

**AMATH 475**

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## 1 Index Notation

### Def: Covariant

Objects that transform under change of basis like the elements of the basis are called **covariant**, and its components have sub-indices.

### Def: Contravariant

Objects that transform under change of basis like components of vectors are called **contravariant** and its components have super-indices.

### Def: Dual Basis & Dual Space

The **dual basis** to  $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is  $\mathbb{B}^* = \{\mathbf{e}^1, \dots, \mathbf{e}^n\}$  and it is the collection of linear operators such that  $\mathbf{e}^i(\mathbf{e}_j) = \delta_j^i$ .  $\mathbb{B}^*$  spans a vector space  $V^*$  called the **dual space** of  $V$ .

## 2 Special Relativity

### 2.1 Postulates of Special Relativity

#### Def: Event

An **event** is an individual point in spacetime, usually labelled which we represent by the tuple  $E \equiv (t, \vec{x})$ ,  $t$  is the *time coordinate* and  $\vec{x} \equiv (x, y, z)$  is the *space coordinates*.

#### Def: Spacetime

**Spacetime** is the set of all events,  $\mathbb{S} = \{E \equiv (t, \vec{x}) : t \in R, \vec{x} \in \mathbb{R}^3\}$ .

#### Def: Reference Frame

A **reference frame**, establishes a spacetime coordinate system  $(t, \vec{x})$ , which is a spatial coordinate system where the position of point-like particles can be specified, and a clock (something that can measure time).

#### Def: Inertial Reference Frame (IRF)

An **inertial reference frame** is a reference frame for which a particle stationary at its origin experience no force (Newton's first law holds).

#### Def: Postulates of Special Relativity 1 (SR): Principle of Relativity

1. **Principle of Relativity:** In the absence of gravity, all the laws of physics are identical in all inertial reference frames (This postulate is also in Galilean relativity).
2. **Speed of Light is Constant and Equal:** The speed of light in vacuum " $c$ " is constant and the same for all inertial reference frames (absence of gravity).

### 2.2 Lorentz Transformation

### Def: Galilean Transformation

Consider two inertial frames  $S \equiv (t, \vec{x})$  and  $S' \equiv (t', \vec{x}')$ , where  $S'$  moves with velocity  $\vec{v} = v\hat{x}$  to the right relative to  $S$ . The **Galilean transformation** (Galilean boost) from  $S$  to  $S'$  is:

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

Note: Galilean transformation would result in mathematical inconsistency (todo).

### Def: Spacetime Interval

Given a particular inertial reference frame that establishes space time coordinates  $(t, \vec{x})$ , the **spacetime interval** between two events  $E_1 \equiv (t_1, \vec{x}_1)$  and  $E_2 \equiv (t_2, \vec{x}_2)$  is  $\Delta s^2 := -c^2\Delta t^2 + \Delta \vec{x}^2 = -c^2(t_2 - t_1)^2 + (\vec{x}_2 - \vec{x}_1)^2$ .

#### Note

In this course we use the signature  $(-, +, +, +)$ .

#### Remark

If the two events are connected by the propagation of light,  $\Delta \vec{x}^2 = c^2\Delta t^2 \Rightarrow \Delta s^2 = -c^2\Delta t^2 + c^2\Delta t^2 = 0$ .

#### Note

Possible transformations between two inertial reference frames: 3 rotations, 3 translations, 1 time shift, 3 boosts.

However, the spatial distance  $\|\vec{x}_2 - \vec{x}_1\|$  is invariant under rotations and translations. Similarly,  $\Delta t = t_2 - t_1$  is invariant under time shift. So the only transformations with significance are the boost.

### 3 Differential Geometry

#### 3.1 Topology

Def: Topology

A **topology on a set**  $X$ ,  $S$  is a collection of open sets of  $X$ , and  $S$  is a subset of the power set of  $X$ .  $S$  satisfies:

- 1)  $\emptyset, X \in S$ .
- 2) Any union of elements of  $S$  is in  $S$ .
- 3) Any finite intersection of elements of  $S$  is in  $S$ .

Def: Topological Space

A **topological space**  $(X, S)$ , is an ordered pair where  $X$  is a set and  $S$  is a topology of  $X$ .

#### 3.2 Manifolds

Def: Homeomorphism

A **homeomorphism** between two topological spaces  $X$  and  $Y$  is a map  $\sigma : X \rightarrow Y$  such that the map is topologically continuous and it's inverse is topologically continuous ( $\phi$  is a bijection).

Def: Hausdorff Space

A topological space  $X$  is a **Hausdorff space** if for all distinct points  $x, y \in X$  there exists neighbourhoods of  $x$  and  $y$ ,  $\mathcal{U}_x, \mathcal{U}_y$  respectively, such that  $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$ .

