Block 1

Introduction to special relativity

1.1 The postulates of special relativity

A reference frame is a spatial coordinate system where the position of pointlike particles can be specified, and a clock.

An inertial reference frame is a frame that satisfies Newton's first law: Free particles keep their state of motion (constant velocity). Two inertial reference frames move with a constant relative velocity.

Postulate 1. *Principle of Relativity:* In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

That is, the equations that describe the laws of physics have to have the same form in every inertial reference frame.

Postulate 2. The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

These two postulates constitute the basis of the theory of relatitvity. Newtonian mechanics can be recovered in the limit $c \to \infty$, i.e. in the limit of instantaneous interactions. Note the importance of Maxwell equations in the advent of special relativity. The first postulate remains unchanged in this new theory. The difference between Galilean and Einsteinian special relativity is that 'all' the laws of physics alluded by the Galilean principle are the laws of mechanics while the Einsteinian one must include Maxwell's electrodynamics, represented by the finiteness and invariance of the speed of light.

In Newtonian mechanics, space is relative: the distance between two non-simultaneous events depend on the inertial reference frame. Indeed, let $\boldsymbol{x}_1(t_1)$ y $\boldsymbol{x}_2(t_2)$ be the position vectors of two events with respect to the inertial frame S. With respect to any other inertial frame S', moving with velocity $\boldsymbol{v}$, the position vector is given by

$$\boldsymbol{x}'(t) = \boldsymbol{x}(t) - \boldsymbol{v}t \tag{1.1.1}$$

and therefore for the two systems under consideration,

$$|\boldsymbol{x}_2'(t_2) - \boldsymbol{x}_1'(t_1)|^2 = |\boldsymbol{x}_2(t_2) - \boldsymbol{x}_1(t_1)|^2 + v^2(t_2 - t_1)^2 - 2(t_2 - t_1)\boldsymbol{v} \cdot [\boldsymbol{x}_2(t_2) - \boldsymbol{x}_1(t_1)]. \tag{1.1.2}$$

However time is absolute: two simultaneous events in an inertial frame are simultaneous in any other inertial frame and, hence $t' = t$. A a consequence, we obtain the law of addition of velocities: if in S a particle has velocity $\boldsymbol{V}$ and S' is moving with velocity $\boldsymbol{v}$ with respect to S, the the velocity of the particle in S' is $\boldsymbol{V}' = \boldsymbol{V} - \boldsymbol{v}$. Indeed, for two times t_1 y $t_2 = t_1 + \mathrm{d}t$,

$$\boldsymbol{V}' = \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \boldsymbol{v} = \boldsymbol{V} - \boldsymbol{v}. \tag{1.1.3}$$

This composition law is incompatible with the universal and finite character of the speed of light. In fact, in special relativity time is relative: two simultaneous events in an inertial frame are necessarily simultaneous in another.

In special relativity, time and space are relative, but not everything is relative. there exist absolute quantities of great importance, the spacetime interval among them.

1.2 Lorentz transformations

Given a particular set of coordinates (inertial reference frame) $(t, \boldsymbol{x})$, we define the *spacetime interval*, Δs^2, between two events $E_1 \equiv (t_1, \boldsymbol{x}_1)$ and $E_2 \equiv (t_2, \boldsymbol{x}_2)$ as

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 = -c^2(t_2 - t_1)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2 \tag{1.2.1}$$

If c is the speed of light described from the frame $(t, \boldsymbol{x})$, for two events $(t_1, \boldsymbol{x}_1)$ and $(t_2, \boldsymbol{x}_2)$ that are connected by the propagation of a ray of light this spacetime interval is zero

$$0 = -c^2(t_2 - t_1)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$$

Let us now consider a different set of coordinates (t', x'). The same events will be represented by different pairs of t' and x'. Namely $E_1 \equiv (t'_1, x'_1)$ and $E_2 \equiv (t'_2, x'_2)$. The spatial separation and time interval between E_1 and E_2 are different from those in the other frame: $\Delta x' \neq \Delta x$, $\Delta t' \neq \Delta t$. If we now impose that the speed of light is also c in the frame S' (as per the postulates of special relativity) the spacetime interval between the two events is still zero. This imposes a relationship between the two frames:

$$0 = -c^2(\Delta t')^2 + (\Delta x')^2 = -c^2(t'_2 - t'_1)^2 + (\boldsymbol{x}'_2 - \boldsymbol{x}'_1)^2 = -c^2(t_2 - t_1)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2 \tag{1.2.2}$$

If the two frames are related by a spatial rotation, it is trivial to see that equation (1.2.1) remains invariant (the distance between Euclidean vectors is invariant under rotations). If the frame (t', x') has a relative velocity $\boldsymbol{v}$ with respect to the frame (t, x) we need to find a transformation between frames that leaves equation (1.2.1) invariant. The coefficients of such transformation can only depend on the relative velocity between the two frames.

Assuming the following:

1. The difference between the two frames is a constant speed $\boldsymbol{v}$ so that the origin of the primed system $\boldsymbol{x}'_{0'} = \boldsymbol{0}$ has the trajectory $\boldsymbol{x}_{0'} = \boldsymbol{v}t$ in the unprimed system
2. The transformation has to be linear.

And further assuming, without loss of generality, that the velocity goes along the direction of the spatial x coordinate: $\boldsymbol{v} = v\boldsymbol{e}_x$ we write the most general linear transformation that satisfies the first assumption of constant speed difference (modulo time shifts) as

$$t' = \gamma(v)[f(v)x + t], \qquad x' = \tilde{\gamma}(v)(x - vt), \qquad y' = y, \qquad z' = z. \tag{1.2.3}$$

Introducing these transformations in (1.2.2) we obtain

$$(c^2\gamma^2 f^2 - \tilde{\gamma}^2 + 1)(x_2 - x_1)^2 + (c^2\gamma^2 - \tilde{\gamma}^2 v^2 - c^2)(t_2 - t_1)^2 + 2(c^2\gamma^2 f + \tilde{\gamma}^2 v)(x_2 - x_1)(t_2 - t_1) = 0$$

since $\gamma, \tilde{\gamma}$ and f do not depend on the times and positions, and this has to be true for all x_1, x_2, t_1, t_2, every coefficient in the left-hand side has to cancel independently. This yields

$$f = -\frac{v}{c^2}, \qquad \tilde{\gamma} = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

therefore we conclude that the relation (1.2.1) is invariant under the following transformations

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \qquad x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z.$$

This is called a *Lorentz transformation.*

If S' moves with an arbitrary velocity $\boldsymbol{v}$ it is very easy to write what the Lorentz transformation would look like by simply using that the transformation only affects time and the component of $\boldsymbol{x}$ parallel to the direction of motion given by the unit vector $\boldsymbol{n} = \boldsymbol{v}/|\boldsymbol{v}|$. The Lorentz transformation between an interial reference frame S and another one S' moving with respect to S with a speed $\boldsymbol{v}$ is therefore written as

$$t' = \gamma\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right), \qquad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} - \gamma\boldsymbol{v}t, \tag{1.2.4}$$

where, as we said

$$\boldsymbol{n} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}, \qquad \gamma(\boldsymbol{v}) = \frac{1}{\sqrt{1 - \boldsymbol{v}^2/c^2}}.$$

To get used to the notation with roman letters for 3-indices $x^i = (x^1, x^2, x^3) \equiv (x, y, z)$, we can express this in terms of components x^i as

$$t' = \gamma\left(t - \frac{v_i x^i}{c^2}\right), \qquad x'^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t \tag{1.2.5}$$

where $v = |\boldsymbol{v}|$.

