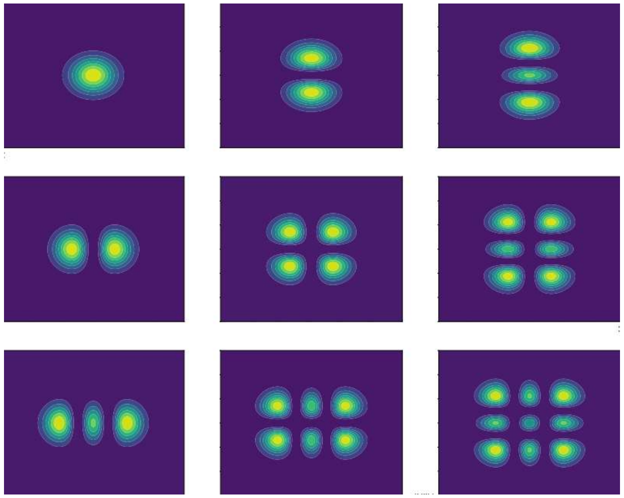
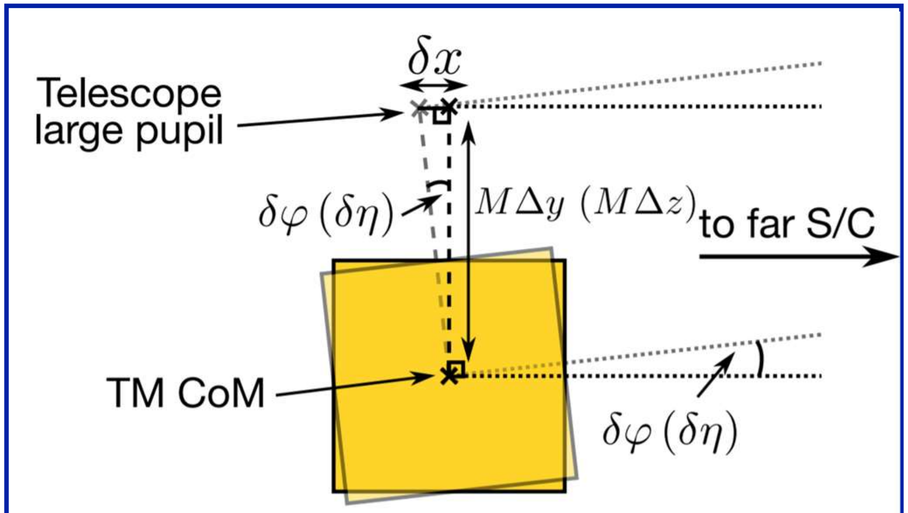
Tilt-to-Length Coupling: Effects of Beam Shape and Photoreceiver Selection on LISA Signals

Paul Edwards



TILT-TO-LENGTH COUPLING IN LISA

- Misalignments between LISA optical bench and bi-directional telescope introduce noise (TTL coupling).
- TTL in the TM interferometry is expected via spacecraft angular jitter relative to the reflected beam.
  - **S/C jitter** $\sim 10\ \mathrm{nrad}/\sqrt{\mathrm{Hz}}$
  - **OB Lateral alignment offset** $\sim 20\ \mu m$
  - **Combined** $\sim 20\ \mathrm{pm}/\sqrt{\mathrm{Hz}}$
- Selection of received and reference beam shape and photoreceiver dimensions can play a critical role in reducing TTL.
- Light path simulation provides an avenue for refining tolerances and the shift-tilt parameter space which may be incorporated into LISANODE.



$$\delta x = M\Delta y \sin(\delta\varphi) + M\Delta z \sin(\delta\eta) \simeq M\Delta y\, \delta\varphi + M\Delta z\, \delta\eta$$

Fig. 1. TTL coupling from S/C jitter coupled to lateral offset.

EXTRACTING THE SIGNAL PHASE

- Beat note signal phase, for received(*RX*) and reference (*LO*) beam :

$$\phi_{beat} = \arg\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{LO}^{*} E_{RX}\; dxdy\right]$$

- Phase of DWS and LPS is calculated for planes of split photodetector with appropriate integration bounds (e.g., half-plane PD with left(*L*) and right(*R*) sides):

$$\Phi_{DWS} = \frac{1}{2}[\phi_R - \phi_L] \qquad \Phi_{LPS} = \frac{1}{2}[\phi_R + \phi_L]$$

- Superposition of HG modes provide a spatial component of electric field. e.g., the received beam(*rec*):

$$E_{rec} = E_{0\ (rec)} e^{i((\omega_{rec} t)+\phi_2)} \sum_{n,m=0} u_{nm}(w_{0\ rec}, z_{0\ rec})$$

HERMITE-GAUSS MODES

$$u_{nm}(x,y,z) = (2^{n+m-1} n! m! \pi)^{-1/2} \frac{1}{w(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(\frac{-ik(x^2+y^2)}{2R_c(z)} - \frac{x^2+y^2}{w(z)^2}\right)$$

- Set of exact solutions to paraxial wave equation using Hermite polynomials

$$\nabla_t^2 u(x,y,z) - 2ik\partial z u(x,y,z) = 0$$

- Orthonormal, s.t.:

$$\int\int dxdy\ u_{nm} u^*_{n'm'} = \delta_{nn'} \delta_{mm'}$$

- Can approximate any paraxial beam as superposition of HG modes (e.g., tilted, shifted tophat and Gaussian)

