Block 3

Introduction to General Relativity

3.1 Spacetime

The spacetime is the set of all the events, and it is described by a pseudo-Riemannian differentiable manifold $\mathcal{M}$ with a Lorentzian metric g. This manifold is defined except for isometries (diffeomorphisms that preserve the metric). More specifically, two pairs of manifolds and metrics $(\mathcal{M}, \mathsf{g})$ and $(\mathcal{M}', \mathsf{g}')$ represent the same spacetime if and only if there exists a diffeomorphism $\theta : \mathcal{M} \to \mathcal{M}'$ such that $\theta_* \mathsf{g} = \mathsf{g}'$. These diffeomorphisms correspond to changes of coordinates.

We will consider only spacetimes that are time-orientable, that is, spacetimes for which there exists a smooth timelike vector field that does not vanish at any point.

A spacetime $(\mathcal{M}, \mathsf{g})$ is analytically continuable if and only if there exits a larger spacetime $(\mathcal{M}', \mathsf{g})$ in which the first one is isometrically embedded. A spacetime is a *maximal analytic continuation* of another spacetime if and only if the former is an analytic continuation of the second that is not analytically continuable. A maximal analytic continuation can be obtained by analytically continuing all the geodesics (this is, letting their affine parameters go to infinity) through the proper changes of coordinates, and so removing any coordinate singularity.

3.1.1 The postulates of General Relativity

The equivalence principle (equality of the inertial mass and gravitational hypercharge) suggests that gravity may be understood as curvature, and the covariance principle (the laws of physics cannot depend on our choice of reference frame) makes us consider that the material fields (massive particles, electromagnetic fields,...) can perhaps be described by the same tensors than in special relativity, only modifying its dynamical equations to accommodate the fact that the spacetime can be curved.

Postulate 0. Gravity can be understood as curvature of the spacetime manifold. The gravitational forces result from geodesic deviation of the free matter fields moving through spacetime.

Postulate 1. (Local causality): The equations of motion of mater fields have to be such that given two points in a (convex) open subset of the spacetime are allowed to exchange signals only if they can be connected by a causal curve (this is, a curve whose tangent vector is not spacelike at any point).

Notice that the metric tensor is determined (except for a conformal factor which may vary along the spacetime) by specifying what points in spacetime are causally connected (exercise!).

Postulate 2. (Conservation of energy and linear momentum): The equations of motion of matter fields are such that there exits a symmetric tensor T^{ab} that we call the *stress-energy tensor*, built from the matter fields, its covariant derivatives and the metric only such that its divergence is zero, that is,

$$\nabla_b T^{ab} = 0. \tag{3.1.1}$$

This postulate fixes the conformal factor that was undetermined from the first postulate except for a multiplicative constant that gives us a choice of units. Indeed, it is easy to see that if the stress-energy tensor has zero divergence for a given metric g, it will not be divergence-less for another metric $\mathsf{g}' = \Omega\mathsf{g}$

In special relativity, the fact that the stress-energy tensor is divergenceless implies the conservation of energy. This can be proved by considering v^b the timelike tangent vector to a family of phsyical trajectories of inertial observers in a flat spacetime. Since parallel transport does not depend on the chosen path in this spacetime, if $\partial_a T^{ab} = 0$ the vector $P^a = -T^{ab}v_b$ which represents the average energy flow measured by these observers, and it is divergence-less. One can apply Gauss's theorem in the region enclosed by two spacelike surfaces of simultaneity to show that this quantity is time independent.

In a generic spacetime, parallel transport depends on the path, and it is not possible to choose any family of observers whose velocities are parallel. That is, there is no vector field such that $\nabla_a v^b = 0$. If such vector field existed, it would automatically satisfy the Killing equation (2.6.1) , and thus be an isometry of the spacetime which is not guaranteed to exist. In fact if that existed that would mean that total energy is consvered, a fact which is incompatible with tidal forces: Tidal forces that appear when considering a non-zero Riemann tensor can do work, and as such violate a naive definition of 'energy conservation'. However, locally, we can have a definition of energy conservation, and that is precisely what equation (3.1.1) imposes.

Postulate 3. (Einstein equations): The equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{8\pi G_{\mathrm{N}}}{c^4}T_{ab}$$

are satisfied in the whole spacetime.

From now on, we are going to work, unless otherwise specified, in natural units. This means that $c = 1$ dimensionless (but we will keep G_{N} full dimensional). This means that Einstein equations take the form

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G_{\mathrm{N}}T_{ab}$$

One can determine the stress-energy tensor corresponding to matter fields from a variational principle in the same way as in special Relativity if we have a Lagrangian that provides the equations of motion for the material field in special relativity. The general 'recipe' would be to replace the flat metric $\eta_{\mu\nu}$ for the curved metric $g_{\mu\nu}$ and the partial derivative ∂_μ by the covariant derivative ∇_μ in the action.

We are not going to focus on how to obtain the stress-energy tensors in this course, but we will see the case of a perfect self-gravitating fluid. Before addressing that here we present the stress energy tensor of a scalar and electromagnetic field that can be obtained respectively from their special relativistic actions.

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\big(\nabla_c\phi\nabla^c\phi + m^2\phi^2\big)$$

for a scalar field ϕ of mass m, and

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right)$$

for an electromagnetic field $F_{ab} := \nabla_a A_b - \nabla_b A_a$.

3.1.2 Dynamics of a single particle: geodesic equation from variational principles II

The simplest case of the equation of motion for a single particle can be derived as a variational principle if we know what the spacetime is. We saw this already in the two previous blocks. If we focus on the motion of a particle going along a timelike trajectory x^μ.

We can derive equation of motion of such a particle out of a stationary action principle. Since free particles follow geodesics, we should obtain that the trajectory $x^\mu(\tau)$ (parametrized in terms of the proper time τ) will folllow the geodesic equation. Consider the case of trying to find a geodesic between two timelike-separated points.

Let the action be

$$S = \int \mathrm{d}s \tag{3.1.2}$$

where $\mathrm{d}s = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$ is the line element. There is a negative sign inside the square root because the curve must be timelike. To get the geodesic equation we must vary this action. To do this let us parametrize this action with respect a parameter λ. Doing this we get:

$$S = \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \tag{3.1.3}$$

We can now go ahead and vary this action with respect to the curve x^μ, keeping the beginning and the end of the trajectory fixed at parameter values times λ_1 and λ_2, i.e,

$$0 = \delta S = \int \mathrm{d}\lambda\, \delta\left(\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\right)\mathrm{d}\lambda = \int \frac{\delta\left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)}{2\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}} \tag{3.1.4}$$

with $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$:

For convenience, let us parameterize the trajectories w.r.t. proper time, τ. If $\lambda = \tau$ then $\mathrm{d}x^\mu/\mathrm{d}\lambda$ becomes $\mathrm{d}x^\mu/\mathrm{d}\tau = \dot{x}^\mu$, which is the four-velocity. Since the four-velocity satisfies that $\dot{x}^\mu \dot{x}_\mu = -1$ (for timelike trajectories) then

$$0 = \int \delta\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)\mathrm{d}\tau. \tag{3.1.5}$$

Using the product rule we get:

$$0 = \int \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta g_{\mu\nu} + g_{\mu\nu}\frac{d\delta x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{d\delta x^\nu}{\mathrm{d}\tau}\right)\mathrm{d}\tau = \int \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha + 2g_{\mu\nu}\frac{d\delta x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{3.1.6}$$