Composition of velocities

As a quick note, we could ask the following question: Let $\boldsymbol{V}$ be the speed of a particle as seen from the frame $S = (t, \boldsymbol{x})$. What would be the speed $\boldsymbol{V}'$ of the particle as measured from the frame $S' = (t', \boldsymbol{x}')$ which moves with a speed $\boldsymbol{v}$ with respect to S. We can readily work out that

$$\boldsymbol{V}' = \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t}\left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right)^{-1}.$$

Differentiating (1.2.4) with respect to t and taking into account that $\boldsymbol{V} = \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$, we get that

$$\boldsymbol{V}' = \frac{\boldsymbol{V} + (\gamma - 1)(\boldsymbol{n}\cdot\boldsymbol{V})\boldsymbol{n} - \gamma\boldsymbol{v}}{\gamma(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{V}}{c^2})}$$

And for illustration, it is convenient to write it in components: If we decompose $\boldsymbol{V}$ into a part parallel to $\boldsymbol{v}$ and a part orthogonal to $\boldsymbol{v}$:

$$\boldsymbol{V} = V_{\parallel}\boldsymbol{n} + \boldsymbol{V}_{\perp},$$

then we get that from the frame S'

$$V'_{\parallel} = \frac{V_{\parallel} - |\boldsymbol{v}|}{1 - \frac{|\boldsymbol{v}|V_{\parallel}}{c^2}}, \qquad \boldsymbol{V}'_{\perp} = \frac{\boldsymbol{V}_{\perp}}{\gamma\left(1 - \frac{|\boldsymbol{v}|V_{\parallel}}{c^2}\right)}$$

So the composition of velocities is also non-trivial for the components that are perpendicular to $\boldsymbol{v}$.

1.3 Line element, proper time and spacelike, timelike and null separation

1.3.1 Classification of spacetime intervals

We can consider the *spacetime interval* between two events $E_1 \equiv (t_1, \boldsymbol{x}_1)$ and $E_2 \equiv (t_2, \boldsymbol{x}_2)$ defined in (1.2.1)

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta \boldsymbol{x}^2.$$

Obviously, this quantity is well defined for whichever pair of events, even if they are not connected by a light signal.

We can classify events according to the following criterion:

- Two events are **Spacelike separated** if the spacetime interval between them is positive, $\Delta s^2 > 0$.
- Two events are **Timelike separated** if the spacetime interval between them is negative, $\Delta s^2 < 0$
- Two events are **Lightlike (or null) separated** if the spacetime interval between them is zero, $\Delta s^2 = 0$

For any E_1 and E_2 timelike separated events there is always a reference frame in which the two events happen at the same position but at different times. For any E_1 and E_2 spacelike separated events there is always a reference frame in which the two events happen at the same time but at different positions. Two events E_1 and E_2 lightlike separated (or null-separated) are always connected by a light-sign. Since the spacetime interval Δs^2 is Lorentz invariant this classification is independent of the reference frame, and so it is absolute.

Given the trajectory of a physical particle moving inertially, we will call *co-moving frame* or *proper frame* to the frame S_p where the particle is at rest. Obviously, all proper reference frames to the same particle share the origin and are at relative rest with respect to each other. The only possible difference between them is a relative rotation.

If the particle is not undergoing inertial motion, the proper reference frame is no longer inertial and it is, in essence, different from the co-moving frame. In that case we call *co-moving frame* S' to the family of inertial reference frames in which the particle is instantaneously at rest (zero instantaneous speed).

1.3.2 Proper time and line element

We define the *propert time* τ of a moving particle as the time measured by a clock that moves with the particle, that is, the time measured in the particle's proper reference frame. Even if the particle is not free, and therefore it is describing non-inertial motion, the co-moving time coincides instantaneously with the proper time.

We can define the differential spacetime interval $\mathrm{d}s^2$ between two differentially-close events $(t, \boldsymbol{x})$ and $(t + \mathrm{d}t, x + \mathrm{d}\boldsymbol{x})$:

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t + \mathrm{d}\boldsymbol{x}^2 \tag{1.3.1}$$

which is invariant under Lorentz transformations. We will call $\mathrm{d}s^2$ the spacetime *line element*.

Let us consider the frame $S = (t, x)$. We want to compute the relationship between the time measured in the frame S, $\mathrm{d}t$ and the co-moving time (that coincides with the proper time of the particle at every instant)

$\mathrm{d}\tau = \mathrm{d}t'$. Since at every instant the comoving frame S' and the proper frame S_p have the same velocity $\boldsymbol{v}$ with respect to S, we can work it out from the Lorentz transformation of times for constant v (1.2.5)

$$\mathrm{d}\tau = \mathrm{d}t' = \gamma\left(1 - \frac{\boldsymbol{v}^2}{c^2}\right)\mathrm{d}t = \gamma^{-1}\mathrm{d}t \tag{1.3.2}$$

The line element and the proper time are, in fact, related, since proper time is just a reparametrization of the parameter s, in fact, from (1.3.1) we know that

$$-\frac{1}{c^2}\frac{\mathrm{d}s^2}{\mathrm{d}t^2} = 1 - \frac{1}{c^2}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)^2$$

and using this we can quickly obtain that

$$\sqrt{-\frac{\mathrm{d}s^2}{c^2}} = \mathrm{d}t\sqrt{-\frac{\mathrm{d}s^2}{c^2\mathrm{d}t^2}} = \mathrm{d}t\sqrt{1 - \frac{1}{c^2}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)^2} = \gamma^{-1}\mathrm{d}t = \mathrm{d}\tau \tag{1.3.3}$$

and this implies that

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2. \tag{1.3.4}$$

1.4 Lorentzian Tensors

Let us consider a reference frame S defined by an orthonormal basis of 3-space $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \boldsymbol{\Upsilon}_3\}$ Υ and a clock which measures time t. every event in spacetime is characterized by a position $\boldsymbol{x}$ and a time t. Roman indices will denote spatial components $\boldsymbol{x} = x^i\boldsymbol{e}_i$ (As opposed to Greek indices notate spacetime indices).

We can consider together the components of the position x^i and the time t as components of a four-vector $\mathsf{x} = (ct, \boldsymbol{x}) = x^\mu\Upsilon_\mu$, where $\mu = 0, 1, 2, 3$ and we have introduced a basis of vectors of a four dimensional space comprised of the ct direction (whose director vector we will notate $\boldsymbol{\Upsilon}_0$) and the vectors in the three directions of space (that we denote $\{\boldsymbol{\Upsilon}_i\}$ with $i = 1, 2, 3$). We call the set $\{\Upsilon_\mu\}$ a *Lorentzian basis*.

Sometimes, when there is no risk of confusion, we will subscribe to the usual convention and abuse notation to write the four-vector x as its components in an arbitrary basis $x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) \equiv (ct, \boldsymbol{x})$. We call these objects four-vectors to distinguish them from spatial three-vectors.

In this new spacetime coordinates, the *line element* (1.3.1) is written as

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + \sum_i(\mathrm{d}x^i)^2 \tag{1.4.1}$$

We introduce a metric tensor so that we can write this line element in a much more compact way in a similar way as we do with Euclidean space

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{1.4.2}$$

Where we have introduced a matrix η (which we will call *metric tensor*) that acts on spacetime vectors. Its components in the basis $\{\Upsilon_\mu\}$ are defined by its action on the elements of the spacetime basis:

$$\eta(\Upsilon_\mu, \Upsilon_\nu) = \eta_{\mu\nu}$$

and by inspection in (1.4.1) we find that

$$\eta_{00} = -1, \qquad \eta_{ij} = \delta_{ij}$$

and all the other entries are zero.

