

# MODELING TILT-TO-LENGTH AND HERMITE-GAUSS MODES IN PYTHON

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A Python module, PauLisa.py, was developed and used to calculate Hermite-Gauss(HG) modes. Beam tilts and misalignments relative to the optical axis can be used to model tilt-to-length(TTL) coupling in LISA. This document presents theory and results of these calculations produced in the Jupyter Notebook IDE. Output of PauLisa.py shows good agreement with theory.

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## 1 HERMITE-GAUSS MODES

Hermite-Gauss (HG) modes represent a set of exact solutions of the paraxial wave equation

$$\nabla_t^2 u(x, y, z) - 2ik\partial_z u(x, y, z) = 0. \quad (1)$$

The Living Reviews article [1] gives the general expression for HG modes as

$$u_{nm}(x, y, z) = (2^{n+m-1}n!m!\pi)^{-1/2} \frac{1}{w(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(\frac{-ik(x^2 + y^2)}{2R_c(z)} - \frac{x^2 + y^2}{w(z)^2}\right). \quad (2)$$

In terms of the Gaussian beam parameter,

$$q(z) = iz_R + z - z_0 = q_0 + z - z_0, \quad (3)$$

HG modes may be expressed as

$$\begin{aligned} u_{nm}(x, y, z) &= u_n(x, z) u_m(y, z) \\ &= \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n! w_0}\right)^{1/2} \left(\frac{q_0}{q(z)}\right)^{1/2} \left(\frac{q_0 q^*(z)}{q_0^* q(z)}\right)^{n/2} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left(\frac{-ik(x^2)}{2q_z}\right) \times u_m(y, z), \end{aligned} \quad (4)$$

where the first three Hermite polynomials are given by

$$H_n(x) = \begin{cases} 1 & (n = 0) \\ 2x & (n = 1) \\ 4x^2 - 2 & (n = 2) \end{cases}$$

The HG modes are orthonormal, such that

$$\int \int dx dy u_{nm} u_{n'm'}^* = \delta_{nn'} \delta_{mm'}. \quad (5)$$

For approximations used in later sections, it is useful to express higher-order modes in terms of the fundamental mode. For  $n = 0, 1, 2$  and  $m = 0$  at the beam waist (letting  $z_0 = 0$ ):

$$u_{00}(x, y, 0) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{1}{w_0}\right) \exp\left[-\left(\frac{x^2 + y^2}{w_0^2}\right)\right], \quad (6)$$

$$u_{10}(x, y, 0) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{2x}{w_0^2}\right) \exp\left[-\left(\frac{x^2 + y^2}{w_0^2}\right)\right], \quad (7)$$

$$u_{20}(x, y, 0) = \left(\frac{2}{\pi}\right)^{1/2} \left[\left(\frac{2\sqrt{2}x^2}{w_0^3}\right) - \frac{\sqrt{2}}{2}\right] \exp\left[-\left(\frac{x^2 + y^2}{w_0^2}\right)\right], \quad (8)$$

Expressing  $u_{10}$  and  $u_{20}$  in terms of  $u_{00}$  at the beam waist,

$$u_{10}(x, y, 0) = \frac{2x}{w_0} u_{00}(x, y, 0) \quad (9)$$

$$u_{20}(x, y, 0) = \left[2\sqrt{2}\left(\frac{x^2}{w_0^2}\right) - \frac{\sqrt{2}}{2}\right] u_{00}(x, y, 0) \quad (10)$$

In PauLisa.py, Eq. 2 is represented by the *calculate* function and Eq. 4 as *calculate\_q*. The outputs of both functions agree exactly. In the appendix, intensity profiles produced by PauLisa.py for  $u_{00}$  to  $u_{33}$  are shown in Fig. A.1.