Def: Topological Manifolds

A **topological manifold** of dimension  $n$ ,  $\mathcal{M}$  is a topological space that is **Hausdorff** and at every point possess an open neighbourhood homeomorphic to  $\mathbb{R}^n$ .

Note: The proper definition also requires that the space is second countable.

Def: Charts

A **chart**  $(\mathcal{U}_\alpha, \phi_\alpha)$ , where  $\phi_\alpha$  is a homeomorphism from an open subset  $\mathcal{U}_\alpha \subseteq \mathcal{M}$  to  $\mathbb{R}^n$ . Note:  $\phi_\alpha : \mathcal{U} \xrightarrow[\text{open}]{} \mathbb{R}^n$ , and  $\phi_\alpha(x) \equiv x^\mu$ ,  $x^\mu$  is the coordinate in  $\mathbb{R}^n$ .

Def: Transition Map

Consider two charts of  $\mathcal{M}$ ,  $(\mathcal{U}_\alpha, \phi_\alpha)$  and  $(\mathcal{U}_\beta, \phi_\beta)$ , where  $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ . The **transition map** is a homeomorphism from  $\phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \rightarrow \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ , defined by  $\phi_\beta \circ \phi_\alpha^{-1}$ .

Def: Smooth Atlas

A **smooth atlas** is a set of charts covering the whole manifold such that the transition map  $\phi_\alpha \circ \phi_\beta^{-1}$  are  $C^\infty$ .

Def: Smooth Manifold

A **smooth manifold** is a topological manifold that has a smooth atlas.

### 3.3 Structures on Manifold

#### 3.3.1 Curves and Functions

Def: Curve on Manifold

A **curve** on a manifold  $\mathcal{M}$  is a smooth and invertible map  $\gamma : \mathbb{R} \rightarrow M$ .  
Note: check if it is actually smooth and invertible.

Def: Functions on Manifold

A **function** on a manifold  $\mathcal{M}$  is a map  $f : M \rightarrow \mathbb{R}$ .  
 $\bar{f} := f \circ \phi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ .  
Note that  $f$  and  $\bar{f}$  are not necessarily invertible,  $f$  is smooth if and only if  $\bar{f}$  is  $C^\infty$ .

#### 3.3.2 Vectors

**Motivation:** We want to be able to define the notion of direction on manifold  $\mathcal{M}$ . To do so we want to relate derivatives and curves to directions.

Def: Coordinate Curves

**Coordinate curves** of a chart are the image of the axes in  $\mathbb{R}^n$  under  $\phi^{-1}$ .

Def: Tangent Vector

The **tangent vector** to  $\gamma$  at  $\tau_0$  is  $\partial_\tau|_{\tau_0} = \frac{d}{d\tau}|_{\tau_0} = \frac{dx^\mu}{d\tau}\partial_\mu|_{\tau_0}$ .  
And  $\Upsilon_\mu(f)|_{\tau_0} = \partial_\mu \bar{f}|_{\tau_0}$ .

Def: Tangent Space  $T_p\mathcal{M}$

The set  $T_p\mathcal{M}$  is the **tangent space** of all the vectors  $\mathbf{v}_p$  at a point  $p \in M$ , and it has dimension equal to  $\dim(\mathcal{M})$ . We call  $T_p\mathcal{M}$  the **tangent space to  $\mathcal{M}$  at  $p \in \mathcal{M}$** .

Def: Coordinate Basis

The set  $\{\Upsilon_{p\mu}\}$  is the **coordinate basis** of  $T_p\mathcal{M}$  is the set of vectors tangent to the coordinate curves of chart  $(\mathcal{U}, \phi)$ .

Note

Every chart comes with a coordinate basis and every coordinate basis defines a chart.

Theorem: Coordinate Basis and Commutativity

A basis of  $\{\Upsilon_\mu\}$   $T_p^*\mathcal{M}$  is a coordinate basis  $\Leftrightarrow [\Upsilon_\mu, \Upsilon_\nu] = 0$ .

Def: Basis Transformation

Consider  $\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$ . Implying  $\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$  with  $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$ .

Def: Vector Field

A **vector field** over  $\mathcal{M}$  is a set of vectors of  $T_p\mathcal{M}$  for each  $p \in \mathcal{M}$  such that their components in any coordinate basis are smooth functions. Note that vector fields follows Leibniz rule.

Def: Composition of Vector Fields

$(\mathbf{v} \circ \mathbf{w})(f) := \mathbf{v}[\mathbf{w}(f)]$ . The composition of vector fields does not obey the Leibniz rule and hence is not a vector field.