Matricially we can represent this metric tensor (abusing the index notation) as

$$\eta \equiv \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will call the inverse matrix (such that the product by $\eta_{\mu\nu}$ yields the identity) is $\eta^{\mu\nu}$. This will be given by

$$(\eta)^{-1} \equiv \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And indeed the product of the two matrices is the identity: $\eta_{\mu\kappa}\eta^{\kappa\nu} = \delta^{\nu}_{\mu}$.

Remember that the postulates of special relativity impose that the line element is invariant. The set of transformations that leave the line element invariant is called the *Poincaré* group.

The Poincaré group consists of the following elements: Spacetime translations, spatial reflexions, time reflexions and the so called proper orthochronus Lorentz group of transformations. The Lorentz group contains pure Lorentz boosts as the ones studied above, and proper spatial rotations. Since thehese transformations leave the line element invariant, they represent the transformations between different inertial reference frames.

Let us introduce the matrices that implement the change between two different Lorentzian bases $\{\Upsilon_\mu\}$ and $\{\Upsilon'_\mu\}$. This is, the matrix Λ that satisfies

$$\Upsilon'_\mu = \Lambda_\mu{}^\nu\,\Upsilon_\nu$$

If the vectors of the base are transformed by applying Λ, it means that that components of the vectors such as the spacetime position vector change with the inverse transposed matrix $\tilde{\Lambda} = (\Lambda^{\mathrm{T}})^{-1}$, whose components are $\tilde{\Lambda}_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu$, that is, the components of a four-vector r^μ are related to the components of the transformed four-vector r'^μ as follows:

$$r'^\mu = \tilde{\Lambda}^\mu{}_\nu r^\nu \tag{1.4.3}$$

This transformation law is what will allow us to define what we call a *contravariant Lorentz vector*. A contravariant vector r is an object whose components transform under the Lorentz group as in equation (1.4.3). As it can be appreciated, it is convenient to use Einstein notation where contravariant indices are superscripts.

On the other hand, we can define *covariant vectors*, which are objects whose components transform like the four-basis vectors, with Λ instead of $\tilde{\Lambda}$ under Lorentz transformations. A covariant vector $\boldsymbol{\omega}$ is an object that transforms under Lorentz transformations as

$$\omega'_\mu = \Lambda_\mu{}^\nu\omega_\nu \tag{1.4.4}$$

From the invariance of the line element (1.4.2) under Lorentz transformations we obtain that

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\tilde{\Lambda}^\rho{}_\mu\tilde{\Lambda}^\sigma{}_\nu. \tag{1.4.5}$$

We can now multiply by Λ twice on both sides of the equation remembering that $\tilde{\Lambda}_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu$, and therefore $\Lambda_\nu{}^\mu \tilde{\Lambda}^\rho{}_\mu = \delta^\rho_\nu$ we obtain that

$$\eta_{\mu\nu}\Lambda_\kappa{}^\mu\Lambda_\beta{}^\nu = \eta_{\rho\sigma}\tilde{\Lambda}^\rho{}_\mu\Lambda_\kappa{}^\mu\tilde{\Lambda}^\sigma{}_\nu\Lambda_\beta{}^\nu \Rightarrow \eta_{\mu\nu}\Lambda_\kappa{}^\mu\Lambda_\beta{}^\nu = \eta_{\rho\sigma}\delta^\rho_\kappa\delta^\sigma_\beta \Rightarrow \eta_{\mu\nu}\Lambda_\kappa{}^\mu\Lambda_\beta{}^\nu = \eta_{\kappa\beta}.$$

Or, for clarity of notation, renaming the indices:

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma. \tag{1.4.6}$$

which is precisely the Lorentz transformation law for the metric tensor. We define a Lorentz 2-covariant tensor as an object that transforms under the Lorentz group as (1.4.6). Under this definition it seems appropriate that we decided to call the matrix η the metric tensor.

Playing similar tricks, (using also the fact discussed above that $\eta_{\mu\kappa}\eta^{\kappa\nu} = \delta^\nu_\mu$) we can use (1.4.5) and (1.4.6) to write the matrices Λ and $\tilde{\Lambda}$ in terms of each other in a very simple and revealing way.

$$\tilde{\Lambda}^\mu{}_\nu = \eta^{\mu\rho}\eta_{\nu\sigma}\Lambda_\rho{}^\sigma, \qquad \Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\tilde{\Lambda}^\rho{}_\sigma \tag{1.4.7}$$

Which shows that the metric tensor allows to establish an isomorphism between the vector spaces of covariant and contravariant (with respect to Lorentz transformations) vectors. To each covariant vector of components r^μ we uniquely associate a contravariant vector of components r_μ through the relationships

$$r_\mu = \eta_{\mu\nu}r^\nu, \qquad r^\mu = \eta^{\mu\nu}r_\nu,$$

which means that we use the metric tensor to "raise and lower" indices. Given the form of the metric tensor for Minkowskian spacetime, the explicit relation between the components of covariant and contravariant vectors is

$$v_0 = -v^0, \qquad v_i = \delta_{ij}v^j$$

This relationship is true for the flat spacetime metric (1.4.2). Notice that we have picked the signature $(-,+,+,+)$ which is the one used by most GR literature.

We define the inner product of two Lorentzian vectors u^μ and v^μ as the real number obtain by the contraction of the two vectors with the metric tensor

$$\mathsf{u}\cdot\mathsf{v} = \eta_{\mu\nu}u^\mu v^\nu = u_\mu v^\mu$$

This inner product is not positive definite. In fact we classify a four-vector u^μ according to their norm with respect of this pseudo-Euclidean inner product as follows.

- v is timelike if and only if $||\mathsf{v}||^2 = v^\mu v_\mu < 0$
- v is spacelike if and only if $||\mathsf{v}||^2 = v^\mu v_\mu > 0$
- v is lightlike (or null) if and only if $||\mathsf{v}||^2 = v^\mu v_\mu = 0$

1.4.1 Four-differentiation

We define the four-gradient as the operator that acts on a scalar function of x^μ and returns a covariant four-vector of components

$$\partial_\mu f = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_i f)$$

the contravariant version of this operator is obtained as usual raising the index

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu}$$

Finally, we introduce the D'Alambertian operator as the four-vector version of the Laplacian

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\partial_t^2 + \boldsymbol{\nabla}^2$$

1.5 The Poincaré group

The Poincaré group consists of the following elements: Spacetime translations, spatial reflexions, time reflexions and the so called proper orthochronus Lorentz group of transformations..

