

AMATH 475

University of Waterloo

Taught by: Eduardo Martin-Martinez

Updated: 2026-01-07

Edward Chang

Contents

1	Index Notation	2
2	Special Relativity	3
2.1	Postulates of Special Relativity	3
2.2	Lorentz Transformation	3
3	Differential Geometry	5
3.1	Topology	5
3.2	Manifolds	5
3.3	Structures on Manifold	6
3.3.1	Curves and Functions	6
3.3.2	Vectors	6
3.3.3	1-Forms	8
3.3.4	Tensors	9
3.3.5	Tensor Operations	9
3.3.6	Integration on Manifold and Differential Forms	10
3.4	Smooth Maps and Diffeomorphisms	12
3.4.1	Smooth Maps: Pullback and Pushforward	12
3.4.2	Diffeomorphisms	13
3.4.3	Lie Derivative	14
3.5	Pseudo-Riemannian Geometry	15
3.5.1	Affine Connection and Covariant Derivative	15

1 Index Notation

Def: Covariant

Objects that transforms under change of basis like the element of the basis are called **covariant**, and its components have sub-indices.

Def: Contravariant

Objects that transforms under change of basis like components of vectors are called **contravariant** and its components have super-indices.

Def: Dual Basis & Dual Space

The **dual basis** to $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is $\mathbb{B}^* = \{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ and it is the collection of linear operators such that $\mathbf{e}^i(\mathbf{e}_j) = \delta_j^i$. \mathbb{B}^* spans a vector space V^* called the **dual space** of V .

2 Special Relativity

2.1 Postulates of Special Relativity

Def: Event

An **event** is an individual point in spacetime, usually labelled which we represent by the tuple $E \equiv (t, \vec{x})$, t is the *time coordinate* and $\vec{x} \equiv (x, y, z)$ is the *space coordinates*.

Def: Spacetime

Spacetime is the set of all events, $\mathbb{S} = \{E \equiv (t, \vec{x}) : t \in R, \vec{x} \in \mathbb{R}^3\}$.

Def: Reference Frame

A **reference frame**, establishes a spacetime coordinate system (t, \vec{x}) , which is a spatial coordinate system where the position of point-like particles can be specified, and a clock (something that can measure time).

Def: Inertial Reference Frame (IRF)

An **inertial reference frame** is a reference frame for which a particle stationary at its origin experience no force (Newton's first law holds).

Def: Postulates of Special Relativity 1 (SR): Principle of Relativity

1. **Principle of Relativity:** In the absence of gravity, all the laws of physics are identical in all inertial reference frames (This postulate is also in Galilean relativity).
2. **Speed of Light is Constant and Equal:** The speed of light in vacuum “ c ” is constant and the same for all inertial reference frames (absence of gravity).

2.2 Lorentz Transformation

Def: Galilean Transformation

Consider two inertial frames $S \equiv (t, \vec{x})$ and $S' \equiv (t', \vec{x}')$, where S' moves with velocity $\vec{v} = v\hat{x}$ to the right relative to S . The **Galilean transformation** (Galilean boost) from S to S' is:

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

Note: Galilean transformation would result in mathematical inconsistency (todo).

Def: Spacetime Interval

Given a particular inertial reference frame that establishes space time coordinates (t, \vec{x}) , the **spacetime interval** between two events $E_1 \equiv (t_1, \vec{x}_1)$ and $E_2 \equiv (t_2, \vec{x}_2)$ is $\Delta s^2 := -c^2\Delta t^2 + \Delta\vec{x}^2 = -c^2(t_2 - t_1)^2 + (\vec{x}_2 - \vec{x}_1)^2$.

Note

In this course we use the signature $(-, +, +, +)$.

Remark

If the two events are connected by the propagation of light, $\Delta\vec{x}^2 = c^2\Delta t^2 \Rightarrow \Delta s^2 = -c^2\Delta t^2 + c^2\Delta t^2 = 0$.

Note

Possible transformations between two inertial reference frames: 3 rotations, 3 translations, 1 times shift, 3 boosts.

However, the spatial distance $\|\vec{x}_2 - \vec{x}_1\|$ is invariant under rotations and translations. Similarly, $\Delta t = t_2 - t_1$ is invariant under time shift. So the only transformations with significance are the boost.

