Generalizing and Describing the Behavior of Fibonacci-like Sequences

17 Pages

1 Introduction and Rationale

I have always been fascinated by identifying and describing patterns, especially those that appear in numbers. I constantly observe these patterns and attempt to describe a relationship between terms so that I can know exactly what to expect next. One object that especially lights this fire is the fibonacci sequence, which always seemed so simple in construction but elusive in significance. I never knew how it connected to its visualizations such as the golden spiral, golden angle, or buildings made with golden proportions. Nor did I know of any methods to explicitly find an arbitrarily-high index in the sequence, if any. I also have an interest in generalizations or extensions of any mathematical concept or method as I do not like being constrained and like observing the new properties that arise. I wondered what factorials for non-integers would look like, so I defined my own version before I discovered the gamma function. Similarly, I now wonder what the fibonacci sequence would behave like if every subsequent term was the sum of the previous three or more terms instead of the sum of the previous two.

This past summer, my mathematical abilities developed greatly and I learned how to treat the subject almost as a form of art, "playing" with new concepts I learned in an attempt to extract a new observation or property. This has allowed me to better understand what intuitive extensions of mathematical concepts would look like and how to deal with their consequences or results more naturally. I also learned about constant recursive sequences, how the fibonacci sequence is just one kind of these, their relationship to homogeneous differential equations, and methods of determining explicit formulas for second-degree recursive relationships. I hope to further develop my mathematical maturity so that in the future I can make significant contributions to the field, and I believe this investigation is a good place to start.

2 Aim and Methodology

In this investigation, I aim to determine the behavior of generalized m-nacci sequences by finding the ratios they approach, which I will call their '**limiting ratio**' as the index of the sequence approaches infinity and finding explicit formulas for their recursive relationships. I will first investigate these properties of the fibonacci sequence, using algebra to show that its limiting ratio is indeed the golden ratio and methods similar to those used to find solutions of homogeneous differential equations to find Binet's Formula. I will then define m-nacci sequences and try to extend methods to these new sequences. I will use computer code to simplify the process of finding explicit formulas and generating high indexes of these formulas to postulate their limiting ratios. I will then use geometric sum formulas to prove these ratios, and derive the ratio that appears when infinitely many previous terms are added.

3 Relevant Mathematical Background

A few fundamental definitions are given below. (3.1) and (3.2) come from Bozlee (2017), and (3.3) and (3.4) comes from a course in sequences and series I took over the summer before my senior year (Geistfeld, 2022). (3.5) and (3.6) are self-developed.

Definition 3.1 (Linear Homogenous Recurrence Relation) *A linear homogenous recurrence relation (which I will call a **linear recurrence**) of degree m, is a recurrence relation of the form:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m}; c_1, c_2, \ldots, c_m \in \boldsymbol{C}$$

With initial terms:

$$a_1, a_2, \ldots, a_m \in \boldsymbol{C}$$

Definition 3.2 (Characteristic Polynomial) *The characteristic polynomial of a linear recurrence of degree m is a polynomial of the form:*

$$f(\lambda) = \lambda^m - c_1\lambda^{m-1} - c_2\lambda^{m-2} - \cdots - c_{m-1}\lambda - c_m$$

Definition 3.3 (Explicit Form) *The explicit form of a linear recurrence is a function of the form:*

$$a_n = d_1\lambda_1^n + d_2\lambda_2^n + \cdots + d_m\lambda_m^n; d_1, d_2, \ldots, d_m \in \boldsymbol{C}$$

Where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the characteristic polynomial. Further, for all $n \in \boldsymbol{Z}^+$, the explicit form of a linear recurrence returns the linear recurrence at that value.

Definition 3.4 (Ratio) *The ratio between terms of a linear recurrence, a_n, is:*

$$r = \frac{a_{n+1}}{a_n}$$

Note that the ratio can also be used as a successor operator to give the next term in a recurrence

$$r = \frac{a_{n+1}}{a_n} \Rightarrow ra_n = a_{n+1} \Rightarrow r^m a_n = a_{n+m}$$

