MIT Research Supplement: A Collection of Mathematical Explorations

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1 Introduction

This document aims to offer a glimpse into the independent mathematical work/exploration I do in my free time and substantiate my deep love for mathematics. While I do hope to present formal mathematics, the guiding text will also include some personal narration to illuminate my motivation and way of thinking aobut the math.

2 Even Powers, Odd Powers, and Modular Arithmetic

2.1 Even powers of 2 and multiples of 3

As I explored the binary number system, I naturally familiarized myself with and memorized the first several powers of 2. As I reviewed the sequence in my head, I noticed that 16, 2^4, is one more than 15. On its own this observation may not seem significant, but it becomes interesting when on considers that 4, 2^2 is one more than 3. Since 3 and 15 are both multiples of 3, I began to wonder where else were there powers of 2 near a multiple of 3, and soon found that 64=2^6 is one more than 63, 256=2^8 is one more than 255, and so on. From these observations I made the following proposition.

Theorem 2.1 *Any positive, even power of 2 is one more than a multiple of 3.*

$$2^{2n} = 3m + 1; n, m \in \boldsymbol{Z}^{+}$$

I knew I could not just claim the above to be true for all n, so I looked into methods of proof. After considering multiple methods of proof, I decided induction would be the most straighforward.

Proof: Indeed, for n=1:

$$2^2 = 4 = 3 + 1$$

Assuming the theorem holds for all n=k:

$$2^{2k} = 3m + 1; k, m \in \mathbf{Z}^{+}$$

We proceed with the inductive step:

$$2^{2(k+1)} = 2^{2k+2} = 4 * 2^{2k}$$

From our assumption:

$$4 * 2^{2k} = 4 * (3m + 1) = 3 * (3m + 1) + 3m + 1 = 3m + 1$$

For some integer m. Thus, by induction, we have proved (2.1) holds for all $n > 0$.

2.2 Even powers of 2 and multiples of 3

After proposing (2.1), I wondered if there was a similar pattern for odd powers of 2, and I found there is! I saw that $2^1 = 2 = 3 - 1$, $2^3 = 8 = 9 - 1$, and so on. From this, I proposed the following:

Theorem 2.2 *Any positive, odd power of 2 is one less than a multiple of 3.*

$$2^{2n-1} = 3m - 1; n, m \in \boldsymbol{Z}^{+}$$

Proof: Indeed, for n=1:

$$2^1 = 2 = 3 - 1$$

Assuming the theorem holds for all n=k:

$$2^{2k-1} = 3m + 1; k, m \in \mathbf{Z}^{+}$$

We proceed with the inductive step:

$$2^{2(k+1)-1} = 2^{2k+1} = 4 * 2^{2k-1}$$

From our assumption:

$$4 * 2^{2k-1} = 4 * (3m - 1) = 3 * (3m - 1) + 3m - 1 = 3m + 1$$

For some integer m. Thus, by induction, we have proved (2.2) holds for all $n > 0$.

While I was satisfied with my conclusion, I could not help but wonder if there was something 'deeper' to this property of even and odd powers of integers, so I looked at the powers of 3, and saw that $3^2 = 9 = 8 + 1$ and $3^3 = 27 = 28 - 1$, which hints towards a similar pattern between powers of 3 and multiples of 4. So I proposed the following:

Theorem 2.3 *Any positive, even power of some integer p is one more than a multiple of p+1.*

$$p^{2n} = (p + 1)m + 1; p, n, m \in \boldsymbol{Z}^{+}$$

Theorem 2.4 *Any positive, odd power of some integer p is one less than a multiple of p+1.*

$$p^{2n-1} = (p+1)m - 1; p, n, m \in \boldsymbol{Z}^{+}$$

I considered proving these by induction as well, but I knew there had to be a better way. As I left the problem to simmer in my head, I continued exploring the beauty of mathematics and came across modular arithmetic from an abstract algebra book i was self-studying from. I soon realized that it could be used to prove (2.3) and (2.4) very succinctly.

Theorem 2.5 *Let p be some positive integer, then the following holds:*

$$p^n \mod (p+1) = (-1)^n$$

Proof: An integer p can be represented as:

$$p \mod (p+1) = -1$$

Using the multiplication property of modular arithmetic:

$$(x \mod a) * (y \mod a) = (x * y) \mod a$$

It follows that

$$x^n \mod a = (x \mod a) * (x \mod a) * \cdots * (x \mod a) = (x \mod a)^n$$

Thus

$$p^n \mod (p+1) = (p \mod (p+1))^n = (-1)^n$$

Taking the modulus of both sides of the expression with respect to (p+1) gives:

$$(p^{2n}) \mod (p+1) = ((p+1)m + 1) \mod (p+1)$$

From (2.5), we have:

$$(p^{2n}) \mod (p+1) = (-1)^{2n} = 1$$

And for the right side, we have:

$$((p+1)m + 1) \mod (p+1) = ((p+1)m) \mod (p+1) + (1) \mod (p+1) = 0 + 1 = 1$$

Thus, (2.3) holds.

Proof of (2.4): Taking the modulus of both sides of the expression with respect to (p+1) gives:

$$(p^{2n+1}) \mod (p+1) = ((p+1)m + 1) \mod (p+1)$$

From (2.5), we have:

$$(p^{2n+1}) \mod (p+1) = (-1)^{2n+1} = -1$$

And for the right side, we have:

$$((p+1)m - 1) \mod (p+1) = ((p+1)m) \mod (p+1) + (-1) \mod (p+1) = 0 - 1 = -1$$

Thus, (2.4) holds.

3 The Reality of Complex Conjugates

As I develop more mathematical maturity, I have learned how to 'play' with new concepts to both make and prove a wider variety of observations. One example is when I learned about the modulus/argument form of complex numbers. The ease they provide when performing rotations in the 2D plane and connection they show between fundamental mathematical constants fascinate me. When I first learned how to express a complex number in modululs-argument form, I immediately tried to derive an expression for their conjugates on my own. I realized that all aspects of a complex number and its conjugate in modulus-argument form are the same except their angles, which are additive inverses.

4 The Behavior of Generalized Fibonacci Sequences

5 Counting Indices of Pascal's Simplex