3 Differential Geometry

3.1 Topology

Def: Topology

A **topology on a set** X, S is a collection of open sets of X , and S is a subset of the power set of X . S satisfies:

- 1) $\emptyset, X \in S$.
- 2) Any union of elements of S is in S .
- 3) Any finite intersection of elements of S is in S .

Def: Topological Space

A **topological space** (X, S) , is an ordered pair where X is a set and S is a topology of X .

3.2 Manifolds

Def: Homeomorphism

A **homeomorphism** between two topological spaces X and Y is a map $\sigma : X \rightarrow Y$ such that the map is topologically continuous and it's inverse is topologically continuous (ϕ is a bijection).

Def: Hausdorff Space

A topological space X is a **Hausdorff space** if for all distinct points $x, y \in X$ there exists neighbourhoods of x and y , $\mathcal{U}_x, \mathcal{U}_y$ respectively, such that $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.

Def: Topological Manifolds

A **topological manifold** of dimension n , \mathcal{M} is a topological space that is **Hausdorff** and at every point possess an open neighbourhood homeomorphic to \mathbb{R}^n .

Note: The proper definition also requires that the space is second countable.

Def: Charts

A **chart** $(\mathcal{U}_\alpha, \phi_\alpha)$, where ϕ_α is a homeomorphism from an open subset $\mathcal{U}_\alpha \subseteq \mathcal{M}$ to \mathbb{R}^n . Note: $\phi_\alpha : \mathcal{U}_\alpha \xrightarrow{\text{open}} M \rightarrow \mathbb{R}^n$, and $\phi_\alpha(x) \equiv x^\mu$, x^μ is the coordinate in \mathbb{R}^n .

Def: Transition Map

Consider two charts of \mathcal{M} , $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$, where $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. The **transition map** is a homeomorphism from $\phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \rightarrow \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, defined by $\phi_\beta \circ \phi_\alpha^{-1}$.

Def: Smooth Atlas

A **smooth atlas** is a set of charts covering the whole manifold such that the transition map $\phi_\alpha \circ \phi_\beta^{-1}$ are C^∞ .

Def: Smooth Manifold

A **smooth manifold** is a topological manifold that has a smooth atlas.

3.3 Structures on Manifold

3.3.1 Curves and Functions

Def: Curve on Manifold

A **curve** on a manifold \mathcal{M} is a smooth and invertible map $\gamma : \mathbb{R} \rightarrow \mathcal{M}$.

Note: check if it is actually smooth and invertible.

Def: Functions on Manifold

A **function** on a manifold \mathcal{M} is a map $f : \mathcal{M} \rightarrow \mathbb{R}$.

$\bar{f} := f \circ \phi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$.

Note that f and \bar{f} are not necessarily invertible, f is smooth if and only if \bar{f} is C^∞ .

3.3.2 Vectors

Motivation: We want to be able to define the notion of direction on manifold \mathcal{M} . To do so we want to relate derivatives and curves to directions.

Def: Coordinate Curves

Coordinate curves of a chart are the image of the axes in \mathbb{R}^n under ϕ^{-1} .

Def: Tangent Vector

The **tangent vector** to γ at τ_0 is $\partial_\tau|_{\tau_0} = \frac{d}{d\tau}|_{\tau_0} = \frac{dx^\mu}{d\tau} \partial_\mu|_{\tau_0}$.
And $\Upsilon_\mu(f)|_{\tau_0} = \partial_\mu \bar{f}|_{\tau_0}$.

Def: Tangent Space $T_p\mathcal{M}$

The set $T_p\mathcal{M}$ is the **tangent space** of all the vectors v_p at a point $p \in M$, and it has dimension equal to $\dim(\mathcal{M})$. We call $T_p\mathcal{M}$ the **tangent space to \mathcal{M} at $p \in \mathcal{M}$** .

Def: Coordinate Basis

The set $\{\Upsilon_{p\mu}\}$ is the **coordinate basis** of $T_p\mathcal{M}$ is the set of vectors tangent to the coordinate curves of chart (\mathcal{U}, ϕ) .

Note

Every chart comes with a coordinate basis and every coordinate basis defines a chart.

Theorem: Coordinate Basis and Commutativity

A basis of $\{\Upsilon_\mu\}$ $T_p^*\mathcal{M}$ is a coordinate basis $\Leftrightarrow [\Upsilon_\mu, \Upsilon_\nu] = 0$.

