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ECE367 - Matrix Algebra and Optimization

Tuesday, September 17, 2024

Problem Set #2
Due Date: Tuesday, October 1, 2024, at 11:59 PM

Homework policy: Problem sets must be turned by the due date. In the following, the course text "Optimization Models" is abbreviated as "OptM" and "Introduction to Applied Linear Algebra" as "IALA".

Problems are categorized as:

- **"Theory" problems:** These are mostly mathematical questions designed to give you deeper insight into the fundamentals of the ideas introduced in this class.
- **"Application" problems:** These questions are designed to expose you to the breadth of application of the ideas developed in class and to introduce you to useful numerical toolboxes. Problems of this sort often ask you to produce plots and discuss your results; said plots and discussions should be included in your submission – think of your submitted solution like a lab book. Your attached code simply provides back-up evidence.
- **"Optional" problems:** Optional problems provide extra practice or introduce interesting connections or extensions. They need not be turned in. We will not grade them, but we will assume you have reviewed and understood the solutions to the optional problems when designing the exams.

Submission and Grading:

- **Submission:** Your submission of the "Theory" and "Application" questions must be uploaded via Quercus by the due date.
- **Grading:** We will grade one problem from the "theory" problems and one problem from the "application" problems. Your score will be the sum of the two grades.
- **For fairness, late submissions will not be accepted!**

Theory Problems

1. *Gram-Schmidt algorithm.* IALA Book, Exercise 5.6.

2. *Projection in Euclidean Space.* In this problem, we use the notation $\text{Proj}_{\mathcal{S}}(x)$ to denote the projection of a vector x onto some set $\mathcal{S}$, which consists of vectors that are of same dimension as x. Consider the following vectors and subspaces.

$$x_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathcal{V}_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$
$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}, \quad \mathcal{V}_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$
$$x_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathcal{V}_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   (a) Compute $\text{Proj}_{\mathcal{V}_i}(x_i)$ for $i = 1, 2, 3$.

   (b) Consider the affine set $\mathcal{A}_i = \{v + b_i \mid v \in \mathcal{V}_i\}$. Compute $\text{Proj}_{\mathcal{A}_i}(x_i)$ for $i = 1, 2, 3$.

   (c) On a 2-d map, sketch the subspace $\mathcal{V}_1$ (a line through the origin) and clearly indicate x_1 and $\text{Proj}_{\mathcal{V}_1}(x_1)$. What is the point on $\mathcal{V}_1$ that is the closest to x_1 in Euclidean sense? On the same axes, sketch $\mathcal{A}_1$ (a line shifted from the origin) and indicate $\text{Proj}_{\mathcal{A}_1}(x_1)$.

   (d) Compute an orthonormal basis $\mathcal{B}_3$ for the subspace $\mathcal{V}_3$ via Gram-Schmidt. Recompute $\text{Proj}_{\mathcal{V}_3}(x_3)$ and $\text{Proj}_{\mathcal{A}_3}(x_3)$ using $\mathcal{B}_3$, and compare with your previous results.

3. *Distance between two parallel hyperplanes.* Find the distance between the two parallel hyperplanes $\mathcal{H}_i$, $i \in [2]$ where $\mathcal{H}_i = \{x \in \mathbb{R}^n | a^T x = b_i\}$. Your solution should be expressed in terms of the problem parameters, i.e., the vector $a \in \mathbb{R}^n$ and the scalars $b_i \in \mathbb{R}$.

4. *Taylor series expansion.* Consider the function $f(x) = -\sum_{l=1}^{m} \log(b_l - a_l^T x)$, where $x \in \mathbb{R}^n$, $b_l \in \mathbb{R}$ and $a_l \in \mathbb{R}^n$. Compute $\nabla f(x)$ and $\nabla^2 f(x)$. Write down the first three terms of the Taylor series expansion of $f(x)$ around some x_0.

5. *Linear dynamical systems.* OptM Book, Exercise 3.4.

Application Problems

1. *First-order approximation of functions.* In this exercise, you will write MATLAB (or Python, Julia, etc) code to plot linear approximations each of three functions $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in [3]$. The three functions are defined pointwise as

$$\begin{aligned} f_1(x,y) &= 2x + 3y + 1, \\ f_2(x,y) &= x^2 + y^2 - xy - 5, \\ f_3(x,y) &= (x-5)\cos(y-5) - (y-5)\sin(x-5). \end{aligned}$$

For each of the above functions, do the following.

(a) Write down the gradient with respect to x and y in closed form. The gradient can be compactly written in the form $\nabla f_i = \left[\frac{\partial f_i}{\partial x} \, \frac{\partial f_i}{\partial y}\right]^T$ for $i = 1, 2, 3$.