Definition 3.5 (Limiting Ratio) *The limiting ratio of a linear recurrence, a_n, is the limit of its ratio as its index approaches ∞:*

$$\Phi = \lim_{n\to\infty} r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Definition 3.6 (m-nacci Sequence) *An m-nacci sequence is a linear recurrence of degree m whose coefficients all equal 1:*

$$a_n^{(m)} = a_{n-1} + a_{n-2} + \cdots + a_{n-m}$$

With initial terms:

$$a_0, a_1, \ldots, a_{m-2} = 0; a_{m-1} = 1$$

And limiting ratio denoted as:

$$\Phi^{(m)} = \lim_{n\to\infty} \frac{a_{n+1}^{(m)}}{a_n^{(m)}}$$

3.1 Method for Deriving the Explicit Formula of a Linear Recurrence

The method I learned for deriving the explicit formula of a general linear recurrence, a_n, of degree m with initial terms, $a_1, \ldots, a_m$, is outlined below (Geistfeld, 2022).

1. Determine the characteristic polynomial of a_n, $f(\lambda)$
2. Determine the roots of $f(\lambda)$: $\lambda_1, \lambda_2, \ldots, \lambda_m$
3. Use the roots of $f(\lambda)$ as the bases of the explicit form of a_n

$$a_n = d_1\lambda_1^n + d_2\lambda_2^n + \cdots + d_m\lambda_m^n$$

4. Use the initial terms of a_n to setup a system of equations

$$
\begin{aligned}
a_1 &= d_1\lambda_1^1 + d_2\lambda_2^1 + \cdots + d_m\lambda_m^1 \\
a_2 &= d_1\lambda_1^2 + d_2\lambda_2^2 + \cdots + d_m\lambda_m^2 \\
&\vdots \\
a_m &= d_1\lambda_1^m + d_2\lambda_2^m + \cdots + d_m\lambda_m^m
\end{aligned}
$$

5. Solve the system for the coefficients $d_1, d_2, \ldots, d_m$ and use them in the explicit form

3.2 Method for Deriving the Limiting Ratio of a Linear Recurrence

The method described below is one I independently developed and after going through some of the Sequences and Series course and the course instructor supported. It aims to determine the Limiting Ratio of a linear recurrence and is outlined below.

1. Express a_n in terms of its lowest-index term and powers of its ratio

$$
\begin{aligned}
a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m} \\
a_{n+m} &= c_1 a_{n+m-1} + c_2 a_{n+m-2} + \cdots + c_m a_n \\
r^m a_n &= c_1 r^{m-1} a_n + c_2 r^{m-2} a_{n-2} + \cdots + c_m a_n
\end{aligned}
$$

2. Factor out a_n and set the expression equal to 0

$$
\begin{aligned}
r^m &= c_1 r^{m-1} + c_2 r^{m-2} + \cdots + c_m \\
r^m - c_1 r^{m-1} - c_2 r^{m-2} - \cdots - c_m &= 0
\end{aligned}
$$

Note how this is analagous to the characteristic polynomial of a_n, $f(\lambda)$

3. Solve for the roots, r (this is the same as asking to solve for the roots of the characteristic polynomial),

4. The root that is the limiting ratio will depend on the nature of the sequence's initial terms and coefficients. Since the objects of study, m-nacci sequences, only have non-negative and real terms, the root that is the limiting ratio will be positive and real.

Theorem 3.7 *The characteristic polynomial of an m-nacci sequence will always only have one positive, real root.*

Proof: We use Descartes' Rule of signs to confirm that there will only ever be one positive, real root for the characteristic polynomial. Decartes' rule of signs states that the maximum number of positive real roots of a polynomial is the number of sign changes, n, in its terms when counting from the highest to lowest (or lowest to highest) degree. Furthermore, it states the number of allowable roots is the number of sign changes, n, minus some even number. By definition, characteristic polynomials of m-nacci sequences always have only one sign change in their terms. So, according to the rule of signs, there is at most one positive, real root of the characteristic polynomial of an m-nacci sequence. Furthermore, subtracting an even number from 1 greater than 0 would not make sense as there cannot be -1 positive roots to a polynomial, so there is only 1 positive, real root to a characteristic polynomial of the form above. So, the limiting ratio of an m-nacci sequence is the only positive, real root of its characteristic polynomial. Q.E.D.

