Block 2

Introduction to differential Geometry

2.1 Differentiable manifolds

The concept of differentiable manifold generalizes that of a two-dimensional surface. Let us formalize this generalization.

A topological manifold is a topological space $\mathcal{M}$ Hausdorff (two distinct points can be separated by disjoint open sets) such that every point possesses an open neighbourhood homeomorphic to $\mathbb{R}^n$ (there is a bi-continuous invertible map between them).

A chart is a pair $(\mathcal{U}, \phi)$ where ϕ is a homeomorphism from an open subset $\mathcal{U}$ of $\mathcal{M}$ to an open subset of $\mathbb{R}^n$. The chart is traditionally recorded as the ordered pair $(\mathcal{U}, \phi)$. An infinitely differentiable atlas is a set of continuous charts $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ and transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ such that their union cover the whole manifold $\mathcal{M}$ and all their transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ (mapping open sets of $\mathbb{R}^n$ to open sets of $\mathbb{R}^n$) are infinitely differentiable in $\mathbb{R}^n$. (See Fig. 2.1) Two atlases are equivalent if and only if their union is an atlas.

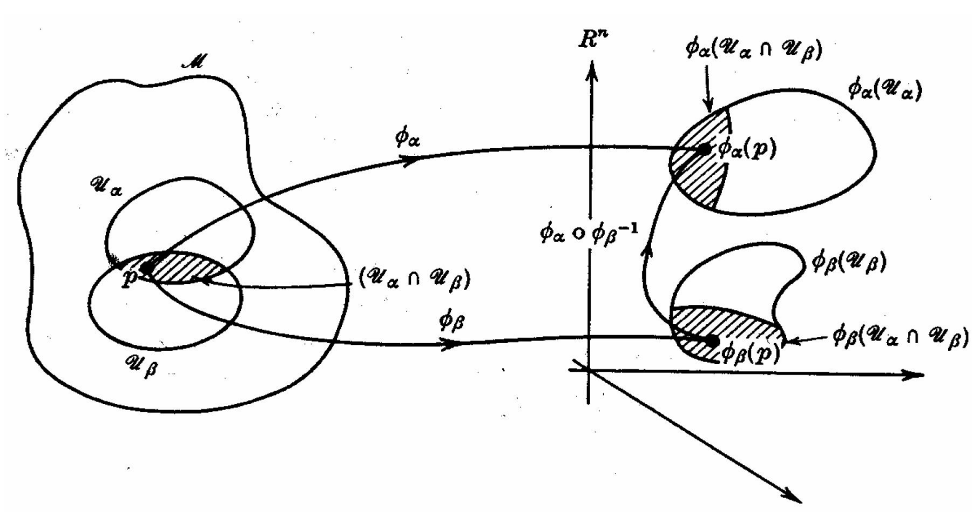

Figure 2.1: Illustration of the condition of compatibility of two charts.

A differentiable manifold (C^∞) is a topological manifold $\mathcal{M}$ together with an infinitely differentiable atlas.

It is obvious that a minimal atlas of $\mathbb{R}^n$ needs only one chart.

Left as Exercise: Show that the smallest atlas for the circumference $\mathbb{S}^1$ must contain two charts.

We will call *coordinates* $x^\mu = (x^1, \dots, x^n)$ of the point $p \in \mathcal{M}$ in a chart $(\mathcal{U}, \phi)$ that contains it to the coordinates of its image $\mathsf{x} = \phi(p)$ in $\mathbb{R}^n$. Because of this, a chart is also commonly called a *local coordinate system.*

A smooth (paracompact) manifold $\mathcal{M}$ is *orientable* if and only if it admits an atlas such that for every pair of overlapping charts $(\mathcal{U}, \phi)$ and $(\mathcal{U}', \phi')$, the Jacobian $\det(\partial x'^\mu / \partial x^\nu)$ is positive. Note that this Jacobian is well defined because the functions $x'^\mu(x^\nu)$ are just another way of writing the C^∞ transition function between overlapping charts as can be seen from the expression

$$\mathsf{x}' = (\phi' \circ \phi^{-1})(\mathsf{x}). \tag{2.1.1}$$

A function $f : \mathcal{M} \to \mathbb{R}$ is smooth if and only if $\bar{f} \coloneqq f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is C^∞.

2.2 Vectors

Given a smooth parametrization $\gamma(s)$ of a smooth curve in $\mathcal{M}$, we define the *tangent vector* to the curve at the point $\gamma(s_0)$ as the operator $\mathsf{v}_{\gamma(s_0)}$ that assigns to each smooth function, $f : \mathcal{M} \to \mathbb{R}$, the number

$$\mathsf{v}|_{\gamma(s_0)}(f) \coloneqq \partial_s (f \circ \gamma)|_{s_0}.$$

In terms of the local coordinates $y^\mu(s)$ of the point $\gamma(s)$, the action of the vector $\mathsf{v}_{\gamma(s_0)}$ on f is written as

$$\mathsf{v}|_{\gamma(s_0)}(f) = \partial_s y^\mu|_{s_0} \partial_\mu \bar{f}|_{y(s_0)}.$$

This means that any vector at the point $p = \gamma(s_0)$ can be written as a linear combination of the vectors $\Upsilon_\mu|_p$ whose action on the functions is just the partial derivative with respect to local coordinates:

$$\Upsilon_\mu|_p(f) = \partial_\mu \bar{f}|_{\phi(p)}.$$

This action on functions is often used to drop the bar from $\bar{f}$ and introduce the notation ∂_μ for the vectors Υ_μ, very common in GR texts, and very convenient in Physics in general.

The set $T_p\mathcal{M}$ of all the vectors v_p at the point $p \in \mathcal{M}$ is a vector space of the same dimension as $\mathcal{M}$. We call $T_p\mathcal{M}$ the *tangent vector space* to $\mathcal{M}$ at the point p. The set of vectors $\{\Upsilon_\mu|_p\}_{\mu=1,2,3,\dots}$ is a *coordinate basis* of $T_p\mathcal{M}$. That is,

$$\mathsf{v}|_p \in T_p\mathcal{M} \Leftrightarrow \mathsf{v}|_p = v^\mu \Upsilon_\mu|_p.$$

To lighten notation, from now on we will not write the $|_p$ unless needed to avoid notational ambiguity.

Notice that a basis of $T_p\mathcal{M}$ need not be coordinate. Through this section of the notes we will use latin indices to notate arbitrary bases $\{\Upsilon_a\}$, not necessarily coordinate bases. We will use greek indices to notate coordinate bases $\{\Upsilon_\alpha\}$, whch in this case represents the coordinate basis associated to the coordinates x^α.

One can prove that the tangent vectors, as operators on functions f satisfy linearity and the Leibniz rule $\mathsf{v}(fg) = f\mathsf{v}(g) + g\mathsf{v}(f)$. Hence, v is a differentiation. We call $\mathsf{v}(f)$ the directional derivative of f.