## 2 SHIFTED BEAM APPROXIMATION

For a small shift of the input axis,  $a \ll w_0$ , in the +x-direction relative to the cavity axis, the shifted  $u_{00}$  mode can be solved in terms of an added  $u_{10}$  mode up to a constant factor:

$$\begin{aligned}
 u_{00}(x-a, y, 0) &= \left(\frac{2}{\pi}\right)^{-1/2} \left(\frac{1}{w_0}\right) \exp\left(-\frac{(x-a)^2 + y^2}{w_0^2}\right) \\
 &= \left(\frac{2}{\pi}\right)^{-1/2} \left(\frac{1}{w_0}\right) \exp\left(-\frac{y^2}{w_0^2}\right) \exp\left(-\frac{(x-a)^2}{w_0^2}\right) \\
 &= u_{00}(x, y, 0) \times \exp\left(\frac{2ax + a^2}{w_0^2}\right) \\
 &= u_{00}(x, y, 0) \left[1 + \frac{2ax}{w_0^2} + \mathcal{O}\left(\frac{a}{w}\right)^2\right] \\
 &\approx u_{00} + \left(\frac{2ax}{w_0^2}\right) u_{00} \\
 &= u_{00}(x, y, 0) + \frac{a}{w_0} u_{10}(x, y, 0) .
 \end{aligned} \tag{11}$$

The shift for mode coefficients  $C_{nm}$  is then

$$a \approx \frac{\Re(C_{10})}{\Re(C_{00})} w_0 . \tag{12}$$

Results for HG(0,0) and HG(1,0) addition are shown in Fig. 2.

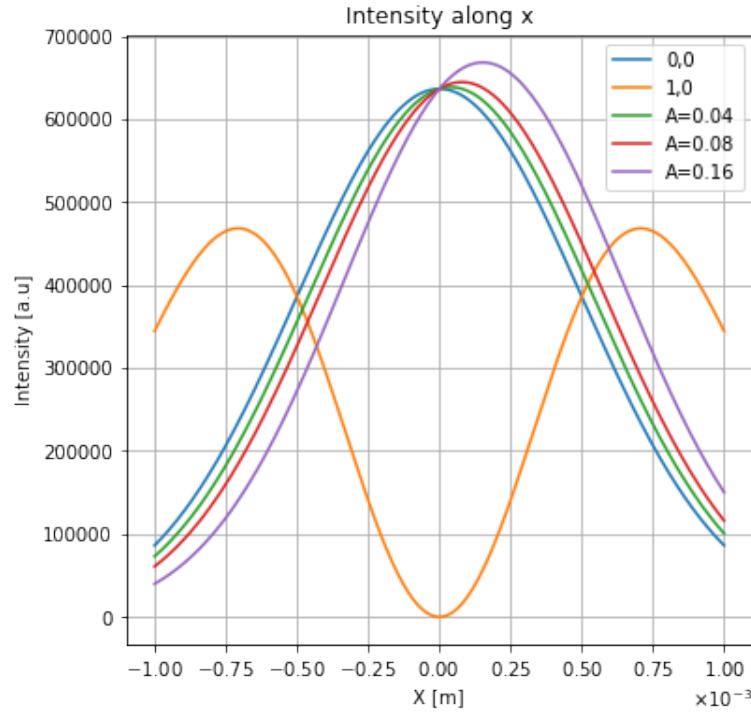


FIG. (2) Intensity at the beam waist and  $y = 0$  for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

(1,0) Scale	Pred. Shift [ $\times 10^{-5}m$ ]	Act. Shift [ $\times 10^{-5}m$ ]	%Error
0.04	4.0	3.9849	0.37
0.08	8.0	7.8998	1.25
0.16	16.0	15.2546	4.65

TABLE (I) Calculated and predicted shift in peaks for varying scales of  $HG_{10}$  added in phase to  $HG_{00}$ .

