CalTech Research Supplement: A Collection of Mathematical Explorations

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1 Introduction

This document aims to offer a glimpse into the independent mathematical research/exploration I do in my free time and substantiate my deep love for mathematics. For brevity, I only present the major results of my explorations with some commentary on their motivation and/or applications.

2 Powers and Modular Arithmetic

(2.1) and (2.2) come from long-standing personal observations I made after thinking about the powers of 2 with reference to their use in computer binary. When generating numbers that consisted of consecutive 1's in binary and converting them to decimal, for example, I noticed how some of them were just shy of a multiple of 3. When I first learned how to do proofs by induction, these were the first two theorems I proved.

Theorem 2.1 *Any positive, even power of 2 is one more than a multiple of 3.*

$$2^{2n} = 3m + 1; n, m \in \mathbf{Z}^+$$

Theorem 2.2 *Any positive, odd power of 2 is one less than a multiple of 3.*

$$2^{2n-1} = 3m - 1; n, m \in \mathbf{Z}^+$$

(2.3) was an immediate observation I made after learning the basics of modular arithmetic and was also one of the first postulates I proved outside of school.

Theorem 2.3 *Let p be some positive integer, then the following holds:*

$$p^n \mod (p+1) = (-1)^n; p, n \in \mathbf{Z}^+$$

(2.4) and (2.5) also come from observations which I made after I tried to extend the ideas in (2.1) and (2.2). I tried proving them by induction in a similar way I did (2.1) and (2.2), but ended up relying heavily on (2.3) instead.

Theorem 2.4 *Any positive, even power of some integer p is one more than a multiple of $p+1$.*

$$p^{2n} = (p+1)m + 1; p, n, m \in \mathbf{Z}^+$$

Theorem 2.5 *Any positive, odd power of some integer p is one less than a multiple of $p+1$.*

$$p^{2n-1} = (p+1)m - 1; p, n, m \in \mathbf{Z}^+$$

(2.6) came from the same observation as (2.1) and (2.2), but did not fully flesh itself out until I suddenly realized it is analogous to factorizations of the polynomial $z^n - 1$, grabbed a pen and my notebook, and left the match of Tetris I was playing to frantically put everything down.

Theorem 2.6 *For all positive integers, b, the following holds:*

$$b^n = (b-1)(b^0 + b^1 + \cdots + b^{n-1}) + 1, ; b, n \in \mathbf{Z}^+$$

Examples where b is a non-integer or negative also hold true, motivating future efforts at proving (2.7).

Postulate 2.7 *For all real numbers, b, the following holds:*

$$b^n = (b-1)(b^0 + b^1 + \cdots + b^{n-1}) + 1, ; b \in \mathbf{R}; b \in \mathbf{Z}^+$$

3 The Behavior of the Generalized Fibonacci Sequence

For my IB Math Analysis Internal Assessment, I am studying the behavior of m-nacci sequences, a generalization of the fibonacci sequence.

Definition 3.1 *An m-nacci sequence is a recursive sequence of the form:*

$$F_n^{(m)} = F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} = \sum_{i=1}^{m} F_{n-i}^{(m)}$$

Remark 3.2 *A system of equations can be derived from an m-nacci sequence:*

$$F_n^{(m)} = F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)}$$
$$F_{n-1}^{(m)} = F_{n-1}^{(m)}$$
$$F_{n-2}^{(m)} = F_{n-2}^{(m)}$$
$$\vdots$$
$$F_{n-m+1}^{(m)} = F_{n-m+1}^{(m)}$$

Which can be represented with a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \\ F_{n-3} \\ \vdots \\ F_{n-m} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \\ F_{n-2} \\ \vdots \\ F_{n-m+1} \end{bmatrix}$$

I use (3.3) in my Internal Assessment to serve as motivation for the construction of the characterisitc polynomial of generalized m-nacci sequences. The roots of the characteristic polynomial are later used in an algorithm to determine an explicit formula for an m-nacci sequence, similar to Binet's formula for the fibonacci sequence.

Theorem 3.3 *The characteristic polynomial of an $m \times m$ matrix of the form*

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & & 1 & 0 \end{bmatrix}$$

is

$$t^m - t^{m-1} - t^{m-2} - \cdots - t - 1 = t^m - \sum_{i=1}^{m} t^{m-i}$$

(3.4) came from an observation I made on the positive, real roots (which I will refer to as the 'leading root') of the characteristic polynomials from (3.3) which are used as bases for the explicit formulas mentioned above. Only the leading roots had a modulus greater than 1 and thus they become predominant as high indices of a sequence are determined. Furthermore, the leading roots approached but never reached 2, so I attempted and succeeded to prove that the limit the leading root approaches as the degree of an m-nacci sequence approaches infinity is 2. For evidence, the explicit formulas for m-nacci sequences of degree 3, 4, and 5, determined by my algorithm, are given below with their leading roots in bold-face:

$$F_n^{(3)} = (0.0994)(\mathbf{1.839})^n + (0.45 - 0.161i)(-0.42 + 0.606i)^n + (0.45 + 0.161i)(-0.420 - 0.606i)^n$$

$$F_n^{(4)} = (0.041)(\mathbf{1.93})^n + (0.272 - 0.174i)(-0.0764 + 0.815i)^n + (0.272 + 0.174i)(-0.0764 - 0.815i)^n + (0.415)(-0.775)^n$$

$$F_n^{(5)} = (0.0183)(\mathbf{1.97})^n + (0.171 - 0.152i)(0.195 + 0.849i)^n + (0.171 + 0.152i)(0.195 - 0.849i)^n + (0.32 - 0.0794i)(-0.678 + 0.459i)^n + (0.32 + 0.0794i)(-0.678 - 0.459i)^n$$