Two different coordinate bases $\{\Upsilon_\mu\}$ and $\{\Upsilon'_\mu\}$ associated to the local coordinates x^μ and x'^μ respectively are related by the expression

$$\Upsilon'_\mu = \Lambda_\mu{}^\nu \Upsilon_\nu \qquad \Leftrightarrow \qquad \partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu \tag{2.2.1}$$

where

$$\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}, \tag{2.2.2}$$

as it can quickly be seen from their definition. If the basis vectors transform as in (2.2.1), then the components of a vector v in the two different coordinate bases v^μ and v'^μ such that $\mathsf{v} = v^\mu \Upsilon_\mu = v'^\mu \Upsilon'_\mu$ are transformed as

$$v'^\mu = \tilde{\Lambda}^\mu{}_\nu v^\nu \tag{2.2.3}$$

where

$$\tilde{\Lambda}^\mu{}_\nu \coloneqq \left(\Lambda^{-1}\right)_\nu{}^\mu = \frac{\partial x'^\mu}{\partial x^\nu}.$$

Tangent vectors are also called *contravariant vectors* because they transform as basis vectors and their components transform under change of coordinate basis with the (transposed) inverse of the matrix Λ characterizing the change of basis. In fact, the transformation law (2.2.3) can be used as an alternate equivalent definition of tangent vector.

A *vector field* over $\mathcal{M}$ is a set of vectors of the tangent space $T_p\mathcal{M}$ per each point $p \in \mathcal{M}$ such that their components in any coordinate basis are smooth.

Given two vector fields v and w, we can define its composition as the linear map $(\mathsf{v} \circ \mathsf{w})(f) \coloneqq \mathsf{v}[\mathsf{w}(f)]$. However this map does not satisfy the Leibniz rule and therefore it is not a vector field:

$$(\mathsf{v}\circ\mathsf{w})(fg) = \mathsf{v}[\mathsf{w}(fg)] = \mathsf{v}[\mathsf{w}(f)g + f\mathsf{w}(g)] = \mathsf{v}[\mathsf{w}(f)g] + \mathsf{v}[f\mathsf{w}(g)] = \mathsf{v}[\mathsf{w}(f)]g + \mathsf{w}(f)\mathsf{v}(g) + \mathsf{v}(f)\mathsf{w}(g) + f\mathsf{v}[\mathsf{w}(g)],$$

that we can write using the definition of composition above as

$$(\mathsf{v}\circ\mathsf{w})(fg) = f\cdot(\mathsf{v}\circ\mathsf{w})(g) + g\cdot(\mathsf{v}\circ\mathsf{w})(f) + \mathsf{w}(f)\cdot\mathsf{v}(g) + \mathsf{v}(f)\cdot\mathsf{w}(g). \tag{2.2.4}$$

where we have used the symbol $\cdot$ to notate the regular multiplication and remove any possible notational confusion with what is applied to what. The Leibniz rule is not satisfied because of the two last terms.

However, let us draw our attention to the fact that, while the composition of two functions is non commutative, the regular product of the two lat terms is. We can therefore eliminate the last two undesirable terms by defining

$$[\mathsf{v},\mathsf{w}] \coloneqq \mathsf{v}\circ\mathsf{w} - \mathsf{w}\circ\mathsf{v}. \tag{2.2.5}$$

This new vector field is called the *commutator* or the *Lie bracket* of the vector fields v and w. It has the following properties (Prove it as an exercise!).

- It is antisymmetric: $[\mathsf{v},\mathsf{w}] = -[\mathsf{w},\mathsf{v}]$.
- It satisfies the *Jacobi identity*: $[\mathsf{u},[\mathsf{v},\mathsf{w}]] + [\mathsf{w},[\mathsf{u},\mathsf{v}]] + [\mathsf{v},[\mathsf{w},\mathsf{u}]] = 0$.

These two properties ensure that the vector space of vector fields, endowed with the commutator as a bilinear map on it, constitutes a *Lie algebra*.

Exercise: show that in a coordinate basis the components of the commutator of two vector fields are

$$[\mathsf{v},\mathsf{w}]^\mu = v^\nu\partial_\nu w^\mu - w^\nu\partial_\nu v^\mu. \tag{2.2.6}$$

2.3 One-forms

A one-form $\boldsymbol{\omega}$ s a real linear functional over the tangent space $T_p\mathcal{M}$. One-forms act on vectors and spit out a real number:

$$\boldsymbol{\omega} : T_p\mathcal{M} \to \mathbb{R}, \qquad \boldsymbol{\omega} : \mathsf{v} \to \langle \boldsymbol{\omega}, \mathsf{v}\rangle$$

Given an arbitrary basis $\{\Upsilon_a\}$ (where a notates here an arbitrary basis, not necessarily coordinate) of vectors in $p \in \mathcal{M}$, there is a unique set of one-forms $\{\Upsilon^a\}$ such that $\langle \Upsilon^a, \Upsilon_b\rangle = \delta^a_b$. This set is linearly independent and forms a basis (called the *dual basis*) of the vector space $T^*_p\mathcal{M}$ of one-forms at $p \in \mathcal{M}$ called the *dual space* or, also, *cotangent vector space* of $\mathcal{M}$ at p.

Given an arbitrary vector $\mathsf{v} = v^a\Upsilon_a$ and a one-form $\boldsymbol{\omega} = \omega_a\Upsilon^a$, the action of $\boldsymbol{\omega}$ on v is

$$\langle \omega, \mathsf{v}\rangle = \omega_a v^a.$$

We call this operation *contraction*. Note that this shows that the tangent vector space $T_p\mathcal{M}$ also acts linearly on $T^*_p\mathcal{M}$, and therefore it's the dual of the latter. In other words $T^{**}_p\mathcal{M} = T_p\mathcal{M}$.

Each function f over $\mathcal{M}$ defines a one-form $\mathrm{d}f|_p$ at the point p that we call the *differential* of f, which is defined as follows:

$$\langle \mathrm{d}f, \mathsf{v}\rangle = \mathsf{v}(f).$$

One first way in which the differential can be thought of is the generalization of the gradient in $\mathbb{R}^n$: the action of $\mathrm{d}f$ on a vector v gives us the directional derivative of f in the direction of v. Given now the coordinate basis $\{\Upsilon_\mu\}$, associated to the coordinates x^μ, and its dual basis $\{\Upsilon^\mu\}$, we see that

$$\langle \mathrm{d}f, \Upsilon_\mu\rangle = \Upsilon_\mu(f) = \partial_\mu f.$$

From this, we can write the components of the one-form $\mathrm{d}f$ in the dual space coordinate basis as

$$\mathrm{d}f = \partial_\mu f \Upsilon^\mu.$$

For this reason, very often we use $\mathrm{d}x^\mu$ to represent the elements Υ^μ of the coordinate basis of one-forms dual to the coordinate basis of vectors $\{\Upsilon_\mu\}$.

The change of coordinate bases of $T^*_p\mathcal{M}$ are given by the expressions

$$\Upsilon'^\mu = \tilde{\Lambda}^\mu{}_\nu \Upsilon^\nu \qquad \Leftrightarrow \qquad \mathrm{d}x'^\mu = \tilde{\Lambda}^\mu{}_\nu \mathrm{d}x^\nu,$$

such that

$$\langle \Upsilon'^\mu, \Upsilon'_\nu\rangle = \langle \Upsilon^\mu, \Upsilon_\nu\rangle = \delta^\mu_\nu.$$

From this, it is easy to see that the components ω_μ of a one-form $\boldsymbol{\omega} = \omega_\mu\Upsilon^\mu$ transform under change of coordinate basis as

$$\omega'_\mu = \Lambda_\mu{}^\nu \omega_\nu.$$

In view of this transformation law, the one-forms are also called *covariant vectors*, since their components transform under change of coordinate basis exactly as the vectors of the coordinate basis of $T_p\mathcal{M}$ (and they themselves transform with the inverse transpose Jacobian).

