Big(\frac{2\pi x\alpha}{\lambda}\Big)u_{10} = i\Big(\frac{4\pi x^2\alpha a}{w_0^2\lambda}\Big)u_{00} \tag{15}$$

Rewriting in terms of u_{00} and u_{20}:

$$i\Big(\frac{4\pi x^2 a\alpha}{w_0^2\lambda}\Big)u_{00} = i\Big(\frac{2\pi\alpha a}{\sqrt{2}\lambda}\Big)\left[u_{20} + \frac{\sqrt{2}}{2}u_{00}\right] \tag{16}$$

Therefore, Eq. 13 is

$$u(x,y,0) = \left[1 + i\Big(\frac{\pi a\alpha}{\lambda}\Big)\right]u_{00} + \left[\Big(\frac{a}{w_0}\Big) + i\Big(\frac{\pi w_0\alpha}{\lambda}\Big)\right]u_{10} + i\Big(\frac{2\pi a\alpha}{\sqrt{2}\lambda}\Big)u_{20} \tag{17}$$

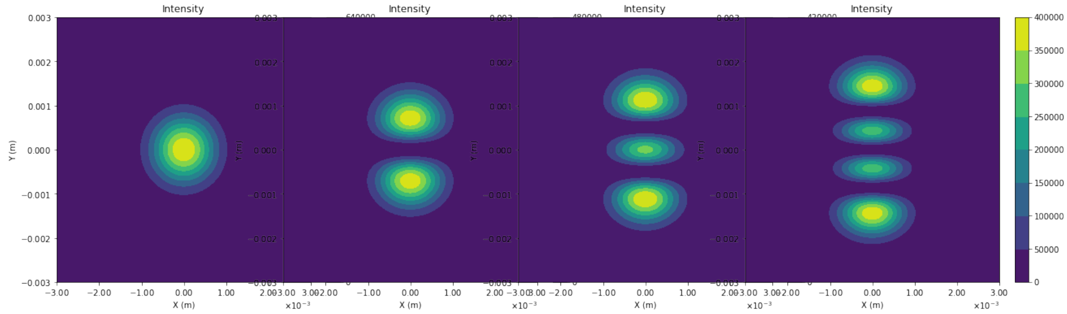
6. MODELING TTL IN LISA'S LONG ARM

Jitter of a tilted beam, as outlined in the previous section, can be used to model TTL coupling in LISA's long arm. Sinusoidal modulation of the terms in Eq. 17 which came as an effect of jitter, scaled in m, yields:

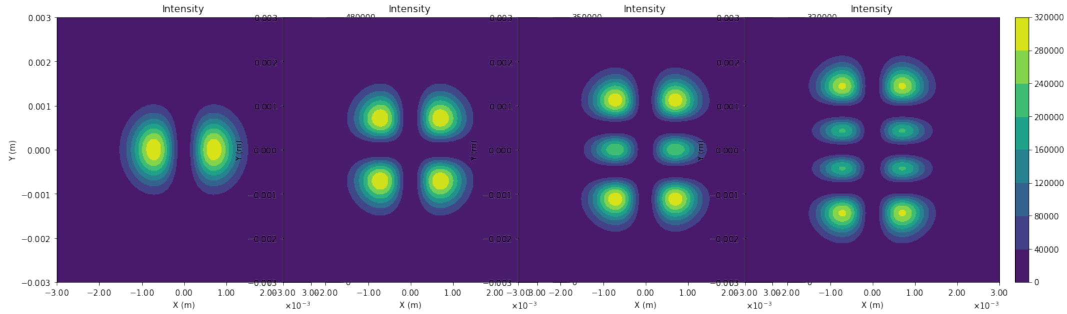
$$u(x,y,0) = \left[u_{00} + \frac{a}{w_0}u_{10}\right] + i\frac{\pi\alpha}{\lambda}\left[w_0 u_{10} + a\Big(u_{00} + \frac{2}{\sqrt{2}}u_{20}\Big)\right] m\sin(\omega t - \phi)\ . \tag{18}$$

Assume an arm of 2.5 Gm, $w_0 = 15$ cm, and photodetector radius = 15 cm.

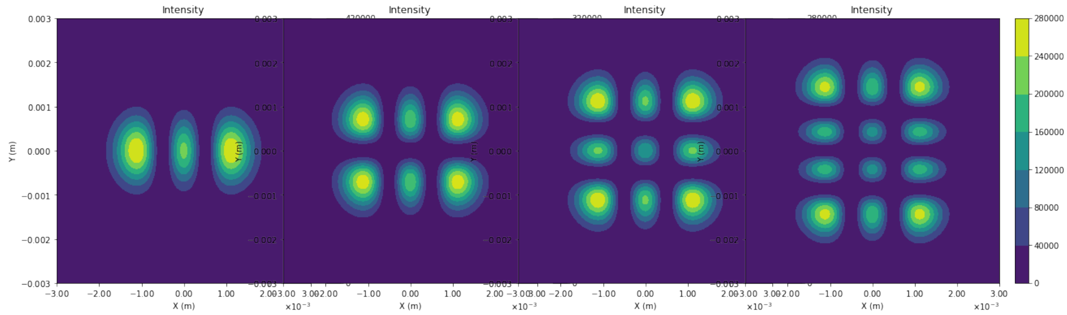
[1] C. Bond, D. Brown, A. Freise, and K. A. Strain, Living Reviews in Relativity **19**, 3 (2017), ISSN 1433-8351, URL https://doi.org/10.1007/s41114-016-0002-8.



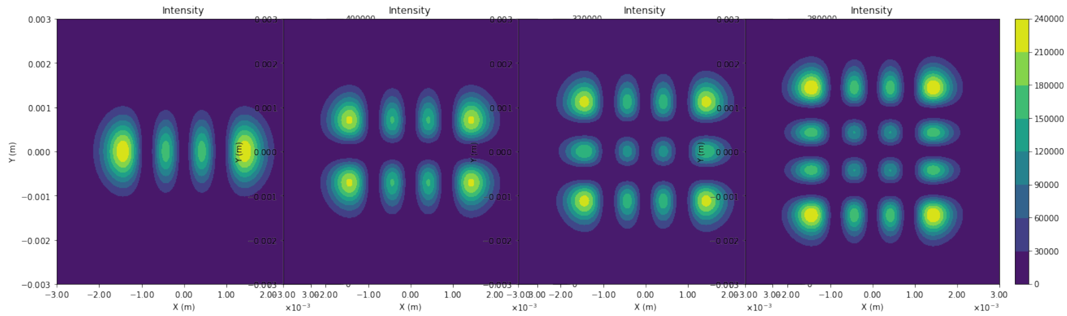
(a) HG(0,0) (b) HG(0,1) (c) HG(0,2) (d) HG(0,3)



(e) HG(1,0) (f) HG(1,1) (g) HG(1,2) (h) HG(1,3)



(i) HG(2,0) (j) HG(2,1) (k) HG(2,2) (l) HG(2,3)



(m) HG(3,0) (n) HG(3,1) (o) HG(3,2) (p) HG(3,3)

FIG. (A.1) Intensity profiles for HG modes u_{00} to u_{33} at the beam waist.