K-th order Shift, *x* transformation

For k-th order shift approximation (note that k is ambiguously used as wavenumber…):

$$\boxed{k >= 2(A+B) + (C+D+E)}$$

$$\boxed{E \equiv F - [2(A+B) + C + D]}$$

Loop over this expression ∀ n and ∀ m in the set of tophat modes:

$$C_{nm} \sum_{F=0}^{k} \sum_{A=0}^{\lfloor F/2 \rfloor} \sum_{B=0}^{\lfloor (F-2A)/2 \rfloor} \sum_{C=0}^{F-2(A+B)} \sum_{D=0}^{F-2(A+B)-C} \{[\frac{1}{A!}(-\frac{a^2}{w^2})^A][\frac{1}{B!}(-\frac{ika^2}{2R_c})^B]$$

$$\times [\frac{1}{C!}(-\frac{2a\boxed{x}}{w^2})^C][\frac{1}{D!}(-\frac{ika\boxed{x}}{R_c})^D] \left[\frac{1}{E!}\sqrt{\frac{n!}{(n-E)!}}(\frac{2ae^{i\Psi}}{w})^E\right] u_{\boxed{n-E},m}\}$$

$$\boxed{C_{nm} x^{C+D} U_{n-E}}$$

$$C_{nm} x U_{n-E}$$

$$X_+^1 C_{nm} U_{n-E+1}$$

$$X_-^1 C_{nm} U_{n-E-1}$$

Max x order = shift * tilt order

Section 3.2 of living doc

(where $a = \frac{\sqrt{2}}{w(z)}$, $y = ax$)

$$\boxed{H_{n+1}(y) = 2yH_n(y) - nH_{n-1}(y)}$$

$$\boxed{xH_n(ax) = \frac{1}{2a}[H_{n+1}(ax) + 2nH_{n-1}(ax)]}$$

$$x^2 H_n(y) = \frac{1}{2a}\boxed{xH_{n+1}(y)} + \frac{n}{a}\boxed{xH_{n-1}(y)}$$

$$\boxed{xH_{n+1}(y)} = \frac{1}{2a}H_{n+2}(y) + \frac{n+1}{a}H_n(y)$$

$$\boxed{xH_{n-1}(y)} = \frac{1}{2a}H_n(\sqrt{a}x) + \frac{n-1}{a}H_{n-2}(y)$$

$$u_{n\pm k,m}(x,y,z) \rightarrow \boxed{C(2^{n\pm k}(n\pm k)!)^{-1/2}H_{n\pm k}\left(\frac{\sqrt{2}x}{w(z)}\right)\exp(i(n\pm k)\psi(z))}$$
$$(2^{m-1}m!\pi)^{-1/2}\frac{1}{w(z)}H_m\left(\frac{\sqrt{2}y}{w(z)}\right)\exp\left(\frac{-ik(x^2+y^2)}{2R_c(z)} - \frac{x^2+y^2}{w(z)^2} + i(m+1)\psi(z)\right)$$

$$C_{n,m} x u_{n,m} = \underbrace{\boxed{\frac{w_0}{2}\left[\sqrt{n+1}\left(1 - i\frac{z}{z_R}\right)\right]}}_{X_+^{1(n)}} C_{n,m} u_{n+1,m} + \underbrace{\boxed{\frac{w_0}{2}\left[\sqrt{n}\left(1 + i\frac{z}{z_R}\right)\right]}}_{X_-^{1(n)}} C_{n,m} u_{n-1,m}$$

$x^2 u_{n,m}$

X count = 2
X count = 1

q1	U_{n-E}					
q2	$X_+^{1(n)}\ U_{n-E+1}$	$X_-^{1(n)}\ U_{n-E-1}$				

$$\begin{tikzcd}
 & \boxed{X_+^{1(n)} U_{n-E+1}} \arrow[r] \arrow[dr] & \boxed{X_+^{1(n)} X_+^{1(n-E+1)} U_{n-E+2}} \\
\boxed{PC_{nm} x^2 u_{n-E,m}} \arrow[ur] \arrow[dr] & & \boxed{\left(X_+^{1(n)} X_+^{1(n-E-1)} U_{n-E} + X_-^{1(n)} X_-^{1(n-E+1)} U_{n-E}\right)} \\
 & \boxed{X_-^{1(n)} U_{n-E-1}} \arrow[ur] \arrow[r] & \boxed{X_-^{1(n)} X_-^{1(n-E-1)} U_{n-E-2}}
\end{tikzcd}$$