2.3.1 Coordinate versus non-coordinate bases

If one considers a set of vector fields that span the tangent space to a manifold at each point (i.e., a basis) it is not always possible to associate local coordinates to it. In other words, not all bases are coordinate bases, not even when restricted to an open set $\mathcal{U} \subset \mathcal{M}$.

To pose the question in a more technical way, what we want to address in this subsection is: given a basis of vector fields $\{\Upsilon_a\}$, can we regard those vectors as tangent to coordinate curves? In other words, is there a chart ϕ such that the vectors Υ_a are tangent to the image under ϕ^{-1} of coordinate curves in $\mathbb{R}^n$? We can answer this with the following theorem:

A set of vector fields $\{\Upsilon_a\}$ is a coordinate basis of vector fields in some open set $\mathcal{U}$ if and only if they mutually commute in $\mathcal{U}$. The proof is involved (for the sufficiency part), so it is not included here.

2.4 Tensors

A tensor T of type (r,s) is a multilinear map that acts on the vector space $(T_p)^r_s\mathcal{M}$, which is defined as the cartesian product of $(T^*_p\mathcal{M})^{\times r}$ and of $(T_p\mathcal{M})^{\times s}$. Tensors of the type $(r,0)$ are called contravariant tensors (their components are r times contravariant), and tensors of the type $(0,s)$ are called covariant tensors (their components are s times covariant). Tensors of type $(0,0)$ are just real numbers (values of functions at p); tensors of type $(1,0)$ are tangent vectors; tensors of type $(0,1)$ are one-forms.

A tensor T is completely characterized by its action on a basis of $(T_p)^r_s\mathcal{M}$:

$$\mathsf{T}(\Upsilon^{a_1},\dots,\Upsilon^{a_r},\Upsilon_{b_1},\dots,\Upsilon_{b_s}) = T^{a_1\dots a_r}{}_{b_1\dots b_s}$$

so that the action of the tensor T on arbitrary one-forms and vectors is given by

$$\mathsf{T}(\boldsymbol{\omega},\boldsymbol{\sigma},\dots,\mathsf{v},\mathsf{w},\dots) = T^{ab\dots}{}_{cd\dots}\,\omega_a\sigma_b\dots v^c w^d\dots.$$

To alleviate notation, the real numbers $T^{ab\dots}{}_{cd\dots}$ are called the components of the tensor T in the basis $\{\Upsilon_a\}$.

Under a change of basis in $T_p\mathcal{M}$ given by $\Upsilon'_a = \Lambda_a{}^b\Upsilon_b$, the components of a tensor transform as tangent vectors per each contravariant index, and as one-forms per each covariant index (as it can be easily proved). For instance

$$T'^{\,ab}_{c} = \Lambda_c{}^d\tilde{\Lambda}^a{}_e\tilde{\Lambda}^b{}_f T_d{}^{ef}$$

Some tensor operations

Tensors form a real vector space with respect to multiplication by a scalar and addition of tensors (component-wise).

- **Symmetrization and Antisymmetrization**: Given a tensor of components $T^{a_1\dots a_r}{}_{c_1\dots c_t b_1\dots b_s}$ we define the symmetrization and antisymmetrization of the s covariant indices respectively as the tensor of components

$$T^{a_1\dots a_r}{}_{c_1\dots c_t(b_1\dots b_s)} := \frac{1}{s!}\sum_\pi T^{a_1\dots a_r}{}_{c_1\dots c_t\pi(b_1)\dots\pi(b_s)}$$

$$T^{a_1\dots a_r}{}_{c_1\dots c_t[b_1\dots b_s]} \coloneqq \frac{1}{s!}\sum_\pi (-1)^\pi T^{a_1\dots a_r}{}_{c_1\dots c_t \pi(b_1)\dots\pi(b_s)}$$

where π represents all the possible permutations $\pi(b_1)\dots\pi(b_s)$ of the indices $b_1\dots b_s$, and $(-1)^\pi$ takes positive values for even permutations and negative values for odd permutations. Analogous definitions apply for the symetrization and antysymmetrization of contravariant indices.

- **Tensor product**: Given two tensors $\mathsf{R} \in (T_p)^r_s\mathcal{M}$ and $\mathsf{T} \in (T_p)^t_q\mathcal{M}$ we define the tensor product $\mathsf{R}\otimes\mathsf{T} \in (T_p)^{r+t}_{s+q}\mathcal{M}$ as the tensor with the components

$$(\mathsf{R}\otimes\mathsf{T})^{a_1\dots a_{r+t}}{}_{b_1\dots b_{s+q}} \coloneqq R^{a_1\dots a_r}{}_{b_1\dots b_s} T^{a_{r+1}\dots a_{r+t}}{}_{b_{s+1}\dots b_{s+q}}$$

- **Contraction**: Given a tensor $\mathsf{T}\in(T_p)^r_s\mathcal{M}$ of components $T^{a_1\dots a_r}{}_{b_1\dots b_s}$, we define the contraction of the first two contravariant and covariant indices as the tensor $\mathsf{R}\in(T_p)^{r-1}_{s-1}\mathcal{M}$ as the tensor of components

$$R^{a_2\dots a_r}{}_{b_2\dots b_s} \coloneqq T^{aa_2\dots a_r}{}_{ab_2\dots b_s}$$

and analogously for any other pair of indices, one covariant and one contravariant.

2.5 Smooth maps and Diffeomorphisms

2.5.1 Smooth maps: pull-back and push-forward

A map $\varphi:\mathcal{M}\to\mathcal{M}'$ is a *smooth map* if and only if given a pair of atlas $\{\mathcal{U}_\alpha,\phi_\alpha\}$ and $\{\mathcal{U}'_\beta,\phi'_\beta\}$ of $\mathcal{M}$ and $\mathcal{M}'$ respectively, the functions $\phi'_\beta\circ\varphi\circ\phi_\alpha^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ are smooth. In other words, if given two sets of local coordinates, the coordinates of the image $p'=\varphi(p)\in\mathcal{M}'$ of a point $p\in\mathcal{M}$ are smooth functions of the coordinates of p.

Notice that the map φ induces a linear map φ^*, which we call *pull-back* between the spaces of functions $\mathcal{F}_{\mathcal{M}'}$ (mapping $\mathcal{M}'$ to $\mathbb{R}$) and $\mathcal{F}_\mathcal{M}$ (mapping $\mathcal{M}$ to $\mathbb{R}$) according to the following rule: given a function $f':\mathcal{M}'\to\mathbb{R}$, its pull-back $\varphi^*f':\mathcal{M}\to\mathbb{R}$ is such that

$$\varphi^*f'(p)\coloneqq f'\circ\varphi(p) = f'(p')$$

Notice that, in general, it is not possible to define an analogous *push-forward* φ_* between the function spaces $\mathcal{F}_{\mathcal{M}'}$ and $\mathcal{F}'_\mathcal{M}$. In fact, given $f:\mathcal{M}\to\mathbb{R}$, the map $\varphi_*f:\mathcal{M}'\to\mathbb{R}$, which would be defined as $\varphi_*f(p')=f\circ\varphi^{-1}(p')$ is not a fuction since we need to use φ^{-1}, which is not guaranteed to exist in general.

