MIT Research Supplement: A Collection of Mathematical Explorations

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November, 2022

1 Introduction

This document aims to offer a glimpse into the independent mathematical research/exploration I do in my free time and substantiate my deep love for mathematics. For brevity, I only present the major results of my explorations with some commentary on motivation and/or applications.

2 Even and Odd Powers and Modular Arithmetic

(2.1) and (2.2) come from long-standing personal observations after thinking about the powers of 2 with reference to their use in computer binary. When I first learned how to do proofs by induction, these were the first two personal theorems I proved.

Theorem 2.1 *Any positive, even power of 2 is one more than a multiple of 3.*

$$2^{2n} = 3m + 1; n, m \in \boldsymbol{Z}^{+}$$

Theorem 2.2 *Any positive, odd power of 2 is one less than a multiple of 3.*

$$2^{2n-1} = 3m - 1; n, m \in \boldsymbol{Z}^{+}$$

(2.3) was an immediate observation I made after learning the basics of modular arithmetic and was also one of the first postulates I proved outside of school.

Theorem 2.3 *Let p be some positive integer, then the following holds:*

$$p^{n} \mod (p+1) = (-1)^{n}; p, n \in \boldsymbol{Z}^{+}$$

(2.4) and (2.5) were also longstanding observations after I tried to extend the ideas in (2.1) and (2.2). I tried proving them by induction but ended up relying heavily on (2.3) instead.

Theorem 2.4 *Any positive, even power of some integer p is one more than a multiple of p+1.*

$$p^{2n} = (p+1)m + 1; p, n, m \in \boldsymbol{Z}^{+}$$

Theorem 2.5 *Any positive, odd power of some integer p is one less than a multiple of p+1.*

$$p^{2n-1} = (p+1)m - 1; p, n, m \in \boldsymbol{Z}^{+}$$

3 The Behavior of the Generalized Fibonacci Sequence

For my IB Math Analysis Internal Assessment, I am studying the behavior of m-nacci sequences, a generalization of the fibonacci sequence.

Definition 3.1 *An m-nacci sequence is a recursive sequence of the form:*

$$F_{n}^{(m)} = F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)}$$

Remark 3.2 *A system of equations can be derived from an m-nacci sequence:*

$$\begin{aligned}
F_{n}^{(m)} &= F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} \\
F_{n-1}^{(m)} &= F_{n-1}^{(m)} \\
F_{n-2}^{(m)} &= F_{n-2}^{(m)} \\
&\vdots \\
F_{n-m+1}^{(m)} &= F_{n-m+1}^{(m)}
\end{aligned}$$

Which can be represented with a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \\ F_{n-3} \\ \vdots \\ F_{n-m} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \\ F_{n-2} \\ \vdots \\ F_{n-m+1} \end{bmatrix}$$

I use (3.3) in my Internal Assessment to serve as motivation for the construction of the characerisitc polynomial of generalized m-nacci sequences. The roots of the characteristic polynomial are later used in an algorithm to determine an explicit formula for an m-nacci sequence, similar to Binet's formula for the fibonacci sequence.

Theorem 3.3 *The characteristic polynomial of an $m \times m$ matrix of the form*

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & 1 & 0 \end{bmatrix}$$

is

$$t^m - t^{m-1} - t^{m-2} - \dots - t - 1$$

(3.4) came from an observation I made on the positive, real roots (which I will refer to as the 'leading root') of the characteristic polynomials from (3.3) used as bases for the explicit formulas mentioned above. Only the leading roots had a modulus greater than 1 and thus they would become predominant as high values of each sequence was determined. Furthermore, the leading roots approached but never reached 2, so I attempted and succeeded to prove that the limit of the leading root as the degree of an m-nacci sequence approached infinity is 2. For evidence, the explicit formulas for m-nacci sequences of degree 3, 4, and 5, are given below with their leading roots in bold-face:

