The Chinese University of Hong Kong
RMSC 4007: Risk Management with Derivatives Concepts

Investigation on Quanto Option

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Abstract

In this study, we aim to conduct in-depth investigations on the complex product - Quanto Option. To achieve this, we will introduce four pricing models, including Black-Scholes Model with constant volatility, Black-Scholes Model with CIR Stochastic Interest Rate Model, Heston Stochastic Volatility Model, and Heston with Vasicek Model. Also, We will explore the risks associated with this product and propose potential strategies for effective risk management, including Greeks hedging, sensitivity analysis, and other types of risks.

1 Product features

1.1 Term sheet

SUMMARY OF OFFERING	
Issuer	ABN AMRO Bank N.V. (incorporated in The Netherlands with its statutory seat in Amsterdam), London branch
Series	Gold American Quanto Call Warrants Series D
Number of Securities	50,000
Underlying	Gold (Bloomberg code: GOLDS Comdty)
Issue Price	EUR 9.047
Entitlement	0.1
Strike Price	USD 1000.00
Expiration Date	21 May 2010
Issue Date	29 November 2007
Settlement	Cash
Settlement Date	5 Business Days following the Valuation Date
Settlement Currency	EUR
Calculation Agent	ABN AMRO Bank N.V., London branch
Principal Agent	ABN AMRO Bank N.V., London branch
Clearing	Clearstream Banking AG, Clearstream Banking S.A, Euroclear Bank S.A.
ISIN	NL0006136723
WKN	AA0SGT

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1.2 Introduction

With the growth in investments in recent years, the popularity of Quanto Option has increased around the world. It allows investors to invest in foreign assets with an underlying asset denominated in foreign currency, while eliminating the impact of currency rate fluctuation because the rate is exchanged at a predetermined rate. Here are some reasons why people buy the Quanto option:

Reduced Volatility: Quanto option is designed to eliminate the impact of unexpected currency fluctuation for the foreign asset, which help reduce the portfolio risks.

Improved risk-Adjusted returns: Quanto option offers a more accessible way to invest in foreign asset without the need for a large amount of capital and transactions. This means that traders can diversify their portfolios and potentially earn higher returns in foreign investment.

For our case, we are investigating an American Floating exchange rate Gold Quanto call option, where the issuer from a European country would like to buy Gold which is issued in a foreign country US. Although the contract is in the American call option, We can apply it to the European option style because there is no advantage in an early exercise of an American call in case of no dividends in the underlying (Kwok, 2022).

Breakdown of the components

Floating Exchange Rate: It refers to the exchange rate between two currencies that fluctuates based on market forces. The value of the exchange rate varies over time.

$$\text{FX rate at time t: } F_t = \frac{\text{Foreign Currency (USD)}}{\text{Domestic Currency (EUR)}}$$

Quanto (Foreign Equity Call struck in foreign currency): It is a type of derivative in which the underlying asset (Gold) and option payoff is calculated in foreign currency (USD), but final Quanto payoff is settled in domestic currency (EUR).

Call Option: A call option is a derivative contract that gives the holder the right to buy the underlying asset (Gold) at the strike price within a specified time period.

1.3 Payoff Structure

The payoff of European Quanto call option with underlying asset Gold denominated in domestic currency (EUR) is:

$$V[F_T, S_T] = F_T \times max(S_T - K^f)$$

where

$$\begin{cases} F_T \text{ is the FX rate } \frac{\text{Foreign Currency (USD)}}{\text{Domestic Currency (EUR)}} \text{ at time T} \\ S_T \text{ is the price index of Gold in foreign currency (USD) at time T} \\ K^f \text{ is the strike price in foreign currency (USD)} \end{cases}$$

Then, we have Quanto option price in domestic (EUR) risk-neutral probability ($\widetilde{Q}$-measure):

$$V[F_t, S_t] = e^{-r^d(T-t)} E^{\widetilde{Q}} \big[F_T \times max(S_T - K^f, 0) \big]$$

Also, by no arbitrage theory, we can express the price in foreign (USD) risk-neutral probability (Q-measure):

$$V[F_t, S_t] = F_t\, e^{-r^f(T-t)} E^{Q} \big[max(S_T - K^f, 0) \big]$$

where r^d and r^f are domestic (EUR) interest rate and foreign (USD) interest rate respectively.