Def: Basis Transformation

Consider $\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$. Implying $\partial'_\mu = \Lambda_\mu^\nu \partial_\nu$ with $\Lambda_\mu^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$.

Def: Vector Field

A **vector field** over \mathcal{M} is a set of vectors of $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ such that their components in any coordinate basis are smooth functions. Note that vector fields follows Leibniz rule.

Def: Composition of Vector Fields

$(v \circ w)(f) := v[w(f)]$. The composition of vector fields does not obey the Leibniz rule and hence is not a vector field.

Note

$(v \circ w)(fg) = f \cdot (v \circ w)(g) + g \cdot (v \circ w)(f) + w(f) \cdot v(g) + v(f) \cdot w(g) \neq f \cdot (v \circ w)(g) + g \cdot (v \circ w)(f)$.

Def: Lie Bracket of Vector Fields

The **Lie brackets of vector field** is a binary operator such that $[\cdot, \cdot] : (A, B) \mapsto AB - BA$. And Lie brackets satisfies:

- 1) Antisymmetry: $[v, w] = -[w, v]$.
- 2) Jacobi Identity: $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$

The Lie bracket of vector fields is a vector field.

Remark

In a coordinate basis, $\{\Upsilon_\mu\} = \{\partial_\mu\}$, $[v, w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$.

3.3.3 1-Forms

Def: 1-Form

A **1-form**, ω , is a real linear functional over $T_p\mathcal{M}$, $\omega : T_p\mathcal{M} \rightarrow \mathbb{R}$ defined by $\omega : T_p\mathcal{M} \rightarrow \mathbb{R}$ defined by $\omega : v \mapsto \langle \omega, v \rangle$. Note: 1-forms are sometimes called covariant vectors.

Def: Cotangent Space and Dual Basis

Given an arbitrary basis $\{e_a\}$ of $T_p\mathcal{M}$. There exists an unique set of 1-forms $\{e^a\}$ such that $\langle e^a, e_b \rangle = \delta_b^a$. This set is linear independent and forms a basis of the **cotangent space** $T_p^*\mathcal{M}$. We call $\{e_a\}$ the **dual basis**.

Note

Elements of $T_p\mathcal{M}$ also acts linearly on $T_p^*\mathcal{M} \Rightarrow T_p^{**}\mathcal{M} = T_p\mathcal{M}$.

Remark

$\langle \omega, v \rangle = \langle \omega_a e^a, v^b e_b \rangle = \omega_a v^b$.
 $e'_a = \Lambda_a^b e_b$ and $e'^a = \tilde{\Lambda}^a_b e^b$.

Def: Differential

Each functions f over \mathcal{M} defines a 1-form $df|_p$ at the point $p \in \mathcal{M}$ that we call the **differential** of f . df is defined by $\langle df, v \rangle = v(f)$. Note that $f : \mathcal{M} \rightarrow \mathbb{R}$.

In a coordinate basis $\{\Upsilon_\mu\}$, $df = \partial_\mu f \Upsilon^\mu = \partial_\mu f \Upsilon^\mu$.

Def: Coordinate Basis (Dual)

The coordinate basis of $T_p^*\mathcal{M}$, $\{\Upsilon^\mu\}$ is often represented by $\{dx^\mu\}$. (Can see by considering $f(x) = x^\mu(x)$, note that this μ is not being summed over.)

Remark

$$\Upsilon'^\mu = \tilde{\Lambda}^\mu{}_\nu \Upsilon^\nu \Leftrightarrow dx'^\mu = \tilde{\Lambda}^\mu{}_\nu dx^\nu.$$

$$\langle \Upsilon'^\mu, \Upsilon'_\nu \rangle = \langle \Upsilon^\mu, \Upsilon_\nu \rangle = \delta^\mu_\nu$$

$$\omega'_\mu = \Lambda_\mu{}^\nu \omega_\nu$$

3.3.4 Tensors

Def: Tensor of Type (r, s)

A **tensor of type (r, s)** is a multilinear map that acts on the vector space $(T_p)_s^r \mathcal{M} = (T_p^* \mathcal{M})^{\times r} \times (T_p \mathcal{M})^{\times s}$. Tensor of type (r, s) are called r -times contravariant and s -times covariant.