1.5.1 The translations group

The translations group consists of all the passive transformations which displace the origin of coordinates. In its active version the group consists of the transformations that displace the position four-vector x^μ by a constant magnitude α^μ. The components of the translated four-vector x'^μ are related to x^μ as

$$x'^\mu = x^\mu + \alpha^\mu \Rightarrow \Delta x^\mu = x'^\mu - x^\mu = \alpha^\mu \tag{1.5.1}$$

Let us assume that a linear differential operator $\hat{O}(\alpha^\mu)$ exists such that one can write the differential translation operation as the action of such operator:

$$x'^\mu = \hat{O}(\alpha^\mu)x^\mu = (\mathbb{1} + \alpha^\nu O^{(1)}_\nu)x^\mu + \mathcal{O}([\alpha^\mu]^2) \Rightarrow \Delta x^\mu = x'^\mu - x^\mu = \alpha^\nu O^{(1)}_\nu x^\mu + \mathcal{O}([\alpha^\mu]^2).$$

If the transformation is infinitesimally small $\alpha^\mu \to \mathrm{d}\alpha^\mu$, $\Delta x^\mu \to \mathrm{d}x^\mu$, then the transformation can be written as

$$\mathrm{d}x^\mu = \mathrm{d}\alpha^\nu O^{(1)}_\nu x^\mu$$

but we also know from (1.5.1) that for a spacetime translation $\mathrm{d}x^\mu = \mathrm{d}\alpha^\mu$, therefore, for a differential translation, it should be satisfied that

$$\mathrm{d}\alpha^\mu = \mathrm{d}\alpha^\nu O^{(1)}_\nu x^\mu$$

It is easy to check that the only linear differential operator $O^{(1)}_\nu$ that satisfies that is the four-gradient ∂_ν, this is:

$$\mathrm{d}\alpha^\mu = \mathrm{d}\alpha^\nu \partial_\nu x^\mu \Rightarrow \mathrm{d}\alpha^\mu = \mathrm{i}\mathrm{d}\alpha^\nu \hat{P}_\nu x^\mu$$

where we have defined the differential operator $\hat{P}_\nu = -\mathrm{i}\partial_\nu$.

Now, since translations are continuous transformations, it is satisfied that we can obtain any finite translaltion as a composition of smaller (even differential) translations. Therefore we can now integrate over α to obtain any finite transformation and obtain the form of $O(\alpha^\mu)$ for any finite spacetime translation:

$$x'^\mu = e^{\mathrm{i}\alpha^\nu \hat{P}_\nu} x^\mu$$

This is the reason why the operators $\hat{P}_\mu$ are the infinitesimal generators of the translations group, and they are intimately related to the total momentum of the system.

These operators obviously commute:

$$[\hat{P}_\mu, \hat{P}_\nu] = 0$$

1.5.2 The Lorentz Group

As we already mentioned before, the Lorentz group is formed for all the matrices Λ that satisfy (1.4.6). This condition already imposes that $\det(\lambda) = \pm 1$.If we restrict ourselves to the proper orthochronus Lorentz group, the determinant will always be positive 1. Additionally, since the transformations cannot contain space or time reflections, ${\Lambda_0}^0 > 0$

Equation (1.4.6) imposes 16 conditions on the 16 entries of Λ. However only 10 of them are independent (4 for which $\mu = \nu$, 3 corresponding to $\mu = 0, \nu \in \{1, 2, 3\}$, 2 corresponding to $\mu = 1, \nu \in \{2, 3\}$ and 1 corresponding to $\mu = 2, \nu = 3$). This leaves 6 free parameters to fully characterize all the Lorentz transformations. These are going to correspond to 3 independent rotations and 3 independent boosts.

The active pure Lorentz boosts $\tilde{\Lambda}$ transform the position three-vector $\boldsymbol{x}$ and the time t at which an event happens in a reference frame S into a different $\boldsymbol{x}'$ and t' in the same reference frame S. The passive Lorentz boosts Λ transforms the coordinates of one single event given by the position three-vector $\boldsymbol{x}$ and the time t in the reference frame S to the coordinates of the same event in the frame S'. This an active transformation corresponding to a boost of velocity v is equivalent to a passive transformation between the two reference frames S and S' specified in (1.2.4) with opposite sign of the velocity, that is,

$$t' = \gamma\left(t + \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right), \qquad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} + \gamma\boldsymbol{v}t, \tag{1.5.2}$$

The matrices $\tilde{\Lambda}$ and Λ corresponding respectively to an active and a passive boost of velocity $\boldsymbol{v}$ are easy to obtain from (1.5.2) and (1.4.7):

$$\begin{aligned} &{\tilde{\Lambda}^0}_0 = \gamma, \qquad &&{\tilde{\Lambda}^0}_i = \gamma v_i/c, \qquad &&{\tilde{\Lambda}^i}_0 = \gamma v^i/c, \qquad &&{\tilde{\Lambda}^i}_j = \delta^i_j + (\gamma-1)v^i v_j/v^2, \\ &{\Lambda^0}_0 = \gamma, &&{\Lambda_0}^i = -\gamma v^i/c, &&{\Lambda_i}^0 = -\gamma v_i/c, &&{\Lambda_i}^j = \delta^j_i + (\gamma-1)v_i v^j/v^2. \end{aligned} \tag{1.5.3}$$

As we can see the three-velocity v_i gives 3 of the six parameters of the Lorentz Transformation. The other three, as discussed above will correspond to rotations.

We can do something similar to what we did with translations with differential Lorentz transformations. Let us consider a differential Lorentz transformation of parameters $\mathrm{d}{\omega_\mu}^\nu$, in that case

$${\Lambda_\mu}^\nu = \delta^\nu_\mu + \mathrm{d}{\omega_\mu}^\nu, \qquad {\tilde{\Lambda}^\mu}_\nu = \delta^\mu_\nu + \mathrm{d}{\tilde{\omega}^\mu}_\nu$$

we can find a relationship between $\mathrm{d}{\tilde{\omega}^\mu}_\nu$ and $\mathrm{d}{\omega_\mu}^\nu$ using the fact that $\tilde{\Lambda} = (\Lambda^{-1})^{\mathrm{T}}$:

$$\delta^\mu_\nu = {\tilde{\Lambda}^\mu}_\rho {\Lambda_\nu}^\rho = \delta^\mu_\nu + \mathrm{d}{\tilde{\omega}^\mu}_\nu + \mathrm{d}{\omega_\nu}^\mu$$

this means that

$$\mathrm{d}\tilde{\omega}_{\mu\nu} = \mathrm{d}\omega_{\mu\nu} = -\mathrm{d}\omega_{\nu\mu}$$

As a consecuence, an active, infinitesimal, Lorentz transformation

$$\mathrm{d}x^\mu = \mathrm{d}{\tilde{\omega}^\mu}_\nu x^\nu$$

is characterized by the six independent components of the anti-symmetric tensor $\mathrm{d}\tilde{\omega}^{\mu}{}_{\nu}$.

For infinitesimal pure boosts of velocities $\mathrm{d}v^i$, we can quickly obtain the non-zero components of this tensor:

$$\mathrm{d}\omega_{0i} = -c^{-1}\mathrm{d}v_i \equiv -\mathrm{d}\xi_i. \tag{1.5.4}$$

We obtained this by expanding (1.5.3) to first order on the velocity. All the other entries are zero.

For rotations, it is easy to see that only the spatial components intervene. That, together with the antisymmetry of $\mathrm{d}\omega_{ij}$ means that we can write it as

$$\mathrm{d}\omega_{ij} = -\epsilon_{ijk}\mathrm{d}\theta^k$$

where $\mathrm{d}\theta^i$ are the three parameters associated to the rotations. One can check that in the way we choose it, $\mathrm{d}\theta^i$ are the angles of rotation round the three spatial axes in counterclockwise direction, as usual.