Integrating by parts the last term and dropping the total derivative (which equals to zero at the boundaries) we get that:

$$0 = \int \mathrm{d}\tau \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\delta x^\mu \frac{d}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)\right) \tag{3.1.7}$$

$$= \int \mathrm{d}\tau \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\delta x^\mu \partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{\mathrm{d}\tau^2}\right) \tag{3.1.8}$$

Simplifying a bit we see that:

$$0 = \int \mathrm{d}\tau \delta x^\mu \left(-2g_{\mu\nu}\frac{d^2x^\nu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\mu g_{\alpha\nu} - 2\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu} \right) \tag{3.1.9}$$

so,

$$0 = \int \mathrm{d}\tau \delta x^\mu \left(-2g_{\mu\nu}\frac{d^2x^\nu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\mu g_{\alpha\nu} - \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu} - \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\partial_\nu g_{\mu\alpha} \right) \tag{3.1.10}$$

multiplying this equation by $-\frac{1}{2}$ we get:

$$0 = \int \mathrm{d}\tau \delta x^\mu \left(g_{\mu\nu}\frac{d^2x^\nu}{\mathrm{d}\tau^2} + \frac{1}{2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right) \right) \tag{3.1.11}$$

from here, we find the following set of Euler-Lagrange equations:

$$g_{\mu\nu}\frac{d^2x^\nu}{\mathrm{d}\tau^2} + \frac{1}{2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right) = 0 \tag{3.1.12}$$

Multiplying by the inverse metric tensor $g^{\mu\beta}$ we get that

$$\frac{d^2x^\beta}{\mathrm{d}\tau^2} + \frac{1}{2}g^{\mu\beta}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{3.1.13}$$

Thus we get the geodesic equation:

$$\frac{\mathrm{d}^2x^\beta}{\mathrm{d}\tau^2} + \Gamma^\beta{}_{\alpha\nu}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \Leftrightarrow \dot{x}^\nu \nabla_\nu \, \dot{x}^\mu = 0 \tag{3.1.14}$$

where the Christoffel symbol in terms of the metric tensor takes the known form

$$\Gamma^\beta{}_{\alpha\nu} = \frac{1}{2}g^{\mu\beta}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right) \tag{3.1.15}$$

3.1.3 Stress-energy tensor of a perfect fluid

Let us consider a small region of spacetime $\mathcal{V}$ which contains a perfect fluid, that is, a fluid with no viscosity that does not a conduct heat. These properties imply that for a comoving observer the fluid is isotropic, as we will see.

Let us understand first the components of the stress energy tensor in special relativity first in a given comoving frame. From the comoving frame:

- T^{00} is the energy density ρ of the fluid as seen from this observer
- T^{0i} represent the energy flow in the direction i. Even if the fluid is not moving with respect to this observer, there could have been an energy flow due to heat conduction. The requirement that the fluid is perfect, and thus it does not conduct heat, implies that $T^{0i} = 0$

- T^{ij} represents the flow of the i-component of the momentum in the direction j, that is, the flow of momentum through the surface perpendicular to Υ_j. Since viscosity is precisely what induces a change of momentum in the direction parallel to the surface, the absence of viscosity means that such a flow cannot exist. This translates into the fact that the non-diagonal components of the spatial part of $T^{\mu\nu}$ cancel in this bases. Additionally, all the diagonal components have to be identical, or otherwise a change of comoving reference frame (say a rotation) would allow us to see perpendicular flows due to viscosity (so we need to demand the non diagonal elements to be zero in all bases, which means that the spatial part of this tensor has to be proportional to the identity). In summary, the absence of viscosity implies that $T^{ij} = p\delta^{ij}$. The quantity p represents the flow of the i-th component of linear momentum through the surface perpendicular to the direction Υ_i. This is therefore a force per unit area, hence the pressure of the fluid.

We know the compoennts of the stress-energy tensor in a very particular frame. We need to write these components in a fully covariant form. In the comoving reference frame, the four-velocity field is simply $u^\mu = (1, 0, 0, 0)$, so we can write the components of the stress energy tensor in covariant form as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}.$$

Since this equation is tensorial, we know how to write the general relativistic version replacing the flat metric by the metric tensor of the curved spacetime and any partial derivative by a covariant derivative:

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab},$$

where v is the velocity field of the fluid.

The condition that the stress-energy tensor is divergence-less implies the following equations of motion:

$$u^a\nabla_a\rho + (\rho + p)\nabla_a u^a = 0 \tag{3.1.16}$$

$$(\rho + p)u^b\nabla_b u^a + (g^{ab} + u^a u^b)\nabla_b p = 0 \tag{3.1.17}$$

These equations are obtained by projecting the conservation law $\nabla_a T^{ab} = 0$ on the velocity u_b and on the directions perpendicular to it respectively. In other words: contracting this conservation law with u_b and $h^c{}_b = \delta^c{}_b + u^c u_b$ respectively.

To interpret the first equation, let us write it in terms of the particle density n in the fluid (number of particles per unit volume) and the specific entropy S (entropy per particle) of the fluid. The first law of thermodynamics tells us that

$$\delta U = NT\delta S - p\delta V$$

where U is the energy, V the volume, T the temperature and N the total number of particles. We can write the first law in terms of the energy density ρ and the particle density n as follows:

$$nT\delta S = \delta\rho - (p + \rho)\delta n/n$$

Dividing this equation by $\delta\tau$ (increment of proper time) and notating $\dot{f} = (\partial f/\partial\tau) = u^a\nabla_a f$ we obtain the expression

$$n^2T\dot{S} = n\dot{\rho} - (p + \rho)\dot{n}.$$

Substituting $\dot{\rho}$ in (3.1.16) by its value in terms of $\dot{S}$ and $\dot{n}$, we get

$$n^2T\dot{S} = -(p + \rho)\nabla_a(nu^a).$$

This equation indicates that the entropy of the fluid is conserved if the particle flow nu^a is conserved. The second equation of motion (3.1.17) relates the acceleration $\dot{u}^a$ of the fluid with the pressure gradients (which are force fields).

Let us consider now a fluid composed of dust (free particles with no interaction whatsoever). In free space such a fluid has zero pressure and then its equations of motion get reduced to

$$\nabla_a(\rho u^a) = 0, \qquad u^b \nabla_b u^a = 0.$$

According with these equations a) energy is conserved and b) every single one of the particles follow geodesics, which is expectable since the fluid is a dust of free particles. We see then that the conservation of the stress-energy tensor contains the laws of (geodesic) motion of free particles.

Very often we consider barotropic fluids, that is, fluids such that the pressure p is a function only of the energy density ρ. This is the case, for example, in cosmological models.

3.1.4 Energy conditions

The energy conditions are a series of additional postulates that are not part of General Relativity per se but that are imposed very often on top of the postulates of GR. There are two main reasons why this is done: 1) the technical requirements of particularly interesting proofs of a nice desired results (e.g. singularity theorems) 2) to avoid 'troublesome' solutions of Einstein equations such as wormholes, warp-drives or time-machines (closed timelike curves).

The different energy conditions are related with the ban of negative energy densities in the matter fields. In particular

- A stress-energy tensor satisfies the *weak energy condition* if and only if $T_{ab}v^a v^b \geq 0$ for any future-directed timelike vector v^a.
- A stress-energy tensor satisfies the *strong energy condition* if and only if $(T_{ab} - T^c{}_c g_{ab}/2)v^a v^b \geq 0$ for any future-directed timelike vector v^a.
- A stress-energy tensor satisfies the *dominant energy condition* if and only if the vector $-T^a{}_b v^b$ is future-directed timelike or lightlike, or it vanishes for any future-directed timelike vector v^a .
- A stress-energy tensor satisfies the *null energy condition* if and only if the vector $T_{ab} k^a k^b \geq 0$ for any future-directed null vector k^a .