2 Pricing models

2.1 Black-Scholes Model

We can use the Black Scholes model to derive the pricing process. The dynamics for Gold price S_t and foreign exchange rate F_t in the foreign physical probability (P-measure) can be represented as follows:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_S dt + \sigma_S dW_S^P(t) \\ \frac{dF_t}{F_t} &= \mu_F dt + \sigma_F dW_F^P(t) \\ E[dW_S^P(t) dW_F^P(t)] &= \rho_{S,F} dt \end{aligned}$$

where

$$\begin{cases} \mu_S,\ \mu_F \text{ are expected returns of Gold index and FX rate respectively;} \\ \sigma_S,\ \sigma_F \text{ are volatility of Gold index and FX rate respectively} \end{cases}$$

When we value derivatives in foreign risk-neutral probability (Q-measure), the dynamics of S_t and F_t become:

$$\begin{aligned} \frac{dS_t}{S_t} &= r^f dt + \sigma_S dW_S^Q(t) \quad \text{or} \quad S_T = S_t\, e^{(r^f - \frac{1}{2}\sigma_S^2)dt + \sigma_S dW_S^Q(t)} \\ \frac{dF_t}{F_t} &= (r^d - r^f) dt + \sigma_F dW_F^Q(t) \quad \text{or} \quad F_T = F_t\, e^{(r^d - r^f - \frac{1}{2}\sigma_F^2)dt + \sigma_F dW_F^Q(t)} \end{aligned}$$

2.1.1 Closed-Formed Solution

Quantos can be priced using a modified Black-Scholes Formula in foreign risk-neutral measure (Q-measure):

$$\begin{aligned} V[F_t, S_t] &= F_t\, e^{-r^f(T-t)} E^Q \big[max(S_T - K^f, 0) \big] \\ &= F_t \times C_{BS}\big(S = S_t, K = K^f, r = r^f, \sigma = \sigma_S \big) \\ &= F_t \times \big[S_t \Phi(d_1) - K^f e^{-r^f(T-t)} \Phi(d_2) \big] \\ \text{where } d_1 &= \frac{ln(\frac{S_t}{K^f}) + (r^f + \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}}, \quad d_2 = d_1 - \sigma_S \sqrt{T-t} \end{aligned}$$

Note that the correlation $\rho_{S,F}$ and exchange rate risk σ_F are not involved in the price formula.

2.1.2 Monte Carlo method

With the use of the Law of Large Numbers, we can estimate the options price by generating a large number of random samples via computer. In this way, we will first assume the stock price follows the Black-Scholes model and then generate a large number of normal variables to simulate the stock prices. The pricing algorithm is shown in the appendices.

2.2 Heston Stochastic Volatility Model

To capture the variability of gold price instead of the constant volatility assumption in the Black-Scholes Model, we can also price the product under the Heston stochastic volatility model. The dynamics of Gold price S_t and variance process V_t in the foreign physical probability (P-measure) can be represented as follows:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_S dt + \sqrt{V_t} dW_S^P(t) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_V^P(t) \\ E[dW_S^P(t) dW_V^P(t)] &= \rho_{S,V} dt \end{aligned}$$

where

$$\begin{cases} V_t \text{ is the instantaneous variance, } \kappa \text{ is the mean reversion rate of variance} \\ \theta \text{ is the long-term variance, } \sigma_V \text{ is the volatility of the variance} \end{cases}$$

When we value derivatives in foreign risk-neutral probability (Q-measure), the dynamics become:

$$\begin{aligned} \frac{dS_t}{S_t} &= r^f dt + \sqrt{V_t} dW_S^Q(t) \quad \text{or} \quad S_T = S_t \, e^{(r^f - \frac{1}{2}V_t)dt + \sqrt{V_t} dW_S^Q(t)} \\ dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma_V \sqrt{V_t} dW_V^Q(t) \quad \text{or} \quad V_T = V_t + \kappa^*(\theta^* - V_t)dt + \sigma_V \sqrt{V_t} dW_V^Q(t) \end{aligned}$$

2.3 Cox–Ingersoll–Ross Stochastic Interest Rate Model

In the above cases, we assumed the interest rate is constant all the time. However, the interest rate varies all the time in the real market, especially during the financial crisis 2007-2009 period where it experienced a huge decline in US 3M Yield. Here we introduce Vasicek stochastic interest rate into our pricing models. The dynamics of Gold price S_t, variance process V_t, and foreign US interest rate r_t^f in the foreign physical probability (P-measure) can be represented as follows:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_S dt + \sigma_S dW_S^P(t) \\ dr_t^f &= a(b - r_t^f)dt + \sigma_{r^f} \sqrt{r_t^f} dW_{r^f}^P(t) \\ E[dW_S^P(t) dW_V^P(t)] &= 0 \end{aligned}$$

where

$$\begin{cases} dW_S^P(t) \perp dW_{r^f}^P(t) \\ a \text{ is the mean reversion rate of interest rate} \\ b \text{ is the long-term interest rate, } \sigma_{r^f} \text{ is the volatility of the interest rate process} \end{cases}$$

Being a safe-haven asset, gold is generally considered to have a low correlation with the market. During the 2008 financial crisis, the gold price remained stable when there is a huge decline in interest rates. So for simplicity, we assume the stock price process is independent with the movement of interest rates.