With some (trivial but lengthy) work, we can show that the general form of an infinitesimal Lorentz transformation is

$$\mathrm{d}x^{\mu} = -\frac{\mathrm{i}}{2}\mathrm{d}\omega^{\rho\sigma}(S_{\rho\sigma})^{\mu}{}_{\nu}\,x^{\nu}. \tag{1.5.5}$$

For every pair of indices ρ, σ, the matrices $S_{\rho\sigma}$ have components

$$(S_{\rho\sigma})^{\mu}{}_{\nu} = -\mathrm{i}(\delta^{\mu}_{\rho}\eta_{\sigma\nu} - \delta^{\mu}_{\sigma}\eta_{\rho\nu})$$

The matrices $S_{\rho\sigma}$ are the *infinitesimal generators of the vector representation of the Lorentz group.* These matrices are intimately related with the total angular momentum of the system.

To alleviate notation we are going to introduce the notation x corresponding to a column vector of four entries with the components of x^{μ}. Using that notation, equation (1.5.5) can be written as

$$\mathrm{d}x^{\mu} = -\frac{\mathrm{i}}{2}\mathrm{d}\omega^{\rho\sigma}(S_{\rho\sigma}\,\mathsf{x})^{\mu}.$$

With this notation at hand we can write the the infinitesimal Lorentz transformation in terms of the infinitesimal parameters $\mathrm{d}\theta^i$ and $\mathrm{d}\xi^i$:

$$\mathrm{d}x^{\mu} = -\mathrm{i}\Big[(\mathrm{d}\omega^{i0}S_{i0} + \frac{1}{2}\mathrm{d}\omega^{ij}S_{ij})\mathsf{x}\Big]^{\mu} = \mathrm{i}\Big[(\mathrm{d}\xi^{i}S_{i0} + \frac{1}{2}\epsilon^{ijk}\mathrm{d}\theta_k S_{ij})\mathsf{x}\Big]^{\mu}$$

if we define the following three-vectors of matrices

$$S^i = \frac{1}{2}\epsilon^{ijk}S_{jk}, \qquad K_i = S_{i0}$$

corresponding to pure rotations and poor boosts respectively, we can write the differential Lorentz transformation in terms of these new infinitesimal generators of rotations and pure boosts as

$$\mathrm{d}x^{\mu} = \mathrm{i}\Big[(\mathrm{d}\xi^{i}K_i + \mathrm{d}\theta_k S^k)\mathsf{x}\Big]^{\mu} = \mathrm{i}\Big[(\mathrm{d}\boldsymbol{\xi}\cdot\boldsymbol{K} + \mathrm{d}\boldsymbol{\theta}\cdot\boldsymbol{S})\mathsf{x}\Big]^{\mu}$$

From the definition of $S_{\mu\nu}$ it is easy to prove that the three vectors of matrices $\boldsymbol{K}$ and $\boldsymbol{S}$ satisfy the following commutation rules

$$[S_i, S_j] = -\mathrm{i}\epsilon_{ijk}S^k, \qquad [S_i, K_j] = -\mathrm{i}\epsilon_{ijk}K^k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}S^k$$

The first one indicates that $\boldsymbol{S}$ commutes like an angular momentum. The second indicates that $\boldsymbol{k}$ is a vector under spatial rotations, and the last one tells us that if we perform a boost, then another one, then we undo the first boost and undo the second, the result is a spatial rotation.

One can actually quickly compute the entries of the $\boldsymbol{S}$ and $\boldsymbol{K}$ matrices:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{1.5.6}$$

Formally integrating equation (1.5.5) over all the free parameters of ω, we obtain the finite version of a (proper orthochronus) Lorentz transformation in terms of its infinitesimal generators

$$\mathsf{x}' = \exp\left(-\frac{\mathrm{i}}{2}\omega^{\rho\sigma}S_{\rho\sigma}\right)\mathsf{x}.$$

Therefore a finite Lorentz transformation can be written as $\mathsf{x}' = \tilde{\Lambda}\mathsf{x}$ where

$$\tilde{\Lambda} = \exp(\mathrm{i}\,\boldsymbol{\theta}\cdot\boldsymbol{S} + \mathrm{i}\,\boldsymbol{\xi}\cdot\boldsymbol{K}), \tag{1.5.7}$$

with the parameters θ^i and ξ_i defined as

$$\theta^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk}, \qquad \xi_i = -\omega_{0i}$$

Since $\boldsymbol{\xi}$ are the boost parameters, they must be functions of the boost speed. From the differnetial relationship (1.5.4) we know that at first order $\mathrm{d}\boldsymbol{\xi} = c^{-1}\mathrm{d}\boldsymbol{v}$, but we have not obtained the value of $\boldsymbol{\xi}$ as a funciton of the boost velocity for finite boosts. To obtain this relationship we apply a finite Lorentz transformation (1.5.7) with $\boldsymbol{\theta} = \boldsymbol{0}$ and we compare it with (1.5.3). This yields

$$\boldsymbol{\xi} = \boldsymbol{n}\,\mathrm{arctanh}\left(\frac{v}{c}\right)$$

where we recall that $\boldsymbol{n} = \boldsymbol{v}/v$ and that $v = |\boldsymbol{v}|$.

As an example, consider the Lorentz transformation $\boldsymbol{\xi} = \xi\,\boldsymbol{e}_1$. In that case it is easy to check that

$$\tilde{\Lambda} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that is, a boost along the x direction is a hyperbolic rotation in the t, x plane, as it can be seen in Fig. 1.1.

Analogously , for $\boldsymbol{\xi} = 0$, $\boldsymbol{\theta} = \theta\,\boldsymbol{e}_3$ we obtain by similar means

$$\tilde{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

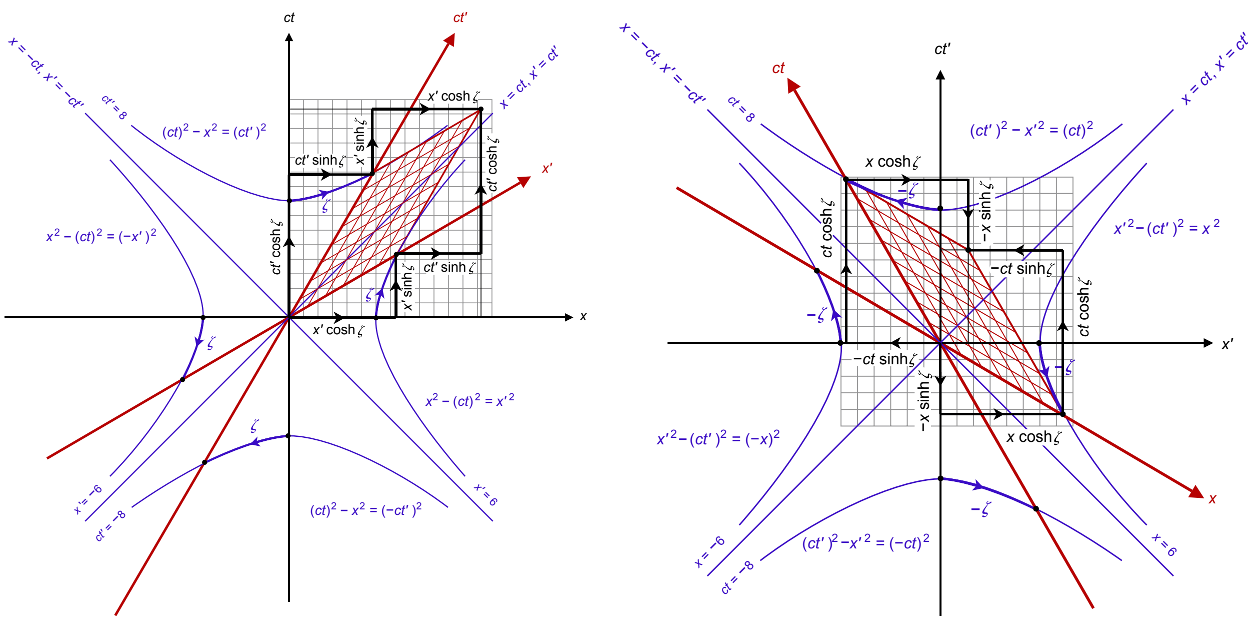

Figure 1.1: Illustration of an example passive Lorentz transformation and its inverse.

which is a rotation around the z axis.