Note

$$(\mathbf{v} \circ \mathbf{w})(fg) = f \cdot (\mathbf{v} \circ \mathbf{w})(g) + g \cdot (\mathbf{v} \circ \mathbf{w})(f) + \mathbf{w}(f) \cdot \mathbf{v}(g) + \mathbf{v}(f) \cdot \mathbf{w}(g) \neq f \cdot (\mathbf{v} \circ \mathbf{w})(g) + g \cdot (\mathbf{v} \circ \mathbf{w})(f).$$

Def: Lie Bracket of Vector Fields

The **Lie brackets of vector field** is a binary operator such that  $[\cdot, \cdot] : (A, B) \mapsto AB - BA$ . And Lie brackets satisfies:

- 1) Antisymmetry:  $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$ .
- 2) Jacobi Identity:  $[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0$

The Lie bracket of vector fields is a vector field.



**Remark**

In a coordinate basis,  $\{\Upsilon_\mu\} = \{\partial_\mu\}$ ,  $[\mathbf{v}, \mathbf{w}]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$ .

### 3.3.3 1-Forms

**Def: 1-Form**

A **1-form**,  $\omega$ , is a real linear functional over  $T_p\mathcal{M}$ ,  $\omega : T_p\mathcal{M} \rightarrow \mathbb{R}$  defined by  $\omega : T_p\mathcal{M} \rightarrow \mathbb{R}$  defined by  $\omega : \mathbf{v} \mapsto \langle \omega, \mathbf{v} \rangle$ . Note: 1-forms are sometimes called covariant vectors.

**Def: Cotangent Space and Dual Basis**

Given an arbitrary basis  $\{\mathbf{e}_a\}$  of  $T_p\mathcal{M}$ . There exists a unique set of 1-forms  $\{\mathbf{e}^a\}$  such that  $\langle \mathbf{e}^a, \mathbf{e}_b \rangle = \delta_b^a$ . This set is linear independent and forms a basis of the **cotangent space**  $T_p^*\mathcal{M}$ . We call  $\{\mathbf{e}_a\}$  the **dual basis**.

**Note**

Elements of  $T_p\mathcal{M}$  also acts linearly on  $T_p^*\mathcal{M} \Rightarrow T_p^{**}\mathcal{M} = T_p\mathcal{M}$ .

**Remark**

$$\langle \omega, \mathbf{v} \rangle = \langle \omega_a \mathbf{e}^a, v^b \mathbf{e}_b \rangle = \omega_a v^b.$$

$$\mathbf{e}'_a = \Lambda_a^b \mathbf{e}_b \text{ and } \mathbf{e}'^a = \tilde{\Lambda}^a_b \mathbf{e}^b.$$

**Def: Differential**

Each functions  $f$  over  $\mathcal{M}$  defines a 1-form  $\mathbf{d}f|_p$  at the point  $p \in \mathcal{M}$  that we call the **differential** of  $f$ .  $\mathbf{d}f$  is defined by  $\langle \mathbf{d}f, \mathbf{v} \rangle = \mathbf{v}(f)$ . Note that  $f : \mathcal{M} \rightarrow \mathbb{R}$ .  
In a coordinate basis  $\{\Upsilon_\mu\}$ ,  $\mathbf{d}f = \partial_\mu f \Upsilon^\mu = \partial_\mu \bar{f} \Upsilon^\mu$ .

**Def: Coordinate Basis (Dual)**

The coordinate basis of  $T_p^*\mathcal{M}$ ,  $\{\Upsilon^\mu\}$  is often represented by  $\{\mathbf{d}x^\mu\}$ . (Can see by considering  $f(\mathbf{x}) = x^\mu(\mathbf{x})$ , note that this  $\mu$  is not being summed over.)

Remark

$$\begin{aligned}\Upsilon'^\mu &= \tilde{\Lambda}^\mu{}_\nu \Upsilon^\nu \Leftrightarrow dx'^\mu = \tilde{\Lambda}^\mu{}_\nu dx^\nu. \\ \langle \Upsilon'^\mu, \Upsilon'_\nu \rangle &= \langle \Upsilon^\mu, \Upsilon_\nu \rangle = \delta^\mu_\nu \\ \omega'_\mu &= \Lambda_\mu{}^\nu \omega_\nu\end{aligned}$$

### 3.3.4 Tensors

Def: Tensor of Type  $(r, s)$

A **tensor of type**  $(r, s)$  is a multilinear map that acts on the vector space  $(T_p)_s \mathcal{M} = (T_p^* \mathcal{M})^{\times r} \times (T_p \mathcal{M})^{\times s}$ . Tensor of type  $(r, s)$  are called  $r$ -times contravariant and  $s$ -times covariant.

Def: Components of Tensor and Action of Tensor

A tensor  $\mathsf{T} \in (T_p)_s \mathcal{M}$  is completely characterized by its action on a basis of  $(T_p)_r \mathcal{M}$ .

