1 Introduction

This document aims to offer a glimpse into the independent mathematical research/exploration I do in my free time and substantiate my deep love for mathematics. For brevity, I only present the major results of my explorations with some commentary on their motivation and/or applications.

2 Powers and Modular Arithmetic

(2.1) and (2.2) come from long-standing personal observations I made after thinking about the powers of 2 with reference to their use in computer binary. When generating numbers that consisted of consecutive 1's in binary and converting them to decimal, for example, I noticed how some of them were just shy of a multiple of 3. When I first learned how to do proofs by induction, these were the first two theorems I proved.

Theorem 2.1 *Any positive, even power of 2 is one more than a multiple of 3.*

$$2^{2n} = 3m + 1; n, m \in \boldsymbol{Z}^{+}$$

Theorem 2.2 *Any positive, odd power of 2 is one less than a multiple of 3.*

$$2^{2n-1} = 3m - 1; n, m \in \boldsymbol{Z}^{+}$$

(2.3) was an immediate observation I made after learning the basics of modular arithmetic and was also one of the first postulates I proved outside of school.

Theorem 2.3 *Let p be some positive integer, then the following holds:*

$$p^{n} \mod (p+1) = (-1)^{n}; p, n \in \boldsymbol{Z}^{+}$$

(2.4) and (2.5) also come from observations which I made after I tried to extend the ideas in (2.1) and (2.2). I tried proving them by induction in a similar way I did (2.1) and (2.2), but ended up relying heavily on (2.3) instead.

Theorem 2.4 *Any positive, even power of some integer p is one more than a multiple of p+1.*

$$p^{2n} = (p+1)m + 1; p, n, m \in \boldsymbol{Z}^{+}$$

Theorem 2.5 *Any positive, odd power of some integer p is one less than a multiple of p+1.*

$$p^{2n-1} = (p+1)m - 1; p, n, m \in \boldsymbol{Z}^{+}$$

(2.6) came from the same observation as (2.1) and (2.2), but did not fully flesh itself out until I suddenly realized it is analagous to factorizations of the polynomial $z^{n} - 1$, grabbed a pen and my notebook, and left the match of Tetris I was playing to frantically put everything down.

Theorem 2.6 *For all positive integers, b, the following holds:*

$$b^{n} = (b-1)(b^{0} + b^{1} + \cdots + b^{n-1}) + 1, ; b, n \in \boldsymbol{Z}^{+}$$

Examples where b is a non-integer or negative also hold true, motivating future efforts at proving (2.7).

Postulate 2.7 *For all real numbers, b, the following holds:*

$$b^{n} = (b-1)(b^{0} + b^{1} + \cdots + b^{n-1}) + 1, ; b \in \boldsymbol{R}; b \in \boldsymbol{Z}^{+}$$

3 The Behavior of the Generalized Fibonacci Sequence

For my IB Math Analysis Internal Assessment, I am studying the behavior of m-nacci sequences, a generalization of the fibonacci sequence.

Definition 3.1 *An m-nacci sequence is a recursive sequence of the form:*

$$F_{n}^{(m)} = F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} = \sum_{i=1}^{m} F_{n-i}^{(m)}$$

Remark 3.2 *A system of equations can be derived from an m-nacci sequence:*

$$\begin{aligned} F_{n}^{(m)} &= F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} \\ F_{n-1}^{(m)} &= F_{n-1}^{(m)} \\ F_{n-2}^{(m)} &= F_{n-2}^{(m)} \\ &\vdots \\ F_{n-m+1}^{(m)} &= F_{n-m+1}^{(m)} \end{aligned}$$

Which can be represented with a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \\ F_{n-3} \\ \vdots \\ F_{n-m} \end{bmatrix} = \begin{bmatrix} F_{n} \\ F_{n-1} \\ F_{n-2} \\ \vdots \\ F_{n-m+1} \end{bmatrix}$$

I use (3.3) in my Internal Assessment to serve as motivation for the construction of the characterisitc polynomial of generalized m-nacci sequences. The roots of the characteristic polynomial are later used in an algorithm to determine an explicit formula for an m-nacci sequence, similar to Binet's formula for the fibonacci sequence.