To obtain all this we have made use of the fact that

$$\exp(\mathrm{i}\xi\,\boldsymbol{n}_\xi\!\cdot\!\boldsymbol{K}) = \mathbb{1} + \mathrm{i}\boldsymbol{n}_\xi\!\cdot\!\boldsymbol{K}\sinh\xi - (\boldsymbol{n}_\xi\!\cdot\!\boldsymbol{K})^2(\cosh\xi - 1)$$
$$\exp(\mathrm{i}\theta\,\boldsymbol{n}_\theta\!\cdot\!\boldsymbol{S}) = \mathbb{1} + \mathrm{i}\boldsymbol{n}_\theta\!\cdot\!\boldsymbol{S}\sin\theta + (\boldsymbol{n}_\theta\!\cdot\!\boldsymbol{S})^2(\cos\theta - 1)$$

In the same fashion as we did for the translations group, we can also express Lorentz transformations in terms of differential operators. Let us introduce the following operator

$$\hat{L}_{\mu\nu} = x_\mu\hat{P}_\nu - x_\nu\hat{P}_\mu = -\mathrm{i}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right).$$

We can now write a differential Lorentz transformation as

$$\mathrm{d}x^\mu = \mathrm{d}\omega^\mu{}_\nu x^\nu = -\frac{\mathrm{i}}{2}\mathrm{d}\omega^{\rho\sigma}\hat{L}_{\rho\sigma}x^\mu$$

as can be readily checked by direct calculation.

The differential operators $\hat{L}_{\mu\nu}$ are differential generators of the Lorentz group, and are intimately related with the orbital angular momentum of the system. They satisfy similar commutation rules as the ones seen above. If we define

$$\hat{L}^k = \frac{1}{2}\epsilon^{ijk}\hat{L}_{ij}, \qquad \hat{K}_i = \hat{L}_{i0}$$

we find that the three-vectors $\boldsymbol{L}$ and $\boldsymbol{K}$ satisfy the following commutation rules

$$[\hat{L}_i, \hat{L}_j] = -\mathrm{i}\epsilon_{ijk}\hat{L}^k, \qquad [\hat{L}_i, \hat{K}_j] = -\mathrm{i}\epsilon_{ijk}\hat{K}^k, \qquad [\hat{K}_i, \hat{K}_j] = -\mathrm{i}\epsilon_{ijk}\hat{L}^k$$

In fact, the matrices $\boldsymbol{S}$ adn $\boldsymbol{K}$ are a vector representation of the Lorentz group generated by the differential operators $\hat{L}_i$ and $\hat{K}_i$, which can be seen in the fact that they act on x in the same way:

$$\hat{L}_i x^\mu = (S_i\mathsf{x})^\mu, \qquad \hat{K}_i x^\mu = (K_i\mathsf{x})^\mu$$

The rest of the commutators of the Poincare group are easy to obtain and reflect that $\hat{P}_\mu$ is a vector under Lorentz transformations.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

We consider the stationary action principle: There exists at least a functional (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained by demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of particles can be written in terms of a function L (called Lagrangian) of the spacetime positions x_n (where $n = 1, 2, 3, ...$ are the particle labels), and the 'velocities' $\mathring{\mathsf{x}}_n$, where the circle-dot represnents derivative with respect to some paramter s, and the parameter s itself:

$$S = \int_{s_1}^{s_2} \mathrm{d}s\, L(s, \mathsf{x}_n, \mathring{\mathsf{x}}_n).$$

As usual, the equations of motion are obtained imposing stationary action under infinitesimal changes $\delta\mathsf{x}_n$ so that the variation is zero at the initial and final positions:

$$\delta S = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta \mathring{x}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2} \tag{1.6.1}$$

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} = 0$$

1.6.2 Conserved quantities and Noether's theorem

If the variation of the action around a physical trajectory under a continuous variation of the positions $\delta\mathsf{x}$ is zero, then the quantity

$$\delta Q = \sum_n \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x^\mu$$

is conserved. That is to say

$$\frac{\mathrm{d}(\delta Q)}{\mathrm{d}s} = 0 \tag{1.6.2}$$

This is easy to prove: We can see from (1.6.1) that the variation of the action of a physical trajectory (satisfying Eurler-Lagrange equations) is exactly

$$\delta S = \delta Q \big|_{s_1}^{s_2} = \delta Q(s_2) - \delta Q(s_1)$$

therefore

$$\delta S = 0 \Rightarrow \delta Q(s_2) = \delta Q(s_1) \forall s_1, s_2 \Rightarrow \delta Q \neq \delta Q(s). \quad \square$$

1.6.3 Four-momentum

Let us assume that the action of a single particle does not depend on its position explicitly. Hence the action would be invariant under arbitrary constant displacements. $\delta\mathsf{x} = \mathsf{n}\delta\alpha$ along the spacetime direction n. Then the quantity

$$\delta Q = \frac{\partial L}{\partial \mathring{x}^\mu} n^\mu \delta\alpha$$

is constant, and hence the projection $\mathsf{n}\cdot\mathsf{p} = n^\mu p_\mu$ of the four-momentum p, defined as

$$p_\mu := \frac{\partial L}{\partial \mathring{x}^\mu}, \tag{1.6.3}$$

is conserved: the projection of the four-momentum along one spacetime direction is the conserved quantity associated with the invariance under spacetime translations in that direction.

Notice that for a system of many particles, the calculation is completely analogous adding a sum over all the particles to all the equations above, and thus the conserved quantity would be the total four-momentum P^μ which is the sum of all the individual particles four-momenta $P^\mu = \sum_n p_n^\mu$.

The quantity cP_0 has dimensions of energy, and it is conserved if the action is invariant under constant time translations $\delta x_n^0 = \delta\alpha^0$. In fact, the system's total energy (by definition, the conserved quantity associated to time translational invariance) is

$$E = cP_0.$$

Analogously, $\boldsymbol{P} = (P^1, P^2, P^3)$ has dimenisons of momentum, and it is conserved under constant spatial translations $\delta\boldsymbol{x} = \delta\boldsymbol{\alpha}$. Therefore $\boldsymbol{P}$ is the total three-momentum of the system (by definition, the conserved quantity associated to invariance under translations of the origin of the reference frame).

1.6.4 Angular momentum

If the action of a particle is invariant under an infinitesimal Lorentz transformation (could be a boost and/or a rotation)

$$\delta x^\mu = \delta\omega^\mu{}_\nu x^\nu,$$

the conserved quantity associated with that symmetry is

$$\delta Q = \frac{\partial L}{\partial \mathring{x}^\mu}\delta x^\mu = \frac{\partial L}{\partial \mathring{x}^\mu} x^\nu \delta\omega^\mu{}_\nu = -\frac{1}{2}\delta\omega^{\mu\nu} J_{\mu\nu}$$

where we have defined the four-angular momentum associated to the Lorentz transformations as

$$J_{\mu\nu} := x_\mu p_\nu - x_\nu p_\mu \tag{1.6.4}$$

where p_μ is given by (1.6.3). The tensor $J_{\mu\nu}$ is therefore a conserved quantity.