However, the map φ does induce a map (which we call *push-forward*) between the tangent spaces $T_p\mathcal{M}$ and $T_p\mathcal{M}'$ according to the following rule: given $\mathsf{v}\in T_p\mathcal{M}$, then $\varphi_*\mathsf{v}$ is a vector of $T_{\varphi(p)}\mathcal{M}'$ such that its action on a function $f'\in\mathcal{F}_{\mathcal{M}'}$ is given by

$$(\varphi_*\mathsf{v})|_{\varphi(p)}(f')\coloneqq \mathsf{v}|_p(\varphi^*f')$$

Same as before, note that we cannot define in general the analogous *pull-back* map on the space of vectors $T_p\mathcal{M}'$ since we would need to use φ^{-1}.

The map φ induces, however, a *pull-back* map between the cotangent spaces $T^*_{\varphi(p)}\mathcal{M}'$ and $T^*_p\mathcal{M}$ according to the following rule: given a one-form $\boldsymbol{\omega}'\in T^*_{\varphi(p)}\mathcal{M}'$, then $\varphi^*\boldsymbol{\omega}'$ is a one-form in $T^*_p\mathcal{M}$ whose action on vectors of $T_p\mathcal{M}$ is given by:

$$\langle\varphi^*\boldsymbol{\omega}',\mathsf{v}\rangle|_p \coloneqq \langle\boldsymbol{\omega}',\varphi_*\mathsf{v}\rangle|_{\varphi(p)}$$

We can quickly prove that the pull-back commutes with the differential, that is, if $f' \in \mathcal{F}'_{\mathcal{M}'}$, then

$$\varphi^*(\mathrm{d}f')|_p = \mathrm{d}(\varphi^* f')|_{\varphi(p)}$$

Indeed:

$$\langle \varphi^*(\mathrm{d}f'), \mathsf{v}\rangle|_p = \langle \mathrm{d}f', \varphi_*\mathsf{v}\rangle|_{\varphi(p)} = (\varphi_*\mathsf{v})|_{\varphi(p)}(f') = \mathsf{v}|_p(\varphi^* f') = \langle \mathrm{d}(\varphi^* f'), \mathsf{v}\rangle|_p$$

From here, t is easy to see that we can define pull-back maps on covariant tensors and push-forward maps on contravariant tensors

2.5.2 Diffeomorphisms

A map $\varphi : \mathcal{M} \to \mathcal{M}'$ is a diffeomorphism if and ony if both φ and its inverse φ^{-1} are smooth bijections.

if φ is a diffeomorphism we can extend the definition of the pull-back φ^*, and push-forward φ_* for any kind of tensors noting that $\varphi_* = (\varphi^{-1})^*$ and that $\varphi_* = (\varphi^{-1})^*$. Also, both maps constitute isomorphisms between the tensor spaces $(T_p)^s_r\mathcal{M}$ and $(T_{\varphi(p)})^s_r\mathcal{M}'$ and as such preserve the type of tensor and all the tensor operations.

Diffeomorphisms between $\mathcal{M}$ and $\mathcal{M}$ (bijective automorphisms) can be interpreted as active coordinate changes in the following sense: given a diffeomorphism φ and a chart $(\mathcal{U}, \phi)$ containing p, and another chart $(\mathcal{U}', \phi')$ containing $p' = \varphi(p)$, the map $\psi = \phi' \circ \varphi$ that assigns p to the coordinates of p', together with the open set $\mathcal{V} = \varphi^{-1}[\varphi(\mathcal{U}) \cap \mathcal{U}']$ is a chart of $\mathcal{M}$ that contains the point p.

Let us focus on the diffeomorphisms mapping from $\mathcal{M}$ to itself. Let us now notate $\gamma_p(s)$ the parametrized curve in $\mathcal{M}$ such that $\gamma_p(0) = p$ and whose tangent vector at every the point $\gamma(s)$ is the vector field k. The vector field k is the generator of a set of local diffeomorphisms (one per sufficiently small value of s) of the form $\varphi_s : \mathcal{U} \to \varphi_s(\mathcal{U})$ such that $\varphi_s(p) = \gamma_p(s)$. This set of local diffeomorphisms is called the *local flow* of k. Since $\varphi_s \circ \varphi_t = \varphi_{s+t}$, local diffeomorphisms constitute a group. The local flow of a vector field k is complete if the associated curve $\gamma_p(s)$ can be extended for all values $s \in \mathbb{R}$.

2.5.3 Lie derivative

As we have already seen, we need the pull-back and push-forward isomorphisms if we want to compare tensors at two different points and hence the value of a tensor field at two different points. Given a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$, it establishes an isomorphism $\varphi^* : (T_{\varphi(p)})^r_s\mathcal{M} \to (T_p)^r_s\mathcal{M}$ between the tensor spaces at $\varphi(p)$ and at p. More explicitly, the tensor at p

$$\left[\varphi^*\big(\mathsf{T}_{\varphi(p)}\big)\right]_p - \mathsf{T}_p \tag{2.5.1}$$

informs us about the difference between the value of a tensor field at $\varphi(p)$ and its value at p (see Fig. 2.2). This is a well-defined quantity because it is a difference of two tensors at the same point p since we have converted $\mathsf{T}_{\varphi(p)}$ to another tensor at p by means of the pull-back. However, this quantity depends on the diffeomorphism φ that we are using to relate both points and, if we consider the local flow φ_t generated by a vector field v, this quantity depends both on v and t.

We define the *Lie derivative* of a tensor field T along the vector field v at the point p through the expression

$$\mathcal{L}_{\mathsf{v}}\mathsf{T}_p \coloneqq \lim_{t\to 0} \frac{\left[\varphi_t^*\big(\mathsf{T}_{\varphi_t(p)}\big)\right]_p - \mathsf{T}_p}{t}. \tag{2.5.2}$$

The Lie derivative has the following properties (Show it!).

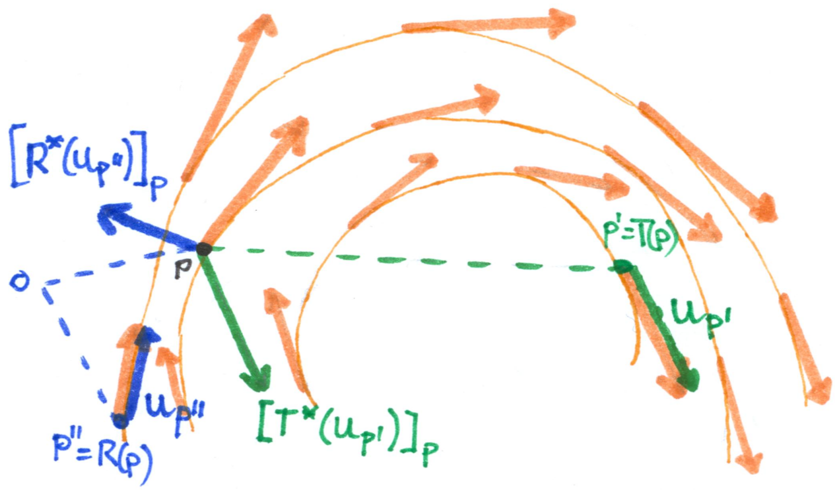

Figure 2.2: As an example, let us consider the diffeomorphisms T and R in $\mathbb{R}^2$ corresponding to a translation by a fix amount (green) and a rotation by a fixed angle around a fixed point o (blue). The figure shows *i)* the values of a vector field (brown) at the images of a point p by these diffeomorphisms, $p' = T(p)$ and $p'' = R(p)$; *ii)* the values of their pull-backs at the original point p.