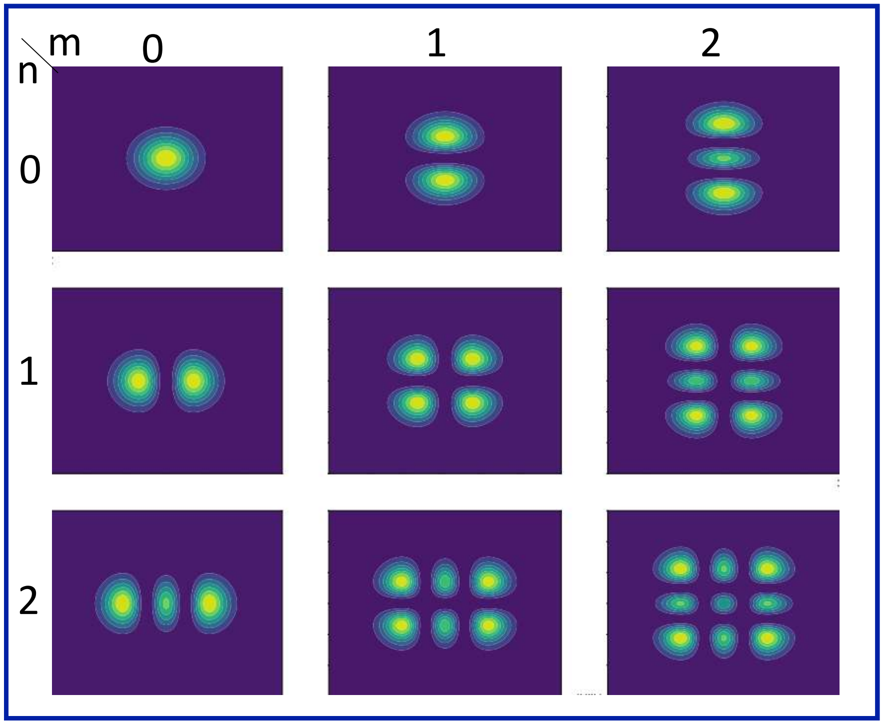


Fig. 2. Intensity profiles of HG modes *(n,m)*= *(0,0)* to *(2,2)*.

HG MODE APPROXIMATION FOR SHIFTED AND TILTED BEAMS

Shift (Lateral Misalignment)

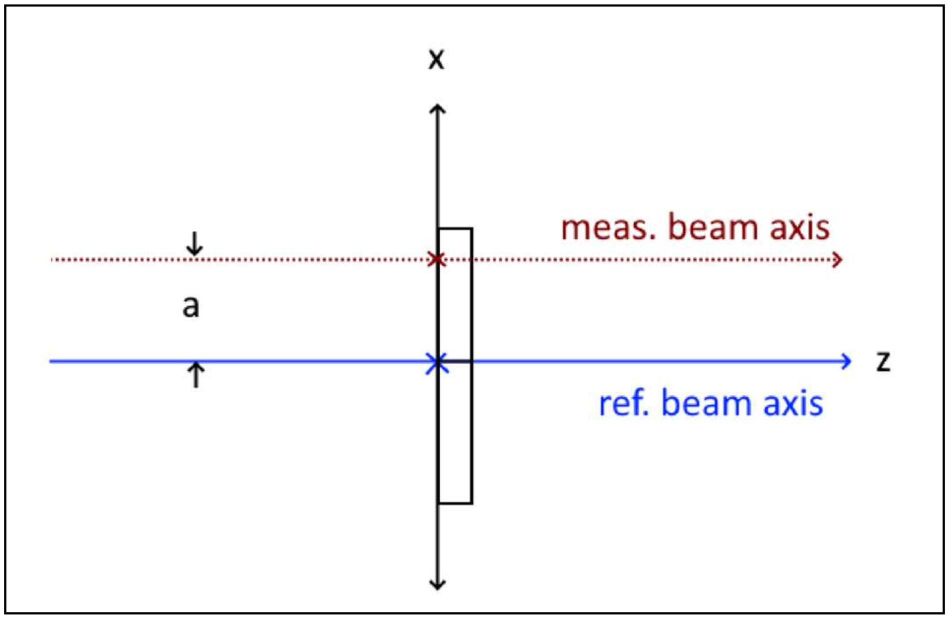


Fig. 3. Measurement beam laterally misaligned in *x* wrt reference beam axis.