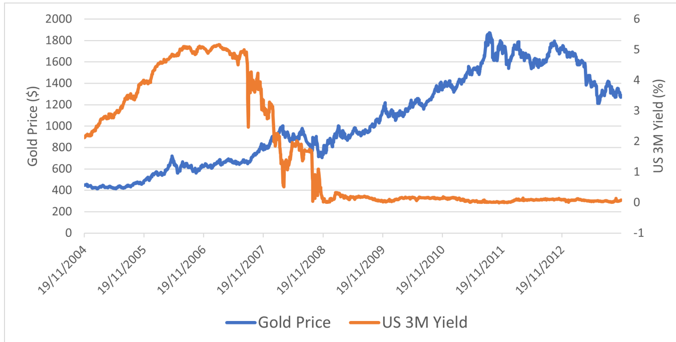

Figure 1: Relationship between Gold Price and US 3M Treasury Yield (2004-2013)

When we value derivatives in foreign risk-neutral probability (Q-measure), the dynamics become:

$$\frac{dS_t}{S_t} = r_t^f dt + \sigma_S dW_S^Q(t) \quad \text{or} \quad S_T = S_t\, e^{(r_t^f - \frac{1}{2}\sigma_S^2)dt + \sigma_S dW_S^Q(t)}$$
$$dr_t^f = a^*(b^* - r^f)dt + \sigma_{r^f} dW_{r_t^f}^Q(t) \quad \text{or} \quad r_T^f = r_t^f + a^*(b^* - r_t^f)dt + \sigma_{r^f} dW_{r^f}^Q(t)$$

2.4 Heston with Vasicek Model

Here we also introduce both stochastic volatility under the Heston Model and stochastic interest rate under the Vasicek Model. The dynamics of Gold price S_t, variance process V_t, and foreign US interest rate r_t^f in the foreign physical probability (P-measure) can be represented as follows:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu_S dt + \sqrt{V_t} dW_S^P(t)\\ dV_t &= \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_V^P(t)\\ dr_t^f &= a(b - r_t^f)dt + \sigma_{r^f} dW_{r^f}^P(t)\\ E[dW_S^P(t) dW_V^P(t)] &= \rho_{S,V} dt\end{aligned}$$

where

$$\begin{cases} dW_S^P(t) \perp dW_{r_t^f}^P(t) \;\text{ and } dW_V^P(t) \perp dW_{r_t^f}^P(t) \\ a \text{ is the mean reversion rate of interest rate} \\ b \text{ is the long-term interest rate, } \sigma_{r_t^f} \text{ is the volatility of the interest rate process}\end{cases}$$

We also assume the stock price process and volatility process are independent of the movement of interest rates for the same reason stated in the section of CIR Model.
When we value derivatives in foreign risk-neutral probability (Q-measure), the dynamics become:

$$\frac{dS_t}{S_t} = r_t^f dt + \sqrt{V_t} dW_S^Q(t) \quad \text{or} \quad S_T = S_t\, e^{(r_t^f - \frac{1}{2}V_t)dt + \sqrt{V_t} dW_S^Q(t)}$$
$$dV_t = \kappa^*(\theta^* - V_t)dt + \sigma_V \sqrt{V_t} dW_V^Q(t) \quad \text{or} \quad V_T = V_t + \kappa^*(\theta^* - V_t)dt + \sigma_V \sqrt{V_t} dW_V^Q(t)$$
$$dr_t^f = a^*(b^* - r_t^f)dt + \sigma_{r^f} dW_{r^f}^Q(t) \quad \text{or} \quad r_T^f = r_t^f + a^*(b^* - r_t^f)dt + \sigma_{r^f} dW_{r^f}^Q(t)$$

3 Parameters Estimation

3.1 Black-Scholes Model

We extract the following parameters, including foreign risk-free rate r^f from the average of US 3M treasury yield and volatility of Gold stock return σ_S (BBG Code: GOLDS Comdty) from the historical period 19th Nov 2004 - 29th Nov 2007.

3.2 CIR Stochastic Interest Rate Model

We implemented the **Maximum Likelihood Estimation (MLE)** to estimate the parameters of the CIR Model. The log-likelihood function is given by:

$$L(a,b,\sigma) = \textstyle\sum_{i=1}^{n} \log\, p_r(t_{i-1}, r_{t_{i-1}}, t_i, r_{t_i})$$

where the transition density function p_r for times s and t with $0 \le s < t$ is:

$$p_r(s, r_s, t, r_t) = \frac{1}{2(\varphi_t - \varphi_s)e^{-bt}}(\frac{r_t e^{bt}}{r_s e^{bs}})^{\frac{1}{2}(\frac{\nu}{2}-1)} \times e^{-\frac{1}{2}\frac{r_s e^{bs} + r_t e^{bt}}{\varphi_t - \varphi_s}} I_{\frac{\nu}{2}-1}(\frac{\sqrt{r_s r_t e^{b(s+t)}}}{\varphi_t - \varphi_s})$$

where $\varphi_t = \varphi_0 + \frac{1}{4}\frac{\sigma^2(e^{bt}-1)}{b}$ and $\nu = \frac{4ab}{\sigma}$ and where

$$I_\nu(\textstyle\sum_{i=0}^{\infty} \frac{1}{i!\Gamma(i+\nu+1)}(\frac{x}{2})^{2i+\nu})$$

is the power series expansion of the modified Bessel function of the first kind. Then, for simplicity, the parameters can be estimated by approximating the transition density of the CIR process with a lognormal distribution (Poulsen, 1999). The approximating log-likelihood function on the set of observed short rates r_{t_i}, for i = 0, 2, ..., n is:

$$l(a,b,\sigma) = -\frac{1}{2}\sum_{i=1}^{n}(log(2\pi v_{t_{i-1}}(t_i)) + log(r_{t_i}) + \frac{(log(r_{t_i}) - \mu_{t_{i-1}}^{(LN)}(t_i))^2}{(\sigma^{(LN)}(t_i))^2})$$