Def: Components of Tensor and Action of Tensor

A tensor $\Upsilon \in (T_p)_s^r \mathcal{M}$ is completely characterized by its action on a basis of $(T_p)_r^s \mathcal{M}$.

The **components** of Υ are $\Upsilon(\Upsilon^{\alpha_1}, \dots, \Upsilon^{\alpha_r}, \Upsilon_{\beta_1}, \dots, \Upsilon_{\beta_s}) = T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}$.

The **action** of Υ on arbitrary 1-forms and vectors is $\Upsilon(\omega, \sigma, \dots, v, w, \dots) = T^{ab \dots}_{cd \dots} \omega_a \sigma_b \dots v^c w^d$.

Def: Transformation of Tensor Components

Examples:

$$1) T'^a{}_b = \Upsilon \left(\tilde{\Lambda}^a{}_c \Upsilon^c, \Lambda_b{}^d \Upsilon_d \right) = \tilde{\Lambda}^a{}_c \Lambda_b{}^d \Upsilon(\Upsilon^c, \Upsilon_d) = \tilde{\Lambda}^a{}_c \Lambda_b{}^d T^c{}_d.$$

$$2) T'{}^a{}_c = \Lambda_c{}^d \tilde{\Lambda}^a{}_e \tilde{\Lambda}^b{}_f T^e{}_d.$$

Question: Ordering of indices?

3.3.5 Tensor Operations

Def: Symmetrization

$$T^{a_1 \dots a_r}_{c_1 \dots c_t(b_1 \dots b_s)} := \frac{1}{s!} \sum_{\pi} T^{a_1 \dots a_r}_{c_1 \dots c_t \pi(b_1) \dots \pi(b_s)}.$$

Examples:

$$1) T_{\alpha(\mu\nu)} = \frac{1}{2!} (T_{\alpha\mu\nu} + T_{\alpha\nu\mu}).$$

$$2) T_{\mu(\nu} R_{\alpha)\beta\gamma} = \frac{1}{2!} (T_{\mu\nu} R_{\alpha\beta\gamma} + T_{\mu\alpha} R_{\nu\beta\gamma}).$$

$$3) T_{(\mu\alpha\beta)} = \frac{1}{3!} (T_{\mu\alpha\beta} + T_{\mu\beta\alpha} + T_{\alpha\mu\beta} + T_{\alpha\beta\mu} + T_{\beta\alpha\mu} + T_{\beta\mu\alpha}).$$

Def: Antisymmetrization

$$T^{a_1 \dots a_r}_{c_1 \dots c_t [b_1 \dots b_s]} := \frac{1}{s!} \sum_{\pi} (-1)^{\pi} T^{a_1 \dots a_r}_{c_1 \dots c_t \pi(b_1) \dots \pi(b_s)}.$$

Examples:

- 1) $T_{[\mu\nu]} = \frac{1}{2!} (T_{\mu\nu} - T_{\nu\mu}).$
- 2) $T_{[\mu\nu]\beta} = \frac{1}{2!} (T_{\mu\nu\beta} - T_{\nu\mu\beta}).$
- 3) $T_{[\mu\alpha\beta]} = \frac{1}{3!} (T_{\mu\alpha\beta} - T_{\mu\beta\alpha} - T_{\alpha\mu\beta} + T_{\alpha\beta\mu} - T_{\beta\alpha\mu} + T_{\beta\mu\alpha}).$

Def: Tensor Product

Let $R \in (T_p)_s^r \mathcal{M}$, $T \in (T_p)_q^t \mathcal{M}$. Then the **tensor product** of R and T is $R \otimes T \in (T_p)_{s+q}^{r+t} \mathcal{M}$ with components $(R \otimes T)^{a_1 \dots a_{r+t}}_{b_1 \dots b_{s+q}} := R^{a_1 \dots a_r}_{b_{s+1} \dots b_{s+q}} \cdot T^{a_{r+1} \dots a_{r+t}}_{b_1 \dots b_q}$.

Def: Contraction

Let $T \in (T_p)_s^r \mathcal{M}$ with components $T^{a_1 \dots a_r}_{b_1 \dots b_s}$, we define the **contraction** of the first covariant and contravariant indices as the tensor $R \in (T_p)_{s-1}^{r-1} \mathcal{M}$ as the tensor of components $R^{a_2 \dots a_r}_{b_2 \dots b_s} := T^{a a_2 \dots a_r}_{ab_1 \dots b_s}$.

