Calculating Hermite-Gauss Modes in Python

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PauLisa.py was used to calculate Hermite-Gauss(HG) modes. This document presents intensity profiles for "pure" modes and phase and intensity slices for higher-order modes. Beam tilts and misalignments relative to the optical axis can be used to model tilt-to-length(TTL) coupling in LISA's long arm.

1. EQUATIONS FOR HG MODES

The Living Review Article [1] gives the general expression for HG modes as

$$u_{nm}(x,y,z) = (2^{n+m-1}n!m!\pi)^{-1/2}\frac{1}{w(z)}H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)H_m\Big(\frac{\sqrt{2}y}{w(z)}\Big)\exp\Big(\frac{-ik(x^2+y^2)}{2R_c(z)} - \frac{x^2+y^2}{w(z)^2}\Big) . \tag{1}$$

In terms of the Gaussian beam parameter,

$$q(z) = iz_R + z - z_0 = q_0 + z - z_0 , \tag{2}$$

this may be expressed as

$$\begin{aligned} u_{nm}(x,y,z) =& u_n(x,z)u_m(y,z) \\ =& \Big(\frac{2}{\pi}\Big)^{1/4}\Big(\frac{1}{2^n n! w_0}\Big)^{1/2}\Big(\frac{q_0}{q(z)}\Big)^{1/2}\Big(\frac{q_0 q*(z)}{q_0 * q(z)}\Big)^{n/2} H_n\Big(\frac{\sqrt{2}x}{w(z)}\Big)\exp\Big(\frac{-ik(x^2)}{2q_z}\Big) \times u_m(y,z) , \end{aligned} \tag{3}$$

where the first three Hermite polynomials are given by

$$H_n(x) = \begin{cases} 1 & (n=0) \\ 2x & (n=1) \\ 4x^2-2 & (n=2) \end{cases}$$

For $n = 0, 1, 2$ and $m = 0$ at the beam waist(letting $z_0 = 0$):

$$u_{00}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{1}{w_0}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{4}$$

$$u_{10}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{2x}{w_0^2}\Big)\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{5}$$

$$u_{20}(x,y,0) = \Big(\frac{2}{\pi}\Big)^{1/2}\Big[\Big(\frac{2\sqrt{2}x^2}{w_0^3}\Big) - \frac{\sqrt{2}}{2}\Big]\exp\Big[-\Big(\frac{x^2+y2}{w_0^2}\Big)\Big] , \tag{6}$$

Expressing u_{10} and u_{20} in terms of u_{00} at the beam waist,

$$u_{10}(x,y,0) = \frac{2x}{w_0}u_{00}(x,y,0) \tag{7}$$

$$u_{20}(x,y,0) = \Big[2\sqrt{2}\Big(\frac{x^2}{w_0^2}\Big) - \frac{\sqrt{2}}{2}\Big]u_{00}(x,y,0) \tag{8}$$

In PauLisa.py, Eq. 1 is represented by the *calculate* function and Eq. 3 as *calculate_q*. The outputs of both functions agree exactly.

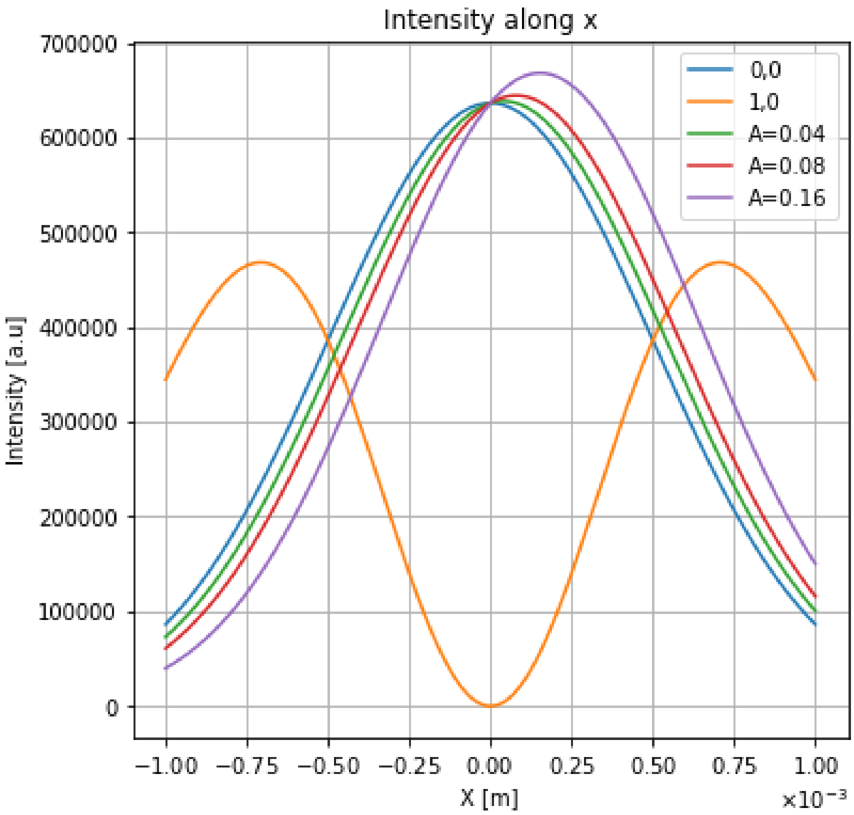


FIG. (1) Intensity at the beam waist and x=0 for HG(0,0) and HG(1,0) modes alongside combined modes. The variable A represents the scale of the HG(1,0) mode, where 1 is the HG(0,0) mode coefficient.