The **components** of  $\mathsf{T}$  are  $\mathsf{T}(\Upsilon^{\alpha_1}, \dots, \Upsilon^{\alpha_r}, \Upsilon_{\beta_1}, \dots, \Upsilon_{\beta_s}) = T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}$ .

The **action** of  $\mathsf{T}$  on arbitrary 1-forms and vectors is  $\mathsf{T}(\omega, \sigma, \dots, \mathbf{v}, \mathbf{w}, \dots) = T^{ab \dots}_{cd \dots} \omega_a \sigma_b \dots v^c w^d$ .

Def: Transformation of Tensor Components

Examples:

$$1) T'^a{}_b = \mathsf{T}(\tilde{\Lambda}^a{}_c \Upsilon^c, \Lambda_b{}^d \Upsilon_d) = \tilde{\Lambda}^a{}_c \Lambda_b{}^d \mathsf{T}(\Upsilon^c, \Upsilon_d) = \tilde{\Lambda}^a{}_c \Lambda_b{}^d T^c{}_d.$$

$$2) T'^{ab}{}_c = \Lambda_c{}^d \tilde{\Lambda}^a{}_e \tilde{\Lambda}^b{}_f T_d{}^{ef}.$$

Question: Ordering of indices?

### 3.3.5 Tensor Operations

Def: Symmetrization

$$T^{a_1 \dots a_r}_{c_1 \dots c_t(b_1 \dots b_s)} := \frac{1}{s!} \sum_{\pi} T^{a_1 \dots a_r}_{c_1 \dots c_t \pi(b_1) \dots \pi(b_s)}.$$

Examples:

$$1) T_{\alpha(\mu\nu)} = \frac{1}{2!} (T_{\alpha\mu\nu} + T_{\alpha\nu\mu}).$$

$$2) T_{\mu(\nu} R_{\alpha)\beta\gamma} = \frac{1}{2!} (T_{\mu\nu} R_{\alpha\beta\gamma} + T_{\mu\alpha} R_{\nu\beta\gamma}).$$

$$3) T_{(\mu\alpha\beta)} = \frac{1}{3!} (T_{\mu\alpha\beta} + T_{\mu\beta\alpha} + T_{\alpha\mu\beta} + T_{\alpha\beta\mu} + T_{\beta\alpha\mu} + T_{\beta\mu\alpha}).$$

Def: Antisymmetrization

$$T_{c_1 \dots c_t [b_1 \dots b_s]}^{a_1 \dots a_r} := \frac{1}{s!} \sum_{\pi} (-1)^{\pi} T_{c_1 \dots c_t \pi(b_1) \dots \pi(b_s)}^{a_1 \dots a_r}.$$

Examples:

$$1) T_{[\mu\nu]} = \frac{1}{2!} (T_{\mu\nu} - T_{\nu\mu}).$$

$$2) T_{[\mu\nu]\beta} = \frac{1}{2!} (T_{\mu\nu\beta} - T_{\nu\mu\beta}).$$

$$3) T_{[\mu\alpha\beta]} = \frac{1}{3!} (T_{\mu\alpha\beta} - T_{\mu\beta\alpha} - T_{\alpha\mu\beta} + T_{\alpha\beta\mu} - T_{\beta\alpha\mu} + T_{\beta\mu\alpha}).$$

Def: Tensor Product

Let  $R \in (T_p)_s^r \mathcal{M}$ ,  $T \in (T_p)_q^t \mathcal{M}$ . Then the **tensor product** of  $R$  and  $T$  is  $R \otimes T \in (T_p)_{s+q}^{r+t} \mathcal{M}$  with components  $(R \otimes T)_{b_1 \dots b_{s+q}}^{a_1 \dots a_{r+t}} := R_{b_1 \dots b_s}^{a_1 \dots a_r} T_{b_{s+1} \dots b_{s+q}}^{a_{r+1} \dots a_{r+t}}$ .

Def: Contraction

Let  $T \in (T_p)_s^r \mathcal{M}$  with components  $T_{b_1 \dots b_s}^{a_1 \dots a_r}$ , we define the **contraction** of the first covariant and contravariant indices as the tensor  $R \in (T_p)_{s-1}^{r-1} \mathcal{M}$  as the tensor of components  $R_{b_2 \dots b_s}^{a_2 \dots a_r} := T_{ab_1 \dots b_s}^{aa_2 \dots a_r}$ .

### 3.3.6 Integration on Manifold and Differential Forms

**Motivation:** In a coordinate basis,  $\omega = \omega_{\mu} dx^{\mu}$ . Notice that a curve  $\gamma$  on the manifold  $\mathcal{M}$  is a submanifold of  $\mathcal{M}$ . The integral of  $\omega$  along the curve  $\gamma$  is  $\int_{\gamma \subset \mathcal{M}} \omega = \int_{\gamma} \omega_{\mu} dx^{\mu} = \int_{s_1}^{s_2} \omega_{\mu} \frac{dx^{\mu}(s)}{ds} ds$ .

