AMATH 475

University of Waterloo

Taught by: Eduardo Martin-Martinez

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Edward Chang

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1 Index Notation

Def: Covariant

Objects that transforms under change of basis like the element of the basis are called **covariant**, and its components have sub-indicies.

Def: Contravariant

Objects that transforms under change of basis like components of vectors are called **contravariant** and its components have super-indicies.

Def: Dual Basis & Dual Space

The **dual basis** to $\mathcal{B} = \{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ is $\mathbb{B}^\star = \left\{\mathsf{e}^1, \ldots, \mathsf{e}^n\right\}$ and it is the collection of linear operators such that $\mathsf{e}^i(\mathsf{e}_j) = \delta^i_j$. $\mathbb{B}^\star$ spans a vector space $V^\star$ called the **dual space** of V.

2 Special Relativity

2.1 Postulates of Special Relativity

Def: Event

An **event** is an individual point in spacetime, usually labelled which we represent by the tuple $E \equiv (t, \vec{x})$, t is the *time coordinate* and $\vec{x} \equiv (x, y, z)$ is the *space coordinates*.

Def: Spacetime

Spacetime is the set of all events, $\mathbb{S} = \left\{E \equiv (t, \vec{x}) : t \in R, \vec{x} \in \mathbb{R}^3\right\}$.

Def: Reference Frame

A **reference frame**, establishes a spacetime coordinate system $(t, \vec{x})$, which is a spatial coordinate system where the position of point-like particles can be specified, and a clock (something that can measure time).

Def: Inertial Reference Frame (IRF)

An **inertial reference frame** is a reference frame for which a particle stationary at it origin experience no force (Newton's first law holds).

Def: Postulates of Special Relativity 1 (SR): Principle of Relativity

1. **Principle of Relativity:** In the absence of gravity, all the laws of physics are identical in all inertial reference frames (This postulate is also in Galilean relativity).
2. **Speed of Light is Constant and Equal:** The speed of light in vacuum "c" is constant and the same for all inertial reference frames (absense of gravity).

2.2 Lorentz Transformation

Def: Galilean Transformation

Consider two inertial frames $S \equiv (t, \vec{x})$ and $S' \equiv (t', \vec{x}')$, where S' moves with velocity $\vec{v} = v\hat{x}$ to the right relative to S. The **Galilean transformation** (Galilean boost) from S to S' is:

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

Note: Galilean transformation would result in mathematical inconsistency (todo).

Def: Spacetime Interval

Given a particular inertial reference frame that establishes space time coordinates $(t, \vec{x})$, the **spacetime interval** between two events $E_1 \equiv (t_1, \vec{x}_1)$ and $E_2 \equiv (t_2, \vec{x}_2)$ is $\Delta s^2 := -c^2\Delta t^2 + \Delta\vec{x}^2 = -c^2\left(t_2 - t_1\right)^2 + \left(\vec{x}_2 - \vec{x}_1\right)^2$.

Note

In this course we use the signature $(-, +, +, +)$.

Remark

If the two events are connected by the propagation of light, $\Delta\vec{x}^2 = c^2\Delta t^2 \Rightarrow \Delta s^2 = -c^2\Delta t^2 + c^2\Delta t^2 = 0$.

Note

Possible transformations between two inertial reference frames: 3 rotations, 3 translations, 1 times shift, 3 boosts.
However, the spatial distance $\|\vec{x}_2 - \vec{x}\|$ is invariant under rotations and translations. Similarly, $\Delta t = t_2 - t_1$ is invariant under time shift. So the only transformations with significance are the boost.

3 Differential Geometry

3.1 Topology

Def: Topology

A **topology on a set** X, S is a collection of open sets of X, and S is a subset of the power set of X. S satisfies:
1) $\varnothing, X \in S$.
2) Any union of elements of S is in S.
3) Any finite intersection of elements of S is in S.

Def: Topological Space

A **topological space** (X, S), is an ordered pair where X is a set and S is a topology of X.

3.2 Manifolds

Def: Homeomorphism

A **homeomorphism** between two topological spaces X and Y is a map $\sigma : X \to Y$ such that the map is topologically continuous and it's inverse is topologically continuous (ϕ is a bijection).

Def: Hausdorff Space

A topological space X is a **Hausdorff space** if for all distinct points $x, y \in X$ there exists neighbourhoods of x and y, $\mathcal{U}_x$,$\mathcal{U}_y$ respectively, such that $\mathcal{U}_x \cap \mathcal{U}_y = \varnothing$.

Def: Topological Manifolds

A **topological manifold** of dimension n, $\mathcal{M}$ is a topological space that is **Hausdorff** and at every point possess an open neighbourhood homeomorphic to $\mathbb{R}^n$.
Note: The proper definition also requires that the space is second countable.