### 3 TILTED BEAM APPROXIMATION

A relative tilt between the input and cavity axis can be expressed as an addition of a first-order mode in quadrature phase,

$$u_{tilt}(x, y, 0) = u_{00}(x, y, 0) + iu_{10}(x, y, 0) . \quad (13)$$

This addition varies the phase of the wave in the x-direction

$$u_{tilt}(x, y, 0) = u_{00}(x, y, 0) \exp(i\phi) . \quad (14)$$

Expressing this in terms of a quadrature phase addition of  $u_{10}$ , with  $\alpha < \frac{\lambda}{w_0\pi}$ :

$$\begin{aligned} u_{tilt(0,0)} &= u_{00} \exp(i\phi) \\ &= u_{00} \exp[i k x \sin(\alpha)] \\ &\approx u_{00} \exp[i k x \alpha] \\ &= u_{00} \exp\left[i \left(\frac{2\pi x \alpha}{\lambda}\right)\right] \\ &\approx u_{00} \left[1 + i \left(\frac{2\pi x \alpha}{\lambda}\right)\right] \\ &= u_{00}(x, y, 0) + i \left(\frac{\pi w_0 \alpha}{\lambda}\right) u_{10} . \end{aligned} \quad (15)$$

Therefore, the predicted angle scales proportionally with the coefficient of the  $u_{10}$  mode,  $C_{10}$ ,

$$\alpha \approx \frac{|\Im(C_{10})|}{\Re(C_{00})} \frac{\lambda}{\pi w_0} \approx \frac{|\Im(C_{10})|}{\Re(C_{00})} \Theta , \quad (16)$$

where  $\Theta = \frac{\pi w_0}{\lambda}$  is the diffraction angle. Results for  $HG_{10}$  quadrature-phase addition are shown in Table 2. Graphical results are shown in Fig. 3.

(1,0) Scale (Imag.)	Pred. Angle [ $\times 10^{-5}$ rad.]	Act. Angle [ $\times 10^{-5}$ rad.]	%Error
0.04	1.3547	1.3518	0.00212
0.08	2.7094	2.6866	0.00840
0.16	5.4189	5.2445	0.03217

TABLE (II) Calculated and expected wavefront angles for varying scales of  $HG_{10}$  added in quadrature phase to  $HG_{00}$ .

By Eq. 15, phase should vary with  $x$  as

$$\frac{d\phi}{dx} \approx \frac{2\pi\alpha}{\lambda} . \quad (17)$$

Figure 4 shows agreement with this approximation.

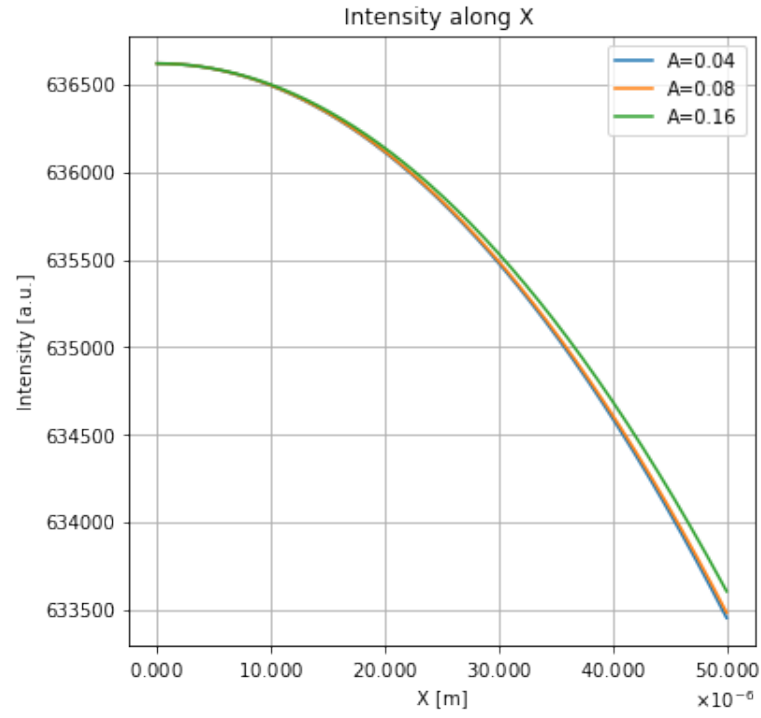


FIG. (3) Intensity at the beam waist and  $y = 0$  for HG(0,0) and HG(1,0) addition in quadrature phase. The variable  $A$  represents the (imaginary) scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