The energy conditions are related by the following implication chain:

$$\text{Strong} \Rightarrow \text{Null} \Leftarrow \text{Weak} \Leftarrow \text{Dominant}$$

Notice that any timelike future-directed vector describes the trajectory of a phsyical observer. For such an observer the momentum density is given by $-T^a{}_b v^b$ and the energy density for such an observer is $T_{ab}v^a v^b$. The weak energy conditions is enough to prevent the existence of wormholes, warp drives or time-machines.

An interesting exercise is to compute the constraints that the energy conditions impose on the density and pressure of a perfect fluid.

Quantum fields (scalar field, electromagnetic, etc) violate energy conditions. For those cases it has been studied how average energy conditions impose bounds on the amount of negative energy they can contain.

3.2 Symmetries and Killing vectors

Solving Einstein equations means solving a system of coupled non-linear partial differential equations. This is complicated.

To simplify the problem we can consider weak gravity: spacetime would be flat except for small perturbations:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

One can prove that in vacuum, the perturbations obey a wave equation: Gravitational waves.

The calculations can also be simplified outside of the weak gravity regime if the physical problem presents symmetries: We learned that for every isometry we can associate a Killing vector ξ^μ, and we know that there is a maximum possible number of symmetries of $\frac{1}{2}n(n+1)$.

The case of flat spacetime $g_{\mu\nu} = \nu_{\mu\nu}$ has all possible symmetries. In 4 dimensions this would be 10. What are the Killing vectors associated to each of them?

- Time translational invariance (the metric does not depend on time):

  Killing vector in Cartesian coordinates:

  - $\mathsf{k}_1 = \partial_t$

- Space translational invariance (the metric does not depend on space x, y, z):

  Killing vectors in Cartesian coordinates:

  - $\mathsf{k}_2 = \partial_x$
  - $\mathsf{k}_3 = \partial_y$
  - $\mathsf{k}_4 = \partial_z$

- Rotational invariance (the metric is invariant under rotations around the axes x, y, z):

  Killing vectors in Cartesian coordinates:

  - $\mathsf{k}_5 \equiv \mathsf{S}_x = y\partial_z - z\partial_y$
  - $\mathsf{k}_6 \equiv \mathsf{S}_y = x\partial_z - z\partial_x$
  - $\mathsf{k}_7 \equiv \mathsf{S}_z = x\partial_y - y\partial_x$

- Boost invariance (the metric is invariant under boosts along the axes x, y, z):

  Killing vectors in Cartesian coordinates:

  - $\mathsf{k}_8 \equiv \mathsf{K}_x = x\partial_t - t\partial_x$
  - $\mathsf{k}_9 \equiv \mathsf{K}_y = y\partial_t - t\partial_y$
  - $\mathsf{k}_{10} \equiv \mathsf{K}_z = z\partial_t - t\partial_z$

A spacetime with the maximum number of symmetries is called 'maximally symmetric'. Those are spaces of constant curvature.

In general, when we have symmetries in the physical setup, we impose the symmetries on the metric first and then plug the metric into Einstein equations to find the undetermined components.

3.3 Schwarzschild metric

Birkoff theorem tells us that the only vacuum solution of Einstein equations with spherical symmetry is also time translational invariant and takes the form (in 3+1D):

$$\mathrm{d}s^2 = -\left(1-\frac{R_\mathrm{S}}{r}\right)\mathrm{d}t^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.3.1}$$

where the notation $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ and R_S is an arbitrary constant called Schwarzschild radius.

Notice that the metric is written in spherical coordinates. For the following sections we are going to use the coordinates in which the spherically symmetric metric is written. Four-positions can be characterized in spherical coordinates of an observer at infinity as follows

$$x^\mu = \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix}. \tag{3.3.2}$$

The components of the metric in the basis associated with these spherical coordinates can be read from the line element since $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$:

$$g_{00} = -\left(1-\frac{R_\mathrm{S}}{r}\right), \qquad g_{11} = \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2\theta, \tag{3.3.3}$$

and all the other components are zero.

This metric is asymptotically flat: when $r\to\infty$ then $\mathrm{d}s^2 \to -\mathrm{d}t^2 + dr^2 + r^2\mathrm{d}\Omega^2 = -dt^2 + \mathrm{dx}^2$. This means that for large values of r the spacetime looks flat.

Can we relate this to the problem of a gravitating spherical mass? It has to be possible since Birkoff theorem says this is the only solution with these symmetries.

When $R_\mathrm{S}/r \ll 1$ then $\left(1-\frac{R_\mathrm{S}}{r}\right)^{-1} = 1 + \frac{R_\mathrm{S}}{r} + \mathcal{O}\left(\frac{R_\mathrm{S}^2}{r^2}\right)$ and we can approximate the metric (3.3.1) by

$$\mathrm{d}s^2 = -\left(1+2\phi\right)\mathrm{d}t^2 + \left(1-2\phi\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.3.4}$$

where $\phi = -\frac{1}{2}\frac{R_\mathrm{S}}{r}$. If we plug in this metric into Einstein equations (far away, so no sources, $T_{\mu\nu}=0$) we obtain that

$$R_{\mu\nu} = 0 \Leftrightarrow \Delta\phi = 0 \tag{3.3.5}$$

where Δ is the Laplacian operator. This reminds of the equation for a gravitational potential in vacuum in newton's gravity. If ϕ is time independent and spherically symmetric then the Laplace equation can be written as

$$\Delta\phi = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi\right) = 0 \Rightarrow \phi = -\frac{A}{r} + B \tag{3.3.6}$$

If we set initial conditions so that this dimensionless potential is zero at infinity then $B=0$. If we want the potential to be attractive then $A>0$.

Now, if we want this potential to be associated linearly with a mass scale then $A \propto M$. Since ϕ is dimensionless then the proportionality constant has to be such that multiplied by M it has units of length.

Those are the dimensions of Newton's constant[1] therefore, through dimensional analysis we get that $A = G_{\rm N} M$ and thus

$$\phi = -\frac{G_{\rm N} M}{r}$$

This tells us that the Schwarzschild metric describes the spacetime generated by a gravitating mass, at least outside of the gravitating mass, and in that case it takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\rm N}M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_{\rm N}M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.3.7}$$

that is, the Schwarzschild radius is $R_{\rm S} = 2G_{\rm N}M$.