Def: Charts

A **chart** $(\mathcal{U}_\alpha, \phi_\alpha)$, where ϕ_α is a homeomorphism from an open subset $\mathcal{U}_\alpha \subseteq \mathcal{M}$ to $\mathbb{R}^n$. Note: $\phi_\alpha : \mathcal{U} \underset{open}{\subseteq} M \to \mathbb{R}^n$, and $\phi_\alpha(\mathsf{x}) \equiv x^\mu$, x^μ is the coordinate in $\mathbb{R}^n$.

Def: Transition Map

Consider two charts of $\mathcal{M}$, $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$, where $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$. The **transition map** is a homeomorphism from $\phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, defined by $\phi_\beta \circ \phi_\alpha^{-1}$.

Def: Smooth Atlas

A **smooth atlas** is a set of charts covering the whole manifold such that the transition map $\phi_\alpha \circ \phi_\beta^{-1}$ are C^∞.

Def: Smooth Manifold

A **smooth manifold** is a topological manifold that has a smooth atlas.

3.3 Structures on Manifold

3.3.1 Curves and Functions

Def: Curve on Manifold

A **curve** on a manifold $\mathcal{M}$ is a smooth and invertible map $\gamma : \mathbb{R} \to M$.
Note: check if it is actually smooth and invertible.

Def: Functions on Manifold

A **function** on a manifold $\mathcal{M}$ is a map $f : M \to \mathbb{R}$.
$\bar{f} := f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$.
Note that f and $\bar{f}$ are not necessarily invertible, f is smooth if and only if $\bar{f}$ is C^∞.

3.3.2 Vectors

Motivation: We want to be able to define the notion of direction on manifold $\mathcal{M}$. To do so we want to relate derivatives and curves to directions.

Def: Coordinate Curves

Coordinate curves of a chart are the image of the axes in $\mathbb{R}^n$ under ϕ^{-1}.

Def: Tangent Vector

The **tangent vector** to γ at τ_0 is $\partial_\tau|_{\tau_0} = \frac{d}{d\tau}|_{\tau_0} = \frac{dx^\mu}{d\tau}\partial_\mu|_{\tau_0}$.
And $\Upsilon_\mu(f)|_{\tau_0} = \partial_\mu \bar{f}|_{\tau_0}$.

Def: Tangent Space $T_p\mathcal{M}$

The set $T_p\mathcal{M}$ is the **tangent space** of all the vectors v_p at a point $p \in M$, and it has dimension equal to $\dim(\mathcal{M})$. We call $T_p\mathcal{M}$ the **tangent space to $\mathcal{M}$ at** $p \in \mathcal{M}$.

Def: Coordinate Basis

The set $\{\Upsilon_{p\,\mu}\}$ is the **coordinate basis** of $T_p\mathcal{M}$ is the set of vectors tangent to the coordinate curves of chart $(\mathcal{U}, \phi)$.

Note

Every chart comes with a coordinate basis and every coordinate basis defines a chart.

Theorem: Coordinate Basis and Commutativity

A basis of $\{\Upsilon_\mu\}$ $T_p^\star\mathcal{M}$ is a coordinate basis $\Leftrightarrow [\Upsilon_\mu, \Upsilon_\nu] = 0$.

Def: Basis Transformation

Consider $\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$. Implying $\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$ with $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$.

Def: Vector Field

A **vector field** over $\mathcal{M}$ is a set of vectors of $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ such that their components in any coordinate basis are smooth functions. Note that vector fields follows Leibniz rule.

Def: Composition of Vector Fields

$(\mathsf{v} \circ \mathsf{w})(f) := \mathsf{v}[\mathsf{w}(f)]$. The composition of vector fields does not obey the Leibniz rule and hence is not a vector field.

Note

$(\mathsf{v} \circ \mathsf{w})(fg) = f \cdot (\mathsf{v} \circ \mathsf{w})(g) + g \cdot (\mathsf{v} \circ \mathsf{w})(f) + \mathsf{w}(f) \cdot \mathsf{v}(g) + \mathsf{v}(f) \cdot \mathsf{w}(g) \neq f \cdot (\mathsf{v} \circ \mathsf{w})(g) + g \cdot (\mathsf{v} \circ \mathsf{w})(f)$.

Def: Lie Bracket of Vector Fields

The **Lie brackets of vector field** is a binary operator such that $[\cdot,\cdot] : (A, B) \mapsto AB - BA$. And Lie brackets satisfies:

1) Antisymmetry: $[\mathsf{v}, \mathsf{w}] = -[\mathsf{w}, \mathsf{v}]$.
2) Jacobi Identity: $[\mathsf{u}, [\mathsf{v}, \mathsf{w}]] + [\mathsf{w}, [\mathsf{u}, \mathsf{v}]] + [\mathsf{v}, [\mathsf{w}, \mathsf{u}]] = 0$

The Lie bracket of vector fields is a vector field.