Def: 2-Form

A **2-form**,  $\sigma$  is an antisymmetric twice covariant (type  $(0, 2)$ ) tensor such that  $\sigma(v, w) = -\sigma(w, v)$  and in a coordinate basis,  $\sigma = \sigma_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$ .

Def: Wedge Product (1-forms)

The **wedge product** is the fully antisymmetric tensor product.  
 $dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu} := 2dx^{[\mu} \otimes dx^{\nu]} = dx^{\mu} \otimes dx^{\nu} - dx^{\nu} \otimes dx^{\mu}$ .

Def:  $k$ -Form

A  **$k$ -form**,  $\sigma$ , is a fully antisymmetric  $k$ -times covariant (type  $(0, k)$ ) tensor.  
 $\sigma = \sigma_{\alpha_1 \dots \alpha_k} dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_k} = \frac{1}{k!} \sigma_{\alpha_1 \dots \alpha_k} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$ .

Def: Exterior Product (Wedge Product)

If  $\sigma$  is a  $k$ -form and  $\omega$  is a  $p$ -form. The **exterior product** of  $\sigma$  and  $\omega$  is their fully antisymmetrized tensor product.

$$(\sigma \wedge \omega)_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_p} = \frac{(k+p)!}{k!p!} \sigma_{[\alpha_1 \dots \alpha_k} \omega_{\beta_1 \dots \beta_p]}.$$

Def: Exterior Derivative

The **exterior derivative**  $d$ , is an operation that acts on a  $k$ -form and return a  $(k+1)$ -form.

$$\begin{aligned} d\sigma &= \frac{1}{k!} d(\sigma_{\alpha_1 \dots \alpha_k}) \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k} \\ &= \frac{1}{k!} \partial_\beta (\sigma_{\alpha_1 \dots \alpha_k}) dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k} \\ &= (k+1) \partial_{[\beta} \sigma_{\alpha_1 \dots \alpha_k]} dx^\beta \otimes dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_k} \end{aligned}$$

Def: Exact

A  $k$ -form,  $\omega$ , is **exact**  $\Leftrightarrow \omega = d\sigma$ , where  $\sigma$  is a  $(k-1)$ -form.

Def: Closed

A  $k$ -form,  $\omega$ , is **closed**  $\Leftrightarrow d\omega = 0$ .

Theorem: Exact  $\Rightarrow$  Closed

$\omega = d\sigma \Rightarrow d\omega = 0$ . However, the converse is not true in general.

$d\omega = 0 \Rightarrow \omega = d\sigma$  locally.

Theorem: Poincaré Lemma

If  $\omega$  is defined in a contractible domain (open subset of a manifold),  $d\omega = 0 \Leftrightarrow \omega = d\sigma$ .

Note

When solving exact ODEs, we are actually have  $dF = 0$  and we check  $\partial_y \partial_x F = \partial_x \partial_y F$  which we are actually checking for closeness.

Remark

If  $\omega$  is exact then  $\omega = df \Rightarrow \int_\gamma \omega = \int_\gamma df = f(s_2) - f(s_1)$ . Also,  $\oint_\gamma \omega = 0$ .

In polar coordinate,  $d\theta$  is actually not exact as  $\oint_\gamma d\theta = 2\pi n \neq 0$ . Also,  $\Theta = d\theta$  is not defined at the origin.

### 3.4 Smooth Maps and Diffeomorphisms

#### 3.4.1 Smooth Maps: Pullback and Pushforward

##### Def: Smooth Map

A map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$  is a **smooth map** if and only if given the atlases  $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$  and  $\{\mathcal{U}'_\alpha, \phi'_\alpha\}$  of  $\mathcal{M}$  and  $\mathcal{M}'$ , the functions  $\phi'_\alpha \circ \varphi \circ \phi_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$  is smooth.

##### Def: Pullback (Function)

The map  $\varphi$  induces a map  $\varphi^* : \mathcal{F}_{\mathcal{M}'} \rightarrow \mathcal{F}_{\mathcal{M}}$  which we call the **pullback** between the spaces of functions  $\mathcal{F}_{\mathcal{M}}$  and  $\mathcal{F}_{\mathcal{M}'}$  according to the following rule:

Given a function  $f' : \mathcal{M}' \rightarrow \mathbb{R}$ , its **pull-back**  $\varphi^* f' : \mathcal{M} \rightarrow \mathbb{R}$  is such that  $\varphi^* f'(p) := f' \circ \varphi(p) = f'(p)$ , with  $p \in \mathcal{M}$ .