3.3.6 Integration on Manifold and Differential Forms

Motivation: In a coordinate basis, $\omega = \omega_\mu dx^\mu$. Notice that a curve γ on the manifold \mathcal{M} is a submanifold of \mathcal{M} . The integral of ω along the curve γ is $\int_{\gamma} \omega = \int_{\gamma} \omega_\mu dx^\mu = \int_{s_1}^{s_2} \omega_\mu \frac{dx^\mu(s)}{ds} ds$.

Def: 2-Form

A **2-form**, σ is an antisymmetric twice covariant (type $(0, 2)$) tensor such that $\sigma(v, w) = -\sigma(w, v)$ and in a coordinate basis, $\sigma = \sigma_{\mu\nu} dx^\mu \otimes dx^\nu$.

Def: Wedge Product (1-forms)

The **wedge product** is the fully antisymmetric tensor product.

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu := 2dx^{[\mu} \otimes dx^{\nu]} = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu.$$

Def: k -Form

A **k -form**, σ , is a fully antisymmetric k -times covariant (type $(0, k)$) tensor.

$$\sigma = \sigma_{\alpha_1 \dots \alpha_k} dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_k} = \frac{1}{k!} \sigma_{\alpha_1 \dots \alpha_k} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}.$$

Def: Exterior Product (Wedge Product)

If σ is a k -form and ω is a p -form. The **exterior product** of σ and ω is their fully antisymmetrized tensor product.

$$(\sigma \wedge \omega)_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_p} = \frac{(k+p)!}{k! p!} \sigma_{[\alpha_1 \dots \alpha_k} \omega_{\beta_1 \dots \beta_p]}.$$

Def: Exterior Derivative

The **exterior derivative** d , is an operation that acts on a k -form and return a $(k+1)$ -form.

$$\begin{aligned} d\sigma &= \frac{1}{k!} d(\sigma_{\alpha_1 \dots \alpha_k}) \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k} \\ &= \frac{1}{k!} \partial_\beta (\sigma_{\alpha_1 \dots \alpha_k}) dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k} \\ &= (k+1) \partial_{[\beta} \sigma_{\alpha_1 \dots \alpha_k]} dx^\beta \otimes dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_k} \end{aligned}$$

Def: Exact

A k -form, ω , is **exact** $\Leftrightarrow \omega = d\sigma$, where σ is a $(k-1)$ -form.

Def: Closed

A k -form, ω , is **closed** $\Leftrightarrow d\omega = 0$.

Theorem: Exact \Rightarrow Closed

$\omega = d\sigma \Rightarrow d\omega = 0$. However, the converse is not true in general.

$d\omega = 0 \Rightarrow \omega = d\sigma$ locally.

Theorem: Poincaré Lemma

If ω is defined in a contractible domain (open subset of a manifold), $d\omega = 0 \Leftrightarrow \omega = d\sigma$.

Note

When solving exact ODEs, we are actually have $dF = 0$ and we check $\partial_y \partial_x F = \partial_x \partial_y F$ which we are actually checking for closeness.

Remark

If ω is exact then $\omega = df \Rightarrow \int_\gamma \omega = \int_\gamma df = f(s_2) - f(s_1)$. Also, $\oint_\gamma \omega = 0$.

In polar coordinate, $d\theta$ is actually not exact as $\oint_\gamma d\theta = 2\pi n \neq 0$. Also, $\Theta = d\theta$ is not defined at the origin.

3.4 Smooth Maps and Diffeomorphisms

3.4.1 Smooth Maps: Pullback and Pushforward

Def: Smooth Map

A map $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ is a **smooth map** if and only if given the atlases $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ and $\{\mathcal{U}'_\alpha, \phi'_\alpha\}$ of \mathcal{M} and \mathcal{M}' , the functions $\phi'_\alpha \circ \varphi \circ \phi_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ is smooth.

Def: Pullback (Function)

The map φ induces a map $\varphi^* : \mathcal{F}_{\mathcal{M}} \rightarrow \mathcal{F}_{\mathcal{M}'}$ which we call the **pullback** between the spaces of functions $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{M}'}$ according to the following rule:

Given a function $f' : \mathcal{M}' \rightarrow \mathbb{R}$, its **pull-back** $\varphi^* f' : \mathcal{M} \rightarrow \mathbb{R}$ is such that $\varphi^* f'(p) := f' \circ \varphi(p) = f'(p)$, with $p \in \mathcal{M}$.