Remark

In a coordinate basis, $\{\Upsilon_\mu\} = \{\partial_\mu\}$, $[\mathsf{v}, \mathsf{w}]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$.

3.3.3 1-Forms

Def: 1-Form

A **1-form**, ω, is a real linear functional over $T_p\mathcal{M}$, $\omega : T_p\mathcal{M} \to \mathbb{R}$ defined by $\omega : \mathsf{T_p}\mathcal{M} \to \mathbb{R}$ defined by $\omega : \mathsf{v} \mapsto \langle \omega, \mathsf{v} \rangle$. Note: 1-forms are sometimes called covariant vectors.

Def: Cotangent Space and Dual Basis

Given an arbitrary basis $\{\mathsf{e}_a\}$ of $T_p\mathcal{M}$. There exists an unique set of 1-forms $\{\mathsf{e}^a\}$ such that $\langle \mathsf{e}^a, \mathsf{e}_b \rangle = \delta^a_b$. This set is linear independent and forms a basis of the **cotangent space** $T^\star_p\mathcal{M}$. We call $\{\mathsf{e}_a\}$ the **dual basis**.

Note

Elements of $T_p\mathcal{M}$ also acts linearly on $T^\star_p\mathcal{M} \Rightarrow T^{\star\star}_p\mathcal{M} = T_p\mathcal{M}$.

Remark

$\langle \omega, \mathsf{v} \rangle = \left\langle \omega_a \mathsf{e}^\mathsf{a}, v^b \mathsf{e}_b \right\rangle = \omega_a v^b$.
$\mathsf{e}'_a = \Lambda_a{}^b \mathsf{e}_b$ and $\mathsf{e}'^a = \tilde{\Lambda}^a{}_b \mathsf{e}^b$.

Def: Differential

Each functions f over $\mathcal{M}$ defines a 1-form $\mathsf{d}f|_p$ at the point $p \in \mathcal{M}$ that we call the **differential** of f. $\mathsf{d}f$ is defined by $\langle \mathsf{d}f, \mathsf{v} \rangle = \mathsf{v}(f)$. Note that $f : \mathcal{M} \to \mathbb{R}$.
In a coordinate basis $\{\Upsilon_\mu\}$, $df = \partial_\mu \bar{f} \Upsilon^\mu = \partial_\mu f \Upsilon^\mu$.

Def: Coordinate Basis (Dual)

The coordinate basis of $T^\star_p\mathcal{M}$, $\{\Upsilon^\mu\}$ is often represented by $\{\mathsf{d}x^\mu\}$. (Can see by considering $f(\mathsf{x}) = x^\mu(\mathsf{x})$, note that this μ is not being summed over.)

Remark

$\Upsilon'^{\mu} = \tilde{\Lambda}^{\mu}{}_{\nu}\Upsilon^{\nu} \Leftrightarrow \mathsf{d}x'^{\mu} = \tilde{\Lambda}^{\mu}{}_{\nu}dx^{\nu}$.
$\langle \Upsilon'^{\mu}, \Upsilon'_{\nu} \rangle = \langle \Upsilon^{\mu}, \Upsilon_{\nu} \rangle = \delta^{\mu}_{\nu}$
$\omega'_{\mu} = \Lambda_{\mu}{}^{\nu}\omega_{\nu}$

3.3.4 Tensors

Def: Tensor of Type (r, s)

A **tensor of type** (r, s) is a multilinear map that acts on the vector space $(T_p)^r_s\mathcal{M} = \left(T^\star_p\mathcal{M}\right)^{\times r} \times (T_p\mathcal{M})^{\times s}$. Tensor of type (r, s) are called r-times contravariant and s-times covariant.

Def: Components of Tensor and Action of Tensor

A tensor $\mathsf{T} \in (T_p)^r_s\mathcal{M}$ is completely characterized by its action on a basis of $(T_p)^s_r\mathcal{M}$.
The **components** of T are $\mathsf{T}\left(\Upsilon^{\alpha_1}, \ldots, \Upsilon^{\alpha_r}, \Upsilon_{\beta_1}, \ldots, \Upsilon_{\beta_s}\right) = T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s}$.
The **action** of T on arbitrary 1-forms and vectors is $\mathsf{T}\left(\omega, \sigma, \ldots, \mathsf{v}, \mathsf{w}, \ldots\right) = T^{ab\ldots}{}_{cd\ldots}\,\omega_a\sigma_b \ldots v^c w^d$.

