61DM HW 1 Problem 1

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1 Problem

Use the principle of mathematical induction to check that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}, n \in \mathbf{Z}^+$$

2 Solution

First, we ensure the proposition is true for $n = 1$:

$$\sum_{j=1}^{1} j^3 = 1^3 = 1$$

$$\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

$$1 = 1$$

So, the proposition holds for $n = 1$. Next, we assume the proposition is true for all $n = k$:

$$\sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4}, k \in \mathbf{Z}^+$$

Now, we show the proposition is true for all $n = k + 1$, that is:

$$\sum_{j=1}^{k+1} j^3 = \frac{(k+1)^2(k+2)^2}{4}, k \in \mathbf{Z}^+$$

Beginning with the left-hand side:

$$\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= \sum_{j=1}^{k} j^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{[by assumption]} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

Which is equal to the right-hand side. Therefore, since the proposition holds for $n = 1$ and all $n = k + 1$, it is true for all $n \in \mathbf{Z}^+$