*P is some factor resulting from everything in the sums other than x

** Cnm and P suppressed in the diagram

$$C_{n,m} x u_{n,m} = \underbrace{\boxed{\frac{w_0}{2}\left[\sqrt{n+1}\left(1 - i\frac{z}{z_R}\right)\right]}}_{X_+^{1(n,m)}} C_{n,m} u_{n+1,m} + \underbrace{\boxed{\frac{w_0}{2}\left[\sqrt{n}\left(1 + i\frac{z}{z_R}\right)\right]}}_{X_-^{1(n,m)}} C_{n,m} u_{n-1,m}$$

$X^2 u_{n,m}$

q1

$X_+^{1(n)} U_{n-E+1}$ / $X_-^{1(n)} U_{n-E-1}$	$X_-^{1(n)} U_{n-E-1}$				

X count = 1
X count = 0

q2

$X_+^{1(n)} X_+^{1(n-E+1)} U_{n-E+2}$	$X_+^{1(n)} X_-^{1(n-E+1)} U_{n-E}$	$X_-^{1(n)} X_+^{1(n-E-1)} U_{n-E}$	$X_-^{1(n)} X_-^{1(n-E-1)} U_{n-E-2}$		

$\boxed{AC_{nm} X^2 u_{n-E,m}}$

$\boxed{X_+^{1(n)} U_{n-E+1}}$

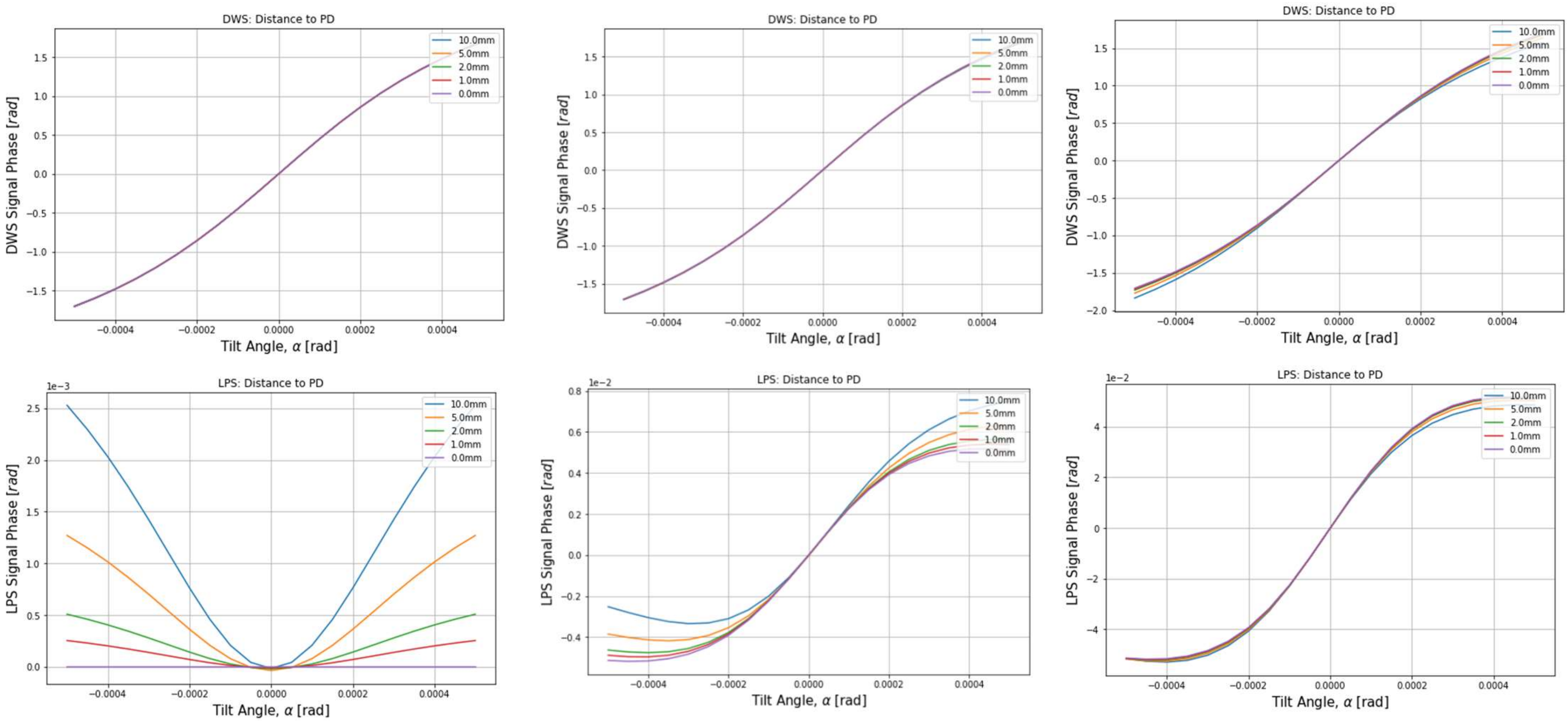
$\boxed{X_-^{1(n)} U_{n-E-1}}$

$\boxed{X_+^{1(n)} X_+^{1(n-E+1)} U_{n-E+2}}$

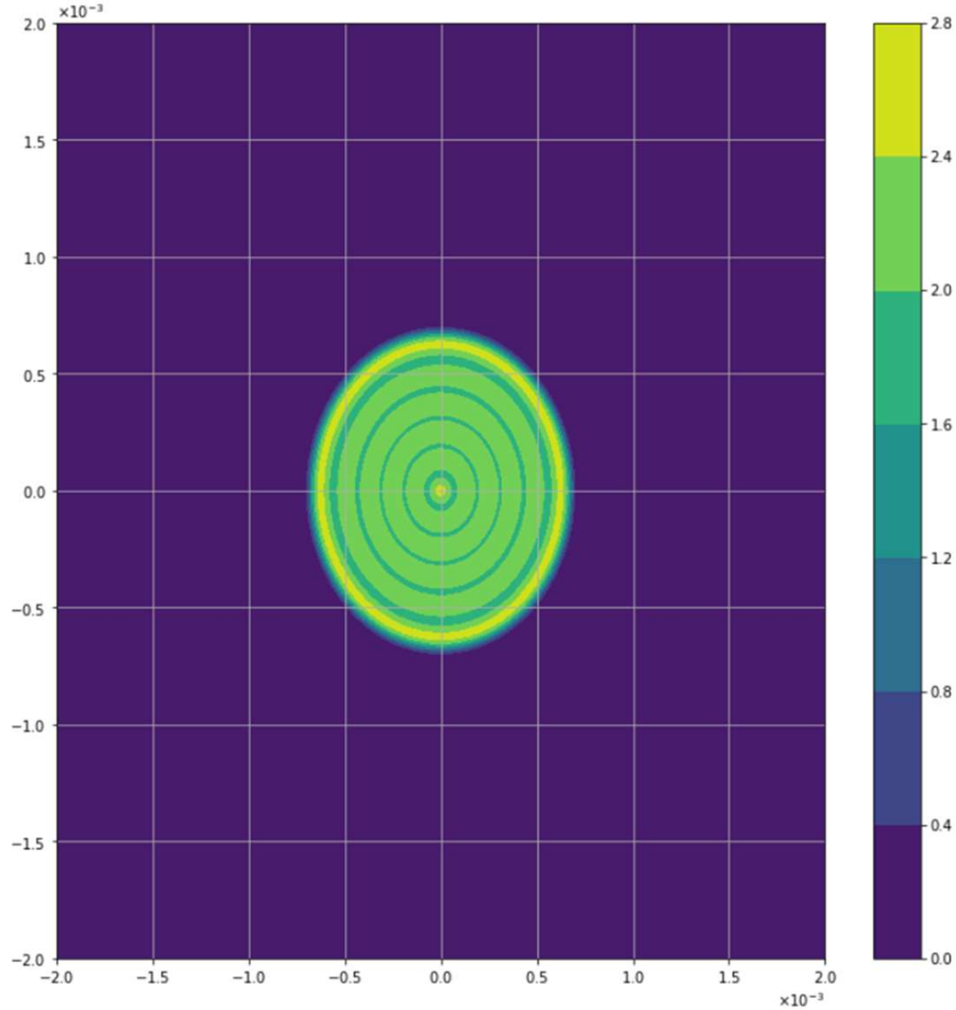
$\boxed{(X_+^{1(n)} X_+^{1(n-E-1)} U_{n-E} + X_-^{1(n)} X_-^{1(n-E+1)} U_{n-E})}$

$\boxed{X_-^{1(n)} X_-^{1(n-E-1)} U_{n-E-2}}$

Original Basis (shift = 0,10,100 [um])



Radius = 1/3 aperture radius = 2/3 mm



(shift = 0,10,100 [um])