Def: Transformation of Tensor Components

Examples:

1) $T'^{a}{}_{b} = \mathsf{T}\left(\tilde{\Lambda}^{a}{}_{c}\Upsilon^{c}, \Lambda_{b}{}^{d}\Upsilon_{d}\right) = \tilde{\Lambda}^{a}{}_{c}\Lambda_{b}{}^{d}\mathsf{T}\left(\Upsilon^{c}, \Upsilon_{d}\right) = \tilde{\Lambda}^{a}{}_{c}\Lambda_{b}{}^{d}T^{c}{}_{d}$.
2) $T'_{c}{}^{ab} = \Lambda_{c}{}^{d}\tilde{\Lambda}^{a}{}_{e}\tilde{\Lambda}^{b}{}_{f}T_{d}{}^{ef}$.

Question: Ordering of indices?

3.3.5 Tensor Operations

Def: Symmetrization

$T^{a_1 \ldots a_r}{}_{c_1 \ldots c_t (b_1 \ldots b_s)} := \frac{1}{s!}\sum_{\pi} T^{a_1 \ldots a_r}{}_{c_1 \ldots c_t \pi(b_1) \ldots \pi(b_s)}$.
Examples:

1) $T_{\alpha(\mu\nu)} = \frac{1}{2!}\left(T_{\alpha\mu\nu} + T_{\alpha\nu\mu}\right)$.
2) $T_{\mu(\nu}R_{\alpha)\beta\gamma} = \frac{1}{2!}\left(T_{\mu\nu}R_{\alpha\beta\gamma} + T_{\mu\alpha}R_{\nu\beta\gamma}\right)$.
3) $T_{(\mu\alpha\beta)} = \frac{1}{3!}\left(T_{\mu\alpha\beta} + T_{\mu\beta\alpha} + T_{\alpha\mu\beta} + T_{\alpha\beta\mu} + T_{\beta\alpha\mu} + T_{\beta\mu\alpha}\right)$.

Def: Antisymmetrization

$T^{a_1\dots a_r}{}_{c_1\dots c_t[b_1\dots b_s]} := \frac{1}{s!}\sum_\pi (-1)^\pi \, T^{a_1\dots a_r}{}_{c_1\dots c_t \pi(b_1)\dots\pi(b_s)}$.
Examples:

1) $T_{[\mu\nu]} = \frac{1}{2!}\left(T_{\mu\nu} - T_{\nu\mu}\right)$.
2) $T_{[\mu\nu]\beta} = \frac{1}{2!}\left(T_{\mu\nu\beta} - T_{\nu\mu\beta}\right)$.
3) $T_{[\mu\alpha\beta]} = \frac{1}{3!}\left(T_{\mu\alpha\beta} - T_{\mu\beta\alpha} - T_{\alpha\mu\beta} + T_{\alpha\beta\mu} - T_{\beta\alpha\mu} + T_{\beta\mu\alpha}\right)$.

Def: Tensor Product

Let $\mathsf{R} \in (T_p)^r_s\,\mathcal{M}$, $\mathsf{T} \in (T_p)^t_q\,\mathcal{M}$. Then the **tensor product** of R and T is $\mathsf{R}\otimes\mathsf{T} \in (T_p)^{r+t}_{s+q}\,\mathcal{M}$ with components $(\mathsf{R}\otimes\mathsf{T})^{a_1\dots a_{r+t}}{}_{b_1\dots b_{s+q}} := R^{a_1\dots a_r}{}_{b_{s+1}\dots b_{s+q}}$.

Def: Contraction

Let $T \in (T_p)^r_s\,\mathcal{M}$ with components $T^{a_1\dots a_r}{}_{b_1\dots b_s}$, we define the **contraction** of the first covariant and contravariant indices as the tensor $\mathsf{R} \in (T_p)^{r-1}_{s-1}\,\mathcal{M}$ as the tensor of components $R^{a_2\dots a_r}{}_{b_2\dots b_s} := T^{aa_2\dots a_r}{}_{ab_1\dots b_s}$.

3.3.6 Integration on Manifold and Differential Forms

Motivation: In a coordinate basis, $\omega = \omega_\mu \mathsf{d}x^\mu$. Notice that a curve γ on the manifold $\mathcal{M}$ is a submanifold of $\mathcal{M}$. The integral of ω along the curve γ is $\int_{\gamma\subset\mathcal{M}} \omega = \int_\gamma \omega_\mu \,\mathsf{d}x^\mu = \int_{s_1}^{s_2} \omega_\mu \frac{dx^\mu(s)}{ds}\, ds$.

Def: 2-Form

A **2-form**, σ is an antisymmetric twice covariant (type $(0,2)$) tensor such that $\sigma(\mathsf{v},\mathsf{w}) = -\sigma(\mathsf{w},\mathsf{v})$ and in a coordinate basis, $\sigma = \sigma_{\mu\nu}\,\mathsf{d}x^\mu \otimes \mathsf{d}x^{nu}$.