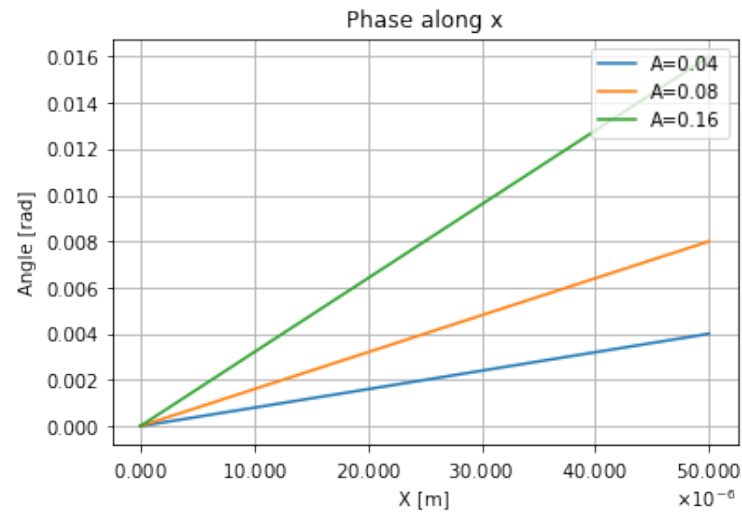


FIG. (4) Phase at the beam waist and  $y = 0$  for HG(0,0) and HG(1,0) modes alongside combined modes. The variable  $A$  represents the (imaginary) scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient. Phase varies with  $x$  as predicted by Eq. 17

## 4 TILTING A MISALIGNED BEAM

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Building on the two previous sections, a tilt to an already shifted beam is

$$\sum_{n,m} u_{nm}(x, y, 0) = u_{00(tilt)}(x, y, 0) + \left(\frac{a}{w_0}\right) u_{10(tilt)}(x, y, 0) . \quad (18)$$

Following the same approximations for a general tilted beam ( $\alpha < \frac{\lambda}{w_0\pi}$ ), the first term is the same as in Eq. 15,

$$u_{00(tilt)}(x, y, 0) = u_{00}(x, y, 0) + i\left(\frac{\pi w_0 \alpha}{\lambda}\right) u_{10} ,$$

while the  $u_{10}$  term in Eq. 18 is:

$$\begin{aligned} \left(\frac{a}{w_0}\right) u_{10(tilt)}(x, y, 0) &= \left(\frac{a}{w_0}\right) u_{10} \exp(i\phi) \\ &= \left(\frac{a}{w_0}\right) u_{10} \exp[ikx \sin(\alpha)] \\ &\approx \left(\frac{a}{w_0}\right) u_{10} \exp[ikx\alpha] \\ &= \left(\frac{a}{w_0}\right) u_{10} \exp\left[i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] \\ &\approx \left(\frac{a}{w_0}\right) u_{10} \left[1 + i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] . \end{aligned} \quad (19)$$

In terms of  $u_{00}$ , the imaginary term in Eq. 19 is:

$$i\left(\frac{2\pi x\alpha}{\lambda}\right) u_{10} = i\left(\frac{4\pi x^2 \alpha a}{w_0^2 \lambda}\right) u_{00} . \quad (20)$$

Rewriting in terms of  $u_{00}$  and  $u_{20}$ :

$$i\left(\frac{4\pi x^2 a \alpha}{w_0^2 \lambda}\right) u_{00} = i\left(\frac{2\pi \alpha a}{\sqrt{2}\lambda}\right) \left[u_{20} + \frac{\sqrt{2}}{2} u_{00}\right] . \quad (21)$$