This also gives us an estimator to see if the Newtonian approximation is good: it will be good if $2\phi \ll 1$ (or equivalently, the strength of relativistic corrections in weak gravity will be given by 2ϕ). Some numbers:

- On the surface of Earth: $2\phi \approx 10^{-9}$
- On the surface of the Sun: $2\phi \approx 10^{-6}$
- On the surface of a neutron star: $2\phi \approx 10^{-1}$

Can we measure GR effects near to Earth or in the Solar system? Yes! we will see some cases

3.3.1 Gravitational redshift

The Scharwzschild metric is time-transnationally invariant thus it has got a timelike killing vector. Let us call it $\mathsf{k} = \partial_t$ where t is the proper time of an observer at infinity (that sees around them Minkowski). We can represent it as a row four-vector:

$$k^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.8}$$

Any observer will have three 'axes' representing their direction of time and their spatial directions. We can associate vectors pointing in these directions and due to curvature, these vectors do not have the same components everywhere. In particular, the timelike vector representing the flow of time (the time axis) for of an observer at a finite constant radius $r = r_i$ can be found from the metric (3.4.1). Since position is constant $\mathrm{d}r = \mathrm{d}\Omega = 0$. A physical observer follows a timelike traejctory so that $\mathrm{d}s^2 = -\mathrm{d}\tau^2$ and so

$$\mathrm{d}\tau^2 = \left(1 - \frac{2G_{\rm N}M}{r_i}\right)\mathrm{d}t^2 \Rightarrow \tau = \sqrt{\left(1 - \frac{2G_{\rm N}M}{r_i}\right)}t + C_i \tag{3.3.9}$$

where C_i is a synchronization constant. We can now determine the timelike vector $\mathsf{u} = \partial_\tau$ giving the flow of time for the observer stationary at r_i. We know that

$$\mathsf{u} = \partial_\tau, \mathsf{k} = \partial_t, \qquad \frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} \Rightarrow \partial_\tau = \frac{\partial t}{\partial \tau}\partial_t. \tag{3.3.10}$$

[1]Newton's constant has units of $[G_{\rm N}] = \frac{[L^3]}{[M][T^2]}$. In natural units ($\hbar = c = 1$), $[T] = [L]$, and $m = [L^{-1}]$, therefore $[G_{\rm N}] = [L]^2$ and $[G_{\rm N}M] = [L]^{-1}$

From (3.3.9) for stationary observers at $r = r_i$:

$$\partial_t = \sqrt{\left(1 - \frac{2G_{\text{N}}M}{r_i}\right)}\,\partial_\tau \Rightarrow \mathsf{u} = \frac{1}{\sqrt{\left(1-\frac{2G_{\text{N}}M}{r_i}\right)}}\mathsf{k} \Rightarrow u^\mu = \left(1-\frac{2G_{\text{N}}M}{r_i}\right)^{-\frac{1}{2}} k^\mu = \begin{pmatrix} \left(1-\frac{2G_{\text{N}}M}{r_i}\right)^{-\frac{1}{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.11}$$

To find the energy of a beam of light (or a particle) of four-momentum p^μ as measured by the observer at r_i will be given (same as in block 1) by the zeroth-component of the four-momentum. To find the zero-th component in the frame of the stationary observer at r_i we need to contract the four-momentum with the vector in the direction of time for this observer, this is u:

$$E_{r=r_i} = {P'}_0(r_i) = -u^\mu(r_i)P_\mu(r_i) = \left(1-\frac{2G_{\text{N}}M}{r_i}\right)^{-\frac{1}{2}} (k^\mu P_\mu)_{r=r_i} \tag{3.3.12}$$

but notice that the four-momentum of the beam of light is tangent to the null-geodesic that the beam of light follows, so we know, from Killing equation that $P^\mu k_\mu = P_\mu k^\mu$ is constant along the geodesic so it has the same constant value for all r along the light trajectory since k^μ is a Killing vector field.

We can compute the ratio of the measured energies for the same particle (or beam of light) as measured by two different observers, one at $r = r_1$ and another at $r = r_2$:

$$\frac{E_{r=r_1}}{E_{r=r_2}} = \left(\frac{1-\frac{2G_{\text{N}}M}{r_1}}{1-\frac{2G_{\text{N}}M}{r_2}}\right)^{-\frac{1}{2}} \frac{(k^\mu u_\mu)_{r=r_1}}{(k^\mu u_\mu)_{r=r_2}} = \left(\frac{1-\frac{2G_{\text{N}}M}{r_1}}{1-\frac{2G_{\text{N}}M}{r_2}}\right)^{-\frac{1}{2}} \tag{3.3.13}$$

which is smaller than 1 if $r_2 > r_1$ (frequency gets reduced, redshift as the beam of light climbs a gravitational portential!) In the Newtonian approximation: $2G_{\text{N}}M/r_i \ll 1$ for both $r = r_1$ and $r = r_2$ so, for wavelengths $\lambda \propto E^{-1}$ we get that

$$\frac{\lambda_2}{\lambda_1} = 1 + \frac{GM}{r_1} - \frac{GM}{r_2} > 1 \tag{3.3.14}$$

So Newtonian gravity already predicts a redshift which, as expected, is the difference of energy of climbing the potential: to establish GR one needs to compare the subleading terms, so it is challenging. A better experimental testbed is needed.

Gravitational time dilation

With this very same calculation we can also see another rather interesting phenomenon: gravitational time dilation. Let us consider the proper time of two different stationary observers at $r = r_1$ and $r = r_2$ and let us compute $d\tau_2/\mathrm{d}\tau_1$. From (3.3.9) we get that

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} = \left(\frac{1-\dfrac{2G_{\text{N}}M}{r_1}}{1-\dfrac{2G_{\text{N}}M}{r_2}}\right)^{-\frac{1}{2}} \tag{3.3.15}$$

Therefore there is a red-shift factor between the ticking of the clocks stationary at different values of r. If the two observers are one on Earth, and the other one at the hight of a geostationary orbit. Converting all to natural units (of length):

- $r_1 = 6.370 \cdot 10^6$ m, $\quad r_2 = 4.215 \cdot 10^7$ m
- $M_{\text{earth}} = 5.972 \cdot 10^{24}$ Kg $\Rightarrow M_{\text{earth}}c^2 = 5.3748 \cdot 10^{41}$ J $\Rightarrow M_{\text{earth}}c/\hbar = 1.699 \cdot 10^{67}$ m^{-1}
- $G_{\text{N}} = 6.674 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2} $\Rightarrow G_{\text{N}}/c^2 = 7.415 \cdot 10^{-28}$ m kg^{-1} $\Rightarrow \hbar G_{\text{N}}/c^3 = 2.612 \cdot 10^{-70}$ m^2

So in natural units (of length):

- $r_1 = 6.370 \cdot 10^6$ m, $\quad r_2 = 4.215 \cdot 10^7$ m
- $M_{\text{earth}} = 1.699 \cdot 10^{67}$ m^{-1}
- $G_{\text{N}} = 2.612 \cdot 10^{-70}$ m^2

Substituting in (3.3.15):

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} = 1 + 5.914 \cdot 10^{-10} \tag{3.3.16}$$

If GPS didn't move, the clocks in the GPS satellites would be faster than the clocks on Earth by (a bit more than) half a nanosecond per second.

3.3.2 Precession of periapses (and in particular, the perihelion of Mercury)

Planets follow geodesics (they are free other than the gravitational interaction). We need to find the geodesics of the Schwarzschild metric. We can simplify the solution of the geodesic equations making use of the symmetries of the problem that will yield conserved quantities. We will proceed as follows: first identify conserved quantities, second we will find the geodesic equations, and third we will solve them with the help of the conserved quantities that we found.

Conserved quantities

- The metric is time-translational invariant: There is a killing vector $\mathsf{k} = \partial_t$. We can associate a conserved quantity (Energy for an observer at infinity) along geodesics that would be equal to the contraction of this Killing vector with the tangent vector v to the geodesic. We know that

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \begin{pmatrix} \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}r}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \end{pmatrix}, \qquad \mathsf{k} = \partial_t \Rightarrow k^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.17}$$

  And therefore the conserved quantity along geodesics (that corresponds to the energy for inertial observers in the asymptotically flat region) can be evaluated to

$$E = k^\mu v_\mu = g_{\mu\nu} k^\mu v^\nu = g_{00} k^0 v^0 = -\left(1 - \frac{2G_{\text{N}}M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{3.3.18}$$

- The metric is also independent of φ, which means that $\mathsf{s} = \partial_\varphi$ is also a Killing vector. This means that the quantity $L = s^\mu v_\mu$ will be conserved along geodesics too. Let us evaluate this quantity:

$$\mathsf{s} = \partial_\varphi \Rightarrow s^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow L = s^\mu v_\mu = g_{\mu\nu} k^\mu v^\nu = g_{33} k^3 v^3 = r^2 \sin^2\theta \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}. \tag{3.3.19}$$

  This quantity corresponds to angular momentum for an observer in the asymptotically flat region of spacetime.