Def: Wedge Product (1-forms)

The **wedge product** is the fully antisymmetric tensor product.
$\mathsf{d}x^\mu \wedge \mathsf{d}x^\nu = -\mathsf{d}x^\nu \wedge \mathsf{d}x^\mu := 2\mathsf{d}x^{[\mu} \otimes \mathsf{d}x^{\nu]} = \mathsf{d}x^\mu \otimes \mathsf{d}x^\nu - \mathsf{d}x^\nu \otimes \mathsf{d}x^\mu$.

Def: k-Form

A **k-form**, σ, is a fully antisymmetric k-times covariant (type $(0,k)$) tensor.
$\sigma = \sigma_{\alpha_1\dots\alpha_k}\mathsf{d}x^{\alpha_1} \otimes \dots \otimes \mathsf{d}x^{\alpha_k} = \frac{1}{k!}\sigma_{\alpha_1\dots\alpha_k}\mathsf{d}x^{\alpha_1} \wedge \dots \wedge \mathsf{d}x^{\alpha_k}$.

Def: Exterior Product (Wedge Product)

If σ is a k-form and ω is a p-form. The **exterior product** of σ and ω is their fully antisymmetrized tensor product.
$(\sigma \wedge \omega)_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_p} = \frac{(k+p)!}{k!p!} \sigma_{[\alpha_1 \dots \alpha_k} \omega_{\beta_1 \dots \beta_p]}$.

Def: Exterior Derivative

The **exterior derivative** d, is an operation that acts on a k-form and return a $(k+1)$-form.

$$\begin{aligned}
\mathsf{d}\sigma &= \frac{1}{k!} \mathsf{d} \left(\sigma_{\alpha_1 \dots \alpha_k} \right) \wedge \mathsf{d}x^{\alpha_1} \wedge \dots \wedge \mathsf{d}x^{\alpha_k} \\
&= \frac{1}{k!} \partial_\beta \left(\sigma_{\alpha_1 \dots \alpha_k} \right) \mathsf{d}x^\beta \wedge \mathsf{d}x^{\alpha_1} \wedge \dots \wedge \mathsf{d}x^{\alpha_k} \\
&= (k+1)\, \partial_{[\beta} \sigma_{\alpha_1 \dots \alpha_k]} \mathsf{d}x^\beta \otimes \mathsf{d}x^{\alpha_1} \otimes \dots \otimes \mathsf{d}x^{\alpha_k}
\end{aligned}$$

Def: Exact

A k-form, ω, is **exact** $\Leftrightarrow \omega = \mathsf{d}\sigma$, where σ is a $(k-1)$-form.

Def: Closed

A k-form, ω, is **closed** $\Leftrightarrow \mathsf{d}\omega = 0$.

Theorem: Exact $\Rightarrow$ Closed

$\omega = \mathsf{d}\sigma \Rightarrow \mathsf{d}\omega = 0$. However, the converse is not true in general.
$\mathsf{d}\omega = 0 \Rightarrow \omega = \mathsf{d}\sigma$ locally.

Theorem: Poincaré Lemma

If ω is defined in a contractible domain (open subset of a manifold), $\mathsf{d}\omega = 0 \Leftrightarrow \omega = \mathsf{d}\sigma$.

Note

When solving exact ODEs, we are actually have $\mathsf{d}F = 0$ and we check $\partial_y \partial_x F = \partial_x \partial_y F$ which we are actually checking for closeness.

Remark

If ω is exact then $\omega = \mathsf{d}f \Rightarrow \int_\gamma \omega = \int_\gamma \mathsf{d}f = f(s_2) - f(s_1)$. Also, $\oint_\gamma \omega = 0$.
In polar coordinate, $\mathsf{d}\theta$ is actually not exact as $\oint_\gamma \mathsf{d}\theta = 2\pi n \neq 0$. Also, $\Theta = \mathsf{d}\theta$ is not defined at the origin.

3.4 Smooth Maps and Diffeomorphisms

3.4.1 Smooth Maps: Pullback and Pushforward

Def: Smooth Map

A map $\varphi : \mathcal{M} \to \mathcal{M}'$ is a **smooth map** if and only if given the atlases$\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ and $\{\mathcal{U}'_\alpha, \phi'_\alpha\}$ of $\mathcal{M}$ and $\mathcal{M}'$, the functions $\phi'_\alpha \circ \varphi \circ \phi_\alpha : \mathbb{R}^n \to \mathbb{R}^{n'}$ is smooth.

Def: Pullback (Function)

The map φ induces a map $\varphi^\star : \mathcal{F}_\mathcal{M} \to \mathcal{F}_{\mathcal{M}'}$ which we call the **pullback** between the spaces of functions $\mathcal{F}_\mathcal{M}$ and $\mathcal{F}_{\mathcal{M}'}$ according to the following rule:
Given a function $f' : \mathcal{M}' \to \mathbb{R}$, its **pull-back** $\varphi^\star f' : \mathcal{M} \to \mathbb{R}$ is such that $\varphi^\star f'(p) := f' \circ \varphi(p) = f'(p)$, with $p \in \mathcal{M}$.