Where $\mu_s^{(LN)}(t) = log(m_s(t)) - \frac{1}{2}(\sigma_s^{(LN)}(t))^2$ and $(\sigma_s^{(LN)}(t))^2 = log(1 + \frac{v_s(t)}{m_s(t)^2})$.
Also, $m_s(t) = abB(s,t) + r_s(1 - bB(s,t))$ and $v_s(t) = \sigma^2(\frac{1}{2}abB(s,t)^2 + bB(s,t)^2 r_s)$ and $B(s,t) = \frac{1-e^{-b(t-s)}}{b}$
The optimization process can be done in R by optim() function OR Excel Solver function. Here is the result:

$$\boxed{a^* = 0.0911,\ b^* = 0.0534,\ \sigma_{r_t} = 0.1078}$$

3.3 Heston Stochastic Volatility Model

We also implemented MLE to estimate the parameters of Heston models. We assumed the stock price process and variance process follow Bivariate normal joint distribution with correlation $\rho_{S,V}$:

$$f(u_t, V_t) = \frac{1}{2\pi\sqrt{V_t}\sigma_V\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho_{S,V}^2)}\left[\left(\frac{u_S(t) - u_S}{\sqrt{V_t}}\right)^2 - 2\rho_{S,V}\left(\frac{u_S(t) - u_S}{\sqrt{V_t}}\right)\left(\frac{V_t - E(V_t)}{\sigma_V}\right) + \left(\frac{V_t - E(V_t)}{\sigma_V}\right)^2\right]\right)$$

where:

$$u_S(t) = \frac{dS_t}{S_t} = u_S dt + \sqrt{V_t} dW_S^P(t)$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_V^P(t)$$
$$V_t = (u_S(t) - u_S)^2 \text{ and } E[V_t] = \kappa(\theta - V_t)$$

We initialize the parameters as follows: θ by taking variance of historical return, κ equals to 1, σ_V equals to standard deviation of realized variance $(u_S(t) - u_S)^2$, $\rho_{S,V}$ equals to 0, and starting variance V_0 equals to θ. We obtain the likelihood function

$$l(\Theta) = \prod_{i=1}^{N} f(u_S(t), V_t) \quad , where \ \Theta = (\kappa, \theta, \sigma_V, V_0, \rho_{S,V}).$$

We use the Excel *Solver* function to get the maximum log-likelihood function $log[l(\Theta)]$ to estimate the parameters of the Heston model. We assume the parameters we get from the physical P-measure is the same as the risk-neutral Q-measure. Here is the result that we get:

$$\boxed{\kappa^* = \kappa = 0.0324, \ \theta^* = \theta = 0.0250, \ \sigma_V = 0.05, \ V_0 = 0.0290, \ \rho_{S,V} = -0.9233}$$

3.4 Heston with Vasicek Model

Similar to CIR and the Heston Model, we also implemented MLE to estimate the parameters for the Vasicek model. We assume the interest rate process follows normal distribution:

$$Z_t = [dr_t - a(b - r_t)] \sim \mathcal{N}(0, \sigma_r^2) = \frac{1}{\sigma_r \sqrt{2\pi}} \exp\left(-\frac{Z_t}{2\sigma_r^2}\right)$$

We initialize the parameters as follows: b by taking the average of the historical return of US 3M treasury yield, a equals to 1, and σ_{r_t} equals to the volatility of Z_t.
We obtain the likelihood function

$$l(\Theta) = \prod_{i=1}^{N} f(Z_t) \quad , where \ \Theta = (a, \ b, \ \sigma_r).$$

Again, we use the Excel *Solver* function to get the maximum log-likelihood function $log[l(\Theta)]$ to estimate the parameters of the Vasicek model. Here is the result that we get:

$$\boxed{a^* = a = 0.005980, \ b^* = b = 4.2986\%, \ \sigma_r = 0.000827}$$

We also have the Expected Interest Rate one year after (29th Sept 2008):

$$E[r_t] = r_0 e^{-at} + b(1 - e^{-at}) = 4.0144\%$$

The parameters of the Heston Model are the same as we obtained in section 3.3.

4 Pricing Results

In the simulation results shown below, we use 10,000 sample paths for each simulation, and each time interval δt equals to 3 month period ($\approx \frac{66}{252}$). The original issue price of this product is €9.047.