4 Describing the Behavior of the Fibonacci Sequence

As an example, we can determine the explicit form and limiting ratio of the Fibonacci Sequence, the second-degree m-nacci sequence, or the 2-nacci sequence:

Definition 4.1 (Fibonacci Sequence) *The second degree m-nacci sequence:*

$$a_n^{(2)} = F_n = F_{n-1} + F_{n-2}; F_0 = 0, F_1 = 1$$

4.1 Explicit form of the Fibonacci Sequence

Using the method outlined in section 3.1, we find the characteristic polynomial of Definition 4.1:

$$\lambda^2 - \lambda - 1$$

With roots:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \Rightarrow$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Which gives the following explicit form

$$F_n = d_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + d_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

This gives the following system:

$$F_0 = 0 = d_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + d_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = d_1 + d_2$$

$$F_1 = 1 = d_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + d_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = d_1 \frac{1+\sqrt{5}}{2} + d_2 \frac{1-\sqrt{5}}{2}$$

Solving gives

$$0 = d_1 + d_2 \Rightarrow d_1 = -d_2$$

$$1 = d_1 \frac{1+\sqrt{5}}{2} + d_2 \frac{1-\sqrt{5}}{2} = -d_2 \frac{1+\sqrt{5}}{2} + d_2 \frac{1-\sqrt{5}}{2} = d_2 \left(\frac{-1-\sqrt{5}+1-\sqrt{5}}{2}\right) = d_2 \left(\frac{-2\sqrt{5}}{2}\right)$$

$$1 = -\sqrt{5} d_2 \Rightarrow d_2 = -\frac{1}{\sqrt{5}} \Rightarrow d_1 = \frac{1}{\sqrt{5}}$$

Thus the explicit form of the Fibonacci Sequence is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Which is indeed Binet's formula for the Fibonacci Sequence (Art of Problem Solving, n.d.).

4.2 Limiting Ratio of the Fibonacci Sequence

The positive root of the Fibonacci sequence's characteristic polynomial is $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.62$, which is indeed the Golden Ratio. This root has a greater magnitude than the other, $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$, and thus becomes dominant as the index approaches infinity. Furthermore, the second root's magnitude is less than 1, so it and the term assoaciated with it, $-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$, approach 0 as the index approaches infinity, which gives λ_1 and its term, $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$, more influence. THerefore, the limiting ratio of the fibonacci sequence is:

$$\Phi^{(2)} = \lambda_1 = \frac{1+\sqrt{2}}{5} \approx 1.62$$

5 Extending to the Tribonacci Sequence

We can use the same methods to find the explicit form and limiting ratio of the tribonacci sequence, the third-degree m-nacci sequence, or the 3-nacci sequence:

$$a_n^{(3)} = T_n = T_{n-1} + T_{n-2} + T_{n-3}; T_0 = T_1 = 0, T_2 = 1$$

With characteirstic polynomial:

$$\lambda^3 - \lambda^2 - \lambda - 1$$

and roots

$$\lambda_1 \approx 1.84, \lambda_2 \approx -0.420 + 0.606i, \lambda_3 \approx -0.420 - 0.606i$$

Which gives the explicit form

$$T_n \approx_1 (1.84)^n + d_2(-0.420 + 0.606i)^n + d_3(-0.420 - 0.606i)^n$$

And the following system:

$$T_0 = 0d_1(1.84)^0 + d_2(-0.420 + 0.606i)^0 + d_3(-0.420 - 0.606i)^0$$
$$T_1 = 0 = d_1(1.84)^1 + d_2(-0.420 + 0.606i)^1 + d_3(-0.420 - 0.606i)^1$$
$$T_2 = 1 = d_1(1.84)^2 + d_2(-0.420 + 0.606i)^2 + d_3(-0.420 - 0.606i)^2$$

Using a python program I wrote to solve systems of equations with complex entries, I obtained the following values for d_1, d_2, d_3:

$$d_1 \approx 0.0994, d_2 \approx 0.45 - 0.161i, d_3 \approx 0.45 + 0.161i$$

Thus, the explicit formula for the Tribonacci Sequence is:

$$T_n \approx (0.0994)(1.84)^n + (0.45 - 0.161i)(-0.42 + 0.606i)^n + (0.45 + 0.161i)(-0.42 - 0.606i)^n$$

And its limiting ratio is

$$\Phi_3 = \lambda_1 \approx 1.84$$

To observe the behavior of each term as the index approaches infinity, we can find the magnitude of each root of the characteristic polynomial gives:

$$\begin{aligned}
|\lambda_1| &\approx |1.84| = 1.84 \\
|\lambda_2| &\approx |-0.420 + 0.606i| = \sqrt{0.420^2 + 0.606^2} \approx 0.737 \\
|\lambda_3| &\approx |-0.420 - 0.606i| = \sqrt{0.420^2 + 0.606^2} \approx 0.737
\end{aligned}$$

Since the second and third roots, λ_2 and λ_3, have a magnitude less than 1, their magnitude approaches 0 as the index approaches infinity, which allows the limiting ratio, λ_1 to 'take over' at higher indices.