Def: Pushforward (Vector)

The map φ induces a map $\varphi_\star : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{M}'$ called the **pushforward** between the tangent spaces $T_p\mathcal{M}$ and $T_p\mathcal{M}'$ according to the following rule:
Given a vector $\mathsf{v} \in T_p\mathcal{M}$ then $\varphi_\star\mathsf{v} \in T_p\mathcal{M}'$ such that its action on a function $f' \in \mathcal{F}_{\mathcal{M}'}$ is $(\varphi_\star\mathsf{v})\,|_{\phi(p)=p'}(f') := \mathsf{v}|_p\,(\varphi^\star f')$.

Remark

In coordinate bases $(x^1, \ldots, x^m)$ and $\left(y^1, \ldots, y^{m'}\right)$:
$([\varphi_\star]\mathsf{v})\,|_{\varphi(p)}(f') = v^i|_p \frac{\partial f'}{\partial y^j} \frac{\partial y^j}{\partial x^i}|_p \Rightarrow ([\varphi_\star]\mathsf{v})\,|_{\varphi(p)} = v^i|_p \frac{\partial y^j}{\partial x^i}|_p \frac{\partial}{\partial y^j}|_{\partial(p)}$.

Remark

If the map φ is not injective, than the pushforward of a vector field does not define a vector field.

Def: Pullback (one-form)

Given a 1-form $\omega \in T_p^\star\mathcal{M}'$ then $\varphi^\star : T^\star_{\varphi(p)}\mathcal{M}' \to T_p^\star\mathcal{M}$ called the **pullback** between the cotangent spaces $T^\star_{\varphi(p)}\mathcal{M}'$ and $T_p^\star\mathcal{M}$ according to the following rule: Given a 1-form $\omega' \in T^\star_{\varphi(p)}\mathcal{M}'$, then $\varphi^\star\omega' \in T_p^\star\mathcal{M}$ whose action on vectors in $T_p\mathcal{M}$ is $\langle\varphi^\star\omega', \mathsf{v}\rangle\,|_p := \langle\omega', \varphi_\star\mathsf{v}\rangle\,|_{\varphi(p)}$.

Remark

$\mathsf{d}\,(\varphi^\star\omega) = \varphi^\star\,(\mathsf{d}\omega)$.

Remark

$\varphi^\star(\alpha \wedge \beta) = (\varphi^\star \alpha) \wedge (\varphi^\star \beta)$.

Remark

In coordinate basis, $\varphi^\star \omega = (\omega_i \circ \varphi)\,\mathsf{d}\left(x^i \circ \varphi\right)$.

Def: Pullback (k-form)

$\varphi^\star \omega = \varphi^\star \left(\frac{1}{k!}\omega_{\alpha_1 \ldots \alpha_k} \bigwedge_{i=1}^{k} \mathsf{d}x^{\alpha_i}\right)$.

3.4.2 Diffeomorphisms

Def: Diffeomorphism

A map $\varphi : \mathcal{M} \to \mathcal{M}'$ is a **diffeomorphism** if and only if φ and φ^{-1} are both smooth bijections.

Remark

For a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$, pullback and pushfoward can be defined for all objects on the manifold as $\varphi_\star = \left(\varphi^{-1}\right)^\star$.

Remark

$[\varphi_\star]\,|_p^{-1} = \left[\varphi_\star^{-1}\right]|_{\varphi(p)}$.
$\varphi_\star f(p) = \left(\varphi^{-1}\right)^\star f(p)$ and $(\varphi^\star \mathsf{v})\,|_p(f) = \mathsf{v}_{\varphi^{-1}(p)} \varphi_\star f$.

Def: Local Flow

Consider a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ and let $\gamma_p(s)$ be the parametrized curve in $\mathcal{M}$ such that $\gamma_p(0) = p$ and its tangent vector at every points $\gamma_p(s)$ is the vector field k. k is the generator of a set of local diffeomorphisms of the form $\varphi_s : \mathcal{U} \to \varphi_s(\mathcal{U})$ such that $\varphi_s(p) = \gamma_p(s)$. This set of local diffeomorphisms is called the **local flow** of k.

Note

1) $\varphi_{-s} \circ \varphi_s(p) = p \Rightarrow \varphi_s^{-1} = \varphi_{-s}$
2) $\varphi_s(p) = \gamma(s)$
3) $\varphi_{s+t}(p) = \gamma(s+t)$
4) $\phi_s(\phi_t(p)) = \varphi_{s+t}(p)$

Hence, local diffeomorphisms forms a group (Lie group). And the local flow of the vector field k is complete if the associated curve $\gamma_p(s)$ can be extended for all values $s \in \mathbb{R}$.