$$F_n^{(3)} = (0.0994)(\mathbf{1.839})^n + (0.45 - 0.161i)(-0.42 + 0.606i)^n + (0.45 + 0.161i) * (-0.420 - 0.606i)^n$$
$$F_n^{(4)} = (0.041) * (\mathbf{1.93})^n + (0.272 - 0.174i) * (-0.0764 + 0.815i)^n + (0.272 + 0.174i) * (-0.0764 - 0.815i)^n + (0.415) * (-0.775)^n$$
$$F_n^{(5)} = (0.0183) * (\mathbf{1.97})^n + (0.171 - 0.152i) * (0.195 + 0.849i)^n + (0.171 + 0.152i) * (0.195 - 0.849i)^n + (0.32 - 0.0794i) * (-0.678 + 0.459i)^n + (0.32 + 0.0794i) * (-0.678 - 0.459i)^n$$

Theorem 3.4 *The ratio between terms of an m-nacci sequence approaches the positive solution to the following expression:*

$$x + x^{-m} = 2$$

In other words,

$$\lim_{m \to \infty} \frac{F_n^{(m)}}{F_{n-1}^{(m)}} = \{x | x + x^{-m} = 2\}$$

4 The Reality of Complex Numbers and Their Conjugates

I developed (4.1) to confirm that the explicit formulas for my Internal Assessment, while containing complex exponential bases, would always return a real number. Notice that the terms that contain complex numbers in each formula come in conjugate pairs.

Theorem 4.1 *The sum of the product of a set of complex numbers and the product of the set of conjugates is a real number:*

$$z_1 * z_2 * \dots * z_n + \bar{z_1} * \bar{z_2} * \dots * \bar{z_n} = 2R \cos(\Phi)$$

Where

$$z_j = a_j + ib_j; z_j \in \boldsymbol{C}; a_j, b_j \in \boldsymbol{R}$$
$$r_j = \sqrt{a_j^2 + b_j^2}; \theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right)$$
$$R = r_1 * r_2 * \dots * r_n; \Phi = \theta_1 + \theta_2 + \dots + \theta_n$$

5 Pascal's Simplex

(5.1) was an extension upon a lesson from a discrete calculus course I took at the Georgia Governor's Honors Program. In the lesson, we used finite sums to derive formulas for figurate numbers such as the triangular, pentagonal, and tetrahedral numbers. I knew there were higher dimensional analogs of the triangle and tetrahderon and wanted to observe how their figurate numbers behaved.

Theorem 5.1 *The following sum:*

$$\sum_{x_2=0}^{x_1}\sum_{x_3=0}^{x_2}\cdots\sum_{x_{m+1}=0}^{x_m} 1$$

Evaluates to:

$$\frac{(x_1+m)^{\underline{m}}}{m!} = \binom{x_1+m}{m}; x_i, m \in \boldsymbol{Z}^+$$

Notice how using $m = 3$ in (5.1) gives the formula for the tetrahedral numbers.

(5.2) is an application of (5.1) I found after studying the multinomial theorem and its extensions on Pascal's Triangle with Pascal's Simplex. I am working on a program that constructs pascal's simplex and (5.2) has served useful in organizing how I will present the entries of high-dimensional simpleces.

Theorem 5.2 *The number of entries or multinomial coefficients in the r-th slice of pascal's n-simplex is given by*

$$\sum_{x_2=0}^{r}\sum_{x_3=0}^{x_2}\cdots\sum_{x_n=0}^{x_{n-1}} 1$$

Which evaluates to:

$$\frac{(r+n-1)^{\underline{n-1}}}{(n-1)!} = \binom{r+m-1}{m-1}; r, m \in \boldsymbol{Z}^+$$

6 Where it All Happens

As I mentioned previously in the portfolio, this work takes place in a notebook that has at this point become a part of my daily life. The image below shows this notebook open to rudimentary work leading to (3.3) (left) and a personal attempt at generalizing the representation of quadratic forms with matrices (right)