Geodesic equations

An easy but lengthy calculation for the Schwarzschild metric yields the following form for the geodesic equations (3.1.14) (Exercise!):

$$\ddot{t} + \frac{2G_{\mathrm{N}}M}{r(r-2G_{\mathrm{N}}M)}\,\dot{r}\,\dot{t} = 0 \tag{3.3.20}$$

$$\ddot{r} + \frac{1}{2}\frac{2G_{\mathrm{N}}M}{r^3}(r-2G_{\mathrm{N}}M)\,\dot{t}^2 - \frac{1}{2}\frac{2G_{\mathrm{N}}M}{r(r-2G_{\mathrm{N}}M)}\,\dot{r}^2 - (r-2G_{\mathrm{N}}M)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) = 0 \tag{3.3.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}(r^2\dot{\theta}) - L\frac{\cos\theta}{\sin\theta}\,\dot{\varphi} = 0 \tag{3.3.22}$$

$$\ddot{\varphi} + \frac{2\,\dot{\varphi}\,\dot{r}}{r} + \frac{2\,\dot{\theta}\,\dot{\varphi}}{\tan\theta} = 0. \tag{3.3.23}$$

There is one more equation that we should not forget: the normalization condition. Since the trajectory of a planet is timelike we know that $v^\mu v_\mu = g_{\mu\nu} v^\mu v^\nu = -1$. Imposing this we get that

$$-\left(1 - \frac{2G_{\mathrm{N}}M}{r}\right)\dot{t}^2 + \left(1 - \frac{2G_{\mathrm{N}}M}{r}\right)^{-1}\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 = -1 \tag{3.3.24}$$

We need to feed these equations with initial conditions for $x^\mu(\tau)$, that is, $x^\mu(0) = \big(t(0), r(0), \theta(0), \varphi(0)\big)$. We quickly realize of the following: $\theta(\tau) = \pi/2$ is a solution of (3.3.22) for all τ. By the theorems guaranteeing the uniqueness of the solution of the DE, $\theta(\tau) = \pi/2$ is the unique solution of (3.3.22) for the initial condition $\theta(0) = \pi/2$. If we start at $\theta = \pi/2$ (equatorial plane) we remain there. In other words, the orbits are contained in a plane (same as with Newtonian gravity). This also means that in this case the equation (3.3.19) can be rewritten as

$$L = r^2\,\dot{\varphi}. \tag{3.3.25}$$

Using the conserved quantities L and E in (3.3.18) and (3.3.19) we can rewrite the normalization constraint (3.3.24) as

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2G_{\mathrm{N}}M}{r}\right)\left(\frac{L^2}{r^2} + 1\right) = 0 \tag{3.3.26}$$

which we can rewrite as

$$\varepsilon = \frac{1}{2}\dot{r}^2 + V(r) \tag{3.3.27}$$

where $\varepsilon = \frac{1}{2}E^2$ is an energy per unit mass and

$$V(r) = \underbrace{\frac{1}{2}}_{V_0} \underbrace{-\frac{G_{\rm N}M}{r} + \frac{L^2}{2r^2}}_{\text{Newtonian}} \underbrace{-\frac{G_{\rm N}ML^2}{r^3}}_{\text{Relativistic}}. \tag{3.3.28}$$

One sees that the prediction from Newtonian mechanics kinetic term, the Newtonian gravitational energy density, the Newtonian centrifugal term and a General Relativistic correction. This correction is small for large r but it becomes important for small radius. Let us see how we solve this differential equation. Multiplying (3.3.28) by $(\dot{\varphi})^{-2} = \frac{r^4}{L^2}$, making the substitution $x = \frac{L^2}{rG_{\rm N}M}$ and finally differentiating with respect to ϕ to obtain a trajectory we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}\varphi^2} - 1 + x = \underbrace{3\frac{G_{\rm N}^2M^2}{L^2}x^2}_{\text{non-Newtonian}} \tag{3.3.29}$$

The left hand side of this equation equated to zero is precisely the Newtonian prediction and it is analytically solvable. The full equation is solvable numerically. So far everything is exact, but since the GR correction to the Newtonian solution is expected to be small we can use perturbation theory to find the leading order correction. Let us write the solution as

$$x = x_0 + x_1 \tag{3.3.30}$$

where x_0 is the Newtonian solution that satisfies

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}\varphi^2} - 1 + x_0 = 0. \tag{3.3.31}$$

This is the equation of an ellipse, so it admits solutions of the form $x_0 = 1 + e\cos\varphi$ where e is the eccentricity of the ellipse $r_0(\varphi)$, this can be seen undoing the change of variables:

$$r_0 = \frac{\frac{L^2}{G_{\rm N}M}}{1 + e\cos\varphi}. \tag{3.3.32}$$

As expected, Newtonian orbits are ellipses. Finding x_1 is complicated, but to leading order in $G_{\rm N}^2M^2/L^2$, we can obtain an approximation. First we write the exact equation (3.3.29) as

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\varphi^2} + x_1 + \underbrace{\frac{\mathrm{d}^2x_0}{\mathrm{d}\varphi^2} + x_0 - 1}_{=0} = 3\frac{G_{\rm N}^2M^2}{L^2}(x_0 + x_1)^2. \tag{3.3.33}$$

and then we keep, as an approximation, only the leading order term x_0 on the right hand side (so far everything was still exact):

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\phi^2} + x_1 \approx 3\frac{G_{\rm N}^2M^2}{L^2}(\underbrace{1 - e\cos\varphi}_{x_0(\varphi)})^2. \tag{3.3.34}$$

The equation is a linear non-homogeneous equation and the homogeneous part does not have independent term. The perturbed solution at leading order can then be written as (exercise)

$$x \approx 1 + e\cos\left[\left(1 - 3\frac{G_{\rm N}^2M^2}{L^2}\right)\varphi\right] \tag{3.3.35}$$

Looking at this solution, since the periapsis happens at $\varphi = 0$, we see that after a full orbit the perihelion advances an angle $\Delta\varphi = 2\pi\left(3\frac{G_{\rm N}^2M^2}{L^2}\right)$.

The angular momentum of the orbit can be written in terms of the longer axis of the ellipse a and the eccentricity e (exercise!):

$$L^2 \approx G_{\rm N} M a(1-e^2) \tag{3.3.36}$$

yielding

$$\Delta\varphi \approx \frac{6\pi G_{\rm N} M}{(1-e^2)a} \tag{3.3.37}$$

For Mercury we obtain that $\Delta\varphi \approx 43''$/century of deviation from the Newtonian prediction.

3.3.3 Gravitational deflection of light

Light follows null-geodesics. For light rays, we have the geodesic equation together with the null-normalization condition.

- Geodesic equation:

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}s} + \Gamma^\mu{}_{\alpha\beta} v^\alpha v^\beta = 0 \tag{3.3.38}$$

  where $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ is a tangent vector to the trajectory $x^\mu(s)$ and s is an affine parameter that parametrizes the trajectory of the 'light particle'. Notice that s is not proper time (which does not make sense for a light trajectory).