Tilt (Angular Misalignment)

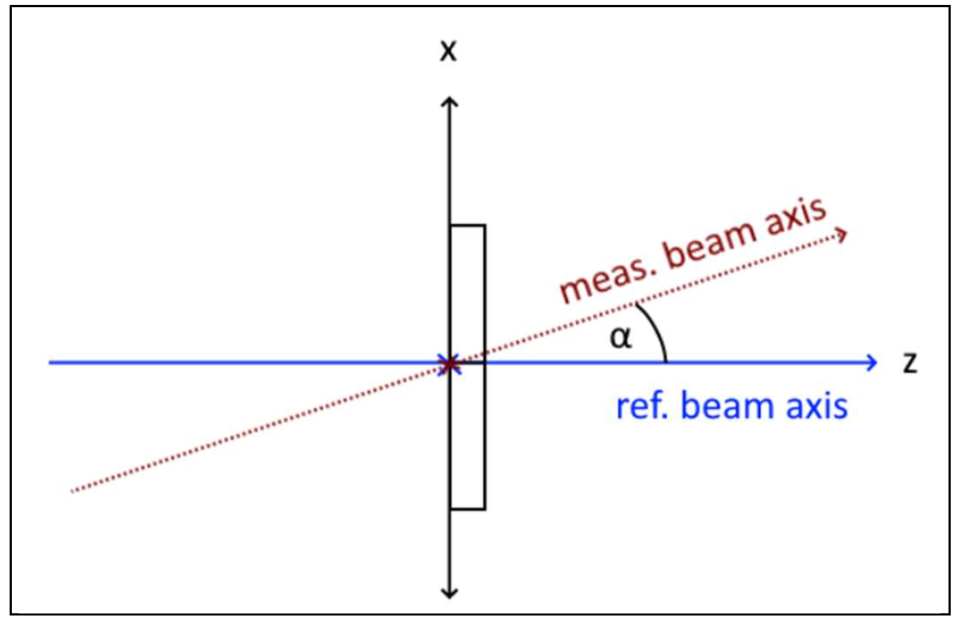


Fig. 4. Tilt of measurement beam by α wrt reference beam.

1ST-ORDER APPROXIMATION

Shift: $a << w_0$

$$
\begin{aligned}
u_{00}(x-a,y,0) &= \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{(x-a)^2+y^2}{w_0^2}\right) \\
&= \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{y^2}{w_0^2}\right)\exp\left(-\frac{(x-a)^2}{w_0^2}\right) \\
&= u_{00}(x,y,0) \times exp\left(\frac{2ax+a^2}{w_0^2}\right) \\
&= u_{00}(x,y,0)\left[1+\boxed{\frac{2ax}{w_0^2}}+\mathcal{O}\left(\frac{a}{w}\right)^2\right] \\
&\approx u_{00}+\left(\frac{2ax}{w_0^2}\right)u_{00} \\
&= u_{00}(x,y,0)+\boxed{\frac{a}{w_0}u_{10}(x,y,0)}
\end{aligned}
$$

Tilt: $\alpha << \frac{\lambda}{\pi w_0}$

$$
\begin{aligned}
u_{tilt(0,0)} &= u_{00}\exp(i\phi) \\
&= u_{00}\exp\left[ikx\sin(\alpha)\right] \\
&\approx u_{00}\exp\left[ikx\alpha\right] \\
&= u_{00}\exp\left[i\left(\frac{2\pi x\alpha}{\lambda}\right)\right] \\
&\approx u_{00}\left[1+\boxed{i\left(\frac{2\pi x\alpha}{\lambda}\right)}\right] \\
&= u_{00}(x,y,0)+\boxed{i\left(\frac{\pi w_0\alpha}{\lambda}\right)u_{10}}
\end{aligned}
$$

Shifted then tilted Gaussian: $$\sum_{n,m}u_{nm}(x,y,0)=\left[1+i\left(\frac{\pi a\alpha}{\lambda}\right)\right]u_{00}+\left[\frac{a}{w_0}+i\left(\frac{\pi w_0\alpha}{\lambda}\right)\right]u_{10}+i\left(\frac{\sqrt{2}\pi a\alpha}{\lambda}\right)u_{20}$$

1ST-ORDER SIGNALS AT HALF-PLANE PD WITH GAPS GAUSSIAN-GAUSSIAN, SHIFTED-TILTED BEAM

$$\phi_R = \arctan\left(\frac{\frac{\pi\alpha}{\lambda}\left[a\left(-\operatorname{erf}\left(\frac{\sqrt{2}r}{w_0}\right)+1+\frac{2\sqrt{2}r\exp\left(-\frac{2r^2}{w_0^2}\right)}{\sqrt{\pi}w_0}\right)+w_0\sqrt{\frac{2}{\pi}}\left(\exp\left(-\frac{2r^2}{w_0^2}\right)\right)\right]}{-\operatorname{erf}\left(\frac{\sqrt{2}r}{w_0}\right)+1+\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\exp\left(\frac{-2r^2}{w_0^2}\right)}\right)$$

$$\phi_L = \arctan\left(\frac{\frac{\pi\alpha}{\lambda}\left[a\left(\operatorname{erf}\left(\frac{\sqrt{2}l}{w_0}\right)+1-\frac{2\sqrt{2}l\exp\left(-\frac{2l^2}{w_0^2}\right)}{\sqrt{\pi}w_0}\right)-w_0\sqrt{\frac{2}{\pi}}\left(\exp\left(-\frac{2l^2}{w_0^2}\right)\right)\right]}{\operatorname{erf}\left(\frac{\sqrt{2}l}{w_0}\right)+1-\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\exp\left(\frac{-2l^2}{w_0^2}\right)}\right)$$

$$\Phi_{DWS} = \frac{1}{2}[\phi_R - \phi_L] \qquad \Phi_{LPS} = \frac{1}{2}[\phi_R + \phi_L]$$