Model	Model Price(in €)	Diff. Market and Model Price(in €)
Black-Scholes Formula	35.34404	26.29704
Black-Scholes Monte Carlo	35.66947	26.62247
Black-Scholes CIR	34.99746	25.95046
Heston	24.8308	15.7838
Heston Vasicek	24.89551	15.84851

Table 1: Summary of price simulation by each model

We find that there is a huge difference between the market price and our model price. Even though Heston Model shows the best result among others, a tremendous price difference still exists. The model still cannot capture accurate market price even when both stochastic volatility and Interest rate is introduced.

Huge drop of US interest rate: During the 2007-2008 financial crisis, the U.S. Federal Reserve made significant cuts to interest rates in an effort to stimulate the economy and mitigate the impact of the crisis. The Federal fund rate shrank from around 5.25% in September 2007 to only around 0-0.2% in December 2008. Therefore, it is not accurate to estimate the US interest rate r^f from the historical period 2004 - 2007 where the rate experienced a bubble rise.

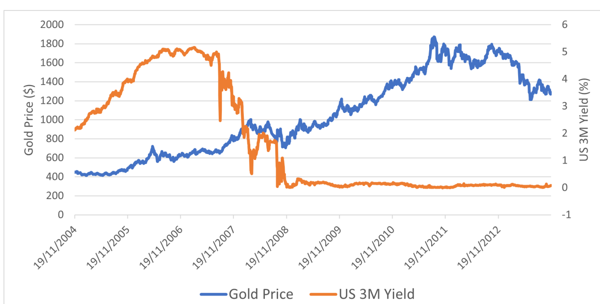

Figure 1: Huge decline of US 3M Treasury Yield since Sept 2007

Hence, we adjusted the historical period for estimation of US 3M treasury yield r^f to 29^{th} Nov 2007 - 21^{th} May 2010, meanwhile keeping the estimation of stock price and its volatility process the same. Below is the model price after such adjustment:

Model	Model Price(in €)	Diff. Market and Model Price(in €)
Black-Scholes Formula	27.51478	18.46778
Black-Scholes Monte Carlo	27.67315	18.62615
Black-Scholes CIR	27.88705	18.84005
Heston	13.82793	4.78093
Heston Vasicek	13.82751	4.77871

Table 2: Summary of price simulation by each model after adjusting estimation of US interest rate

Now the price from all models have improved significantly, where Heston with Vasicek Model and Heston Model show the best result to keep the price difference within €5, showing the introduction of stochastic volatility in Heston model and stochastic interest rate in Vasicek improves pricing performance. Notice that the price difference from CIR model and BS formula are similarly large, showing that the model's accuracy do not improve distinctly after introducing stochastic interest rate in CIR model.

Liquidity Issues: The financial crisis caused many financial institutions to face financial distress, leading to a reduction in their trading activities. This reduction in liquidity had a significant impact on the pricing of financial products. Pricing models often assume the availability of sufficient liquidity for smooth trading and market efficiency. However, during the crisis, liquidity dried up as market participants became more risk-averse and hesitant to engage in transactions. The lack of liquidity resulted in lower prices for financial products, as buyers were scarce and sellers had difficulty finding counterparties.

5 Risk Analysis

Below are the results of the Greek summary in each pricing model:

Model	**Delta Δ**	**Gamma Γ**
Black-Scholes Formula	0.2458634	0.001025863
Black-Scholes Monte Carlo	0.2501959	0.001160953
Black-Scholes CIR	0.2495167	0.001034692
Heston	0.3482798	0.001697839
Heston Vasicek	0.3487109	0.001971891

Table 3: Greek Summary (estimation of US interest rate from 2004 - 2007)

5.1 Delta

Notice that in all pricing models, we obtain a range of $+0.24$ to $+0.35$ Delta Δ. To effectively hedge delta, it is necessary to take into account both the exposure to the foreign underlying asset and currency rate. For example, we could take a short position in the foreign Gold stock which offsets any potential price movements in the stock. Additionally, the currency exposure (when we hedge by shorting stock) can be hedged by entering into a currency forward contract. This contract allows for the exchange of one currency for another at a predetermined rate and future date. By doing so, any fluctuations in currency exchange rates can be mitigated.

However, it is important to be aware of certain limitations when implementing delta hedging strategies. Transaction costs and liquidity constraints can impact the effectiveness of the hedge. Moreover, it is crucial to recognize that the delta we calculated relies on certain model assumptions. These model assumptions come from the model we used for pricing. Deviations from these assumptions can introduce additional risks and impact the effectiveness of the hedge. Additionally, it is important to evaluate risks associated with the counterparty risk when dealing with currency forward contract during financial crisis.