- Normalization condition for a lightlike trajectory :

$$v^\mu v_\mu = g_{\mu\nu} v^\mu v^\nu = -\left(1-\frac{2G_{\rm N}M}{r}\right)\dot{t}^2 + \left(1-\frac{2G_{\rm N}M}{r}\right)^{-1}\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 = 0 \tag{3.3.39}$$

  where the dot denotes differentiation with respect to s.

Same as before, we work on a plane $\theta(\tau) = \pi/2$. We have the conserved quantities:

$$E = -\left(1-\frac{2G_{\rm N}M}{r}\right)\dot{t}, \qquad L = r^2\sin^2\theta\dot{\varphi} = r^2\dot{\varphi}. \tag{3.3.40}$$

Using these conserved quantities and multiplying by $-\left(1-\frac{2G_{\rm N}M}{r}\right)$, we can rewrite the normalization constraint (3.3.39) as

$$E^2 - \dot{r}^2 - \left(1-\frac{2G_{\rm N}M}{r}\right)\frac{L^2}{r^2} = 0 \Rightarrow \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = E^2 - \left(1-\frac{2G_{\rm N}M}{r}\right)\frac{L^2}{r^2} \tag{3.3.41}$$

This radial equation can be rewritten in terms of an effective attractive potential:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = E^2 - V(r) \tag{3.3.42}$$

where

$$V(r) = \left(1-\frac{2G_{\rm N}M}{r}\right)\frac{L^2}{r^2}. \tag{3.3.43}$$

This potential has a maximum at $r = 3G_{\rm N}M$. This means that there is a constant r solution (a photon orbit!) for $r = 3G_{\rm N}M$. However it is unstable (maximum, not minimum). There are no stable closed orbits. The rest of the solutions are scattering (either colliding with the mass or being bounced out).

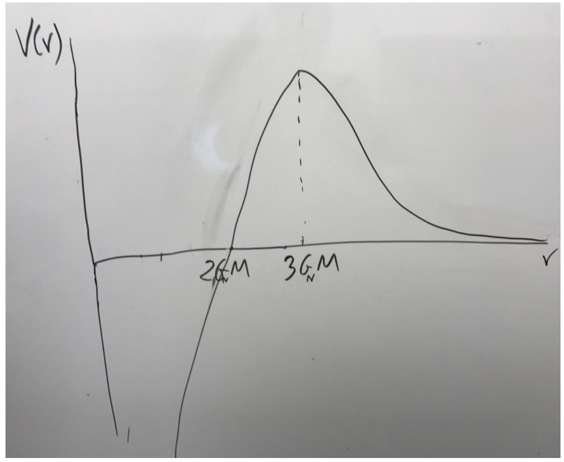

Figure 3.1: Effective potential.

For the scattering case we can use that $L^2 = r^4 \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2$ and substitute in (3.3.42) to yield:

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 = \frac{L^2}{r^4}\left(E^2 - V(r)\right)^{-1} \Rightarrow \frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm\frac{L}{r^2}\left(E^2 - V(r)\right)^{-\frac{1}{2}} \tag{3.3.44}$$

The two branches (the sign) gives the direction of the orbit: at the minimum distance $r = r_0$ the sign changes. To obtain the deflection angle we need to integrate this equation:

$$\Delta\varphi = \int_{\infty}^{r_0}\mathrm{d}r\left[-\frac{L}{r^2}\left(E^2 - V(r)\right)^{-\frac{1}{2}}\right] + \int_{r_0}^{\infty}\mathrm{d}r\frac{L}{r^2}\left(E^2 - V(r)\right)^{-\frac{1}{2}} = 2\int_{r_0}^{\infty}\mathrm{d}r\frac{L}{r^2}\left(E^2 - V(r)\right)^{-\frac{1}{2}} \tag{3.3.45}$$

This integral can be evaluated for $G_{\mathrm{N}} M\frac{E}{L} \ll 1$:

$$\Delta\varphi = \pi + \frac{4G_{\mathrm{N}}E}{L} + \mathcal{O}\left(G_{\mathrm{N}}^2 M^2\frac{E^2}{L^2}\right) \tag{3.3.46}$$

For rays of light grazing the surface of the Sun $\frac{4G_{\mathrm{N}}ME}{L} \approx 1.7''$

Newtonian Gravity predicts gravitational deflection of light, but gives the wrong prediction for the deflection angle by a factor 2. Eddington performed the experiment in a 1919 solar eclipse. Relativity nailed the results. Einstein became (TV) famous.

To see the Newtonian prediction, you can check, for instance: https://arxiv.org/pdf/physics/0508030.pdf

3.4 Schwarzschild Black Holes

Let us look again at the Schwarszchild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{R_{\mathrm{S}}}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_{\mathrm{S}}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.4.1}$$

We quickly realize that in (t, r, θ, φ) coordinates the metric appears to present two singularities: one when $r = R_{\mathrm{S}}$ and another when $r = 0$.

Let us first analyze what happens at $r = R_\mathrm{S}$. As a first approach let us consider three physical observers. Alice is falling from a radius r_0 in a radial trajectory $\dot\phi = \dot\theta = 0$, Bob watching the scene from far away: stationary in the asymptotically flat region at $r \to \infty$. Carol is stationary at a constant radius $r = r_\mathrm{C}$. The observer at infinity has proper coordinates (t, r, θ, ϕ). Alice has proper time τ. Let us describe the trajectory of Alice.

The normalization condition $v^\mu v_\mu = -1$ gives

$$-\left(1-\frac{R_\mathrm{S}}{r}\right)\dot t^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\dot r^2 = -1. \tag{3.4.2}$$

We also have the conserved quantity (3.3.18): $E = k^\mu v_\mu = -\left(1-\frac{R_\mathrm{S}}{r}\right)\dot t$. Substituting in (3.4.2):

$$-\left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}E^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\dot r^2 = -1. \tag{3.4.3}$$

Alice starts falling radially inward at t_0, with no initial speed, from r_0, which means that $r(t_0) = r_0, \dot r(t_0) = 0$. From the normalization condition that allows us to determine the conserved quantity E:

$$E^2 = \left(1-\frac{2G_\mathrm{N}M}{r_0}\right) \tag{3.4.4}$$

we can pick either value. If we consider that Alice and Bob were initially together (Alice falls from infinity) $r_0 \to \infty \Rightarrow E = \pm 1$, substituting in (3.4.3):

$$1-\dot r^2 = 1-\frac{R_\mathrm{S}}{r} \Rightarrow \dot r = \pm\sqrt{\frac{R_\mathrm{S}}{r}}. \tag{3.4.5}$$

Since Allice is falling inwards we pick the negative branch:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{R_\mathrm{S}}{r}}. \tag{3.4.6}$$

We can now integrate (3.4.6) to find how much time Alice takes to go from Carol ($r = r_\mathrm{C}$) to the Schwarzschild radius ($r = R_\mathrm{S}$).