Theorem 3.3 *The characteristic polynomial of an $m \times m$ matrix of the form*

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & 1 & 0 \end{bmatrix}$$

is

$$t^m - t^{m-1} - t^{m-2} - \dots - t - 1 = t^m - \sum_{i=1}^{m} t^{m-i}$$

(3.4) came from an observation I made on the positive, real roots (which I will refer to as the 'leading root') of the characteristic polynomials from (3.3) which are used as bases for the explicit formulas mentioned above. Only the leading roots had a modulus greater than 1 and thus they become predominant as high indices of a sequence are determined. Furthermore, the leading roots approached but never reached 2, so I attempted and succeeded to prove that the limit the leading root approaches as the degree of an m-nacci sequence approaches infinity is 2. For evidence, the explicit formulas for m-nacci sequences of degree 3, 4, and 5, determined by my algorithm, are given below with their leading roots in bold-face:

$$F_n^{(3)} = (0.0994)(\mathbf{1.839})^n + (0.45 - 0.161i)(-0.42 + 0.606i)^n + (0.45 + 0.161i)(-0.420 - 0.606i)^n$$
$$F_n^{(4)} = (0.041)(\mathbf{1.93})^n + (0.272 - 0.174i)(-0.0764 + 0.815i)^n + (0.272 + 0.174i)(-0.0764 - 0.815i)^n + (0.415)(-0.775)^n$$
$$F_n^{(5)} = (0.0183)(\mathbf{1.97})^n + (0.171 - 0.152i)(0.195 + 0.849i)^n + (0.171 + 0.152i)(0.195 - 0.849i)^n + (0.32 - 0.0794i)(-0.678 + 0.459i)^n + (0.32 + 0.0794i)(-0.678 - 0.459i)^n$$

Theorem 3.4 *The ratio between terms of an m-nacci sequence approaches the positive, real solution to the following expression:*

$$x + x^{-m} = 2$$

Thus,

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{F_n^{(m)}}{F_{n-1}^{(m)}} = \lim_{m\to\infty} x + x^{-m} = 2 \Rightarrow x = 2$$

4 Pascal's Simplex

(5.2) was an extension upon a lesson from a discrete calculus course I took at the Georgia Governor's Honors Program. In the lesson, we used finite sums to derive formulas for figurate numbers such as the triangular, pentagonal, and tetrahedral numbers. I knew there are higher dimensional analogs of the triangle and tetrahderon and wanted to observe how their figurate numbers behaved.

Definition 4.1 *The falling exponent is defined as:*

$$x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1); x \in \boldsymbol{R}^+, n \in \boldsymbol{Z}^+$$

Theorem 4.2 *The following sum:*

$$\sum_{x_2=0}^{x_1} \sum_{x_3=0}^{x_2} \dots \sum_{x_{m+1}=0}^{x_m} 1$$

Evaluates to:

$$\frac{(x_1+m)^{\underline{m}}}{m!} = \binom{x_1+m}{m}; x_i, m \in \boldsymbol{Z}^+$$

Notice how using $m = 3$ in (5.2) gives the formula for the tetrahedral numbers. In general, the expression gives the figurate number for the $(x+1)$th m-simplex.

(5.3) is an application of (5.2) I found after studying the multinomial theorem and its extensions on Pascal's Triangle with Pascal's Simplex. I am working on a program that constructs Pascal's Simplex and (5.3) has served useful in organizing how I will present the entries of high-dimensional simpleces.

Theorem 4.3 *The number of entries or multinomial coefficients in the r-th slice of pascal's m-simplex is given by*

$$\sum_{x_2=0}^{r} \sum_{x_3=0}^{x_2} \dots \sum_{x_m=0}^{x_{m-1}} 1$$

Which evaluates to:

$$\frac{(r+m-1)^{\underline{m-1}}}{(m-1)!} = \binom{r+m-1}{m-1}; r, m \in \boldsymbol{Z}^+$$