1a i) $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}$ $\det(A-\lambda I)=0 = \det\begin{bmatrix}3-\lambda&1\\2&2-\lambda\end{bmatrix}=0 \to (3-\lambda)(2-\lambda)=2$

$6-3\lambda+\lambda^2-2\lambda=2 \to \lambda^2-5\lambda+4=0$ $\underline{\lambda_1=1\ \ \lambda_2=4}$

ii) $(A-\lambda_i I)v_i=0 \to \begin{bmatrix}3-1&1\\2&2-1\end{bmatrix}v_1=0$ $\begin{bmatrix}2&1\\2&1\end{bmatrix}v_1=0 \to \begin{bmatrix}2&1\\0&0\end{bmatrix}v_1=0$

$2v_1^{(1)}+v_1^{(2)}=0$

$\begin{bmatrix}3-4&1\\2&2-4\end{bmatrix}v_2=0$ $\begin{bmatrix}-1&1\\2&-2\end{bmatrix}v_2=0$ $-v_2^{(1)}+v_2^{(2)}=0$, $-2v_2^{(1)}+2v_2^{(2)}=0$ $v_1^{(1)}=-\frac{1}{2}v_1^{(2)}$

$\underline{v_2=\begin{bmatrix}1\\1\end{bmatrix}}$ $v_1=\begin{bmatrix}1\\-2\end{bmatrix}$

iii) $AM(1)=1$ Since quadratic roots distinct $AM(4)=1$ Since quadratic roots distinct

$GM(1)=1$ Since $\dim N(\begin{bmatrix}2&1\\2&1\end{bmatrix})=1$ $GM(4)=1$ Since $\dim N(\begin{bmatrix}-1&1\\2&-2\end{bmatrix})=1$

iv) $\forall$ eigenvalues $GM=AM$ $\therefore$ nondefective, $\therefore$ diagonalizable

$V=\begin{bmatrix}1&1\\-2&1\end{bmatrix}$ $\Lambda=\begin{bmatrix}1&0\\0&4\end{bmatrix}$

b i) $\begin{bmatrix}0-\lambda&1\\1&0-\lambda\end{bmatrix}$ $\det A-\lambda I=0=(-\lambda)^2-1^2=\lambda_1=1\ \ \lambda_2=-1$

ii) $\begin{bmatrix}-1&1\\1&-1\end{bmatrix}v_1=0$ $-v_1^{(1)}+v_1^{(2)}=0$ $\underline{v_1=\begin{bmatrix}1\\1\end{bmatrix}}$ $\begin{bmatrix}1&1\\1&1\end{bmatrix}v_2=0$ $v_2^{(1)}+v_2^{(2)}=0$ $v_2=\begin{bmatrix}1\\-1\end{bmatrix}$

iii) $AM(1)=1$ Since quadratic roots distinct $AM(-1)=1$ Since quadratic roots distinct

$GM(1)=1$ Since $\dim N(\begin{bmatrix}-1&1\\1&-1\end{bmatrix})=1$ $GM(-1)=1$ Since $\dim N(\begin{bmatrix}1&1\\1&1\end{bmatrix})=1$

iv) $\forall$ eigenvalues $GM=AM$ $\therefore$ nondefective, $\therefore$ diagonalizable

$V=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ $\Lambda=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Ci) $(-\lambda)^2-(1)(-1)=0 \to \lambda^2=-1 \to \lambda_1=j\ \ \lambda_2=-j$

ii) $\left[\begin{array}{cc|c}-j&1&0\\-1&-j&0\end{array}\right] \xrightarrow{\cdot j} \left[\begin{array}{cc|c}1&j&0\\-1&-j&0\end{array}\right] \to \left[\begin{array}{cc|c}1&j&0\\0&0&0\end{array}\right]$ $1v_1+jv_2=0$, $v_1=-jv_2$ $V_1=\begin{bmatrix}1\\j\end{bmatrix}$

$\left[\begin{array}{cc|c}j&1&0\\-1&j&0\end{array}\right] \xrightarrow{\cdot -j} \left[\begin{array}{cc|c}1&-j&0\\-1&j&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-j&0\\0&0&0\end{array}\right]$ $1v_1-jv_2=0$, $v_1=jv_2$ $V_2=\begin{bmatrix}j\\1\end{bmatrix}$

iii) $AM(j)=1$ Since quadratic roots distinct $AM(-j)=1$ Since quadratic roots distinct

$GM(j)=1$ Since $\dim N(\begin{bmatrix}-j&1\\-1&-j\end{bmatrix})=1$ $GM(-j)=1$ Since $\dim N(\begin{bmatrix}j&1\\-1&j\end{bmatrix})=1$

iv) $\forall$ eigenvalues $GM=AM$ $\therefore$ nondefective, $\therefore$ diagonalizable

$V=\begin{bmatrix}1&j\\j&1\end{bmatrix}$ $\Lambda=\begin{bmatrix}j&0\\0&-j\end{bmatrix}$

d i) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\det A - \lambda I = (1-\lambda)\cdot \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} - 0\cdot\cancel{\ } + 0\cdot\cancel{\ }$

$0 = (1-\lambda)\cdot\left((1-\lambda)^2 - 0\right) \rightarrow 0 = (1-\lambda)^3$ $\underline{\lambda = 1}$

ii) $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} v_1 + v_3 = 0 \\ v_1 = -v_3 \end{array}$ $v = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ x_2 also free var

iii) $AM(1) = 3$ since degree 3 with 1 root.