- It is linear.
- It preserves the type, symmetries, and contraction because φ^* does.
- It satisfies Leibniz rule (and therefore it is a derivation):

$$\mathcal{L}_\mathsf{v}(\mathsf{T}\otimes\mathsf{S}) = \mathcal{L}_\mathsf{v}\mathsf{T}\otimes\mathsf{S} + \mathsf{T}\otimes\mathcal{L}_\mathsf{v}\mathsf{S}. \tag{2.5.3}$$

The Lie derivative of a function f along a vector field v is given by the action of the vector on the function:

$$\mathcal{L}_\mathsf{v} f = \mathsf{v}(f), \tag{2.5.4}$$

as can be directly seen from the definition, since for sufficiently small values of t, $\varphi_t^* f|_p = f[\gamma_p(t)] = f(p) + t\mathsf{v}(f)|_p$.

The Lie derivative of a vector is

$$\mathcal{L}_\mathsf{v}\mathsf{u} = [\mathsf{v},\mathsf{u}] = -\mathcal{L}_\mathsf{u}\mathsf{v}, \tag{2.5.5}$$

i.e., $\mathcal{L}_\mathsf{v}\mathsf{u}$ is a vector such that, acting on functions f, gives:

$$(\mathcal{L}_\mathsf{v}\mathsf{u})(f) = \mathsf{v}[\mathsf{u}(f)] - \mathsf{u}[\mathsf{v}(f)]. \tag{2.5.6}$$

To prove this statement, we can choose a coordinate chart around p such that the vector v generate a flow along the coordinate x^1, i.e. $\mathsf{v} = \Upsilon_1$. Then from the definition of Lie derivative, we see that $\mathcal{L}_\mathsf{v}\mathsf{u} = \partial_1\mathsf{u}$ whose components in the basis Υ_μ are $\partial_1 u^\mu$. On the other hand,

$$\begin{aligned}[\mathsf{v},\mathsf{u}](f) = [\Upsilon_1, u^\mu\Upsilon_\mu](f) = \Upsilon_1(u^\mu\partial_\mu f) - u^\mu\Upsilon_\mu(\partial_1 f) =\\ = \partial_1 u^\mu\partial_\mu f = \partial_1 u^\mu\Upsilon_\mu(f) = (\partial_1\mathsf{u})(f),\end{aligned} \tag{2.5.7}$$

that is, $[\mathsf{v},\mathsf{u}]=\partial_1\mathsf{u}$, which means that both expressions are equal. The components of $\mathcal{L}_\mathsf{v}\mathsf{u}$ in local coordinates are (Left as exercise)

$$(\mathcal{L}_\mathsf{v}\mathsf{u})^\mu = v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu. \tag{2.5.8}$$

The Lie derivative of a one-form $\mathcal{L}_\mathsf{v}\boldsymbol{\omega}$ is such that for any vector u

$$\langle\mathcal{L}_\mathsf{v}\boldsymbol{\omega},\mathsf{u}\rangle = \mathcal{L}_\mathsf{v}\langle\boldsymbol{\omega},\mathsf{u}\rangle - \langle\boldsymbol{\omega},\mathcal{L}_\mathsf{v}\mathsf{u}\rangle. \tag{2.5.9}$$

Its components in local coordinates are (Exercise)

$$(\mathcal{L}_\mathsf{v}\boldsymbol{\omega})_\mu = v^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu v^\nu. \tag{2.5.10}$$

(Exercise: Calculate the components of the Lie derivative of an arbitrary tensor in a coordinate basis.)

Since the Lie derivative of a tensor depends not only on the vector field v at the point p but also in a neighbourhood (it also depends on its derviatives at p), the Lie derivative is not appropriate as a generalisation of the directional derivative (or, equivalently, of the partial derivative). However, the Lie derivative allows us to decide whether a tensor is invariant under diffeomorphisms in some direction, depending on whether its Lie derivative along that direction vanishes or not. In other words, $\mathcal{L}_\mathsf{v}\mathsf{T}=0$ if and only if the tensor T is conserved along the flow generated by v.

2.5.4 Affine connection and covariant derivative

An affine connection ∇ is a rule through which we assign to every tensor field T of type (r,s) (r times contravariant, s times covariant) and components $T^{bc\dots}{}_{de\dots}$ another tensor field $\nabla\mathsf{T}$ of type $(r,s+1)$ and components

$$(\nabla T)^{bc\dots}{}_{de\dots a} \equiv \nabla_a T^{bc\dots}{}_{de\dots} \equiv T^{bc\dots}{}_{de\dots;a}$$

called the *covariant derivative* of T, which satsifies the follwoing properties

- Linearity
- Leibniz rule
- Commutes with the contraction
- Over functions, it is just the differential $\nabla f = \mathrm{d}f$

There is infinitely many different affine connections and as such infinitely ways to build a covariant derivative. We define the directional covariant derivative of T in the direction v as the tensor $\nabla_\mathsf{v}\mathsf{T}$ of type (r,s) whose components are $v^a\nabla_a T^{bc\dots}{}_{de\dots}$.

We can introduce the components of the connection ∇ in an arbitrary basis $\{\Upsilon_a\}$ by characterizing its action on the elements of the basis:

$$\Gamma^a{}_{bc} \coloneqq (\nabla\Upsilon_c)^a{}_b = \langle\Upsilon^a, \nabla_{\Upsilon_c}\Upsilon_b\rangle$$

One can prove (exercise) that under change arbitrary of basis $\Upsilon'_{a'} = \Lambda_{a'}{}^a\Upsilon_a$, $\Upsilon'^{a'} = \tilde{\Lambda}^{a'}{}_a\Upsilon^a$, where $\tilde{\Lambda}^{a'}{}_a = (\Lambda^{-1})_a{}^{a'}$, the components $\Gamma^a{}_{bc}$ of the connection ∇ are transformed as follows:

$$\Gamma^{a'}{}_{b'c'} = \tilde{\Lambda}^{a'}{}_a\Lambda_{b'}{}^b\Lambda_{c'}{}^c\Gamma^a{}_{bc} + \tilde{\Lambda}^{a'}{}_a\Lambda_{c'}{}^c\Upsilon_c(\Lambda_{b'}{}^a).$$

The first term would be the transformation corresponding to a tensor, but the second term makes the connection not transform like a tensor. Notice two that this second term does not depend explicit on the components of the connection in a given basis, but only in the change of basis itself. This means that the difference between two affine connections does transform like a tensor.