Def: Pushforward (Vector)

The map φ induces a map $\varphi_* : T_p \mathcal{M} \rightarrow T_{\varphi(p)} \mathcal{M}'$ called the **pushforward** between the tangent spaces $T_p \mathcal{M}$ and $T_{\varphi(p)} \mathcal{M}'$ according to the following rule:

Given a vector $v \in T_p \mathcal{M}$ then $\varphi_* v \in T_{\varphi(p)} \mathcal{M}'$ such that its action on a function $f' \in \mathcal{F}_{\mathcal{M}'}$ is $(\varphi_* v)|_{\varphi(p)}(f') := v|_p (\varphi^* f')$.

Remark

In coordinate bases (x^1, \dots, x^m) and $(y^1, \dots, y^{m'})$:
 $([\varphi_*]v)|_{\varphi(p)}(f') = v^i|_p \frac{\partial f'}{\partial y^j} \frac{\partial y^j}{\partial x^i}|_p \Rightarrow ([\varphi_*]v)|_{\varphi(p)} = v^i|_p \frac{\partial y^j}{\partial x^i}|_p \frac{\partial}{\partial y^j}|_{\partial(p)}$.

Remark

If the map φ is not injective, than the pushforward of a vector field does not define a vector field.

Def: Pullback (one-form)

Given a 1-form $\omega \in T_p^* \mathcal{M}'$ then $\varphi^* : T_{\varphi(p)}^* \mathcal{M}' \rightarrow T_p^* \mathcal{M}$ called the **pullback** between the cotangent spaces $T_{\varphi(p)}^* \mathcal{M}'$ and $T_p^* \mathcal{M}$ according to the following rule: Given a 1-form $\omega' \in T_{\varphi(p)}^* \mathcal{M}'$, then $\varphi^* \omega' \in T_p^* \mathcal{M}$ whose action on vectors in $T_p \mathcal{M}$ is $\langle \varphi^* \omega', v \rangle|_p := \langle \omega', \varphi_* v \rangle|_{\varphi(p)}$.

Remark

$d(\varphi^* \omega) = \varphi^*(d\omega)$.

Remark

$$\varphi^*(\alpha \wedge \beta) = (\varphi^*\alpha) \wedge (\varphi^*\beta).$$

Remark

$$\text{In coordinate basis, } \varphi^*\omega = (\omega_i \circ \varphi) d(x^i \circ \varphi).$$

Def: Pullback (k -form)

$$\varphi^*\omega = \varphi^* \left(\frac{1}{k!} \omega_{\alpha_1 \dots \alpha_k} \wedge_{i=1}^k dx^{\alpha_i} \right).$$

3.4.2 Diffeomorphisms

Def: Diffeomorphism

A map $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ is a **diffeomorphism** if and only if φ and φ^{-1} are both smooth bijections.

Remark

For a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$, pullback and pushforward can be defined for all objects on the manifold as $\varphi_* = (\varphi^{-1})^*$.

Remark

$$[\varphi_*]_p^{-1} = [\varphi_*^{-1}]_{\varphi(p)}. \\ \varphi_* f(p) = (\varphi^{-1})^* f(p) \text{ and } (\varphi^* v)|_p(f) = v_{\varphi^{-1}(p)} \varphi_* f.$$

Def: Local Flow

Consider a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ and let $\gamma_p(s)$ be the parametrized curve in \mathcal{M} such that $\gamma_p(0) = p$ and its tangent vector at every points $\gamma_p'(s)$ is the vector field \mathbf{k} . \mathbf{k} is the generator of a set of local diffeomorphisms of the form $\varphi_s : \mathcal{U} \rightarrow \varphi_s(\mathcal{U})$ such that $\varphi_s(p) = \gamma_p(s)$. This set of local diffeomorphisms is called the **local flow** of \mathbf{k} .