$GM(1) = 2$ since we have 2 free variables $\rightarrow \dim N\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 2$

iv) not diagonalizable since $GM < AM$ so it is defective.

2a) $A(p+q) = (pq^T + qp^T)(p+q)$

$= pq^Tp + qp^Tp + pq^Tq + qp^Tq$

$= p(p^Tq) + q(1) + p(1) + q(p^Tq)$

$= cp + q + p + cq$

$= p(1+c) + q(1+c)$ $\therefore p+q$ is an eigenvector of A with

$= (1+c)(p+q)$ eigenvalue $1+c$

$A(p-q) = (pq^T + qp^T)(p-q)$

$= pq^Tp + qp^Tp - pq^Tq - qp^Tq$

$= p(p^Tq) + q(1) - p(1) - q(p^Tq)$

$= cp + q - p - cq$

$= p(c-1) + q(1-c)$ $\therefore p-q$ is an eigenvector of A with

$= p(c-1) - q(c-1)$ eigenvalue $c-1$

$= (c-1)(p-q)$

b) $Ax = pq^Tx + qp^Tx = p(q^Tx) + q(p^Tx)$

$N(A) = \{x \in \mathbb{R}^n \mid q^Tx = 0 \text{ and } p^Tx = 0\}$

$\hookrightarrow$ all vectors $\perp$ to p and q in $\mathbb{R}^n$

Since p, q linearly indep., they span a 2d subspace of $\mathbb{R}^n$

$\therefore \dim N(A) = n-2$

Rank(A):

Using rank-nullity theorem

$$\text{rank}(A) = n - \text{nullity}(A)$$
$$= n-(n-2)$$
$$= \underline{2} \qquad \therefore \text{rank} A = 2$$

c) $A = Q \Lambda Q^T$

Since rank $A = 2$, only eigenvectors are $p+q$, $q-q$

Normalize:

$$(p+q)^T(p+q) = p^Tp + p^Tq + q^Tp + q^Tq = \|p\|^2 + 2c + \|q\|^2 = 2+2c$$
$$(p-q)^T(p-q) = p^Tp - p^Tq - q^Tp + q^Tq = 2-2c$$

$$\therefore\ u_1 = \frac{p+q}{\sqrt{2+2c}} \qquad u_2 = \frac{p-q}{\sqrt{2-2c}}$$

$$Q = \begin{bmatrix} | & | & | & & | \\ u_1 & u_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix}$$

where $v_3 \cdots v_n$ are obtained via gram schmidt and all cols of Q are orthogonal

$$\Lambda = \begin{bmatrix} 1+c & 0 & & \\ 0 & c-1 & & \\ & & 0 & \\ & & & \ddots \ 0 \end{bmatrix} \quad (\text{Since Rank } A = 2)$$

$$A = Q \Lambda Q^T$$
$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \overset{0}{\lambda_3} v_3 v_3^T + \cdots + \overset{0}{\lambda_n} v_n v_n^T$$
$$= \frac{(1+c)}{2+2c}(p+q)(p+q)^T - \frac{1-c}{2-2c}(p-q)(p-q)^T$$
$$= \frac{1}{2}(\cancel{pp^T} + pq^T + qp^T + \cancel{qq^T}) - \frac{1}{2}(\cancel{pp^T} - pq^T - qp^T + \cancel{qq^T})$$
$$= pq^T + qp^T$$

it works!

d) $\lambda_1 = p^T q + \|p\|\|q\|$ $\quad\quad$ $\lambda_2 = p^T q - \|p\|\|q\|$