2. INTENSITY PROFILES

In the appendix, intensity profiles produced by PauLisa.py for u_{00} to u_{33} are shown in Fig. A.1.

3. SHIFTED BEAM

For a small shift of the input axis, $a<<w_0$, in the +x-direction relative to the cavity axis, the shifted u_{00} mode can be solved in terms of an added u_{10} mode up to a constant factor:

$$\begin{aligned} u_{00}(x-a,y,0) =& \Big(\frac{2}{\pi}\Big)^{-1/2}\Big(\frac{1}{w_0}\Big)\exp\Big(-\frac{(x-a)^2+y^2}{w_0^2}\Big) \\ =& \Big(\frac{2}{\pi}\Big)^{-1/2}\Big(\frac{1}{w_0}\Big)\exp\Big(-\frac{y^2}{w_0^2}\Big)\exp\Big(-\frac{(x-a)^2}{w_0^2}\Big) \\ =& u_{00}(x,y,0)\times exp\Big(\frac{2ax+a^2}{w_0^2}\Big) \\ =& u_{00}(x,y,0)\Big[1+\frac{2ax}{w_0^2}+\mathcal{O}\Big(\frac{a}{w}\Big)^2\Big] \\ \approx& u_{00}+\Big(\frac{2ax}{w_0^2}\Big)u_{00} \\ =& u_{00}(x,y,0)+\frac{a}{w_0}u_{10}(x,y,0)\ . \end{aligned} \tag{9}$$

Results for HG(0,0) and HG(1,0) addition are shown in Fig. 1.

(1,0) Scale	Pred. Shift[$\times 10^{-5}m$]	Act. Shift[$\times 10^{-5}m$]	%Error
0.04	4.0	3.9849	0.37
0.08	8.0	7.8998	1.25
0.16	16.0	15.2546	4.65

4. TILTED BEAM

A relative tilt between the input and cavity axis can be expressed as an addition of a first-order mode in quadrature phase,

$$u_{tilt}(x,y,0) = u_{00}(x,y,0) + \frac{ia\pi w_0}{\lambda} u_{10}(x,y,0) \ . \tag{10}$$

This addition varies the phase of the wave in the x-direction

$$u_{tilt}(x,y,0) = u_{00}(x,y,0)\exp(i\phi) \ . \tag{11}$$

Expressing this in terms of a quadrature phase addition of u_{10}, with $\alpha << \frac{\lambda}{w_0 \pi}$:

$$\begin{aligned} u_{tilt(0,0)} =& u_{00}\exp(i\phi) \\ =& u_{00}\exp\left[ikx\sin(\alpha)\right] \\ \approx& u_{00}\exp\left[ikx\alpha\right] \\ =& u_{00}\exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ \approx& u_{00}\left[1 + i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ =& u_{00}(x,y,0) + i\Big(\frac{\pi w_0 \alpha}{\lambda}\Big)u_{10} \ . \end{aligned} \tag{12}$$

5. TILTING A MISALIGNED BEAM

Building on the two previous sections, a tilt to an already shifted beam is

$$u(x,y,0) = u_{tilt(0,0)} + \Big(\frac{a}{w_0}\Big)u_{tilt(1,0)} \ . \tag{13}$$

Following the same approximations for a general tilted beam ($\alpha << \frac{\lambda}{w_0 \pi}$), the first term is the same as in Eq. 9, while the second term is:

$$\begin{aligned} \Big(\frac{a}{w_0}\Big)u_{tilt(1,0)}(x,y,0) =& \Big(\frac{a}{w_0}\Big)u_{10}\exp(i\phi) \\ =& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\sin(\alpha)\right] \\ \approx& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[ikx\alpha\right] \\ =& \Big(\frac{a}{w_0}\Big)u_{10}\exp\left[i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \\ \approx& \Big(\frac{a}{w_0}\Big)u_{10}\left[1 + i\Big(\frac{2\pi x\alpha}{\lambda}\Big)\right] \end{aligned} \tag{14}$$

In terms of u_{00}, the second term is:

$$i\Big(\frac{2\pi x\alpha}{\lambda}\Big)u_{10} = i\Big(\frac{4\pi x^2\alpha a}{w_0^2\lambda}\Big)u_{00} \tag{15}$$

Rewriting in terms of u_{20}:

$$i\Big(\frac{4\pi x^2 a\alpha}{w_0^2\lambda}\Big)u_{00} = i\Big(\frac{2\alpha a}{\sqrt{2}\lambda}\Big)\left[u_{20} + \frac{\sqrt{2}}{2}u_{00}\right] \tag{16}$$