If we define $\delta\theta^i = -\frac{1}{2}\epsilon^{ijk}\delta\omega_{jk}$ and $\delta\xi_i = \delta\omega_{i0}$ as we did before, then we can write

$$\delta Q = \delta\boldsymbol{\theta}\cdot\boldsymbol{J} + \delta\boldsymbol{\xi}\cdot\boldsymbol{K}$$

where the angular momentum $\boldsymbol{J}$ associated to spatial rotations and the vector $\boldsymbol{K}$ associated to boosts can be extracted directly from $J_{\mu\nu}$:

$$J^i = \frac{1}{2}\epsilon^{ijk} J_{jk}, \qquad K_i = J_{i0}$$

Therefore, if the action is invariant under spatial rotations around a certain axis $\boldsymbol{n}$ parametrized by $\delta\boldsymbol{\theta} = \boldsymbol{n}\delta\theta$ then the component of the three-angular momentum is conserved in that direction ($\boldsymbol{n}\cdot\boldsymbol{J}$). If the action is invariant under boosts $\delta\boldsymbol{\xi} = \boldsymbol{n}\delta\xi$ then the component of the three-vector $\boldsymbol{K}$ is conserved in that direction ($\boldsymbol{n}\cdot\boldsymbol{K}$).

The four-angular momentum is of special relevance in systems with more than one particle. In this case if the action is invariant under Lorentz transformations the total four-angular momentum

$$J^{\mu\nu} = x_n^\mu p_n^\nu - x_n^\nu p_n^\mu$$

is conserved. Notice that what is seen as conservation of angular momentum in a given reference frame, would be in general seen as a combination conservation of the vector $\boldsymbol{K}$ and $\boldsymbol{J}$ in a different frame.

1.6.5 Free particle dynamics

We need to choose an action functional for a relativistic particle.

1. Must be invariant under Lorentz transformations so that it is independent of the reference frame
2. Must coincide with the non-relativistic action in the limit $c \to \infty$

We propose the following action

$$S = mc\int \mathrm{d}s = -mc^2\int \mathrm{d}\tau = -mc^2\int \mathrm{d}t\frac{\mathrm{d}\tau}{\mathrm{d}t} = -mc^2\int\frac{\mathrm{d}t}{\gamma} = -mc^2\int \mathrm{d}t\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}.$$

where we will call the constant m (with units of mass) the particle's *invariant mass* and $(t,\boldsymbol{x})$ is an arbitrary inertial frame. where we have used that

$$\mathrm{d}\tau = \frac{1}{c}\mathrm{d}s = \frac{1}{c}\sqrt{-\eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\mathrm{d}t = \gamma^{-1}\mathrm{d}t. \tag{1.6.5}$$

One can see that this action is

$$S = -mc^2\int \mathrm{d}t\left[1-\frac{\boldsymbol{v}^2}{2c^2}+\mathcal{O}\left(\frac{\boldsymbol{v}^4}{c^4}\right)\right]$$

So in the low-speed limit we reobtain the non-relativistic action (except an irrelevant constant factor) and the invariant mass gets identified with the particle's non-relativistic inertial mass in that limit.

Under this assumption, the Lagrangian of the particle in an arbitrary frame is

$$L = -mc\sqrt{-\eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}} = -mc^2\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}. \tag{1.6.6}$$

Particularizing, in the particle's comoving[1] frame it is $L = -mc^2$. The Euler-Lagrange equations in an arbitrary frame are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\boldsymbol{v}) = 0.$$

[1]Since the particle is free, it is inertial and comoving and proper frame are the same.

The three-momentum is obtained through the variational derivative of the action with respect to the velocity, as usual:

$$\boldsymbol{p} = \frac{\delta S}{\delta \boldsymbol{v}} = \frac{\partial L}{\partial \boldsymbol{v}} = m\gamma \boldsymbol{v},$$

and the Hamiltonian, as usual, via a Legendre transform of the Lagrangian where $\boldsymbol{v} \to \boldsymbol{v}(\boldsymbol{p})$:

$$H = (\boldsymbol{p}\cdot\boldsymbol{v} - L)_{\boldsymbol{v}\to\boldsymbol{v}(\boldsymbol{p})} = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}.$$

We also appreciate that the action is invariant under spacetime translations, and as such, the four-momentum is a conserved quantity. Let us define the *four-velocity* $\dot{x}^\mu$as the derivative of the four-position of the particle with respect to proper time:

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}(ct, \boldsymbol{x}) = \gamma(c, \boldsymbol{v})$$

where when we wrote in terms of compentes with respect to the generic frame $(t, \boldsymbol{x})$ we used that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\boldsymbol{v}.$$

Notice that we can quickly compute the norm of the four-velocity to be

$$\dot{x}^\mu \dot{x}_\mu = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = -c^2$$

where we have used (1.3.4) in the last step. The four velocity of a massive particle is thus the normalized future-directed timelike tangent four-vector to the world line (spacetime trajectory) of the particle.

Knowing that the four-momentum is a four vector, there is an easy way to obtain it. The Lagrangian (1.6.6) in the proper frame of the particle can be written explicitly in terms of the four-velocity from (1.6.6) as

$$L = -mc\sqrt{\dot{x}^\mu\dot{x}_\mu} = -mc\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}. \tag{1.6.7}$$

The four-momentum of the free particle is the variation of the action with respect to the four-velocity:

$$p_\mu = \frac{\delta S}{\delta \dot{x}^\mu} = \frac{\partial L}{\partial \dot{x}^\mu} = m\dot{x}_\mu \Rightarrow p^\mu = m\dot{x}^\mu = m\gamma(c, \boldsymbol{v})$$

where the last implication makes use of the fact that this is indeed a four-vector.

In an arbitrary frame, the time component of the four momentum is proportional to the energy of the particle $cp^0 = E$. This can be seen easily in the particle's proper frame, where $\boldsymbol{v} = \boldsymbol{0}$ and $\gamma = 1$. in that case $p^0 = mc$ and therefore

$$E = mc^2$$

To see that the zero-th component of the four momentum is E/c in any reference frame, we use the frame independent quantity (Lorentz scalar)

$$p^\mu p_\mu = m^2\dot{x}^\mu\dot{x}_\mu = -m^2c^2 \Rightarrow -m^2c^2 = -(p^0)^2 + \boldsymbol{p}^2 \Rightarrow E = cp^0 = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}.$$

which coincides with the Hamiltonian of the particle obtained above. We can also express it in terms of the velocity using that $\boldsymbol{p}^2 = m^2\gamma^2\boldsymbol{v}^2$ and thus

$$E = mc^2\sqrt{1 + \frac{\gamma^2}{c^2}\boldsymbol{v}^2} = mc^2 + \frac{1}{2}m\boldsymbol{v}^2 + \mathcal{O}\left(\frac{\boldsymbol{v}^4}{c^4}\right)$$

which is the total energy: rest energy plus kinetic energy from the point of view of newtonian mechanics in the limit $c \to \infty$.