5.1 Proving the output will always be real

Initially, the explicit form for the tribonacci sequence I derived concerned me as I was aware complex numbers could appear as the roots of a polynomial, but I did not know they would be so prevalent. I was concerned their presence would introduce complex numbers in the sequence the explicit form generates, which would not agree with the tribonacci sequence, but I realized the two terms that contain complex numbers contain only conjugates (both the coefficient and exponential base in the second term are conjugates of their counterparts in the third). Thus we can show that the sum of conjugate complex numbers returns a real, and that conjugation is distributive across complex numbers, to show that my explicit form will always return a real number.

Theorem 5.1 *The sum of a complex number, $z = a + bi$, is a real number.*

$$z + \bar{z} = 2a$$

To prove (5.1), we break up the complex number and its conjugate into their complex and real parts.

$$z + \bar{z} = a + bi + a - bi = (a + a) + (bi - bi) = 2a + 0 = 2a$$

We have shown the two sides of the equation are equal and therefore proved (5.1), QED. Since complex conjugates sum to a real number, we must now show that conjugation is distributive.

Theorem 5.2 *Complex conjugation is distributive over multiplication for any number of complex numbers.*

$$\overline{z_1} \cdot \overline{z_2} \cdot \dots \cdot \overline{z_n} = \overline{z_1 \cdot z_2 \cdot \dots \cdot z_n}$$

To prove Theorem 5.1, induction is used. First, we show our proposition is true for $n = 2$. Let z_1 and z_2 be complex numbers:

$$z_1 = a + bi, z_2 = c + di, a, b, c, d \in \mathbf{R}$$

We want to show that

$$\overline{z_1} \cdot \overline{z_2} = \overline{z_1 \cdot z_2}$$

Performing the left operation gives

$$\overline{z_1} \cdot \overline{z_2} = (a - bi)(c - di) = ac - adi - bci - bd = (ac - bd) - (ad + bc)i$$

Perfoming the right operation gives:

$$\overline{z_1 \cdot z_2} = \overline{(a + bi)(c + di)} = \overline{ac + adi + bci - bd} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i$$

Thus, the proposition holds true for $n = 2$ Now, we assume the proposition is true for all $n = k$

$$\overline{z_1} \cdot \overline{z_2} \cdot \dots \cdot \overline{z_k} = \overline{z_1 \cdot z_2 \cdot \dots \cdot z_k}$$

And prove it is true for all $n = k + 1$

$$\overline{z_1} \cdot \overline{z_2} \cdot \cdots \cdot \overline{z_k} \cdot \overline{z_{k+1}} = \overline{z_1 \cdot z_2 \cdot \cdots \cdot z_k \cdot z_{k+1}}$$

Let $z_1 \cdot z_2 \cdot \cdots \cdot z_k$ be denoted as ζ_1 and z_{k+1} be denoted as ζ_2. Let ζ_1 and ζ_2 be complex numbers:

$$\zeta_1 = a + bi, \zeta_2 = c + di, a, b, c, d \in \mathbf{R}$$

So, we want to prove

$$\overline{\zeta_1} \cdot \overline{\zeta_2} = \overline{\zeta_1 \cdot \zeta_2}$$

Performing the left operation gives

$$\overline{\zeta_1} \cdot \overline{\zeta_2} = (a - bi)(c - di) = ac - adi - bci - bd = (ac - bd) - (ad + bc)i$$

Perfoming the right operation gives:

$$\overline{\zeta_1 \cdot \zeta_2} = \overline{(a + bi)(c + di)} = \overline{ac + adi + bci - bd} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i$$

Therefore, the proposition holds for all $n = k + 1$, and, combining with the base case, we have proved (5.2), QED.

By (5.2), we know the sets of conjugate complex numbers present in the second and third terms of the explicit form for the tribonacci sequence will multiply to give conjugate complex numbers themselves and, by (5.1) will sum to a real number. In fact, since complex roots of a polynmial always come in conjugate pairs, we know their associated pairs of terms in the explicit forms of higher-degree m-nacci sequences will also sum to a real number. Therefore, complex outputs are not a risk in my method of explicit forms for m-nacci sequences.