Theorem 3.4 *The ratio between terms of an m-nacci sequence approaches the positive, real solution to the following expression:*

$$x + x^{-m} = 2$$

Thus,

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{F_n^{(m)}}{F_{n-1}^{(m)}} = \lim_{m \to \infty} x + x^{-m} = 2 \Rightarrow x = 2$$

4 The Reality of Complex Numbers and Their Conjugates

I developed (4.1) to confirm that the explicit formulas for my Internal Assessment (such as $F_n^{(3)}$ from above), while containing complex exponential bases, would always return a real number. Notice that the terms that contain complex numbers in each explicit formula come in conjugate pairs.

Theorem 4.1 *The sum of the product of a set of complex numbers and the product of the set of conjugates is a real number:*

$$z_1 z_2 \ldots z_n + \bar{z_1} \bar{z_2} \ldots \bar{z_n} = \prod_{i=1}^{n} z_i + \prod_{i=1}^{n} \bar{z_i} = 2R \cos(\Phi)$$

Where

$$z_j = a_j + i b_j; z_j \in \boldsymbol{C}; a_j, b_j \in \boldsymbol{R}$$

$$r_j = \sqrt{a_j^2 + b_j^2}; \theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right)$$

$$R = r_1 r_2 \ldots r_n = \prod_{i=1}^{n} r_i; \Phi = \theta_1 + \theta_2 + \cdots + \theta_n = \sum_{i=1}^{n}$$

5 Pascal's Simplex

(5.2) was an extension upon a lesson from a discrete calculus course I took at the Georgia Governor's Honors Program. In the lesson, we used finite sums to derive formulas for figurate numbers such as the triangular, pentagonal, and tetrahedral numbers. I knew there are higher dimensional analogs of the triangle and tetrahderon and wanted to observe how their figurate numbers behaved.

Definition 5.1 *The falling exponent is defined as:*

$$x^{\underline{n}} = x(x-1)(x-2)\ldots(x-n+1); x \in \boldsymbol{R}^{+}, n \in \boldsymbol{Z}^{+}$$

Theorem 5.2 *The following sum:*

$$\sum_{x_2=0}^{x_1} \sum_{x_3=0}^{x_2} \cdots \sum_{x_{m+1}=0}^{x_m} 1$$

Evaluates to:

$$\frac{(x_1 + m)^{\underline{m}}}{m!} = \binom{x_1 + m}{m}; x_i, m \in \boldsymbol{Z}^{+}$$

Notice how using $m = 3$ in (5.2) gives the formula for the tetrahedral numbers. In general, the expression gives the figurate number for the $(x+1)$th m-simplex.

(5.3) is an application of (5.2) I found after studying the multinomial theorem and its extensions on Pascal's Triangle with Pascal's Simplex. I am working on a program that constructs Pascal's Simplex and (5.3) has served useful in organizing how I will present the entries of high-dimensional simpleces.

Theorem 5.3 *The number of entries or multinomial coefficients in the r-th slice of pascal's m-simplex is given by*

$$\sum_{x_2=0}^{r} \sum_{x_3=0}^{x_2} \cdots \sum_{x_m=0}^{x_{m-1}} 1$$

Which evaluates to:

$$\frac{(r + m - 1)^{\underline{m-1}}}{(m-1)!} = \binom{r + m - 1}{m - 1}; r, m \in \boldsymbol{Z}^{+}$$

6 A First Look into Real Analysis

During the first semester of my senior year, I began self-studying real analysis and after playing with definitions and their implications, I proved the generalized triangle inequality using an induction technique similar to the one I used to prove (4.1).

Theorem 6.1 *For all $x_i \in \boldsymbol{R}$, the following holds.*

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n| \Rightarrow |\sum_{i=1}^{n} x_i| \le \sum_{i=1}^{n} |x_i|$$

Analysis is far more fun than internet communities would lead me to believe and has become one of my favorite fields of math. I am currently working on a proof that shows (6.1) holds for all complex numbers as well.

7 Where it All Happens

As I mentioned previously in the portfolio (and in the preface to (2.6)), this work takes place in a notebook that has at this point become a part of my daily life. So here I present, the book! The image below shows it open to rudimentary work leading to (3.3) (left) and a personal attempt at generalizing the representation of quadratic forms of any number of variables with matrices (right).