Note

- 1) $\varphi_{-s} \circ \varphi_s(p) = p \Rightarrow \varphi_s^{-1} = \varphi_{-s}$
- 2) $\varphi_s(p) = \gamma(s)$
- 3) $\varphi_{s+t}(p) = \gamma(s+t)$
- 4) $\phi_s(\phi_t(p)) = \varphi_{s+t}(p)$

Hence, local diffeomorphisms forms a group (Lie group). And the local flow of the vector field \mathbf{k} is complete if the associated curve $\gamma_p(s)$ can be extended for all values $s \in \mathbb{R}$.

3.4.3 Lie Derivative

Motivation: Let $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ be a diffeomorphism, φ establishes an isomorphism $\varphi^* : (T_{\varphi(p)})_s^r \mathcal{M} \rightarrow (T_p)_s^r \mathcal{M}$ for $p \in \mathcal{M}$. The tensor $[\varphi^*(T_{\varphi(p)})]_p - T|_p$ provides information about the difference between the value of a tensor field at $\varphi(p)$ and at p .

Def: Lie Derivative

Consider a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ and its pullback $\varphi^* : (T_{\varphi(p)})_s^r \mathcal{M} \rightarrow (T_p)_s^r \mathcal{M}$. Consider $\varphi \equiv \varphi_t$ be the local flow generated by a vector field \mathbf{v} . The **Lie derivative** of a tensor T along the (curve generated by) vector field \mathbf{v} at the point $p \in \mathcal{M}$ is $(\mathcal{L}_v T)|_p := \lim_{t \rightarrow 0} \frac{[\varphi_t^*(T_{\varphi_t(p)})]_p - T_p}{t}$.

Note

t is a parameter along a curve generated by \mathbf{v} .

Intuitively, the Lie derivative is to:

- 1) Take the tensor at t steps away from point $p \in \mathcal{M}$ (along the curve generated by \mathbf{v}).
- 2) Move the tensor to point p by using the pullback.
- 3) Compute the difference between the two tensors at p .
- 4) Divide the difference of the two tensors by t then take the limit $t \rightarrow 0$.

Acknowledgement: This intuition is from [here](#). One can also think of Lie Derivative as something that measures the variation of a tensor field when moving

Remark

Lie Derivative have the properties:

- 1) Linearity.
- 2) Preserves the tensor type, symmetries, and contractions because φ^* does.
- 3) Satisfies Leibniz rule under tensor product and contraction: $\mathcal{L}_v(T \otimes S) = (\mathcal{L}_v T) \otimes S + T \otimes (\mathcal{L}_v S)$.
- 4) Cartan identity: The Lie derivative of a k -form field along the vector field \mathbf{v} follows $\mathcal{L}_v = i_v d + d i_v$ (took from JiayuePhysics).

Def: Lie Derivative of Scalar Field

For a scalar field f , $\mathcal{L}_v f = v(f)$, and in a coordinate basis, $\mathcal{L}_v f = v^\nu \partial_\nu f$.

Def: Lie Derivative of Vector Field

For a vector field w , $\mathcal{L}_v w = [v, w]$, and in a coordinate basis, $\mathcal{L}_v w = (v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \partial_\mu$.

Def: Lie Derivative of 1-Form Field

For a 1-form field ω in a coordinate basis, $\mathcal{L}_v \omega = (v^\nu (\partial_\nu \omega_\mu) + \omega_\nu (\partial_\mu v^\nu)) dx^\mu$.

Ex: More Lie Derivatives

$$\omega = \omega_{\mu\nu} dx^\mu \otimes dx^\nu \Rightarrow \mathcal{L}_v \omega = (v^\alpha (\partial_\alpha \omega_{\mu\nu}) + \omega_{\alpha\nu} (\partial_\mu v^\alpha) + \omega_{\mu\alpha} (\partial_\nu v^\alpha)) dx^\mu \otimes dx^\nu.$$

$$T = T_\mu^\nu dx^\mu \otimes \partial_\nu \Rightarrow \mathcal{L}_v T = (v^\alpha (\partial_\alpha T_\mu^\nu) + T_\alpha^\nu (\partial_\mu v^\alpha) - T_\mu^\alpha (\partial_\alpha v^\nu)) dx^\mu \otimes \partial_\nu.$$

$$T = T_{\alpha\beta}^{\mu\nu} dx^\alpha \otimes dx^\beta \otimes \partial_\mu \otimes \partial_\nu.$$

$$\Rightarrow \mathcal{L}_v T = \left(v^\gamma \partial_\gamma T_{\alpha\beta}^{\mu\nu} + T_{\gamma\beta}^{\mu\nu} \partial_\alpha v^\gamma + T_{\alpha\gamma}^{\mu\nu} \partial_\beta v^\gamma - T_{\alpha\beta}^{\gamma\nu} \partial_\gamma v^\mu - T_{\alpha\beta}^{\mu\gamma} \partial_\gamma v^\nu \right) dx^\alpha \otimes dx^\beta \otimes \partial_\mu \otimes \partial_\nu.$$