(b) For each function produce a 2-D contour plot indicating the level sets of each function in the range $-2 \leq x, y \leq 3.5$ (i.e., make three plots). An example of a contour plot is illustrated in Fig 2.28 of OptM (the second sub-figure). You may find **meshgrid** and **contour** commands in MATLAB useful. Please refer the MATLAB documentation for further details. In addition, compute the the gradient at the point $(x, y) = (1, 0)$ for each function. On your contour plots also plot the direction of the gradient and the tangent line to the level sets. Your resulting plot should be analogous to Fig 2.29 of OptM.

(c) For the same point $(x, y) = (1, 0)$ where, plot the 3-D linear approximation of the function. Since we are considering only the first derivative, the approximation should be the tangent plane at the specified point. Your plot for $f_2(x, y)$ should look something like Fig. 1. The function approximation is plotted as a tangent plane to the surface plot of $f_2(x, y)$. You may find **meshgrid** and **meshc** (or **mesh**) commands in MATLAB useful. Include your code for all sections.

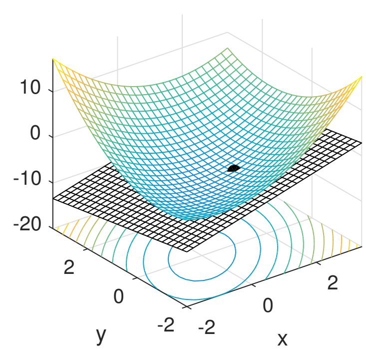

Figure 1: Example plot and approximating tangent plane.

Note: We recommend you design these plotting scripts as functions (in MATLAB, Python, Julia) so that you can reuse them for to plot approximations for different non-linear functions (or for these functions at different points). In either case, make sure to attach your code.

2. *Second-order approximation of functions.* In this exercise you extend your code from the problem "First-order approximation of functions" to second order. (As before you are welcome

to use Matlab or Python or the software of your choice.) The objective is, as before, to write code to plot approximations each of (the same) three functions $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in [3]$, but now the approximation is quadratic rather than linear. To recall, the three functions are defined pointwise as

$$\begin{aligned} f_1(x, y) &= 2x + 3y + 1, \\ f_2(x, y) &= x^2 + y^2 - xy - 5, \\ f_3(x, y) &= (x - 5)\cos(y - 5) - (y - 5)\sin(x - 5). \end{aligned}$$

For each of the above functions, do the following.

(a) Write down the gradient and Hessian with respect to x and y in closed form. To recall the gradient and Hessian are compactly denoted as ∇f_i and $\nabla^2 f_i$ for $i \in [3]$ where

$$\nabla f_i = \begin{bmatrix} \frac{\partial f_i}{\partial x} \\ \frac{\partial f_i}{\partial y} \end{bmatrix} \quad \nabla^2 f_i = \begin{bmatrix} \frac{\partial^2 f_i}{\partial^2 x} & \frac{\partial^2 f_i}{\partial x \partial y} \\ \frac{\partial^2 f_i}{\partial y \partial x} & \frac{\partial^2 f_i}{\partial^2 y} \end{bmatrix}.$$

(b) For each function, do the following two things. First, produce a 2-D contour plot indicating the level sets of each function in the range $-2 \leq x, y \leq 3.5$ and plot the direction of the gradient and tanget to the level set at the point (x,y) = 1,0). (Note: you have already produced these plots in the previous problem, we are just reproducing them here to help with visualization). Second, For the same point $(x, y) = (1, 0)$ plot the 3-D quadratic approximation of the function.

(c) Now, repeat the previous plot for point $(x, y) = (-0.7, 2)$ and $(x, y) = (2.5, -1)$.

(d) Comment on where your approximations are accurate and where they are not (if anywhere) for the three functions. Discuss what the reason is behind your observations.

Note: We recommend you design these plotting scripts as functions so that you can reuse them to plot approximations for different non-linear functions (or for theses functions at different points). In either case, make sure to attach your code.

Optional Problems

1. *Practice computing nullspace, range, and rank.*

   (a) Find rank(A) and the dimension of, and a basis for, each of the four subspaces $\mathcal{R}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A)$, $\mathcal{N}(A^T)$ when

   $$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (b) Find rank(B) and the dimension of, and a basis for, each of the four subspaces $\mathcal{R}(B)$, $\mathcal{R}(B^T)$, $\mathcal{N}(B)$, $\mathcal{N}(B^T)$ when

   $$B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

   (b) Find rank(C) and the dimension of, and a basis for, each of the four subspaces $\mathcal{R}(C)$, $\mathcal{R}(C^T)$, $\mathcal{N}(C)$, $\mathcal{N}(C^T)$ when

   $$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 6 & 9 \\ 1 & 4 & 4 & 4 \\ 1 & 0 & 10 & 4 \end{bmatrix}.$$

2. *Rank and nullspace.* OptM Book, Exercise 3.6.