5.2 Gamma

Notice that in all pricing models, we obtain a range of +0.001 to +0.002 Gamma Γ. To effectively hedge Gamma, similar to delta hedging, it is important to consider both the underlying asset exposure and the currency exposure. One possible approach to hedge underlying asset exposure is through Dynamic Delta Hedging. This strategy involves continuously adjusting the Delta hedge position as the underlying asset's price changes. By doing so, the hedge position is rebalanced to maintain a neutral delta, which helps offset potential losses due to changes in the underlying asset's price and helps mitigate the impact of Gamma risk. Another way to hedge the Gamma is by entering into volatility derivatives, such as VIX options or Variance Swaps. These instruments allow investors to directly trade and hedge against changes in market volatility of the underlying asset.

Similar to Delta hedging, transaction costs, and liquidity constraints can impact the effectiveness of the hedge. Additionally, it is crucial to consider the market risk and counterparty risk associated with the volatility derivatives used for Gamma hedging during financial crisis.

5.3 Credit Risk

Credit Rating	Moody's	S&P	Fitch
Long Term Rating	A1	A	A

Investors may face the counterparty default risk of the product issuer. The table above shows the credit ratings assigned by the Big Three credit rating agencies to ABN AMRO Bank in 2007. Although the issuer had moderately good credit ratings assigned by the Big Three credit rating agencies, these ratings may not accurately reflect the issuer's creditworthiness. This is due to the fact that in 2007, financial institutions and products received high credit ratings despite underlying risks that were not adequately captured. As a result, investors may underestimate the credit risk associated with the product and issuer.

5.4 Model Risk

The model assumptions of each model may cause the model risk. The Black-Scholes model assumes a constant volatility and risk-free rate throughout the life of the options, which is not always realistic. It also assumes the return of the underlying asset follows normal distribution. However, empirical evidence suggests that stock returns often exhibit fat tails and skewness, meaning that extreme events occur more frequently than predicted by a normal distribution.

In the Heston model, the sensitivity of parameters to the initial data set and the assumption of a continuous sample path for the stock price can lead to model risk. The parameter estimation process in the Heston model relies on historical data, and if the initial data set is not representative of the true market dynamics, it can lead to inaccurate parameter estimates. Additionally, assuming a continuous sample path for the stock price may not hold in real-world scenarios, where sudden jumps or discontinuities can occur which leads to inaccurate risk assessments.

In both CIR and Vasicek model, the parameters are highly sensitive to the initial data set used for estimation, and small changes in values can lead to significant differences in model outputs. In our project, both models assume independent dynamics between stock prices and the US yield, which may not capture all the intricacies of real-world relationships.

6 Conclusion

To summarize, we believe all pricing models do not reflect the real market dynamics when we estimate the US risk free rate based on the historical period before the contract started (2004-2007). After adjusting the US risk-free rate estimation period to 2007-2010, we found that the Heston Model and the Heston with Vasicek Model perform relatively better than the Black-Scholes Model and CIR Model. We believe such pricing difference is due to the decline of the US interest rate and liquidity pressure during the 2008 financial crisis.

7 Limitations

Firstly, there is a limitation in sourcing options data since the product is "old", it is difficult to collect options data from the past 15 years. Therefore, we cannot calibrate parameters for our Heston model. We are using historical data for parameter estimation instead and we can determine the accuracy of the pricing model solely based on the issuer price of this Quanto Option.

Secondly, generally speaking, the short rate fits better in stochastic interest rate models such as the Vasicek and CIR model. However, we use a 3-month US treasury yield as the risk-free rate instead of a 1-day short rate. It is because, during financial crises, short-term money markets can experience significant volatility and liquidity disruptions, which can distort the 1-day short rate, making it an unreliable measure of the true risk-free rate.

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9 Appendices

From foreign P-measure to foreign Q-measure (under BS Model)

$$\begin{aligned}
\frac{dS_t}{S_t} &= \mu_S dt + \sigma_S dW_S^P(t) \\
&= (\mu_S + r^f - r^f)dt + \sigma_S dW_S^P(t) \\
&= r^f dt + \sigma_S \left[dW_S^P(t) + \frac{\mu_S - r^f}{\sigma_S} dt \right] \\
&= r^f dt + \sigma_S dW_S^Q(t) \quad \left(\text{where } dW_S^Q(t) = dW_S^P(t) + \frac{\mu_S - r^f}{\sigma_S} dt \right) \\
S_T &= S_t \, e^{(r^f - \frac{1}{2}\sigma_S^2)dt + \sigma^S dW_S^Q(t)} \\
\frac{dF_t}{F_t} &= \mu_F dt + \sigma_F dW_F^P(t) \\
&= \left[\mu_F + (r^d - r^f) - (r^d - r^f) \right] dt + \sigma_F dW_F^P(t) \\
&= (r^d - r^f)dt + \sigma_F \left[dW_F^P(t) + \frac{\mu_F - (r^d - r^f)}{\sigma_F} dt \right] \\
&= (r^d - r^f)dt + \sigma_F dW_F^Q(t) \quad \left(\text{where } dW_F^Q(t) = dW_F^P(t) + \frac{\mu_F - (r^d - r^f)}{\sigma_F} dt \right) \\
F_T &= F_t \, e^{(r^d - r^f - \frac{1}{2}\sigma_F^2)dt + \sigma^F dW_F^Q(t)}
\end{aligned}$$