As a handy example we could write the components of the covariant derivative in terms of the components of the connection for vectors and one-forms:

$$\nabla_a \omega_b = \Upsilon_a(\omega_b) - \Gamma^c{}_{ba}\omega_c, \qquad \nabla_a v^b = \Upsilon_a(v^b) + \Gamma^b{}_{ca}v^c$$

We will say that an affine connection is symmetric (or torsion-free) if and only if it acts symmetrically on functions f: $\nabla_a\nabla_b f = \nabla_b\nabla_a f$. It can be proved (left as exercise) that if the connection is symmetric then, in an arbitrary basis it has to be satisfied that $2\Gamma^a{}_{[bc]}\Upsilon_a = [\Upsilon_c, \Upsilon_b]$. It can also be proved that this implies that in a coordinate basis, the components of a symmetric connection satisfy that

$$\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}.$$

Curiosity

Notice an interesting feature: let us consider a fixed particular reference coordinate basis $\{\Upsilon_\alpha\}$ (as opposed to a generic basis $\{\Upsilon_a\}$), then we can define the *ordinary covariant derivative* $\eth$ *associated to the particular coordinate basis* $\{\Upsilon_\alpha\}$ as the covariant derivative that given a tensor T, the components of the tensor $\eth\mathsf{T}$ are given by just the conventional partial derivative: $\partial_\alpha T^{\beta\gamma\dots}{}_{\delta\varepsilon\dots}$. Since we are claiming this is a covariant derivative this must mean that $\partial_\alpha T^{\beta\gamma\dots}{}_{\delta\varepsilon\dots}$ is a tensor. What's the trick? To understand better this particular covariant derivative we can focus on the action of this covariant derivative on a vector. Let us write the vectors of an arbitrary base $\{\Upsilon_a\}$ in terms of the concrete basis $\{\Upsilon_a\}$: $\Upsilon_\alpha = \Lambda_\alpha{}^a\Upsilon_a$, and its dual $\Upsilon^\alpha = \tilde{\Lambda}^\alpha{}_a\Upsilon^a$. Then the components of the ordinary covariant derivative associated to the reference basis $\{\Upsilon_\alpha\}$ of a vector v in an arbitrary basis $\{\Upsilon_a\}$, are given by

$$(\eth\mathsf{v})_a{}^b = \eth_a v^b = \Lambda_a{}^\alpha\tilde{\Lambda}^b{}_\beta\partial_\alpha v^\beta.$$

Notice that of course this is not the transformed ordinary partial derivative, which would be $\Lambda_a{}^\alpha\partial_\alpha(\tilde{\Lambda}^b{}_\beta v^\beta)$, which would not be a covariant derivative: The ordinary covariant derivative associated to the reference basis $\{\Upsilon_\alpha\}$ does not correspond to the partial derivative in any other basis other than the one to which it is associated. What this statement is telling us is that we can always find a covariant derivative so that its action on the **components** of a given tensor in a **particular** basis is given by simple partial differentiation. That would not be the case in any other basis.

2.5.5 Parallel transport

Given a parametrization of the curve $\gamma(s)$ whose tangent vector is $\mathsf{w}(s)$, we define the *covariant derivative* of a tensor T *along* $\gamma(s)$ as the directional covariant derivative of T along the direction of the tangent vector $\mathsf{w}(s)$, in other words: $\nabla_{\mathsf{w}}\mathsf{T}$.

We will say that T is parallel-transported along the curve parametrized by $\gamma(s)$ if and only if $\nabla_{\mathsf{w}}\mathsf{T} = 0$. Parallel transport establishes an isomorphism between the tangent tensor spaces defined at each point of $\gamma(s)$.

Geodesics

A curve γ is a *geodesic curve* if and only if it admits a parametrization $\gamma(s)$ whose tangent vector $\mathsf{w}(s) = \partial_s$ is such that $\nabla_{\mathsf{w}}\mathsf{w}$ is parallel (proportional) to w. Writing it explicitly in an arbitrary basis: $w^a \nabla_a w^b \propto w^b$.

If γ is a geodesic curve, it is always possible to find a reparametrization $\gamma(t) := \gamma[s(t)]$ such that its tangent vector $\mathsf{v}(t) = \partial_t$ is parallel-transported along γ. That is $\nabla_{\mathsf{v}}\mathsf{v} = 0$. Such a parameter is called the *affine parameter*, and it is unique modulo multiplication and addition of constants (this can be thought as choice of 'units' and origin of the parameter respectively). The particular parametrization $\gamma(t)$ where t is an affine parameter is called *geodesic*. Given a coordinate system y^μ, the geodesics satisfy the following equations

$$\dot{y}^\nu \nabla_\nu \dot{y}^\mu = 0 \quad \Leftrightarrow \quad \ddot{y}^\mu + \Gamma^\mu{}_{\nu\rho}\,\dot{y}^\nu \dot{y}^\rho = 0,$$

where the dot notates the derivative with respect to the affine parameter s: $\dot{y} := \mathrm{d}y/\mathrm{d}s$.

A geodesic is said to be *complete* if and only if its affine parameter covers the whole real line. A differentiable manifold $\mathcal{M}$ is said to be *geodesically complete* if and only if all its geodesics are complete.

We define the *exponential map* of $p \in \mathcal{M}$ as the smooth map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ such that to each vector v in p it assigns the point $\exp_p(\mathsf{v}) \in \mathcal{M}$ that is at an affine parametric distance of unity from p along the geodesic that contains p and has as tangent vector v. $\mathcal{M}$ is geodesically complete if and only if the exponential map is defined for all vectors of $T_p\mathcal{M}$ for all the points of $\mathcal{M}$.

2.5.6 Curvature: Riemann and Ricci tensors

Let f and ω_c be a scalar function and a one-form respectively. For a symmetric connection, the action of the antisymmetrized double covariant derivative over the product $f\omega_c$ is

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c,$$

that is, $\nabla_a\nabla_b - \nabla_b\nabla_a$ acts linearly on one-forms. This means that $\nabla\nabla' - \nabla'\nabla$ is a tensor $R_{abc}{}^d$ of type $(1,3)$:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c := R_{abc}{}^d\omega_d. \tag{2.5.11}$$

We call this tensor the Riemann tensor.

Since we defined the Riemann tensor from the action of $(\nabla_a\nabla_b - \nabla_b\nabla_a)$ on one-forms, we can now find the action of $(\nabla_a\nabla_b - \nabla_b\nabla_a)$ with a vector v from the action of $(\nabla_a\nabla_b - \nabla_b\nabla_a)$ on the scalar $v^c\omega_c$. This yields:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^d = -R_{abc}{}^d v^c. \tag{2.5.12}$$

Similarly from the properties of the covariant derivative, we can prove the following properties for the Riemann tensor:

- It is antisymmetric in the first two indices: $R_{abc}{}^d = -R_{bac}{}^d$
- Its antisymmetric part with respect to the first three indices is zero: $R_{[abc]}{}^d = 0$
- It satisfies Bianchi's identity: $\nabla_{[e}R_{ab]c}{}^d = 0$

The components of the Riemann tensor in a coordinate basis can be written in terms of the components of the connection in such a basis:

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\sigma}{}_{\lambda\mu}.$$

We define the *Ricci tensor* R_{ac} as the contraction of the second and fourth indices of the Riemann tensor (also called the trace with respect to the second and fourth indices):

$$R_{ac} \coloneqq R_{abc}{}^{b}.$$

2.5.7 Geodesic deviation

In general relativity, geodesic deviation describes the tendency of objects to approach or recede from one another while moving under the influence of a spatially varying gravitational field. Put another way, if two objects are set in motion along two initially parallel trajectories, the presence of a tidal gravitational force will cause the trajectories to bend towards or away from each other, producing a relative acceleration between the objects. The geodesic deviation equation relates the Riemann curvature tensor to the relative acceleration of two neighboring geodesics.