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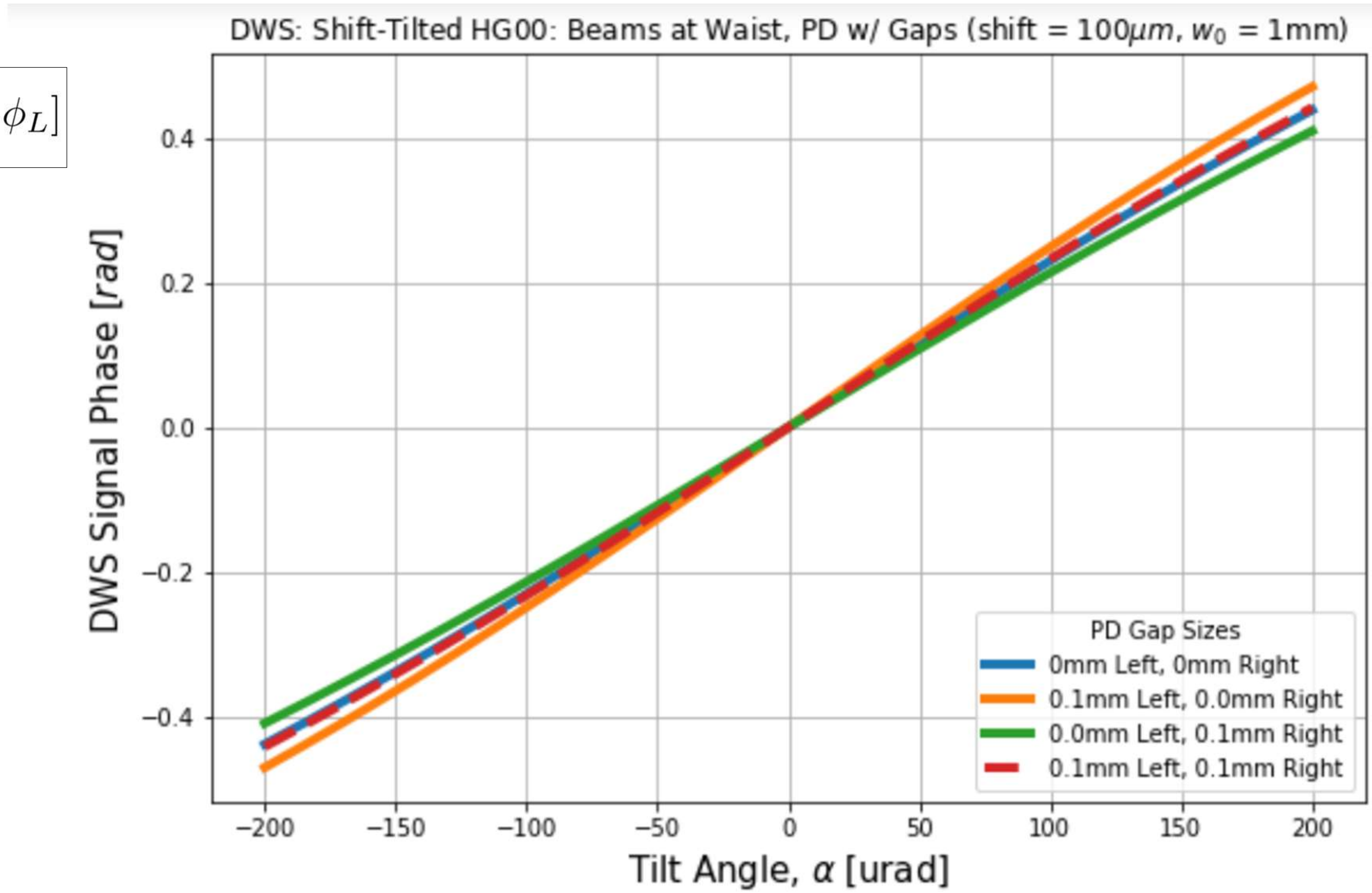


Fig. 5. DWS signal phase, first-order with gaps in half-plane PD.

2ND ORDER APPROXIMATION IN SHIFT

$$u_{00}(x-a,y,0) = \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{(x-a)^2+y^2}{w_0^2}\right)$$

$$= \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{y^2}{w_0^2}\right)\exp\left(-\frac{(x-a)^2}{w_0^2}\right)$$

$$= \left(\frac{2}{\pi}\right)^{-1/2}\left(\frac{1}{w_0}\right)\exp\left(-\frac{y^2}{w_0^2}\right)\exp\left(-\frac{x^2}{w_0^2}\right)\exp\left(-\frac{(a^2-2ax)}{w_0^2}\right)$$

$$= u_{00}(x,y,0)\times exp\left(\frac{2ax-a^2}{w_0^2}\right)$$

$$= u_{00}(x,y,0)\left[1+\frac{2ax}{w_0^2}+\boxed{\frac{a^2(2x^2-w_0^2)}{w_0^4}}+\mathcal{O}\frac{a}{w}^3\right]$$

$$\approx u_{00}+\frac{2ax}{w_0^2}u_{00}+\boxed{a^2\left(\frac{2x^2}{w_0^4}-\frac{1}{w_0^2}\right)u_{00}}$$

$$= u_{00}(x,y,0)+\frac{a}{w_0}u_{10}(x,y,0)+\boxed{a^2\left(\frac{1}{\sqrt{2}w_0^2}u_{20}-\frac{1}{2w_0^2}u_{00}\right)}$$