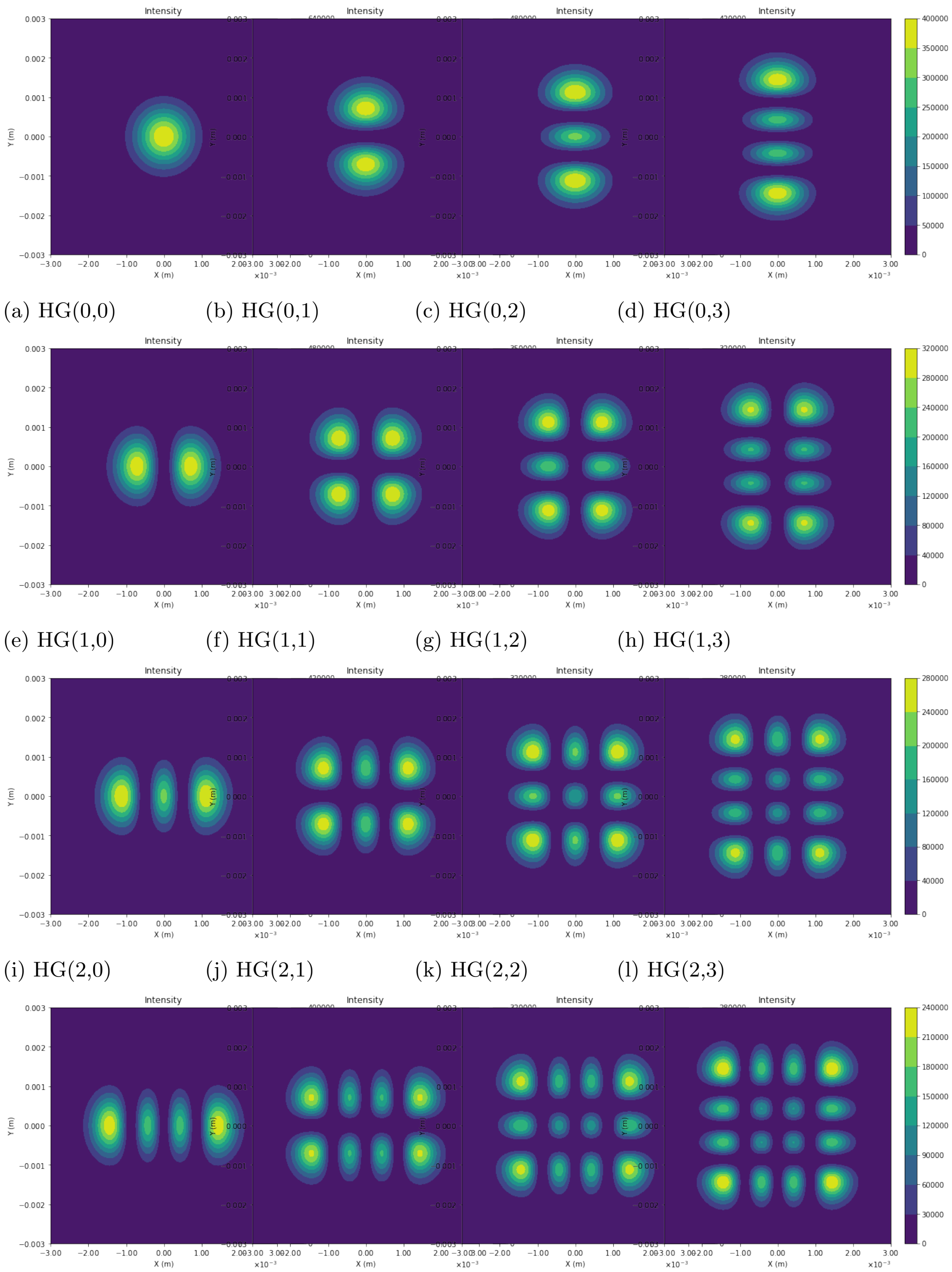
Therefore, Eq. 13 is

$$u(x,y,0) = \left[1 + i\Big(\frac{a\alpha}{\lambda}\Big)\right]u_{00} + \Big(\frac{a}{w_0}\Big)u_{10} + i\Big(\frac{2a\alpha}{\sqrt{2}\lambda}\Big)u_{20} \tag{17}$$

6. MODELING TTL IN LISA'S LONG ARM

Jitter of a tilted beam, as outlined in the previous section, can be used to model TTL coupling in LISA's long arm

[1] C. Bond, Living Reviews in Rel. **19:3** (2016), URL https://link.springer.com/article/10.12942%2Flrr-2010-1.



(a) HG(0,0) (b) HG(0,1) (c) HG(0,2) (d) HG(0,3)

(e) HG(1,0) (f) HG(1,1) (g) HG(1,2) (h) HG(1,3)

(i) HG(2,0) (j) HG(2,1) (k) HG(2,2) (l) HG(2,3)

(m) HG(3,0) (n) HG(3,1) (o) HG(3,2) (p) HG(3,3)

FIG. (A.1) Intensity profiles for HG modes u_{00} to u_{33} at the beam waist.