Therefore, Eq. 18 for a shifted then tilted beam of fundamental mode at the waist is

$$\sum_{n,m} u_{nm}(x, y, 0) = \left[1 + i\left(\frac{\pi a \alpha}{\lambda}\right)\right] u_{00} + \left[\frac{a}{w_0} + i\left(\frac{\pi w_0 \alpha}{\lambda}\right)\right] u_{10} + i\left(\frac{\sqrt{2}\pi a \alpha}{\lambda}\right) u_{20} . \quad (22)$$



## 5 TILT-TO-LENGTH COUPLING

LISA uses PD's to measure interference of a received beam with a reference beam, resulting in beat notes which are used to determine phase difference ( $\Delta\phi = \phi_1 - \phi_2$ ) of the two beams. Fields of the reference and received beam defined with a basis ( $w_0, z_0$ ) are, respectively:

$$E_{ref} = E_{0(ref)} e^{i((\omega_{ref}t) + \phi_1)} u_{00}(w_{0ref}, z_{0ref}) . \quad (23)$$

$$E_{rec} = E_{0(rec)} e^{i((\omega_{rec}t) + \phi_2)} \sum_{n,m=0} u_{nm}(w_{0rec}, z_{0rec}) , \quad (24)$$

These fields produce a power at a single PD (assuming the PD area is large relative to the beam)

$$P_{pd} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E^* E \, dx dy, \quad (25)$$

where the electric field is the sum of the received and reference fields. Let the spatial component of the received beam be given by Eq. 22. By orthonormality of HG modes, only the imaginary part of the received beam's  $u_{00}$  mode contributes to an overall phase shift. The sum of the electric fields of the received and reference beam of the same basis is, effectively,

$$E = E_{0(ref)} e^{i((\omega_{ref}t) + \phi_1)} u_{00} + E_{0(rec)} e^{i((\omega_{rec}t) + \phi_2)} u_{00} \left[ 1 + i \left( \frac{\pi a \alpha}{\lambda} \right) \right] , \quad (26)$$

and, condensing the  $E_0$  terms,

$$E^* E = u_{00}^* u_{00} |E_0|^2 \left\{ 2 + \left( \frac{\pi a \alpha}{\lambda} \right)^2 + e^{i\Delta\phi} \left[ 1 - i \frac{\pi a \alpha}{\lambda} \right] + e^{-i\Delta\phi} \left[ 1 + i \frac{\pi a \alpha}{\lambda} \right] \right\} . \quad (27)$$

Neglecting time-independent terms and  $u_{00}$  (factors by orthonormality), and letting  $\Delta\omega = \omega_{0ref} - \omega_{0rec}$ :

$$\begin{aligned} E^* E &= |E_0|^2 \left\{ e^{i(\Delta\omega t + \Delta\phi)} \left[ 1 - i \frac{\pi a \alpha}{\lambda} \right] + e^{-i(\Delta\omega t + \Delta\phi)} \left[ 1 + i \frac{\pi a \alpha}{\lambda} \right] \right\} \\ &= |E_0|^2 \left\{ \left[ e^{i(\Delta\omega t + \Delta\phi)} + e^{-i(\Delta\omega t + \Delta\phi)} \right] + i \frac{\pi a \alpha}{\lambda} \left[ e^{-i(\Delta\omega t + \Delta\phi)} - e^{i(\Delta\omega t + \Delta\phi)} \right] \right\} \\ &= 2|E_0|^2 \left\{ \cos(\Delta\omega t + \Delta\phi) + \frac{\pi a \alpha}{\lambda} \sin(\Delta\omega t + \Delta\phi) \right\} . \end{aligned} \quad (28)$$