From foreign Q-measure to domestic $\widetilde{Q}$-measure (under BS Model)

By no arbitrage rule,

$$\begin{aligned}
e^{-r^d(T-t)} E^{\widetilde{Q}} \left[F_T V[F_T, S_T] \right] &= F_t e^{-r^f(T-t)} E^Q \left[V[F_T, S_T] \right] \\
e^{-r^d(T-t)} E^{\widetilde{Q}} \left[F_T \left(\frac{S_T}{F_T} \right) \right] &= F_t e^{-r^f(T-t)} E^Q \left[\frac{S_T}{F_T} \right] \quad \left(\text{Let } V[F_T, S_T] = \frac{S_T}{F_T}, \text{ where } E[dW_S^Q(t) dW_F^Q(t)] = -\rho_{S,F} dt \right) \\
E^{\widetilde{Q}}[S_T] &= F_t e^{(r^d - r^f)dt} E^Q \left[\frac{S_T}{F_T} \right] \\
&= F_t e^{(r^d - r^f)dt} E^Q \left[\frac{S_t \, e^{(r^f - \frac{1}{2}\sigma_S^2)dt + \sigma^S dW_S^Q(t)}}{F_t \, e^{(r^d - r^f - \frac{1}{2}\sigma_F^2)dt + \sigma^F dW_F^Q(t)}} \right] \\
&= S_t e^{r^f(T-t)} E^Q \left[e^{-\frac{1}{2}(\sigma_S^2 - \sigma_F^2)dt + \sigma_S dW_S^Q(t) - \sigma_F dW_F^Q(t)} \right]
\end{aligned}$$

By Girsanov's Theorem,

$$\begin{aligned}
\eta_t &= e^{\frac{1}{2}\sigma_F^2 dt - \sigma_F dW_F^Q(t)} = E^Q \left[\frac{d\widetilde{Q}}{dQ} | \mathcal{F}_t \right] \\
d\widetilde{W}_F^Q(t) &= dW_F^Q(t) - \sigma_F dt \\
dW_F^Q(t) &= d\widetilde{W}_F^Q(t) + \sigma_F dt \\
dW_S^Q(t) &= \sqrt{1 - \rho_{S,F}} dW\perp_S^Q(t) + (-\rho_{S,F}) dW_F^Q(t) \\
&= \sqrt{1 - \rho_{S,F}} dW\perp_S^Q(t) - \rho_{S,F}(d\widetilde{W}_F^Q(t) + \sigma_F dt) \\
&= d\widetilde{W}_S^Q(t) - \rho_{S,F}\sigma_F dt
\end{aligned}$$

We continue to calculate $E^{\widetilde{Q}}[S_T]$,

$$\begin{aligned}
E^{\widetilde{Q}}[S_T] &= S_t e^{r^f(T-t)} E^{\widetilde{Q}}\left[e^{-\frac{1}{2}(\sigma_S^2-\sigma_F^2)dt+\sigma_S\left(d\widetilde{W}_S^Q(t)-\rho_{S,F}\sigma_F dt\right)-\sigma_F\left(d\widetilde{W}_F^Q(t)+\sigma_F dt\right)}\right] \\
&= S_t e^{r^f(T-t)} E^{\widetilde{Q}}\left[e^{-\frac{1}{2}(\sigma_S^2+2\rho_{S,F}\sigma_S\sigma_F+\sigma_F^2)dt+\sigma_S d\widetilde{W}_S^Q(t)-\sigma_F d\widetilde{W}_F^Q(t)}\right] \\
&= S_t e^{(r^f-\rho S,F\sigma_S\sigma_F)dt} \text{ (By M.G.F of normal random variable)}
\end{aligned}$$

Then we finally get stock price process under $\widetilde{Q}$-measure

$$\begin{aligned}
\frac{dS_t}{S_t} &= (r^f-\rho_{S,F}\sigma_S\sigma_F)dt+\sigma_S d\widetilde{W}_S^Q(t) \\
\text{or, } d\log(S_t) &= \left(r^f-\rho_{S,F}\sigma_S\sigma_F-\frac{\sigma_S^2}{2}\right)dt+\sigma_S d\widetilde{W}_S^Q(t) \text{ (By Itô's lemma)}
\end{aligned}$$

Black-Scholes closed-form solution:

$$\begin{aligned}
V[F_t,S_t] &= F_t\, e^{-r^f(T-t)} E^Q\left[max(S_T-K^f,0)\right] \\
&= F_t \times \left[S_t N(d_1) - K^f e^{-r^f(T-t)} N(d_2)\right] \\
\text{where } d_1 &= \frac{ln(\frac{S_t}{K^f})+(r^f+\frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}}, \quad d_2 = d_1-\sigma_S\sqrt{T-t}
\end{aligned}$$