- According to Alice:

$$\Delta\tau = -\int_{r_\mathrm{C}}^{R_\mathrm{S}} \mathrm{d}r\sqrt{\frac{r}{R_\mathrm{S}}} = \frac{2}{3}\left(R_\mathrm{S}^{\frac{3}{2}} - r_\mathrm{C}^{\frac{3}{2}}\right) < \infty \tag{3.4.7}$$

- According to Bob: The time dilation factor between Alice and Bob is obtained from differentiation with respect to τ of (3.4.1) for a radial trajectory ($\mathrm{d}\Omega = 0$)

$$-\,\mathrm{d}\tau^2 = -\left(1-\frac{R_\mathrm{S}}{r}\right)\mathrm{d}t^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 \Rightarrow -1 = -\left(1-\frac{R_\mathrm{S}}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2$$
$$\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{\left[\left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]^2 + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}} = \sqrt{\left(1-\frac{R_\mathrm{S}}{r}\right)^{-2}\frac{R_\mathrm{S}}{r} + \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1}} = \left(1-\frac{R_\mathrm{S}}{r}\right)^{-1} \tag{3.4.8}$$

and therefore

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\sqrt{\frac{R_\mathrm{S}}{r}}\left(1-\frac{R_\mathrm{S}}{r}\right). \tag{3.4.9}$$

Recalling that Bob's proper time is t and integrating this:

$$\Delta t = -\int_{r_{\rm C}}^{R_{\rm S}} \mathrm{d}r \sqrt{\frac{r}{R_{\rm S}}}\left(1-\frac{R_{\rm S}}{r}\right)^{-1} = \infty \tag{3.4.10}$$

Alice reaches $r = R_{\rm S}$ in a finite amount of proper time, but she takes an infinite amount of time according to an observer at infinity. Bob will observe a slower and slower (and redder and redder) Alice as she falls.

3.4.1 Event horizon

But what happens at $r = R_{\rm S}$? is that a singularity? will Alice's geodesic just end there? Let us see. First let us see if there is a problem with the curvature. Let us compute an estimate of the curvature at the horizon. We can compute the Kretschmann scalar (exercise!), which is the full contraction of the Riemann tensor:

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12R_{\rm S}^2}{r^6} \tag{3.4.11}$$

which is a scalar and as such its value at a point is coordinate independent. We see that the curvature is singular at $r = 0$ (definitely something bad happens there). However, $r = R_{\rm S}$ is not special: curvature there is finite (and not particularly large in general).

Why does the metric (3.4.1) have divergent coefficients for $r = R_{\rm S}$? Perhaps, it was our fault: our choice of coordinates (t, r, θ, φ) is not great around the horizon! In fact, we can define a new set of coordinates (T, X, θ, φ) (called Kruskal-Szekeres coordinates) as follows:

$$T = \left(\frac{r}{R_{\rm S}}-1\right)^{1/2} e^{r/2R_{\rm S}} \sinh\left(\frac{t}{2R_{\rm S}}\right), \qquad X = \left(\frac{r}{R_{\rm S}}-1\right)^{1/2} e^{r/2R_{\rm S}} \cosh\left(\frac{t}{2R_{\rm S}}\right) \tag{3.4.12}$$

for the exterior region $r > R_{\rm S}$ and

$$T = \left(1-\frac{r}{R_{\rm S}}\right)^{1/2} e^{r/2R_{\rm S}} \cosh\left(\frac{t}{2R_{\rm S}}\right), \qquad X = \left(1-\frac{r}{R_{\rm S}}\right)^{1/2} e^{r/2R_{\rm S}} \sinh\left(\frac{t}{2R_{\rm S}}\right) \tag{3.4.13}$$

for the interior region $0 < r < R_{\rm S}$. From this it follows that the radial coordinate r is given in terms of T and X by the implicit relationship:

$$T^2 - X^2 = \left(1-\frac{r}{2R_{\rm S}}\right)e^{r/R_{\rm S}} \tag{3.4.14}$$

In these new coordinates the metric is given by

$$\mathrm{d}s^2 = \frac{4R_{\rm S}^3}{r}e^{-r/R_{\rm S}}(-dT^2 + dX^2) + r^2 d\Omega^2. \tag{3.4.15}$$

The location of the point corresponding to $r = R_{\rm S}$ in these coordinates is given by $T = \pm X$. Note that the metric is perfectly well defined and non-singular at this point. Therefore there is 'no drama' at this point, the spacetime does not break and indeed Alice's geodesic can be continued past the Schwarzschild radius. The curvature singularity that we identified before ($r = 0$) is located at $T^2 - X^2 = 1$ in these coordinates.

However, there is something interesting happening at the Schwarzschild radius. Coming back to Schwarzschild coordinates and looking at (3.4.1), we see that in the interior region, the vector ponting in the radial direction $\mathsf{r} = \partial_r$ has negative norm!:

$$r^\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow r^\mu r_\mu = \left(1-\frac{R_{\rm S}}{r}\right)^{-1} \tag{3.4.16}$$

r^μ is spacelike for $r > R_\mathrm{S}$, but it becomes timelike $r < R_\mathrm{S}$ past the Schwarzschild radius. In the same fashion, the vector that was pointing the direction of time for observers living in the flat spacetime region at infinity $\mathsf{t} = \partial_t$ becomes spacelike!:

$$t^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow t^\mu t_\mu = -\left(1 - \frac{R_\mathrm{S}}{r}\right) \tag{3.4.17}$$

t^μ is timelike for $r > R_\mathrm{S}$, but it becomes spacelike $r < R_\mathrm{S}$ past the Schwarzschild radius. What is time for an observer outside the Schwarzschild radius, becomes space for an observer inside. Any timelike geodesic inside the Schwarzschild radius reaches $r = 0$ in a finite amount of time. Observers inside the Schawzschild radius are separated from the point $r = 0$ by a timelike interval: the singularity is in their futures, not somewhere else (it's actually everywhere!). The singularity is unavoidable. No signal can ever reach the point $r = R_\mathrm{S}$ again: that point is in the geodesic's past. We call the surface $r = r_s$ the event horizon, since no signals can make it back from the region $r < R_\mathrm{S}$ once crossed.

For a radially infalling observer such as Alice, the time it takes to fall from the Schwarzschild radius to the singularity is (exercise!) $t \approx 10^{-5} M/M_\mathrm{sun}$ where M is the mass of the black hole. For the largest known black hole, that gives Alice almost two days and a half to enjoy the interior of the black hole.

3.5 Penrose diagrams

The aim of Penrose diagrams is to graphically bring the infinity to a finite distance without changing the causal structure (e.g., using conformal transformations) and in such a way that light rays always follow straight lines with a slope $\pm\pi/4$.

Penrose diagrams are two-dimensional sections, therefore we must be sure that they are *totally geodesic*, i.e., that if a geodesic is tangent to it somewhere then it is tangent everywhere.

3.5.1 Minkowski spacetime

Let us start with the metric of the Minkowski spacetime $\mathbb{M}$

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2 \mathrm{d}\Omega_2^2,$$

change to null radial coordinates u', v' and introduce $\tilde{u}$, $\tilde{v}$ defined by

$$\left.\begin{array}{l} \tan\tilde{u} = u' = t' - r', \\ \tan\tilde{v} = v' = t' + r' \end{array}\right\} \qquad \text{with} \qquad \left\{\begin{array}{c} -\infty < u' \le v' < \infty, \\ -\pi/2 < \tilde{u} \le \tilde{v} < \pi/2. \end{array}\right.$$

The Minkowski line element acquires the form

$$\mathrm{d}s^2 = \Omega^{-2}\mathrm{d}\tilde{s}^2, \qquad \Omega = 2\cos\tilde{u}\cos\tilde{v},$$

where the new conformal metric $\mathrm{d}\tilde{s}^2$ is

$$\begin{aligned} \mathrm{d}\tilde{s}^2 &= -4\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \sin^2(\tilde{v} - \tilde{u})\mathrm{d}\Omega_2^2 \\ &= -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{r}^2 + \sin^2\tilde{r}\,\mathrm{d}\Omega_2^2, \end{aligned}$$