3.4.3 Lie Derivative

Motivation: Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism, φ establishes an isomorphism $\varphi^\star : \left(T_{\varphi(p)}\right)^r_s \mathcal{M} \to (T_p)^r_s \mathcal{M}$ for $p \in \mathcal{M}$. The tensor $\left[\varphi^\star\left(T_{\varphi(p)}\right)\right]|_p - T|_p$ provides information about the difference between the value of a tensor field at $\varphi(p)$ and at p.

Def: Lie Derivative

Consider a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ and its pullback $\varphi^\star : \left(T_{\varphi(p)}\right)^r_s \mathcal{M} \to (T_p)^r_s \mathcal{M}$. Consider $\varphi \equiv \varphi_t$ be the local flow generated by a vector field v. The **Lie derivative** of a tensor T along the (curve generated by) vector field v at the point $p \in \mathcal{M}$ is $(\mathcal{L}_{\mathsf{v}}\mathsf{T})|_p := \lim_{t\to 0} \frac{\left[\varphi_t^\star\left(\mathsf{T}_{\varphi_t(p)}\right)\right]_p - T_p}{t}$.

Note

t is a parameter along a curve generated by v.
Intuitively, the Lie derivative is to:
1) Take the tensor at t steps away from point $p \in \mathcal{M}$ (along the curve generated by v).
2) Move the tensor to point p by using the pullback.
3) Compute the difference between the two tensors at p.
4) Divide the difference of the two tensors by t then take the limit $t \to 0$.

Acknowledgement: This intuition is from here. One can also think of Lie Derivative as something that measures the variation of a tensor field when moving

Remark

Lie Derivative have the properties:
1) Linearity.
2) Preserves the tensor type, symmetries, and contractions because $\varphi^\star$ does.
3) Satisfies Leibniz rule under tensor product and contraction: $\mathcal{L}_{\mathsf{v}}(\mathsf{T} \otimes \mathsf{S}) = (\mathcal{L}_{\mathsf{v}}\mathsf{T}) \otimes \mathsf{S} + \mathsf{T} \otimes (\mathcal{L}_{\mathsf{v}}\mathsf{S})$.
4) Cartan identity: The Lie derivative of a k-form field along the vector field v follows $\mathcal{L}_{\mathsf{v}} = i_{\mathsf{v}}\mathsf{d} + \mathsf{d}i_v$ (took from JiayuePhysics).

Def: Lie Derivative of Scalar Field

For a scalar field f, $\mathcal{L}_{\mathsf{v}} f = \mathsf{v}(f)$, and in a coordinate basis, $\mathcal{L}_{\mathsf{v}} f = v^\nu \partial_\nu f$.

Def: Lie Derivative of Vector Field

For a vector field w, $\mathcal{L}_{\mathsf{v}} \mathsf{w} = [\mathsf{v}, \mathsf{w}]$, and in a coordinate basis, $\mathcal{L}_{\mathsf{v}} \mathsf{w} = \left(v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu\right) \partial_\mu$.

Def: Lie Derivative of 1-Form Field

For a 1-form field ω in a coordinate basis, $\mathcal{L}_{\mathsf{v}} \omega = \left(v^\nu \left(\partial_\nu \omega_\mu\right) + \omega_\nu \left(\partial_\mu v^\nu\right)\right) \mathsf{d}x^\mu$.

Ex: More Lie Derivatives

$\omega = \omega_{\mu\nu} \mathsf{d}x^\mu \otimes \mathsf{d}x^\nu \Rightarrow \mathcal{L}_{\mathsf{v}} \omega = \left(v^\alpha \left(\partial_\alpha \omega_{\mu\nu}\right) + \omega_{\alpha\nu} \left(\partial_\mu v^\alpha\right) + \omega_{\mu\alpha} \left(\partial_\nu v^\alpha\right)\right) \mathsf{d}x^\mu \otimes \mathsf{d}x^\nu$.
$\mathsf{T} = T_\mu{}^\nu \mathsf{d}x^\mu \otimes \partial_\nu \Rightarrow \mathcal{L}_{\mathsf{v}} \mathsf{T} = \left(v^\alpha \left(\partial_\alpha T_\mu{}^\nu\right) + T_\alpha{}^\nu \left(\partial_\mu v^\alpha\right) - T_\mu{}^\alpha \left(\partial_\alpha v^\nu\right)\right) \mathsf{d}x^\mu \otimes \partial_\nu$.
$\mathsf{T} = T_{\alpha\beta}{}^{\mu\nu} \mathsf{d}x^\alpha \otimes \mathsf{d}x^\beta \otimes \partial_\mu \otimes \partial_\nu$.
$\Rightarrow \mathcal{L}_{\mathsf{v}} \mathsf{T} = \left(v^\gamma \partial_\gamma T_{\alpha\beta}{}^{\mu\nu} + T_{\gamma\beta}{}^{\mu\nu} \partial_\alpha v^\gamma + T_{\alpha\gamma}{}^{\mu\nu} \partial_\beta v^\gamma - T_{\alpha\beta}{}^{\gamma\nu} \partial_\gamma v^\mu - T_{\alpha\beta}{}^{\mu\gamma} \partial_\gamma v^\nu\right) \mathsf{d}x^\alpha \otimes \mathsf{d}x^\beta \otimes \partial_\mu \otimes \partial_\nu$.