3.5 Pseudo-Riemannian Geometry

3.5.1 Affine Connection and Covariant Derivative

Motivation: While Lie derivative is a derivative, we can't use it as a generalised gradient, as it doesn't change the type of the tensor. So we need to explore other derivatives. We take the derivative of a function f as $\frac{\partial f}{\partial x^\mu} = \partial_\mu f$. And for a vector $v = v^\mu \Upsilon_\mu$, $\frac{\partial v}{\partial x^\mu}$ is a bad notation for the derivative of the vector because both the components and basis of v can change. What we want to do is to write $\frac{\partial \Upsilon}{\partial x^\mu}$ in terms of the bases ($\frac{\partial \Upsilon}{\partial x^\mu} = \Gamma^\rho_{\nu\mu} \Upsilon_\rho$ with $\Gamma^\rho_{\nu\mu}$ describing how basis changes at different point).

Def: Affine Connection, Covariant Derivative

An **affine connection** $\nabla : (T_p \mathcal{M})_s^r \rightarrow (T_p \mathcal{M})_{s+1}^r$ is a rule through which we assign to each tensor T of type (r, s) and components $T^{bc\dots}_{de\dots}$ another tensor field ∇T of type $(r, s+1)$ and components $(\nabla T)^{bc\dots}_{de\dots;a} \equiv \nabla_a T^{bc\dots}_{de\dots} \equiv T^{bc\dots}_{de\dots;a}$. We call ∇T the **covariant derivative** of T .

Remark

Affine connection follows the properties:

- 1) Linearity.
- 2) Leibniz rule.
- 3) Commutes with contraction.
- 4) Over functions f , it is the differential $\nabla f = df$.

Note

There are infinitely many affine connections and infinite ways to build covariant derivative.

Def: Directional Covariant Derivative

We define the **directional covariant derivative** of T in the direction of vector v as the tensor $\nabla_v T$ of type (r, s) of components $(\nabla_v T)^{bc\dots}_{de\dots} = v^a \nabla_a T^{bc\dots}_{de\dots}$. $(\nabla_v T)^{bc\dots}_{de\dots} = \langle \nabla T, v \rangle$.

Def: Coefficient of the Connection

The **coefficient of the connection** in an arbitrary basis $\{\Upsilon_a\}$ is

$$\Gamma^a_{bc} := (\nabla \Upsilon_b)^a_c = \langle \Upsilon^a, \nabla_{\Upsilon_c} \Upsilon_b \rangle \Rightarrow \nabla_c \Upsilon_b := \nabla_{\Upsilon_c} \Upsilon_b := \Gamma^a_{bc} \Upsilon_a.$$

Γ^a_{bc} is the a -th components of the covariant derivative of the b -th basis vector in the direction of the c -th basis vector.

Remark

The notation for covariant derivative $\nabla_\mu v^\nu = (\nabla v)_\mu^\nu$. Also...

Note

Another convention is $\Gamma^a_{bc} := (\nabla \Upsilon_c)^a_b = \langle \Upsilon^a, \nabla_{\Upsilon_b} \Upsilon_c \rangle$.

$$\nabla_a v^b = \Upsilon_a(v^b) + v^c \Gamma^b_{ca}$$

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + v^\sigma \Gamma^\nu_{\sigma\mu}$$

$$\nabla_a \omega_b = \Upsilon_a(\omega_b) - \omega_c \Gamma^c_{ba}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_{\nu\mu} - \omega_\sigma \Gamma^\sigma_{\nu\mu}$$

$$\text{Note: } \frac{\partial v}{\partial x^\mu} = \left(\frac{\partial v^\nu}{\partial x^\mu} + v^\sigma \Gamma^\nu_{\sigma\mu} \right) \Upsilon_\nu.$$