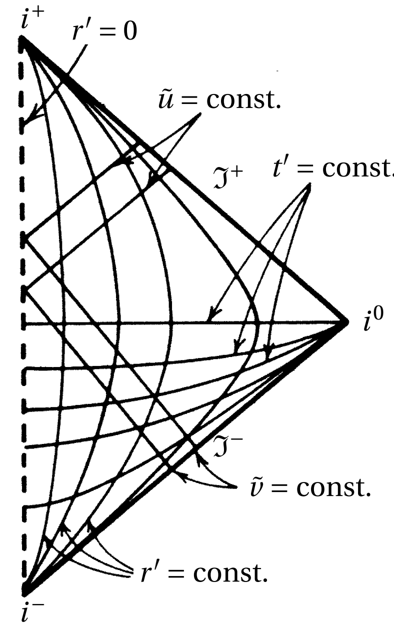

Figure 3.2: Penrose diagram of Minkowski spacetime.

with $\quad \tilde{t} = \tilde{v} + \tilde{u}, \quad \tilde{r} = \tilde{v} - \tilde{u}.$

This figure represents the Penrose diagram of the spacetime $\tilde{\mathbb{M}}$, called *conformal compactification* of Minkowski spacetime, which includes not only the conformal Minkowski spacetime but also its conformal infinities. Each point defines a two-sphere, except for i^0 e $i^\pm$ that are just points.

- i^0 defines the spatial infinity.
  It corresponds to $\quad r' = \infty,\ t'$ finite, $\quad \tilde{u} = -\pi/2,\ \tilde{v} = \pi/2.$
  Spatial geodesics start and end at i^0.
- $i^\pm$ are the past $(-)$ and future $(+)$ timelike infinities.
  They correspond to $t' = \pm\infty,\ r'$ finite, $\quad \tilde{u} = \tilde{v} = \pm\pi/2.$
  Timelike geodesics start at i^- and end at i^+.
- $\mathfrak{I}^\pm$ are the past $(-)$ and future $(+)$ null infinities.
  They correspond to $r' = \infty,\ t' = \pm\infty$ with $r' \mp t'$ finite.
  Null geodesics start at $\mathfrak{I}^-$ and end at $\mathfrak{I}^+$.

3.5.2 Black holes and event horizons (in Schwarzschild spacetime)

From now on, we will take $R_{\mathrm{S}} = 2G_{\mathrm{N}}M$ and we take $M > 0$, which can be interpreted as the mass of the black hole.

Consider the region $r > 2G_{\mathrm{N}}M$, introduce the so-called *tortoise radial coordinate* $r^* = r + 2G_{\mathrm{N}}M \log|1 - r/2G_{\mathrm{N}}M| \in \mathbb{R}$ and define the *Eddington-Finkelstein null coordinates*

$$u = t - r^*, \qquad v = t + r^*, \qquad -\infty < u, v < +\infty.$$

It is easy to see that the curves $\dot{u} = 0$ y $\dot{v} = 0$ are null radial geodesics whose affine parameters reach a finite value at $r = 2G_{\rm N}M$. Therefore, we introduce the *Kruskal-Szeckeres coordinates*

$$u' = -e^{-u/4G_{\rm N}M}, \qquad v' = e^{v/4G_{\rm N}M},$$

in terms of which, the metric acquires the form

$$\mathrm{d}s^2 = -\frac{32G_{\rm N}^3M^3}{r}e^{-r/2G_{\rm N}M}\mathrm{d}u'\mathrm{d}v' + r^2\mathrm{d}\Omega_2^2,$$

where r is implicitly defined by $u'v' = (1 - r/2G_{\rm N}M)e^{r/2G_{\rm N}M}$.

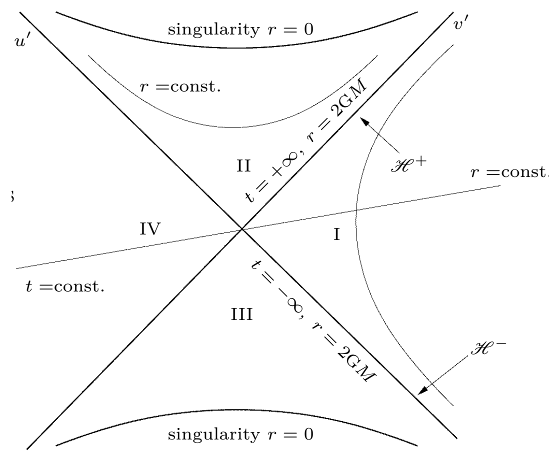

Figure 3.3: E-F diagram of the maximal analytical extension of Schwarzschild spacetime.

The range of u' and v' (defined in the region $r > 2G_{\rm N}M$) is $u' < 0$, $v' > 0$. We thus see that this spacetime admits an extension since the null geodesics parametrized by u' and v' artificially end at finite values of their parameters. If we extend u' and v' to the whole real line, we obtain the Kruskal maximal extension of the Schwarzschild spacetime. Since $r > 0$, u' and v' are constrained by the condition $u'v' > 1$.

Penrose diagram

Let us now introduce new coordinates defined by

$$\tan\tilde{u} = u', \qquad \tan\tilde{v} = v', \qquad -\pi/2 < \tilde{u}, \tilde{v}, \tilde{u} + \tilde{v} < \pi/2, \tag{3.5.1}$$

in terms of which, the line element becomes

$$\mathrm{d}s^2 = \Omega^{-2}\mathrm{d}\tilde{s}^2, \qquad \Omega^2 = re^{r/2M}(\cos\tilde{u}\cos\tilde{v})^2/8M^3, \qquad \mathrm{d}\tilde{s}^2 = -4\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + r^2\Omega^2\mathrm{d}\Omega_2^2. \tag{3.5.2}$$

The figure above shows the Penrose diagram of the Kruskal spacetime. We see that, as mentioned in efsec:naked, this spacetime has a visible singularity but it is an initial singularity which does not affect predictability. Indeed, it is globally hyperbolic (any spatial surface that joins $i^0_{\rm L}$ and $i^0_{\rm R}$ is a Cauchy surface) and it is also strongly asymptotically predictable: The closure of the intersection of the spacetime itself with the causal past of the future null infinity is the union of regions I, III and IV, which is globally hyperbolic. Therefore, the Kruskal spacetime has no naked singularities nor has the Schwarzschild spacetime.

In an asymptotically flat spacetime, we call a black hole a region where null geodesics cannot reach future null infinity i^+.

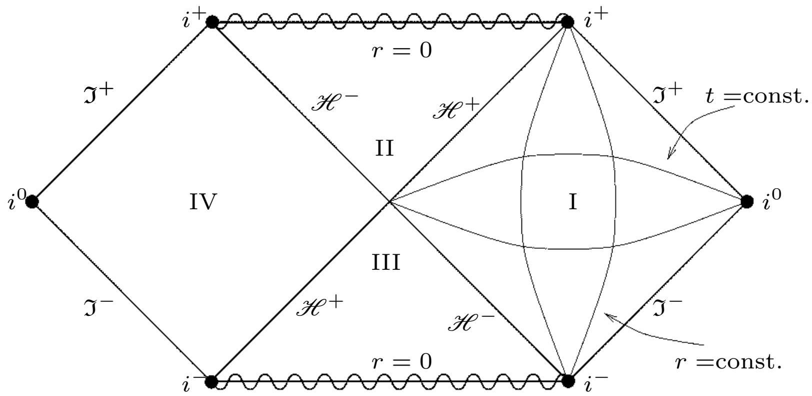

Figure 3.4: Penrose diagram of Kruskal spacetime.

3.6 Warp drives 101: The Alcubierre spacetime

Special lecture!