6 Applying to Higher-Degree Sequences

I wrote a python script that finds an explicit form for any m-nacci sequence and ran it with a few values above those we have seen so far. The 4-nacci and 5-nacci sequences are shown below.

$$a_n^{(4)} = (0.0410) * (1.93)^n + (0.272 - 0.174i) * (-0.0764 + 0.815i)^n +$$
$$(0.272 + 0.174i) * (-0.0764 - 0.815i)^n + (0.415) * (-0.775)^n$$
$$a_n^{(5)} = (0.0183) * (1.97)^n + (0.171 - 0.152i) * (0.195 + 0.849i)^n +$$
$$(0.171 + 0.152i) * (0.195 - 0.849i)^n + (0.32 - 0.0794i) * (-0.678 + 0.459i)^n +$$
$$(0.32 + 0.0794i) * (-0.679 - 0.459i)^n$$

To verify they work for at least low indices, the first 12 terms in each of the 4-nacci and 5-nacci sequences were calculated by hand along with the above formulas, and are shown below.

$$a_n^{(4)} = 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108$$
$$a_n^{(5)} = 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61$$

Comparison betwen both lists of values for each sequence showed no discrepancies, which verifies their efficacy at least for low indices. It is also interesting to note that the first few non-zero indices of each sequence list the powers of two, before veering off in a different direction. This indicates a relationship between m-nacci sequences and the powers of 2.

By inspection, the limiting ratio of the 4-nacci sequence is $\Phi_4 \approx 1.93$, and the limiting ratio of the 5-nacci sequence is $\Phi_5 \approx 1.97$ as these are the positive, real roots in their explicit formulas. The limiting ratios for m-nacci sequences of degree 2-8 are summarized below:

Table 1: The Limiting Ratio of m-nacci Sequences of Degree 2 through 8

m	2	3	4	5	6	7	8
Φ_m	1.618	1.839	1.928	1.966	1.984	1.991	1.996

The table suggests that the limiting ratio of an m-nacci sequence approaches 2 as the degree approaches infinity, which is the relationship the initial terms shown above hint to.

Theorem 6.1 *The limiting ratio of an m-nacci sequence as m approaches infinity is 2.*

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{a_{n+1}^{(m)}}{a_n^{(m)}} = \lim_{m\to\infty}\Phi^{(m)} = 2$$

Proof: To start, we recall the defition for the ratio between two terms in an m-nacci sequence

$$r = \frac{a_{n+1}^{(m)}}{a_n^{(m)}}$$

We note that because of the ratio can be used as a successor operator, converting the n-th term to the $(n+1)$-th term:

$$r = \frac{a_{n+1}^{(m)}}{a_n^{(m)}} \Rightarrow r a_n^{(m)} = a_{n+1}^{(m)}$$

Furthermore, we can use it to do the opposite, converting the n-th term to the $(n+1)$-th term:

$$r = \frac{a_{n+1}^{(m)}}{a_n^{(m)}} \Rightarrow r = \frac{a_n^{(m)}}{a_{n-1}^{(m)}} \Rightarrow \frac{a_n^{(m)}}{r} = a_{n-1}^{(m)}$$

With this, we can rewrite an m-nacci sequence as:

$$a_n^{(m)} = a_{n-1}^{(m)} + a_{n-2}^{(m)} + \cdots + a_{n-m}^{(m)} \Rightarrow$$
$$a_n^{(m)} = \frac{a_n^{(m)}}{r} + \frac{a_n^{(m)}}{r^2} + \cdots + \frac{a_n^{(m)}}{r^m}$$

Which I call the reciprocal form of an m-nacci sequence. Furthermore, we divide both sides by the place-holder term, $a_n^{(m)}$ to obtain:

$$1 = \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^m}$$

Which I call the stripped reciprocal form of an m-nacci sequence. Note that the stripped reciprocal form of an m-nacci sequence is another way of writing its characteristic equation:

$$
\begin{aligned}
1 &= \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^m} \Rightarrow \\
r^m &= r^{m-1} + r^{m-2} + \cdots + r + 1 \Rightarrow \\
r^m &- r^{m-1} - r^{m-2} - \cdots - r - 1 = 0
\end{aligned}
$$