2ND ORDER APPROXIMATION IN TILT

$$
\begin{aligned}
u_{00(tilt))}(x,y,0) &= u_{00}\exp(i\phi) \\
&= u_{00}\exp\left[ikx\sin(\alpha)\right] \\
&\approx u_{00}\exp\left[ikx(\alpha - \frac{\alpha^3}{6})\right] \\
&\approx u_{00}[1 + \alpha ikx \boxed{- \frac{1}{2}\alpha^2(kx)^2}
\end{aligned}
$$

2ND-ORDER APPROXIMATION: DWS SIGNAL PHASE

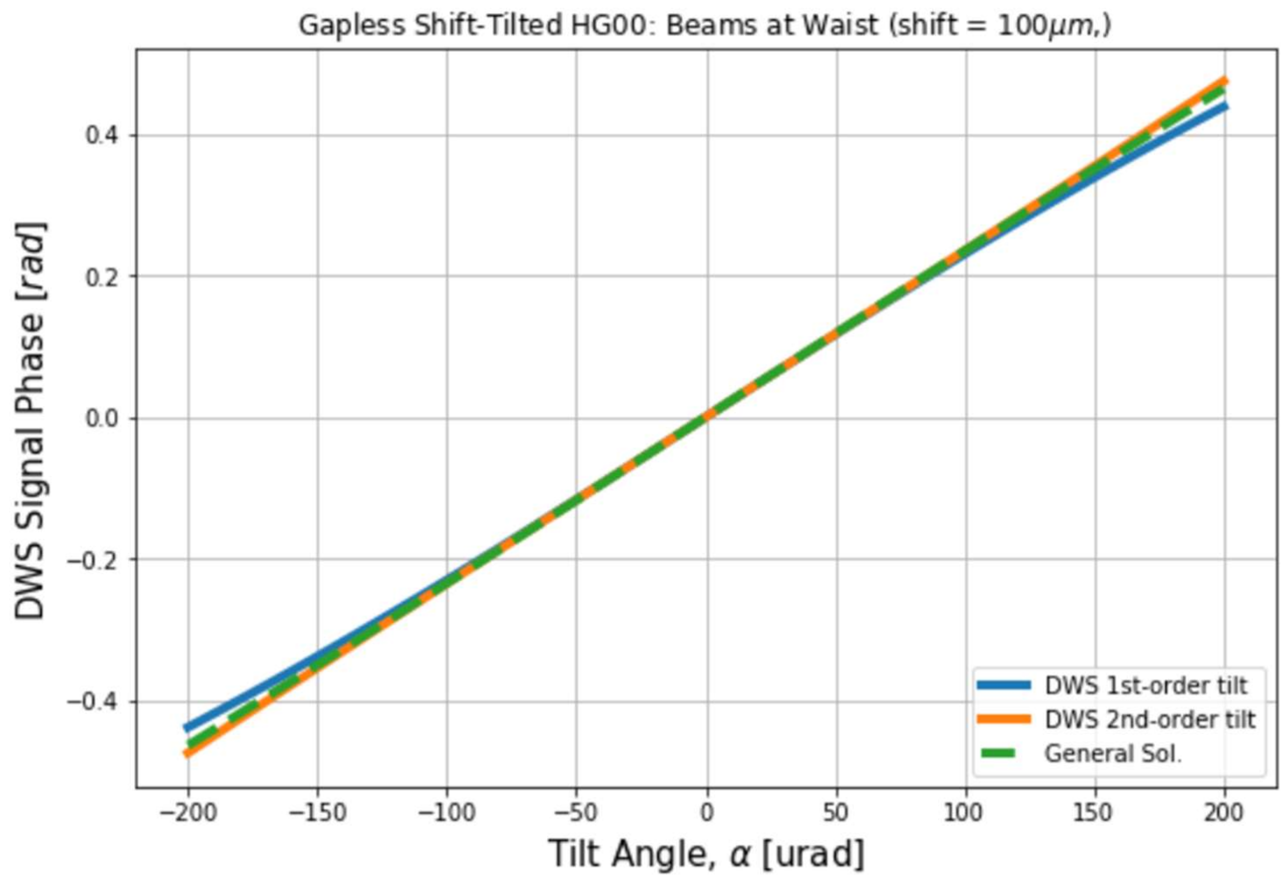
Shift

$$\frac{1}{2}\left[\arctan\left[\frac{\frac{\pi\alpha}{\lambda}\left[a+\sqrt{\frac{2}{\pi}}w_0\right]}{1+\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\boxed{-a^2\frac{1}{2w_0^2}}}\right]-\arctan\left[\frac{\frac{\pi\alpha}{\lambda}\left[a-\sqrt{\frac{2}{\pi}}w_0\right]}{1-\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\boxed{-a^2\frac{1}{2w_0^2}}}\right]\right]$$