$v_1 = p + \frac{\|p\|}{\|q\|} q$ $\quad\quad$ $v_2 = p - \frac{\|p\|}{\|q\|} q$

$$\|v_1\|^2 = \left(p + \frac{\|p\|}{\|q\|} q\right)^T \left(p + \frac{\|p\|}{\|q\|} q\right) = p^T p + 2\frac{\|p\|}{\|q\|} p^T q + \left(\frac{\|p\|}{\|q\|}\right)^2 q^T q = 2\|p\|^2 + 2\frac{\|p\|}{\|q\|} c$$

$$u_1 = \frac{p + \left(\frac{\|p\|}{\|q\|}\right) q}{\sqrt{2\|p\|^2 + 2p^T q \cdot \frac{\|p\|}{\|q\|}}}$$

$$\|v_1\|^2 = \left(p - \frac{\|p\|}{\|q\|} q\right)^T \left(p - \frac{\|p\|}{\|q\|} q\right) = p^T p - 2\frac{\|p\|}{\|q\|} p^T q - \left(\frac{\|p\|}{\|q\|}\right)^2 q^T q = 2\|p\|^2 - 2\frac{\|p\|}{\|q\|} c$$

$$u_2 = \frac{p - \frac{\|p\|}{\|q\|} q}{\sqrt{2\|p\|^2 - 2p^T q \frac{\|p\|}{\|q\|}}}$$

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

3a) $(2-\lambda)^2-1=\lambda^2-4\lambda+3=0 \quad \lambda_1=1,\ \lambda_2=3$

$\lambda_1>0,\ \lambda_2>0$ ∴ Positive definite ∴ ellipse

$\lambda_1=1: \quad \begin{bmatrix}1&1\\1&1\end{bmatrix}v=0 \rightarrow v=\begin{bmatrix}1\\-1\end{bmatrix} \quad \|v_1\|=\sqrt{2}$

$\lambda_2=3: \quad \begin{bmatrix}-1&1\\1&-1\end{bmatrix}v=0 \rightarrow v=\begin{bmatrix}1\\1\end{bmatrix} \quad \|v_2\|=\sqrt{2}$

$A=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

$y=Q^Tx \qquad x^TAx=y^TQ^TAQy=y^T\Lambda y=\lambda_1 x_1^2+\lambda_2 y_2^2\le 1$

$1x_1^2+3x_2^2\le 1$

↳ Standard ellipse form

b) $(1-\lambda)^2-1=0 \quad \lambda^2-2\lambda=0=\lambda(\lambda-2) \quad \lambda_1=0 \quad \lambda_2=2$

$\lambda_1=0$

$\begin{bmatrix}1&-1\\-1&1\end{bmatrix}v=0 \quad v_1=\begin{bmatrix}1\\1\end{bmatrix}$

$\|v_1\|=\sqrt{2}$

$\lambda_2=2$

$\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}v=0 \quad v_2=\begin{bmatrix}1\\-1\end{bmatrix}$

$\|v_2\|=\sqrt{2}$

$Q=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix} \qquad \Lambda=\begin{pmatrix}0&0\\0&2\end{pmatrix}$

$y=Q^Tx \quad x^TAx=(Qy)^TA(Qy)=y^TQ^TAQy=y^T\Lambda y=2y_2^2\le 1 \rightarrow y_2=\frac{\pm 1}{\sqrt{2}}$

y_1 free variable

∴ Shape is infinite strip bounded by $-\frac{1}{\sqrt{2}}\le y_2\le\frac{1}{\sqrt{2}}$

c) $-(1-\lambda)^2 = 0 \quad \lambda = -1$

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Pick $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$X^T A x = y^T \Lambda y = -y_1^2 - x_2^2 \leq 1$$

$y_1^2 + y_2^2 \geq -1$ true for all vals

$\therefore$ entire plane $\mathbb{R}^2$

$\therefore$ Shape is entire $\mathbb{R}^2$ plane

4a) $\text{tr}(A) = \text{tr}(Q \Lambda Q^T) = \text{tr}(\Lambda Q^T Q) = \text{tr}(\lambda) = \sum_{i=1}^{n} \lambda_i$

$$\|A\|_F = \sqrt{\sum A_{ij}^2} = \sqrt{\text{tr}(A^T A)} \to A = A^T \to \sqrt{\text{tr}(A^2)}$$

$$A^2 = (Q \Lambda Q^T)(Q \Lambda Q^T) = Q \Lambda^2 Q^T$$

$$= \sqrt{\text{tr}(Q \Lambda^2 Q^T)} = \sqrt{\text{tr}(\Lambda^2)} = \sqrt{\sum_{i=1}^{n} \lambda_i^2}$$

b) A PSD $\therefore \lambda_i \geq 0 \ \forall i = 1, \dots, n$

let $r = \text{rank}\, A$ be # of non-zero eigenvalues

WLoG: assume first r eigenvalues non zero and remaining $n-r$ are 0.

let $u = [\lambda_1, \lambda_2 \dots \lambda_r]^T$ $\quad v = \underbrace{[1, 1, \dots, 1]}_{r \text{ ones}}$

Cauchy-Schwartz:

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$$

$\langle u, v \rangle = \sum_{i=1}^{r} \lambda_i = \text{tr}(A)$

$\langle v, v \rangle = r$

$\langle u, u \rangle = \sum_{i=1}^{r} \lambda_i^2 = \|A\|_F^2$

$$\therefore (\text{tr}(A))^2 \leq \|A\|_F^2 \cdot \text{rank}(A)$$