For the reciprocal form of an m-nacci sequence, taking the limit of the index as it approaches infinity then turns the ratios into the limiting ratio that corresponds with the degree of the sequence:

$$
\begin{aligned}
\lim_{n\to\infty} a_n^{(m)} &= \frac{a_n^{(m)}}{r} + \frac{a_n^{(m)}}{r^2} + \cdots + \frac{a_n^{(m)}}{r^m} \Rightarrow \\
a_n^{(m)} &= \frac{a_n^{(m)}}{\Phi^{(m)}} + \frac{a_n^{(m)}}{(\Phi^{(m)})^2} + \cdots + \frac{a_n^{(m)}}{(\Phi^{(m)})^m}
\end{aligned}
$$

The limiting stripped reciprocal form is thus:

$$
\begin{aligned}
a_n^{(m)} &= \frac{a_n^{(m)}}{\Phi^{(m)}} + \frac{a_n^{(m)}}{(\Phi^{(m)})^2} + \cdots + \frac{a_n^{(m)}}{(\Phi^{(m)})^m} \Rightarrow \\
1 &= \frac{1}{\Phi^{(m)}} + \frac{1}{(\Phi^{(m)})^2} + \cdots + \frac{1}{(\Phi^{(m)})^m}
\end{aligned}
$$

Representing the Fibonacci sequence in its limiting stripped reciprocal form gives:

$$
\begin{aligned}
a_n^{(2)} &= a_{n-1}^{(2)} + a_{n-2}^{(2)} \Rightarrow a_n^{(2)} = \frac{a_n^{(2)}}{r} + \frac{a_n^{(2)}}{r^2} \Rightarrow \\
\lim_{n\to\infty} a_n^{(2)} &= \frac{a_n^{(2)}}{r} + \frac{a_n^{(2)}}{r^2} \Rightarrow a_n^{(2)} = \frac{a_n^{(2)}}{\Phi^{(2)}} + \frac{a_n^{(2)}}{(\Phi^{(2)})^2} \Rightarrow \\
1 &= \frac{1}{\Phi^{(2)}} + \frac{1}{(\Phi^{(2)})^2}
\end{aligned}
$$

Note that this represents a geometric sum with initial term $\frac{1}{(\Phi^{(2)})^2}$, common ratio $\Phi^{(2)}$, and sum 1. Using the geomtric sum formula gives:

$$
\begin{aligned}
S_n &= \frac{u_1(r^n - 1)}{r - 1} \\
S_2 = 1 &= \frac{\frac{1}{(\Phi^{(2)})^2}((\Phi^{(2)})^2 - 1)}{\Phi^{(2)} - 1}
\end{aligned}
$$

Solving for $\Phi^{(2)}$ gives:

$$\begin{aligned} 1 &= \frac{\frac{1}{(\Phi^{(2)})^2}((\Phi^{(2)})^2 - 1)}{\Phi^{(2)} - 1} = \frac{(1 - \frac{1}{(\Phi^{(2)})^2})}{\Phi^{(2)} - 1} \Rightarrow \\ \Phi^{(2)} - 1 &= 1 - \frac{1}{(\Phi^{(2)})^2} \Rightarrow \\ \Phi^{(2)} + \frac{1}{(\Phi^{(2)})^2} &= 2 \end{aligned}$$

The same process of using the limiting stripped reciprocal form of an m-nacci to solve for $\Phi^{(m)}$ can be applied to any degree sequence, indeed:

$$\begin{aligned} 1 &= \frac{1}{\Phi^{(m)}} + \frac{1}{(\Phi^{(m)})^2} + \cdots + \frac{1}{(\Phi^{(m)})^m} \Rightarrow \\ 1 &= \frac{\frac{1}{(\Phi^{(m)})^m}((\Phi^{(m)})^m - 1)}{\Phi^{(m)} - 1} = \frac{(1 - \frac{1}{(\Phi^{(m)})^m})}{\Phi^{(m)} - 1} \Rightarrow \\ \Phi^{(m)} - 1 &= 1 - \frac{1}{(\Phi^{(m)})^m} \Rightarrow \\ \Phi^{(m)} + \frac{1}{(\Phi^{(m)})^m} &= 2 \end{aligned}$$

The solution to which for any m will give the limiting ratio of the m-nacci sequence of that degree. Since we know $\Phi^{(m)}$ will always be positive and real, taking the limit:

$$\lim_{m\to\infty} \Phi^{(m)} + \frac{1}{(\Phi^{(m)})^m} = 2$$

Gives:

$$\lim_{m\to\infty} \Phi^{(m)} + \frac{1}{(\Phi^{(m)})^m} = 2 \Rightarrow$$
$$\Phi^{(\infty)} + 0 = 2 \Rightarrow \Phi^{(\infty)} = 2$$

Which proves (6.1), Q.E.D

Which makes sense as it continues the pattern of powers of 2 being present in the first few terms of the 4-nacci and 5-nacci sequences in the beginning of section 6.