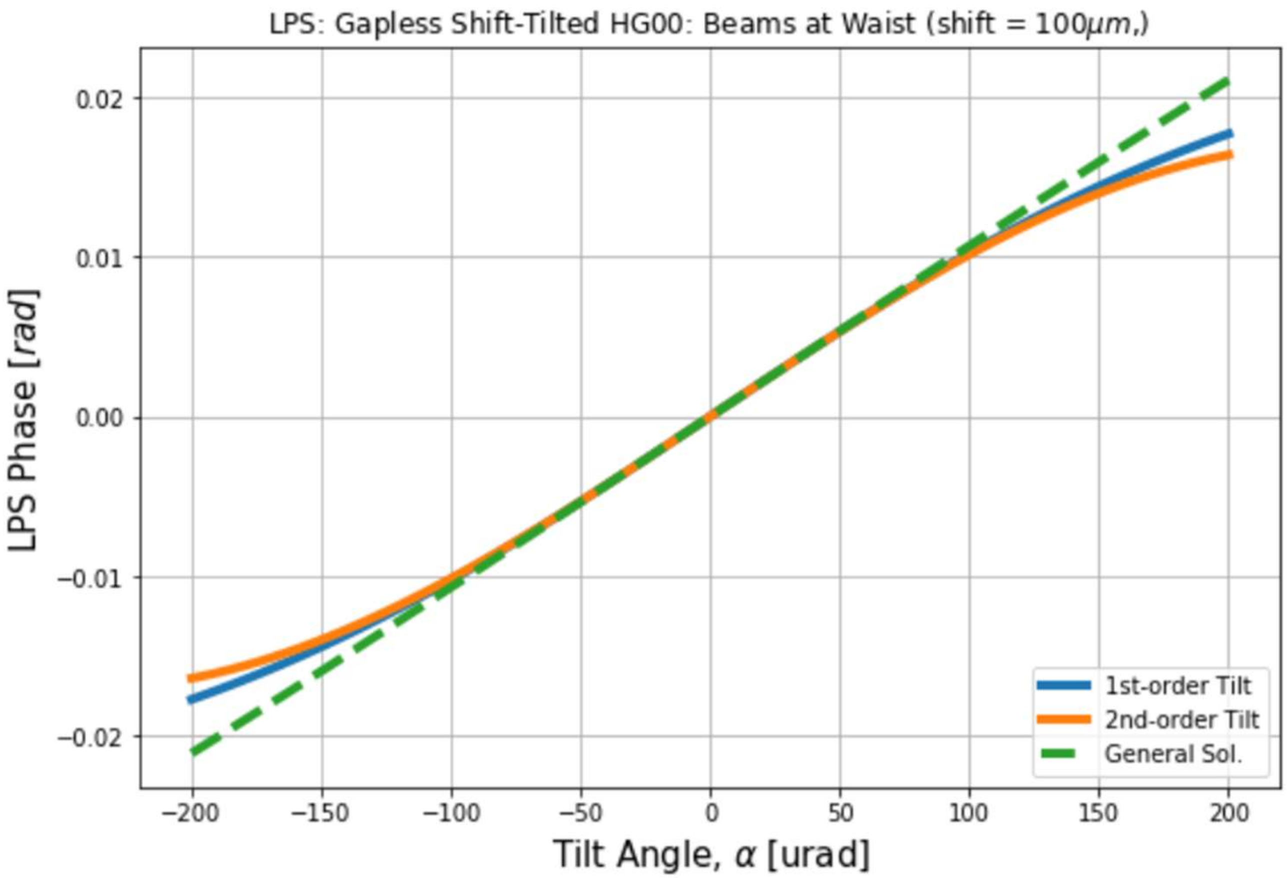
Tilt

$$\frac{1}{2}\left[\arctan\left[\frac{\frac{\pi\alpha}{\lambda}\left[a+\sqrt{\frac{2}{\pi}}w_0\right]}{1+\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\boxed{-\frac{1}{16}\alpha^2k^2[w_0^2+aw_0(\frac{1}{4}\sqrt{\frac{2}{\pi}})]}}\right]-\arctan\left[\frac{\frac{\pi\alpha}{\lambda}\left[a-\sqrt{\frac{2}{\pi}}w_0\right]}{1-\sqrt{\frac{2}{\pi}}\frac{a}{w_0}\boxed{-\frac{1}{16}\alpha^2k^2[w_0^2-aw_0(\frac{1}{4}\sqrt{\frac{2}{\pi}})]}}\right]\right]$$

