Two Quick Proofs as an Expression of Affection for Mathematics

Edwin Trejo Balderas

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1 Introduction

This document aims to offer a glimpse into the independent math work I do in my free time as, unfortunately, I do not believe the activities section on the WISE application offers me enough space to do that. I also hope it serves as sufficient proof of my deep love for mathematics and interest in MIT's course 18. In this document, I prove that every positive even power of 2 is one more than a multiple of 3 and every positive odd power of 2 is one less than a multiple of 3 using mathematical induction. I came up with the propositions presented by accident, simply wondering about the powers of 2 and the numbers near them, which is how many of my propositions originate. This creativity that math allows is what I mean when I say it is an art.

2 Observations and Propositions

The first few powers of 2 (starting at $n = 1$) are shown below:

$$a_n = 2^n = 2, 4, 8, 16, 32, 64, \ldots$$

Notice how adding 1 to each of 2, 8, and 32 gives a multiple of 3:

$$2 + 1 = 3 = 3(1)$$

$$8 + 1 = 9 = 3(3)$$

$$32 + 1 = 33 = 3(11)$$

Also notice how 2, 8, and 32 are odd powers of 2:

$$2 = 2^1, 8 = 2^3, 32 = 2^5$$

From this, I propose this holds for all odd powers of 2:

Proposition 2.1

$$2^{2n-1} + 1 = 3(x) \textit{ such that } x \textit{ is an integer for all } n = 1, 2, 3, \ldots$$

Alternatively, we notice how subtracting 1 from each of 4, 16, and 64 gives a multiple of 3:

$$4 - 1 = 3 = 3(1)$$

$$16 - 1 = 15 = 3(5)$$

$$64 - 1 = 63 = 3(21)$$

Also notice how 4, 16, and 64 are even powers of 2:

$$4 = 2^2, 16 = 2^4, 64 = 2^6$$

From this, I propose this holds for all odd powers of 2:

Proposition 2.2

$$2^{2n} - 1 = 3(x) \textit{ such that } x \textit{ is an integer for all } n = 1, 2, 3, \ldots$$

3 Proof of Proposition 1

To prove the following:

$$2^{2n-1} + 1 = 3(x) \text{ such that x is an integer for all } n = 1, 2, 3, \ldots$$

We first must show it is true for the case where $n = 1$, indeed:

$$2^{2(1)-1} + 1 = 2 + 1 = 3$$

Now, assuming $n = k$ for all $k = 1, 2, 3, \ldots$, we have:

$$2^{2k-1} + 1 = 3(x) \text{ such that x is an integer for all } k = 1, 2, 3, \ldots$$

It follows that this is also true for all $n = k + 1$:

$$2^{2(k+1)-1} + 1 = 3(x)$$

$$2^{2(k+1)-1} + 1 = 2^{2k-1+2} + 1 = 2^2 2^{2k-1} + 1 = 4(2^{2k-1}) + 1 = 3(2^{2k-1}) + 2^{2k-1} + 1$$

From our assumption, we know that $2^{2k-1} + 1$ is in fact a multiple of 3, so we can replace it with $3m$ where m is some integer:

$$3(2^{2k-1}) + 3m = 3(2^{2k-1} + m)$$

Since both factors of 3 are integers, we have shown that the above expression is indeed a multiple of 3, and have proven proposition 1

4 Proof of Proposition 2

The proof of the second proposition, shown below, follows a very similar progression as the first.

$$2^{2n} - 1 = 3(x) \text{ such that x is an integer for all } n = 1, 2, 3, \ldots$$

For $n = 1$, indeed:

$$2^2 - 1 = 4 - 1 = 3$$

Assuming $n = k$ for all $k = 1, 2, 3, \ldots$, we have:

$$2^{2k} - 1 = 3(x) \text{ such that x is an integer for all } k = 1, 2, 3, \ldots$$

It follows that this is also true for all $n = k + 1$:

$$2^{2(k+1)} - 1 = 3(x)$$

$$2^{2(k+1)} - 1 = 2^2 2^k - 1 = 4(2^k) - 1 = 3(2^k) + 2^k - 1 = 3(2^k) + 3m = 3(2^k + m)$$

Thus proving proposition 2.

5 Bonus Proof

Since all even powers of 2, 2^{2n}, can be simplified to powers of 4, 4^n, we can see that, using the proof of proposition 2, the following holds:

$$4^n - 1 = 3(x) \text{ such that x is an integer for all } n = 1, 2, 3, \ldots$$

6 Extensions

Currently, I am working towards generalizing these two theorems to the rest of the integers, such as even/odd powers of 3 being 1 greater/less than multiples of 4, or rather even/odd powers of n being 1 greater/less than multiples of $n + 1$. I am also working to see if this holds in reverse, where all even/odd powers of n could be 1 greater/less than multiples of $n - 1$. Thank you so much for reading! :)