Notice that if one considers the frame S' of an observer with four-velocity u, the energy of a particle with four-momentum p can be quickly computed as

$$E' = cP'^0 = -u'^\mu P'_\mu = -u^\mu P_\mu \tag{1.6.8}$$

Let us again consider the full expression for the total energy of a particle in a given frame

$$E = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2} \tag{1.6.9}$$

and consider two different limits:

- **Ultrarrelativistic limit:** The kinetic term inside the square root is much larger than the rest energy of the particle: $pc \gg mc^2$, then
$$E \approx pc$$
- **Deep non-relativistic limit:** The rest energy is much larger than the kinetic energy of the particle: $mc^2 \gg pc$, then
$$E \approx mc^2$$

1.7 Accelerated observers and the Rindler metric

1.7.1 Four-acceleration

We define the four-acceleration of a single particle as the derivative of the four-velocity with respect to proper time

$$b^\mu = \ddot{x}^\mu = \frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\tau}$$

Using that in an arbitrary reference frame $(t, \boldsymbol{x})$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\boldsymbol{v}\cdot\boldsymbol{a} \tag{1.7.1}$$

where $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$, we can quickly obtain that

$$b^0 = \frac{\gamma^4}{c}\boldsymbol{v}\cdot\boldsymbol{a}, \qquad \boldsymbol{b} = \frac{\gamma^4}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v} + \gamma^2\boldsymbol{a} \tag{1.7.2}$$

and that

$$b^\mu b_\mu = \gamma^4\left[\frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})^2 + \boldsymbol{a}^2\right] \geq 0$$

therefore b^μ is a spacelike four-vector. In fact, it is easy to prove that the four-acceleration and the four-velocity are orthogonal:

$$b^\mu\dot{x}_\mu = \frac{\mathrm{d}\dot{x}_\mu}{\mathrm{d}\tau}\dot{x}^\mu = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{x}^\mu\dot{x}_\mu) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}c^2 = 0.$$

1.7.2 Constantly accelerated frame: Rindler coordinates

From, for example, (1.7.2), a constantly accelerated observer has a four-acceleration $b^\mu = (0, \boldsymbol{a})$ in its proper reference frame $(\tau, \boldsymbol{\xi})$. The observer has a proper acceleration

$$a = \sqrt{\eta^{\mu\nu} b_\mu b_\nu} = \sqrt{b^\mu b_\mu} = |\boldsymbol{a}|$$

Imagine that we describe the trajectory of the particle from the co-moving frame $(t, \boldsymbol{x})$ at time $t = 0$ where we assume the velocity of the particle is originally $\boldsymbol{v}(0) = \boldsymbol{0}$ in this frame, the relativistic version of Newton's second law gives the equations of motion, which can be written as

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}t} = m b^\mu \Rightarrow m \frac{\mathrm{d}(\gamma \boldsymbol{v})}{\mathrm{d}t} = m\boldsymbol{a} \Rightarrow \boldsymbol{a} = \frac{\mathrm{d}(\gamma \boldsymbol{v})}{\mathrm{d}t}$$

without loss of generality let us consider the acceleration to be in the direction x. We can write the equation of motion in the (initially) co-moving inertial reference frame as

$$a = \frac{\mathrm{d}\gamma}{\mathrm{d}t} v + \frac{\mathrm{d}v}{\mathrm{d}t}\gamma$$

Using (1.7.1) we get that

$$a\,\mathrm{d}t = \gamma\left(\gamma^2 \frac{v^2}{c^2} + 1\right)\mathrm{d}v \Rightarrow a\,\mathrm{d}t = \frac{\mathrm{d}v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

which we can integrate twice using the fact that we imposed that both the velocity and x were zero at $t = 0$. This yields

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \Rightarrow x = \frac{c^2}{a}\left[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1\right] \tag{1.7.3}$$

We can now combine this equation with the relationship between proper time τ and the co-moving time at $t = 0$, t and integrate again:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \gamma^{-1} = \sqrt{1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2} = \sqrt{1 - \frac{1}{c^2}\frac{a^2t^2}{1 + \left(\frac{at}{c}\right)^2}} \Rightarrow \tau = \frac{c}{a}\mathrm{asinh}\left(\frac{at}{c}\right) \Rightarrow t = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

which, using the properties of the hyperbolic functions and substituting in (1.7.3) yields finally

$$x = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], \qquad t = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

Finally, we make the change of frame to the coordinates (τ, ξ) so that $\xi = 0$ is described in the (initially comoving) (t, x) frame by these equations of motion. This yields the following reference change

$$t = \left(\frac{c}{a} + \frac{\xi}{c}\right)\sinh\left(\frac{a\tau}{c}\right), \qquad x = \left(\frac{c^2}{a} + \xi\right)\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a}. \tag{1.7.4}$$

The coordinates (τ, ξ) proper (physically meaningful time intervals and distances) for an accelerated observer are called Rindler coordinates.

Directly from these expressions we see that the domain of these coordinates is

$$-\infty < \tau < \infty, \qquad -\frac{c^2}{a} \leq \xi < \infty$$

And as such, notice that, as shown in Fig. 1.2, a set of Rindler coordinates only covers a wedge of flat spacetime. In particular events for which $x < -\frac{c^2}{a}$ are not represented in the Rindler map.

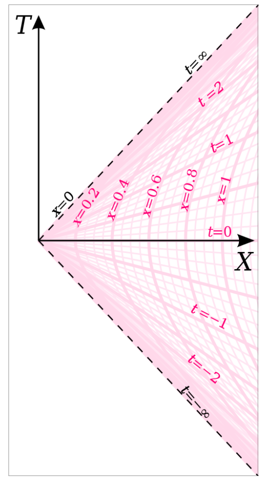

Figure 1.2: Illustration a set of Rindler coordinates normalized in natural units $\left(t = \frac{a\tau}{c}, x = \xi + \frac{c^2}{a}\right)$ as compared with trasnlated Minkowski coordinates $\left(T = ct, X + \frac{c^2}{a}\right)$. These Rindler coordinates only cover the right wedge of spacetime.

1.7.3 Flat spacetime in Rindler coordinates

We can now express the line element $\mathrm{d}s^2$ as a function of the proper coordinates (τ, ξ) of a constantly accelerated observer. Using the transformation (1.7.4), and assuming as above that the direction of motion is the x direction in the comoving frame at rest, we obtain that

$$\mathrm{d}s^2 = -c\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \rightarrow \mathrm{d}s^2 = -\left(1 + \frac{a\xi}{c^2}\right)^2 c^2\,\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

let us focus on the reduced space where the motion is happening. In the (τ, ξ) plane, we can compute the trajectory of rays of light. The equation of motion is easily obtained from the line element when $\mathrm{d}s^2 = 0$. It yields

$$0 = -\left(1 + \frac{a\xi}{c^2}\right)^2 c^2\,\mathrm{d}\tau^2 + \mathrm{d}\xi^2 \Rightarrow \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \left(1 + \frac{a\xi}{c^2}\right) c$$

A first observation is that the speed of light is not c. I will be time dependent and it is only c at the point that the light touches the accelerated frame's origin. In particular it can be much larger than c or much smaller than c. Integrating this with the initial condition that at $\tau = 0$ the pulse of light is at $\xi = 0$ (light emmited by the accelerated observer) we obtain that

$$\xi(\tau) = \frac{c^2}{a}\left[\xi_c \exp\left(\frac{a\tau}{c}\right) - 1\right].$$

This means that the trajectory of rays of light is curved in the accelerated frame's adapted coordinates.