On demodulating and neglecting  $\Delta\phi$  at the phasemeter, the I signal is

$$\begin{aligned} I &= E^* E \times \cos(\Delta\omega t) \\ &= 2|E_0|^2 \left\{ \cos(\Delta\omega t) + \frac{\pi a \alpha}{\lambda} \sin(\Delta\omega t) \right\} \times \cos(\Delta\omega t) \\ &= |E_0|^2 \left\{ 1 + \cos(2\Delta\omega t) + \frac{\pi a \alpha}{\lambda} \sin(2\Delta\omega t) \right\} , \end{aligned} \quad (29)$$

and the Q signal is

$$\begin{aligned} Q &= E^* E \times \sin(\Delta\omega t) \\ &= 2|E_0|^2 \left\{ \cos(\Delta\omega t) + \frac{\pi a \alpha}{\lambda} \sin(\Delta\omega t) \right\} \times \sin(\Delta\omega t) \\ &= |E_0|^2 \left\{ \frac{\pi a \alpha}{\lambda} (1 - \cos(2\Delta\omega t)) + \sin(2\Delta\omega t) \right\} , \end{aligned} \quad (30)$$

Therefore, after a low pass filter, the phase is linear in shift and tilt

$$\begin{aligned}
 \Phi &= \arctan\left(\frac{Q}{I}\right) \\
 &= \arctan\left(\frac{\frac{\pi a \alpha}{\lambda}}{1}\right) \\
 &\approx \frac{\pi a \alpha}{\lambda} .
 \end{aligned} \tag{31}$$

Compared to a single misalignment of a beam (Eq. 15), which only has a phase shift contribution via a  $u_{10}$  term, the shifted and tilted beam has phase shift via a  $u_{00}$  term in quadrature phase to the reference beam. Moreover, the S/C rotating about the COM of the TM means the beam with a waist centered on the TM also appears to rotate about the TM CoM, resulting in no TTL coupling as the wavefronts at the receiving S/C appear spherically centered on the TM CoM. The result of a misaligned then tilted beam, however, is tilt-to-length coupling, where jitter has coupled into length.

Section C of the LISA Payload Description Document (PDD) outlines expected TTL couplings. Spacecraft jitter is assigned  $\sim 10$  nrad/ $\sqrt{\text{Hz}}$ . Lateral alignment offset with optical bench (OB) is  $\sim 20$   $\mu\text{m}$ . These combine for  $\sim 20$  pm/ $\sqrt{\text{Hz}}$ .

## 6 MODULATING TTL OVER LISA'S LONG ARM

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Jitter of a tilted beam, as outlined in the previous section, can be used to model TTL coupling in LISA's long arm. Sinusoidal modulation of the terms in Eq. 22 which came as an effect of jitter, scaled in  $m$ , yields:

$$u(x, y, 0) = \left[ u_{00} + \frac{a}{w_0} u_{10} \right] + i \frac{\pi \alpha}{\lambda} \left[ a \left( u_{00} + \sqrt{2} u_{20} \right) + w_0 u_{10} \right] m \sin(\omega t - \phi) . \quad (32)$$

Assume an arm of 2.5 Gm,  $w_0 = 15$  cm, and PD radius = 15 cm.

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- [1] C. Bond, D. Brown, A. Freise, and K. A. Strain, Living Reviews in Relativity **19**, 3 (2017), ISSN 1433-8351, URL <https://doi.org/10.1007/s41114-016-0002-8>.