##### Def: Pushforward (Vector)

The map  $\varphi$  induces a map  $\varphi_* : T_p \mathcal{M} \rightarrow T_{\varphi(p)} \mathcal{M}'$  called the **pushforward** between the tangent spaces  $T_p \mathcal{M}$  and  $T_{\varphi(p)} \mathcal{M}'$  according to the following rule:

Given a vector  $\mathbf{v} \in T_p \mathcal{M}$  then  $\varphi_* \mathbf{v} \in T_{\varphi(p)} \mathcal{M}'$  such that its action on a function  $f' \in \mathcal{F}_{\mathcal{M}'}$  is  $(\varphi_* \mathbf{v})|_{\varphi(p)}(f') := \mathbf{v}|_p(\varphi^* f')$ .

##### Remark

In coordinate bases  $(x^1, \dots, x^m)$  and  $(y^1, \dots, y^{m'})$ :

$$([\varphi_*] \mathbf{v})|_{\varphi(p)}(f') = v^i|_p \frac{\partial f'}{\partial y^j} \frac{\partial y^j}{\partial x^i}|_p \Rightarrow ([\varphi_*] \mathbf{v})|_{\varphi(p)} = v^i|_p \frac{\partial y^j}{\partial x^i}|_p \frac{\partial}{\partial y^j}|_{\varphi(p)}.$$

##### Remark

If the map  $\varphi$  is not injective, then the pushforward of a vector field does not define a vector field.

##### Def: Pullback (one-form)

Given a 1-form  $\omega \in T_p^* \mathcal{M}'$  then  $\varphi^* : T_{\varphi(p)}^* \mathcal{M}' \rightarrow T_p^* \mathcal{M}$  called the **pullback** between the cotangent spaces  $T_{\varphi(p)}^* \mathcal{M}'$  and  $T_p^* \mathcal{M}$  according to the following rule: Given a 1-form  $\omega' \in T_{\varphi(p)}^* \mathcal{M}'$ , then  $\varphi^* \omega' \in T_p^* \mathcal{M}$  whose action on vectors in  $T_p \mathcal{M}$  is  $\langle \varphi^* \omega', \mathbf{v} \rangle|_p := \langle \omega', \varphi_* \mathbf{v} \rangle|_{\varphi(p)}$ .

##### Remark

$$d(\varphi^* \omega) = \varphi^*(d\omega).$$

Remark

$$\varphi^*(\alpha \wedge \beta) = (\varphi^*\alpha) \wedge (\varphi^*\beta).$$

Remark

$$\text{In coordinate basis, } \varphi^*\omega = (\omega_i \circ \varphi) dx^i \circ \varphi.$$

Def: Pullback (k-form)

$$\varphi^*\omega = \varphi^*\left(\frac{1}{k!}\omega_{\alpha_1\dots\alpha_k}\wedge_{i=1}^k dx^{\alpha_i}\right).$$

### 3.4.2 Diffeomorphisms

Def: Diffeomorphism

A map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$  is a **diffeomorphism** if and only if  $\varphi$  and  $\varphi^{-1}$  are both smooth bijections.

Remark

For a diffeomorphism  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ , pullback and pushforward can be defined for all objects on the manifold as  $\varphi_* = (\varphi^{-1})^*$ .

Remark

$$[\varphi_*]_p^{-1} = [\varphi_*^{-1}]_{\varphi(p)}.$$

$$\varphi_*f(p) = (\varphi^{-1})^*f(p) \text{ and } (\varphi^*v)_p(f) = v_{\varphi^{-1}(p)}\varphi_*f.$$

Def: Local Flow

Consider a diffeomorphism  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and let  $\gamma_p(s)$  be the parametrized curve in  $\mathcal{M}$  such that  $\gamma_p(0) = p$  and its tangent vector at every points  $\gamma_p(s)$  is the vector field  $k$ .  $k$  is the generator of a set of local diffeomorphisms of the form  $\varphi_s : \mathcal{U} \rightarrow \varphi_s(\mathcal{U})$  such that  $\varphi_s(p) = \gamma_p(s)$ . This set of local diffeomorphisms is called the **local flow** of  $k$ .

Note

- 1)  $\varphi_{-s} \circ \varphi_s(p) = p \Rightarrow \varphi_s^{-1} = \varphi_{-s}$
- 2)  $\varphi_s(p) = \gamma(s)$
- 3)  $\varphi_{s+t}(p) = \gamma(s+t)$
- 4)  $\phi_s(\phi_t(p)) = \varphi_{s+t}(p)$

Hence, local diffeomorphisms forms a group (Lie group). And the local flow of the vector field  $\mathbf{k}$  is complete if the associated curve  $\gamma_p(s)$  can be extended for all values  $s \in \mathbb{R}$ .