Delta (Black-Scholes closed-form):

$$\frac{\partial V}{\partial S} = F_t N(d_1)$$

Gamma (Black-Scholes closed-form):

$$\frac{\partial V}{\partial S} = \frac{F_t N'(d_1)}{S\sigma_s\sqrt{T-t}}$$

Algorithm 1 Monte Carlo Method for Black-Scholes Model with constant volatility

1. Generate $Z^{(i)} \sim N(0,1)$
2. Set $S_T^{(i)} = S_t \exp\left((r-\frac{\sigma_S^2}{2})(T-t)+\sigma_S\sqrt{T-t}Z^{(i)}\right)$
3. Repeat steps 1 to 2 for N = 10000 times
4. Compute options price $V[F_t,S_t] = F_t\left(\frac{1}{N}\sum_{i=1}^N e^{r(T-t)}max(S_T^{(i)}-K_f,\ 0)\right)$

Algorithm 2 Monte Carlo Method for Black-Scholes Model with CIR Stochastic Interest Rates

1. Generate $Z_1^{(i)}$ *and* $Z_2^{(i)} \sim N(0,1),\ Z_1^{(i)} \perp Z_2^{(i)}$
2. Set $r_{t+\delta t}^{(i)} = r_t^{(i)} + a(b - r_t^{(i)})\delta t + \sigma_r \sqrt{r_t^{(i)}} \sqrt{\delta t} Z_2^{(i)}$
3. Set $S_{t+\delta t}^{(i)} = S_t \exp\left((r_t^{(i)} - \frac{\sigma_S^2}{2})\delta t + \sigma_S \sqrt{\delta t} Z_1^{(i)}\right)$
4. Compute $\bar{r}^{(i)} = \frac{\delta t}{T-t} \sum_{\tau=t}^{T} r_\tau^{(i)}$
5. Repeat steps 1 to 4 for N = 10000 times.
6. Compute options price $V[F_t, S_t] = F_t \left(\frac{1}{N} \sum_{i=1}^{N} e^{\bar{r}^{(i)}(T-t)} max(S_T^{(i)} - K_f,\ 0)\right)$

Algorithm 3 Monte Carlo Method for the Heston Stochastic Volatility Model

1. Generate $Z_1^{(i)}$ and $Z_3^{(i)} \sim N(0,1)$ with correlation ρ, where $\tilde{Z}^{(i)} = \rho Z_1^{(i)} + \sqrt{1-\rho^2} Z_3^{(i)}$;
2. Set $V_{t+\delta t}^{(i)} = V_t^{(i)} + \kappa(\theta - V_t^{(i)})\delta t + \sigma_V \sqrt{V_t^{(i)}} \sqrt{\delta t} \tilde{Z}^{(i)}$;
3. Set $S_{t+\delta t}^{(i)} = S_t^{(i)} \exp\left((r - \frac{V_t^{(i)}}{2})\delta t + \sqrt{V_t^{(i)}} \sqrt{\delta t} Z_1^{(i)}\right)$;
4. Repeat steps 1 to 4 for N = 10000 times.
5. Compute options price $V[F_t, S_t] = F_t \left(\frac{1}{N} \sum_{i=1}^{N} e^{r(T-t)} max(S_T^{(i)} - K_f,\ 0)\right)$

Algorithm 4 Monte Carlo Method for Heston with Vasicek Model

1. Generate $Z_1^{(i)}$ and $Z_3^{(i)} \sim N(0,1)$ with correlation ρ, where $\tilde{Z}^{(i)} = \rho Z_1^{(i)} + \sqrt{1-\rho^2} Z_3^{(i)}$;
2. Generate $Z_2^{(i)} \sim N(0,1),\ Z_2^{(i)} \perp Z_1^{(i)} and\ Z_2^{(i)} \perp Z_3^{(i)}$
3. Set $r_{t+\delta t}^{(i)} = r_t^{(i)} + a(b - r_t^{(i)})\delta t + \sigma_r \sqrt{r_t^{(i)}} \sqrt{\delta t} Z_2^{(i)}$
4. Set $V_{t+\delta t}^{(i)} = V_t^{(i)} + \kappa(\theta - V_t^{(i)})\delta t + \sigma_V \sqrt{V_t^{(i)}} \sqrt{\delta t} \tilde{Z}^{(i)}$;
5. Set $S_{t+\delta t}^{(i)} = S_t^{(i)} \exp\left((r_t^{(i)} - \frac{V_t^{(i)}}{2})\delta t + \sqrt{V_t^{(i)}} \sqrt{\delta t} Z_1^{(i)}\right)$;
6. Compute $\bar{r}^{(i)} = \frac{\delta t}{T-t} \sum_{\tau=t}^{T} r_\tau^{(i)}$;
7. Repeat steps 1 to 4 for N = 10000 times.
8. Compute options price $V[F_t, S_t] = F_t \left(\frac{1}{N} \sum_{i=1}^{N} e^{\bar{r}^{(i)}(T-t)} max(S_T^{(i)} - K_f,\ 0)\right)$