Consider a one-parametric family of geodesics $\gamma_s(t)$. We define it such that for each value of s, γ_s is a geodesic parametrized by the afine parameter t such that the map $(t, s) \in \mathbb{R}^2 \to \gamma_s(t) \in \mathcal{M}$ is a smooth bijection with smooth inverse between $\mathbb{R}^2$ and a submanifold of $\mathcal{M}$ of dimension 2. Then we can use (t, s) as coordinates of the two-dimensional submanifold generated by the curves $\gamma_s(t)$. Let t be the vector field of vectors tangent to the family of geodesics. Then t satisfies the geodesic equation $t^a \nabla_a t^b = 0$. Let us consider a vector $\mathsf{z} = \partial_s$ tangent to the curve γ_s for each constant value of t. This vector can be interpreted graphically as the vector joining two points (corresponding to the same value of t) of two neighbouring geodesics (separated an amount ds). Note that t and z commute since they are vectors of a coordinate basis (t, s), and therefore $t^a \nabla_a z^b = z^a \nabla_a t^b$

The vector $v^a = t^b \nabla_b z^a$ (variation of z in the direction of the vector field t tangent to γ_s) gives us the velocity of separation between two very clsoe geodesics, and $a^a = t^b \nabla_b v^a$ their relative acceleration (the acceleration with which they recede from or close to each other). One can check that the acceleration vector satisfies the geodesic deviation equation:

$$a^a = t^c \nabla_c (t^b \nabla_b z^a) = -R_{cbd}{}^{a} z^b t^c t^d.$$

This equation yields the necessary and sufficient condition so that two geodesic initially parallel, stay parallel (they do not experience relative acceleration). This condition is that the Riemann tensor is zero along the geodesics.

If the Riemann tensor is zero in the whole manifold we will call the connection from which the Riemann tensor comes from a *flat connection.*

The Riemann tensor also determines when parallel transport of a vector is independent of the chosen paths in regions sufficiently small. In other words, it determines when, parallel-transporting a vector v along a closed curve γ sufficiently small, the vector does not change. Let $y^\mu(t)$ such that $y^\mu(0) = y^\mu(1)$ the coordinates of a closed curve γ. Then it is verified that

$$\Delta v^\mu = v^\mu(1) - v^\mu(0) = \frac{1}{2} R_{\sigma\rho\nu}{}^{\mu}|_{\gamma(0)} v^\nu(0) \int_0^1 \mathrm{d}t y^{[\rho} \dot{y}^{\sigma]}.$$

2.6 Pseudo-Riemannian manifolds

2.6.1 Metric tensor

A metric tensor g in a differentiable manifold is a doubly-covariant (type $(0,2)$) symmetric tensor field. Its components in an arbitrary base $\{\Upsilon_a\}$ are $g_{ab} = \mathsf{g}(\Upsilon_a, \Upsilon_b)$. In a coordinate basis, the metric tensor can be written as $\mathsf{g} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. We will also use the notation

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

to represent the metric tensor, where $\mathrm{d}s$ is called the *line element*. That is to say, the role played by the Lorentz covariant tensor $\eta_{\mu\nu}$ will now be played by $g_{\mu\nu}$.

With the introduciton of a metric, we can define the a norm of a vector v as

$$\|\mathsf{v}\| = \sqrt{g_{ab}v^a v^b},$$

and given two vectors of of non-zero norm v^a and w^a, we define the angle α between them through the following expression:

$$\cos\alpha = \frac{g_{ab}v^a w^b}{\sqrt{g_{cd}v^c v^d}\sqrt{g_{ef}w^e w^f}}.$$

Two vectors v and w are orthogonal if and only if $g_{ab}v^a w^b = 0$. Notice that the notion of orthogonality is well defined even for vectors of zero norm.

A metric is degenerate at a point p if and only if there exists some vector of the tangent space to $\mathcal{M}$ that is perpendicular to all the vectors of the tangent manifold. This happens if and only if the matrix of the components of the metric $\mathsf{g} = \{g_{\mu\nu}\}$ is singular in that point in any basis.

If the metric is not degenerate, there is a unique doubly-covariant tensor whose components in an arbitrary base are $g_{\mu\nu}$ and such that $\mathsf{g}^{-1} = \{g^{\mu\nu}\}$ is the inverse matrix of the matrix g. That is: $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$.

The introduction of this metric gives us an additional structure of the differential manifold $\mathcal{M}$ by establishing an isomorphism between the tantgent space $T_p\mathcal{M}$ and cotangent space $T^*_p\mathcal{M}$ of the manifold $\mathcal{M}$ at every point p. This allows us to raize and lower indices in the same way that $\eta_{\mu\nu}$ allowed us to do it for flat spacetime:

$$v_\mu = g_{\mu\nu}v^\nu, \qquad v^\mu = g^{\mu\nu}v_\nu$$

With the difference that, of course, g will not necessarily be represented by a diagonal matrix.

Similarly, given a tensor T_{ab} of type (0,2), we can associate (through the metric) a unique tensor $T^a{}_b = g^{ac}T_{cb}$ of type $(1,1)$ and another one $T_{ab} = g^{ac}g^{bd}T_{cd}$ of type $(2,0)$ that we will consider as different representations of the same object. Similarly g_{ab}, g^{ab} and δ^a_b can be considered as different representations of the metric tensor g.

We call the 'metric signature' the number (counted with multiplicity) of positive, negative and zero eigenvalues of the real symmetric matrix g_{ab} of the metric tensor in any basis. The signature thus classifies the metric up to a choice of basis. The signature is often denoted by an explicit list of signs of eigenvalues such as $(+,-,-,-)$ or $(-,+,+,+)$. We will call a Lorentzian metric in this course a metric with the signature $(-,+,+,+)$. This choice separates the vectors of the tangent space $T_p\mathcal{M}$ in three groups (which define what we call the manifold's causal structure)

- v is timelike if and only if $\|\mathsf{v}\|^2 = g_{\mu\nu}v^\mu v^\nu = v^\mu v_\mu < 0$
- v^μ is spacelike if and only if $\|\mathsf{v}\|^2 = g_{\mu\nu}v^\mu v^\nu = v^\mu v_\mu > 0$
- v^μ is lightlike (or null) if and only if $\|\mathsf{v}\|^2 = g_{\mu\nu}v^\mu v^\nu = v^\mu v_\mu = 0$

In every point $p \in \mathcal{M}$ we call *causal future* of p to all the points in $T_p\mathcal{M}$ that are not spacelike separated from p. If it is possible to continuously associate a causal future to all the points in the manifold we call $\mathcal{M}$ a temporally orientable manifold. If a manifold $\mathcal{M}$ is temporally orientable there exists a smooth vector field t^μ that does not become null in any point and that is timelike in the whole manifold $\mathcal{M}$.

Lastly, we would call a curve lightlike, spacelike or timelike if its tangent vector is respectively lightlike, spacelike or timelike in all the points of the curve.

It can be proved that all non-compact manifolds admit a Lorentzian metric. However it is possible to find compact manifolds that do not admit a Lorentzian metric (such as a sphere, for instance).

2.6.2 The Levi-Civita connection

In a differentiable manifold we can independently define a connection and a metric un related to each other. However, there is a unique torsion-free (symmetric) connection, called the Levi-Civita connection, which is *compatible with the metric*, that is, that the covariant derivative of the metric $\nabla \mathsf{g} = 0$, or in components $\nabla_a g_{bc} = 0$.