# A INTENSITY PLOTS

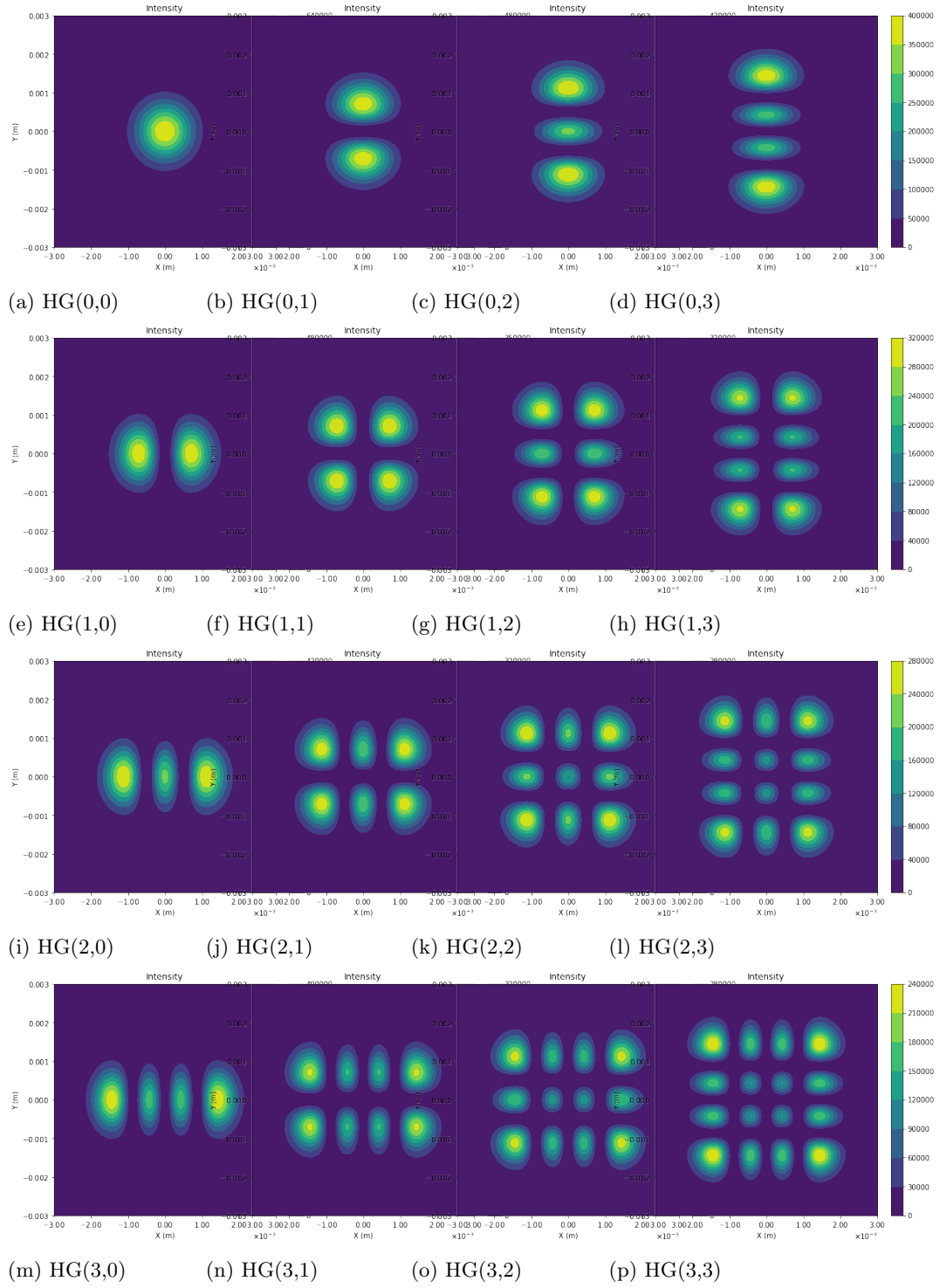


FIG. (A.1) Intensity profiles for HG modes  $u_{00}$  to  $u_{33}$  at the beam waist.

## B ACRONYMS

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### ACRONYMS

**CoM** *center of mass*

**HG** *Hermite – Gauss*

**PD** *photodetector*

**PDD** *Payload Definition Document*

**S/C** *spacecraft*

**TM** *test mass*

**TTL** *tilt to length*