7 Conclusion and Evaluation

I have defined a generalized version of the Fibonacci Sequence for which each successive term is the sum of a given number of previous terms and shown a fairly robust method of finding explicit formulas for these sequences. I have also defined the characteristic equation for linear recurrences and shown that, for m-nacci sequences, the ratio that terms approach is the only positive, real root of its characteristic polynomial. These ratios can also be found for any given m-nacci sequence by solving the following equation for $\Phi^{(m)}$:

$$\Phi^{(m)} + \frac{1}{(\Phi^{(m)})^m} = 2$$

This expression also helps prove that the limiting ratio as the degree of the m-nacci sequence approaches infinity approaches 2.

Unfortunately, there is little rigorous evidence to prove the efficacy of the methods I have developed, as they are for the most part driven by intuition. Furthermore, there is no definitive proof of the existance and uniqueness of the limiting ratio of the generalized m-nacci sequences. The lack of proof for both the eficacy of my methods and existence of the limiting ratios does imply that my final conclusion is true, only if the afforementioned aspects are true.

8 Avenues for Future Research

I have determined two main paths I can take with my future research relating to this paper, the first of which is making improvements to my existing methods. For example, I could attempt to show that the only root of the characteristic polynomial for an m-nacci sequence that has a magnitude greater than 1 is its positive real root, and that all other roots have a magnitude less than 1. This would prove definitively that, for all degrees, the positive, real root of the characteristic polynomial for an m-nacci sequence will become dominant over the other roots and thus become the limiting ratio. Another improvement I could make is to attempt to find a way to eliminate the need for complex numbers in the explicit formulas of linear recurrences whose characterisitc equations have complex roots, similar to the use of Euler's formula to deal with the existence of complex roots of the characteristice equation for linear homogenous differencial equations. Finding a method to represent the explicit form of a linear recurrence with only real numbers would also make it easier to prove more rigorously the existance and uniqueness of the limiting ratio of m-nacci sequences and linear recurrences in general, and an improved method of finding them, such as how de Spindel (n.d.) does so for the general second degree linear recurrence:

$$a_n = pa_{n-1} + qa_{n-2}; p, q \in \mathbf{R}^+$$

The other path is to apply methods of discrete calculus to linear recurrences, such as the finite difference, to observe what properties arise, perhaps finding more analogs between linear recurrences and linear homogenous differential equations.

9 Reflection

Overall, the process of this papaer taught me much about the nature and process of more rigorous mathematical research. I learned how to become more comfortable developing my own definitions and exploring them to see what properties arise, such as the fact that the the characteristic polynomials of the m-nacci sequences I defined will always only have on positive, real root. I also learned how to take advantage of the properties of my own definitions in my proofs, such as when I took the limit to infinity of the reciprocal form of an m-nacci sequence to obtain the limiting stripped reciprocal form, for which I could then treat the limiting ratio as a constant to solve for. Another aspect of mathematical research I observed was the great care needed in making assumptions. Without solid evicence, ideally a proof, to back up an assumption, any conclusions determined using that assumption cannot be taken to be true in all cases. I find this does not discount the value of the conclusion, but rather adds motivation at proving its condition, which would reveal two conclusions at once.

I also learned more about the continuity of mathematical research. The paper answered many questions, but made me consider new ones, some of which are discussed in Section 8, which I hope to pursue in my independent research. There were many other aspects of mathematics that I considered incorporating but did not have the space to do so. Something particularly interesting to me is attempting to use matrices to represent systems generated by a linear recurrence, and finding their eigenvalues and eigenvectors to determine the limiting ratios of the linear recurrence or other long-term behaviors. Regarding other perspectives, there is no debate over the validity of my methods and conclusions; Wolfram (1997) agrees that the limiting ratio of m-nacci sequences are the real roots of characteristic equations with coefficients of 1.

10 Works Cited

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