This condition implies that in a coordinate basis, the components of the connection must satisfy

$$\Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho} = \partial_\mu g_{\nu\rho}.$$

From this expression we see that $\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} = 2\Gamma_{\rho\mu\nu}$ and therefore we find an explicit expression for the coefficients of the Levi-Civita connection (Christoffel relationships):

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

This connection is the only symmetric solution of the equation $\nabla_\mu g_{\nu\rho} = 0$.

The norm and the angle between two vectors is always preserved under parallel transport with the Levi-Civita connection. As a consequence, geodesics always are timelike, spacelike or null curves.

Revisiting the Riemann tensor

On top of the symmetries that the Riemann tensor already has because of the fact that it is associated to an affine connection, the compatibility with the metric imposes one more symmetry (prove it!): it is antisymmetric with respect with the second pair of (covariant) indices:

$$R_{abcd} = -R_{abdc}.$$

Let us review now all the symmetries that the Riemann tensor associated to the Levi-Civita connection:

- Antisymmetric with respect to the two first indices $R_{abcd} = -R_{bacd}$

- Antisymmetric with respect to the two last indices $R_{abcd} = -R_{abdc}$
- Its antisymmetric part with respect to the first three indices is zero: $R_{[abc]d} = 0$
- This makes the Ricci Tensor symmetric: $R_{ab} = R_{ba}$

Taking into account this symmetries, for a manifold $\mathcal{M}$ of dimension n, it is easy to see that the Riemann tensor has $n^2(n^2-1)/12$ algebraically independent components. If $n \geq 3$ the Ricci tensor has $n(n+1)/2$ algebraically independent components. For $n = 1$ we have that both the Riemann and the Ricci tensors are identically zero and for $n = 2$ the Riemann and the Ricci tensor both have only one independent component, and in $n = 3$ they both have 6 independent components. In particular for $n \leq 3$ The Riemann tensor is completely determined by the Ricci tensor. $n = 4$ is the lowest-dimensional case where knowledge of the Ricci does not fully characterize the Riemann tensor. In particular, for $n = 4$ the Ricci tensor determines half of the components of the Riemann tensor (10), and the other half is independent.

We define the scalar curvature R as the trace (contraction) of the Ricci tensor

$$R \coloneqq R_a^a = g^{ab} R_{ab}.$$

We define the *Weyl tensor* as the tensor C_{abcd} which has eactly the same symmetries of the Riemann, but it is traceless, this is $C^a{}_{bad} = 0$:

$$C_{abcd} = R_{abcd} + \frac{2}{n-2}\Big(g_{a[d}R_{c]b} + g_{b[c}R_{d]a}\Big) + \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b}$$

The Ricci tensor has information about how the volumes of 'boxes' change in the presence of curvature when traveling along curves (we can think of tidal forces), and the Weyl tensor contains information about how the shape of the 'boxes' change in the presence of curvature when traveling along curves. The difference is only relevant for 4 dimensions or more.

Finally, we define the *Einstein Tensor* as

$$G_{ab} \coloneqq R_{ab} - \frac{1}{2} R g_{ab}.$$

G_{ab} is a symmetric tensor, and also (from Bianchi's identity), it has zero divergence $\nabla_a G^a{}_b = 0$.

2.6.3 Geodesics as a variational principle

Given a parametric spacelike curve $\gamma(t)$ parametrized by t and whose tangent vector is v, we define its *proper length* between the points $\gamma(0)$ and $\gamma(1)$ as

$$l = \int_0^1 \mathrm{d}t\, \|v\| = \int_0^1 \mathrm{d}t\, |g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu|^{\frac{1}{2}}$$

where $y^\mu(t)$ are the coordinates of $\gamma(t)$ and $\dot{y}^\mu(t)$ the components of its tangent vector. This length is independent of the parametrization, as it is easy to test. If the curve is light-like, it's length is zero. If the curve is timelike, we would call this integral τ and call it *proper time*.

We can ask which curves make the proper length (or proper time) stationary under small variations of the curve keeping the beginning and the end constant. We can use variation calculus to easily obtain that the condition for a curve to yield a stationary proper length:

$$\dot{y}^\nu \nabla_\nu \dot{y}^\mu = \dot{y}^\mu \frac{\mathrm{d}}{\mathrm{d}t} \log\|\dot{y}\|$$

which is the equation of a geodesic curve in temrs of a non-affine parameter. An adequate reparametrization would transform it in a geodesic parametrization.

In particular, we can easily prove that the geodesics are the curves that make the following action stationary:

$$S = \int_0^1 \mathrm{d}t\, \|v\|^2 = \int_0^1 \mathrm{d}t\, |g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu|$$

2.6.4 Isometries

Proper isometries

A diffeomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is an isometry if and only if it preserves the metric tensor in the whole manifold $\mathcal{M}$. That is, if and only if $\varphi_* \mathsf{g} = \mathsf{g}$. The one-parametric group of diffeomorphisms φ_t generated by the vector field k is a group of isometries if an only if it satisfies the Killing equation

$$\nabla_a k_b + \nabla_b k_a = 0. \tag{2.6.1}$$

A vector field k satisfying (2.6.1) is called *Killing vector field.*

Let v^a be the tangent vector of a geodesic, and k^a a killing vector. Then $k_a v^a$ is constant along the geodesic. Indeed:

$$v^b \nabla_b (k_a v^a) = v^a v^b \nabla_b k_a + k_a v^b \nabla_b v^a = 0.$$

This means that for geodesics, the quantities:

$$Q = k^a v_a$$

are conserved quantities (and we will use them to find the expressions of conserved energies, angular momenta, etc).

For Killing vectors it is satisfied that

$$\nabla_a \nabla_b k_c = R_{cba}{}^d k_d.$$

This means that a differentiable manifold has, at most, $n(n+1)/2$ independent Killing vectors.

Stationary and static manifolds

A Lorentzian manifold is stationary if and only if there exists a timelike Killing vector $\mathsf{k} = \partial_t$. Then the most general stationary metric in a coordinate basis adapted to that Killing vector (has the Killing vector as the timelike element of the basis) $\{\mathsf{k}, \Upsilon^i\}$ with $i = 1, 2, 3, \ldots n-1$ is if the form $g_{\mu\nu}(x^i)$ (does not depend on t explicitly).

A Lorentzian manifold is static if and only if it is stationary and there also exists a spacelike hypersurface orthogonal to the timelike Killing vector $\mathsf{k} = \partial_t$. This is equivalent (Frobenius theorem) to the condition

$$k_{[a}\nabla_b k_{c]} = 0.$$

For static Lorentizan manifolds, in any coordinate system such that t is one of them, the metric is invariant under t-reversal and the metric is such that $g_{0i} = 0$.

Conformal isometries

A diffeomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is a conformal isometry if and only if there exits a function Ω such that $\varphi_* \mathsf{g} = \Omega \mathsf{g}$. The one-parametric group of diffeomorphisms φ_t generated by the vector field k is a group of conformal isometries if an only if it satisfies the conformal Killing equation

$$\nabla_a k_b + \nabla_b k_a = \frac{2}{n}(\nabla_c k^c) g_{ab}. \tag{2.6.2}$$

A vector field k satisfying (2.6.2) is called *conformal Killing vector field.*