### 3.4.3 Lie Derivative

**Motivation:** Let  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  be a diffeomorphism,  $\varphi$  establishes an isomorphism  $\varphi^* : (T_{\varphi(p)})^r_s \mathcal{M} \rightarrow (T_p)^r_s \mathcal{M}$  for  $p \in \mathcal{M}$ . The tensor  $[\varphi^*(T_{\varphi(p)})]_p - T|_p$  provides information about the difference between the value of a tensor field at  $\varphi(p)$  and at  $p$ .

Def: Lie Derivative

Consider a diffeomorphism  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and its pullback  $\varphi^* : (T_{\varphi(p)})^r_s \mathcal{M} \rightarrow (T_p)^r_s \mathcal{M}$ . Consider  $\varphi \equiv \varphi_t$  be the local flow generated by a vector field  $\mathbf{v}$ . The **Lie derivative** of a tensor  $\mathbf{T}$  along the (curve generated by) vector field  $\mathbf{v}$  at the point  $p \in \mathcal{M}$  is  $(\mathcal{L}_{\mathbf{v}}\mathbf{T})|_p := \lim_{t \rightarrow 0} \frac{[\varphi_t^*(\mathbf{T}_{\varphi_t(p)})]_p - \mathbf{T}_p}{t}$ .

Note

$t$  is a parameter along a curve generated by  $\mathbf{v}$ .

Intuitively, the Lie derivative is to:

- 1) Take the tensor at  $t$  steps away from point  $p \in \mathcal{M}$  (along the curve generated by  $\mathbf{v}$ ).
- 2) Move the tensor to point  $p$  by using the pullback.
- 3) Compute the difference between the two tensors at  $p$ .
- 4) Divide the difference of the two tensors by  $t$  then take the limit  $t \rightarrow 0$ .

Acknowledgement: This intuition is from [here](#). One can also think of Lie Derivative as something that measures the variation of a tensor field when moving

Remark

Lie Derivative have the properties:

- 1) Linearity.
- 2) Preserves the tensor type, symmetries, and contractions because  $\varphi^*$  does.
- 3) Satisfies Leibniz rule under tensor product and contraction:  $\mathcal{L}_{\mathbf{v}}(\mathbf{T} \otimes \mathbf{S}) = (\mathcal{L}_{\mathbf{v}}\mathbf{T}) \otimes \mathbf{S} + \mathbf{T} \otimes (\mathcal{L}_{\mathbf{v}}\mathbf{S})$ .
- 4) Cartan identity: The Lie derivative of a  $k$ -form field along the vector field  $\mathbf{v}$  follows  $\mathcal{L}_{\mathbf{v}} = i_{\mathbf{v}}d + di_{\mathbf{v}}$  (took from JiayuePhysics).

**Def: Lie Derivative of Scalar Field**

For a scalar field  $f$ ,  $\mathcal{L}_v f = v(f)$ , and in a coordinate basis,  $\mathcal{L}_v f = v^\nu \partial_\nu f$ .

**Def: Lie Derivative of Vector Field**

For a vector field  $w$ ,  $\mathcal{L}_v w = [v, w]$ , and in a coordinate basis,  $\mathcal{L}_v w = (v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \partial_\mu$ .

**Def: Lie Derivative of 1-Form Field**

For a 1-form field  $\omega$  in a coordinate basis,  $\mathcal{L}_v \omega = (v^\nu (\partial_\nu \omega_\mu) + \omega_\nu (\partial_\mu v^\nu)) dx^\mu$ .

**Ex: More Lie Derivatives**

$$\begin{aligned} \omega &= \omega_{\mu\nu} dx^\mu \otimes dx^\nu \Rightarrow \mathcal{L}_v \omega = (v^\alpha (\partial_\alpha \omega_{\mu\nu}) + \omega_{\alpha\nu} (\partial_\mu v^\alpha) + \omega_{\mu\alpha} (\partial_\nu v^\alpha)) dx^\mu \otimes dx^\nu. \\ T &= T_\mu{}^\nu dx^\mu \otimes \partial_\nu \Rightarrow \mathcal{L}_v T = (v^\alpha (\partial_\alpha T_\mu{}^\nu) + T_\alpha{}^\nu (\partial_\mu v^\alpha) - T_\mu{}^\alpha (\partial_\alpha v^\nu)) dx^\mu \otimes \partial_\nu. \\ T &= T_{\alpha\beta}{}^{\mu\nu} dx^\alpha \otimes dx^\beta \otimes \partial_\mu \otimes \partial_\nu. \\ \Rightarrow \mathcal{L}_v T &= (v^\gamma \partial_\gamma T_{\alpha\beta}{}^{\mu\nu} + T_{\gamma\beta}{}^{\mu\nu} \partial_\alpha v^\gamma + T_{\alpha\gamma}{}^{\mu\nu} \partial_\beta v^\gamma - T_{\alpha\beta}{}^{\gamma\nu} \partial_\gamma v^\mu - T_{\alpha\beta}{}^{\mu\gamma} \partial_\gamma v^\nu) dx^\alpha \otimes dx^\beta \otimes \partial_\mu \otimes \partial_\nu. \end{aligned}$$