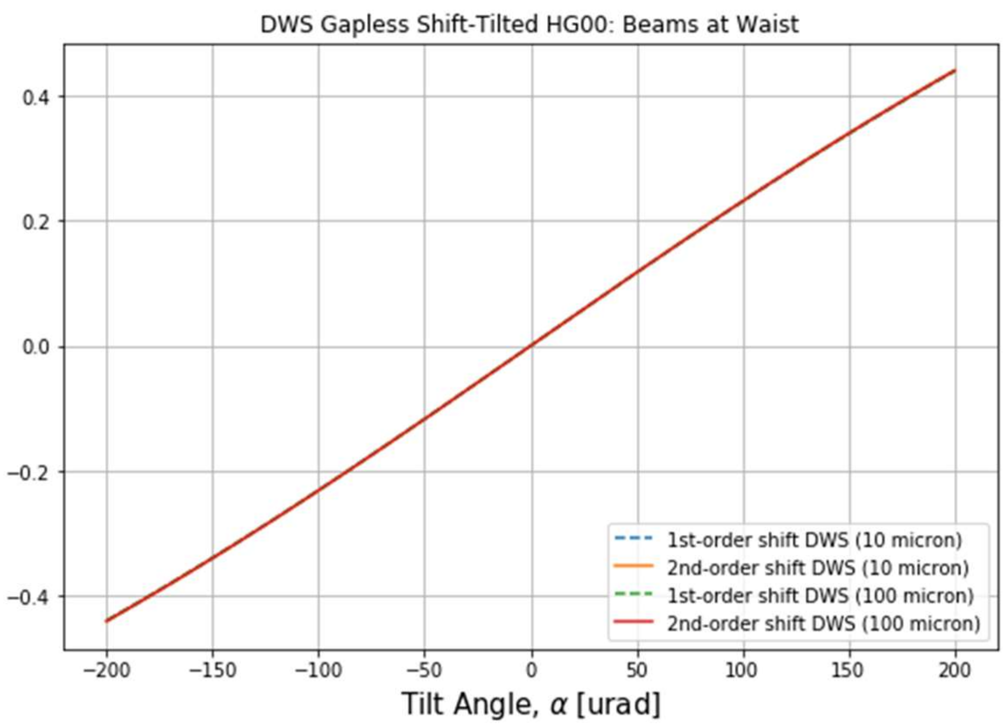
2ND -ORDER TILT : DWS

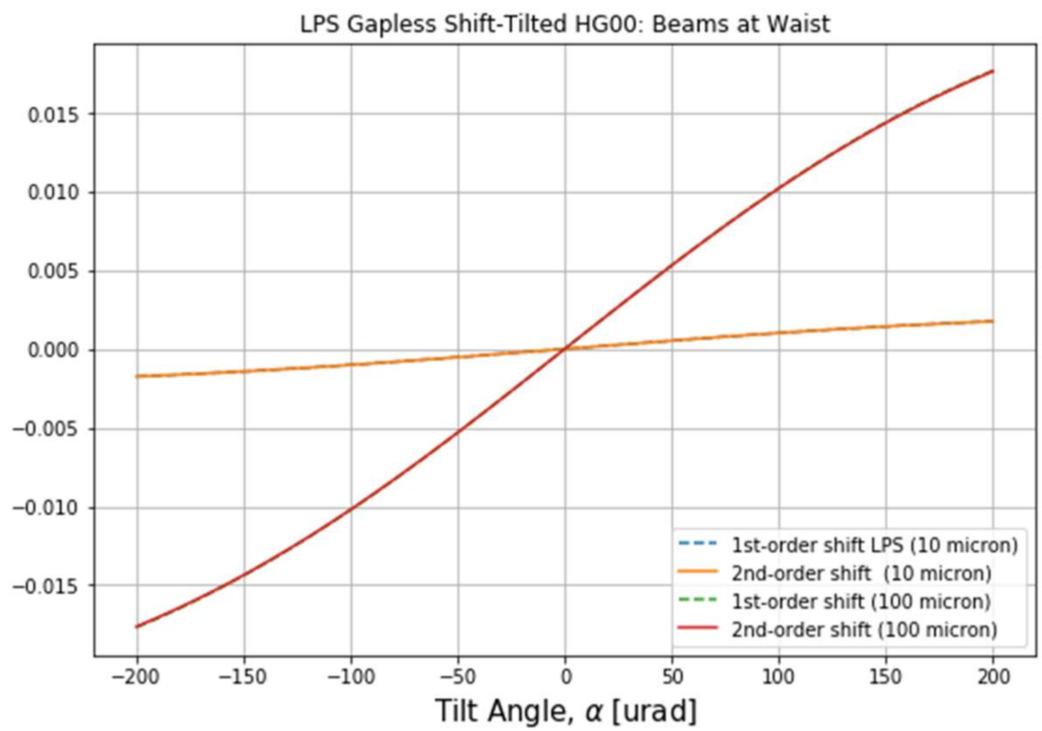


2ND -ORDER TILT : LPS

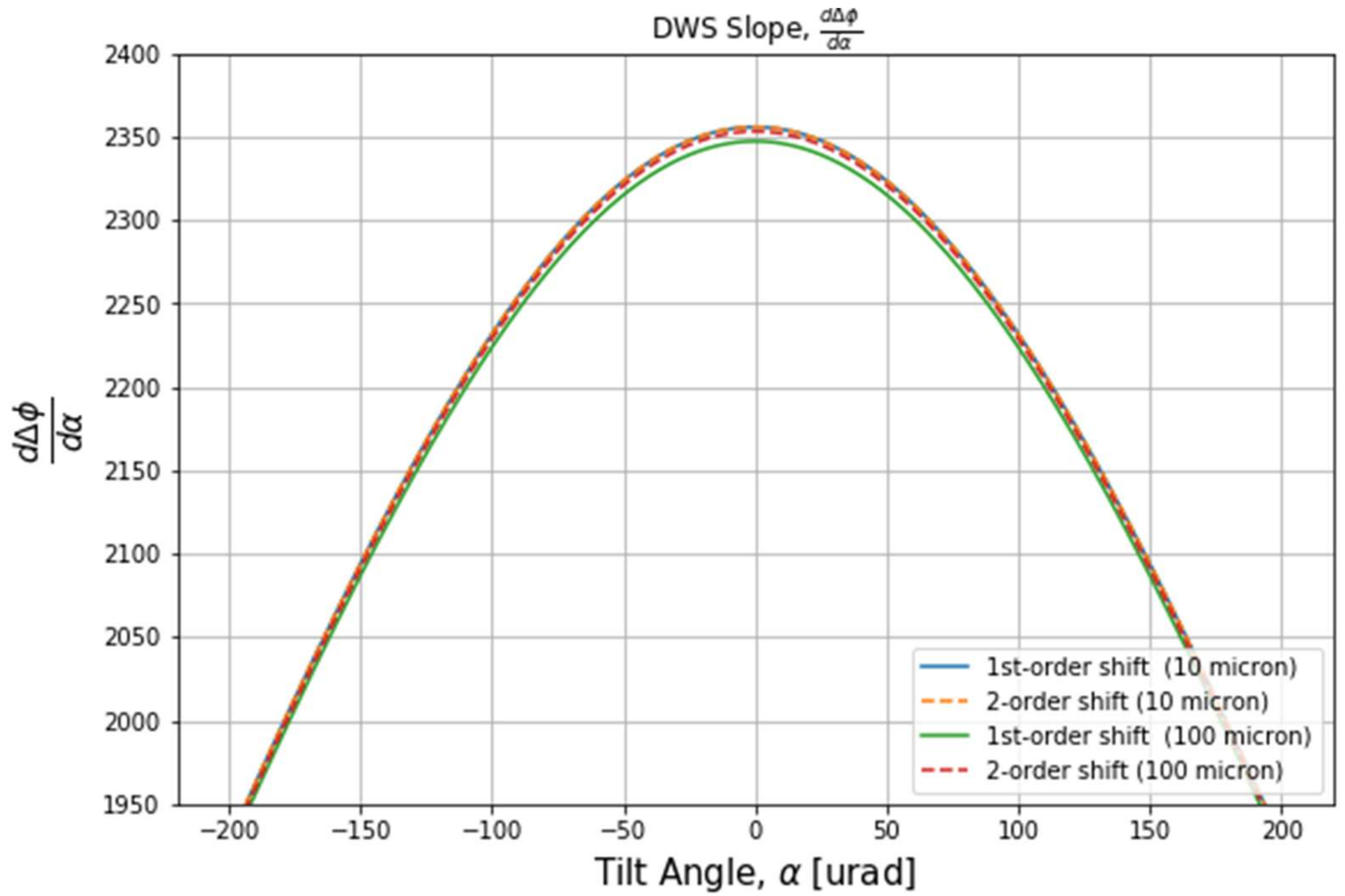


2ND -ORDER SHIFT : DWS





2ND -ORDER SHIFT : DWS

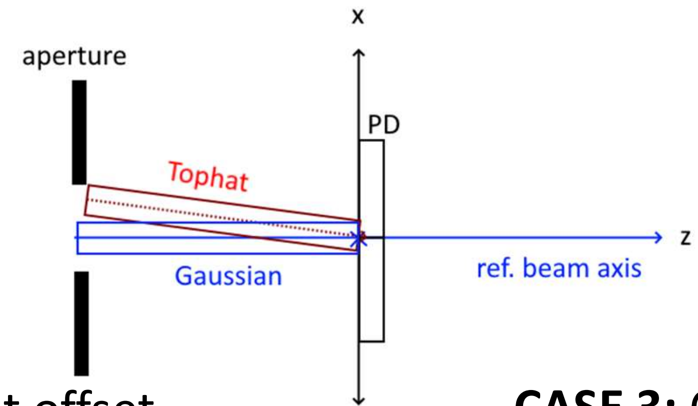


FUTURE DEVELOPMENTS

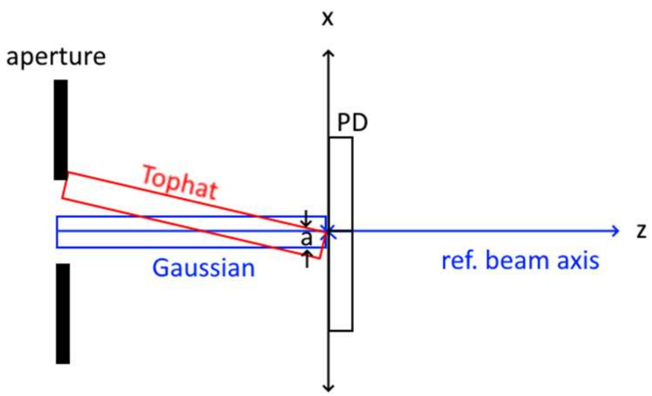
- Third-order (tilt)
- Tophat-Gaussian interference
- Tophat-Tophat (?)
- Changing Center of Rotation (CoR) relative to PD and aperture

FUTURE DEVELOPMENTS

CASE 1: CoR on PD, no offset



CASE 2: CoR on PD, but offset



CASE 3: CoR on aperture