3.5 Pseudo-Riemannian Geometry

3.5.1 Affine Connection and Covariant Derivative

Motivation: While Lie derivative is a derivative, we can't use it as a generalised gradient, as it doesn't change the type of the tensor. So we need to explore other derivatives. We take the derivative of a function f as $\frac{\partial f}{\partial x^\mu} = \partial_\mu f$. And for a vector $\mathsf{v} = v^\mu \Upsilon_\mu$, $\frac{\partial \mathsf{v}}{\partial x^\mu}$ is a bad notation for the derivative of the vector because both the components and basis of v can change. What we want to do is to write $\frac{\partial \Upsilon}{\partial x^\mu}$ in terms of the bases ($\frac{\partial \Upsilon}{\partial x^\mu} = \Gamma^\rho{}_{\nu\mu} \Upsilon_\rho$ with $\Gamma^\rho{}_{\nu\mu}$ describing how basis changes at different point).

Def: Affine Connection, Covariant Derivative

An **affine connection** $\nabla : \left(T_p\mathcal{M}\right)^r_s \to \left(T_p\mathcal{M}\right)^r_{s+1}$ is a rule through which we assign to each tensor T of type (r, s) and components $T^{bc\ldots}{}_{de\ldots}$ another tensor field $\nabla\mathsf{T}$ of type $(r, s+1)$ and components $(\nabla\mathsf{T})^{bc\ldots}{}_{de\ldots a} \equiv \nabla_a T^{bc\ldots}{}_{de\ldots} \equiv T^{bc\ldots}{}_{de\ldots;a}$. We call $\nabla\mathsf{T}$ the **covariant derivative** of T.

Remark

Affine connection follows the properties:
1) Linearity.
2) Leibniz rule.
3) Commutes with contraction.
4) Over functions f, it is the differential $\nabla f = \mathsf{d}f$.

Note

There are infinitely many affine connections and infinite ways to build covariant derivative.

Def: Directional Covariant Derivative

We define the **directional covariant derivative** of T in the direction of vector v as the tensor $\nabla_{\mathsf{v}}\mathsf{T}$ of type (r, s) of components $(\nabla_{\mathsf{v}}\mathsf{T})^{bc\ldots}{}_{de\ldots} = v^a \nabla_a T^{bc\ldots}{}_{de\ldots}$. $(\nabla_{\mathsf{v}}\mathsf{T})^{bc\ldots}{}_{de\ldots} = \langle \nabla \mathsf{T}, \mathsf{v}\rangle$.

Def: Coefficient of the Connection

The **coefficient of the connection** in an arbitary basis $\{\Upsilon_a\}$ is

$$\Gamma^a{}_{bc} := (\nabla \Upsilon_b)^a{}_c = \langle \Upsilon^a, \nabla_{\Upsilon_c}\Upsilon_b\rangle \Rightarrow \nabla_c \Upsilon_b := \nabla_{\Upsilon_c}\Upsilon_b := \Gamma^a{}_{bc}\Upsilon_a.$$

$\Gamma^a{}_{bc}$ is the a-th components of the covariant derivative of the b-th basis vector in the direction of the c-th basis vector.

Remark

The notation for covariant derivative $\nabla_\mu v^\nu = (\nabla \mathsf{v})_\mu{}^\nu$. Also...

Note

Another convention is $\Gamma^a{}_{bc} := (\nabla \Upsilon_c)^a{}_b = \langle \Upsilon^a, \nabla_{\Upsilon_b}\Upsilon_c\rangle$.

$\nabla_a v^b = \Upsilon_a(v^b) + v^c \Gamma^b{}_{ca}$

$\nabla_\mu v^\nu = \partial_\mu v^\nu + v^\sigma \Gamma^\nu{}_{\sigma\mu}$

$\nabla_a \omega_b = \Upsilon_a(\omega_b) - \omega_c \Gamma^c{}_{ba}$

$\nabla_\mu \omega_\nu = \partial_\mu \omega_{nu} - \omega_\sigma \Gamma^\sigma{}_{\nu\mu}$

Note: $\frac{\partial \mathsf{v}}{\partial x^\mu} = \left(\frac{\partial v^\nu}{\partial x^\mu} + v^\sigma \Gamma^\nu{}_{\sigma\mu}\right)\Upsilon_\nu$.