## 3.5 Pseudo-Riemannian Geometry

### 3.5.1 Affine Connection and Covariant Derivative

**Motivation:** While Lie derivative is a derivative, we can't use it as a generalised gradient, as it doesn't change the type of the tensor. So we need to explore other derivatives. We take the derivative of a function  $f$  as  $\frac{\partial f}{\partial x^\mu} = \partial_\mu f$ . And for a vector  $v = v^\mu \Upsilon_\mu$ ,  $\frac{\partial v}{\partial x^\mu}$  is a bad notation for the derivative of the vector because both the components and basis of  $v$  can change. What we want to do is to write  $\frac{\partial \Upsilon}{\partial x^\mu}$  in terms of the bases ( $\frac{\partial \Upsilon}{\partial x^\mu} = \Gamma^\rho{}_{\nu\mu} \Upsilon_\rho$  with  $\Gamma^\rho{}_{\nu\mu}$  describing how basis changes at different point).

**Def: Affine Connection, Covariant Derivative**

An **affine connection**  $\nabla : (T_p \mathcal{M})_s^r \rightarrow (T_p \mathcal{M})_{s+1}^r$  is a rule through which we assign to each tensor  $T$  of type  $(r, s)$  and components  $T^{bc\dots}_{de\dots}$  another tensor field  $\nabla T$  of type  $(r, s+1)$  and components  $(\nabla T)^{bc\dots}_{de\dots a} \equiv \nabla_a T^{bc\dots}_{de\dots} \equiv T^{bc\dots}_{de\dots; a}$ . We call  $\nabla T$  the **covariant derivative** of  $T$ .

**Remark**

Affine connection follows the properties:

- 1) Linearity.
- 2) Leibniz rule.
- 3) Commutes with contraction.
- 4) Over functions  $f$ , it is the differential  $\nabla f = df$ .



Note

There are infinitely many affine connections and infinite ways to build covariant derivative.

Def: Directional Covariant Derivative

We define the **directional covariant derivative** of  $\mathsf{T}$  in the direction of vector  $\mathsf{v}$  as the tensor  $\nabla_{\mathsf{v}}\mathsf{T}$  of type  $(r, s)$  of components  $(\nabla_{\mathsf{v}}\mathsf{T})^{bc\dots}_{de\dots} = v^a \nabla_a T^{bc\dots}_{de\dots} \cdot (\nabla_{\mathsf{v}}\mathsf{T})^{bc\dots}_{de\dots} = \langle \nabla\mathsf{T}, \mathsf{v} \rangle$ .

Def: Coefficient of the Connection

The **coefficient of the connection** in an arbitrary basis  $\{\Upsilon_a\}$  is

$$\Gamma^a_{bc} := (\nabla\Upsilon_b)^a_c = \langle \Upsilon^a, \nabla\Upsilon_c \Upsilon_b \rangle \Rightarrow \nabla_c \Upsilon_b := \nabla\Upsilon_c \Upsilon_b := \Gamma^a_{bc} \Upsilon_a.$$

$\Gamma^a_{bc}$  is the  $a$ -th components of the covariant derivative of the  $b$ -th basis vector in the direction of the  $c$ -th basis vector.

Remark

The notation for covariant derivative  $\nabla_{\mu}v^{\nu} = (\nabla\mathsf{v})_{\mu}^{\nu}$ . Also...

Note

Another convention is  $\Gamma^a_{bc} := (\nabla\Upsilon_c)^a_b = \langle \Upsilon^a, \nabla\Upsilon_b \Upsilon_c \rangle$ .

$$\nabla_a v^b = \Upsilon_a(v^b) + v^c \Gamma^b_{ca}$$

$$\nabla_{\mu} v^{\nu} = \partial_{\mu} v^{\nu} + v^{\sigma} \Gamma^{\nu}_{\sigma\mu}$$

$$\nabla_a \omega_b = \Upsilon_a(\omega_b) - \omega_c \Gamma^c_{ba}$$

$$\nabla_{\mu} \omega_{\nu} = \partial_{\mu} \omega_{\nu} - \omega_{\sigma} \Gamma^{\sigma}_{\nu\mu}$$

$$\text{Note: } \frac{\partial \mathsf{v}}{\partial x^{\mu}} = \left( \frac{\partial v^{\nu}}{\partial x^{\mu}} + v^{\sigma} \Gamma^{\nu}_{\sigma\mu} \right) \Upsilon_{\nu}.$$