1.

$$z^{(k)} = a^{(k)} - \sum_{i=1}^{k-1} \overbrace{\langle a^{(k)}, q^{(i)} \rangle q^{(i)}}^{\mathrm{Proj}_i (a_k)} \qquad q^{(k)} = \frac{z^{(k)}}{\|z^{(k)}\|}$$

$$z^{(1)} = a^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad q^{(1)} = a^{(1)}$$

$$z^{(2)} = a^{(2)} - \langle a^{(2)}, q^{(1)} \rangle q^{(1)} \qquad q^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= a^{(2)} - 1 \cdot q^{(1)}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$z^{(3)} = a^{(3)} - \langle a^{(3)}, q^{(1)} \rangle q^{(1)} - \langle a^{(3)}, q^{(2)} \rangle q^{(2)}$$

$$= a^3 - 1 \cdot q^{(1)} - 1 \cdot q^{(2)}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad q^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

it can be shown that each $\langle a^{(k)}, q^{(i)} \rangle$ for all $i = 1 \cdots k-1$ will be 1 since they only share 1 entry position valued at 1 for each.

This means the sum becomes $\sum_{i=1}^{k-1} q^{(i)}$ where $q^{(i)}$ has a 1 at the ith element.
$z^{(k)} = a^{(k)} - \sum_{i=1}^{k-1} z^{(i)}$ gives $z^{(k)}$ where $z^{(k)}$ has a single entry of 1 in the k^{th} position: $\begin{bmatrix} 0,0,\cdots,0,1,0,\cdots,0,0,0 \end{bmatrix}^T$ (positions $1,2,\cdots,k-1,k,k+1,\cdots,n$)
the l_2 norm of such a vector is 1 so $q^{(k)} = z^{(k)}$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I \in \mathbb{R}^{n \times n}$$

$q_1, \cdots, q_n$ forms an orthonormal basis for $\mathrm{Span}\{a_1, \cdots, a_n\}$ implying $a_1, \cdots, a_n$ are LI.
Since $\mathrm{span}\{q_1, \cdots, q_n\} = \mathrm{Span}\{a_1, \cdots, a_n\}$ and $a_1, \cdots, a_n$ LI, $a_1, \cdots, a_n$ forms a basis for A.

2a) $\text{Proj}_{V_1}(X_1)$ $\quad V_1 = \text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ $\quad X_1 = \begin{bmatrix}3\\-1\end{bmatrix}$

$X - X^* = X - \alpha u_1 \quad \langle X - \alpha u_1, V_1 \rangle = 0 \quad \langle X, V_1 \rangle = \alpha \langle V_1, V_1 \rangle$

$(3 + -2) = \alpha(5+1)$

$X^* = 0.2\,V_1 = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0.2\\0.4\end{bmatrix} \qquad \alpha = \frac{1}{5}$

$\text{Proj}_{V_2}(X_2) \quad X - X^* = X - \alpha_1 V_1 - \alpha_2 V_2$

$X^* = \sum_{i=1}^{2} \alpha_i v^{(i)} \qquad \langle X - X^*, v^{(j)} \rangle = 0 \ \forall\, j \in \{1,2\} \qquad \langle X, v^{(j)} \rangle = \langle \sum_{i=1}^{2} \alpha_i v^{(i)}, v^{(j)} \rangle$

$$\begin{bmatrix}\langle v^{(1)}, v^{(1)} \rangle & \langle v^{(2)}, v^{(1)} \rangle \\ \langle v^{(1)}, v^{(2)} \rangle & \langle v^{(2)}, v^{(2)} \rangle\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}\langle X, v^{(1)} \rangle \\ \langle X, v^{(2)} \rangle\end{bmatrix} \rightarrow \begin{bmatrix}4 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}10\\-2\end{bmatrix}$$

$$= \begin{bmatrix}4 & -2\\0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}10\\3\end{bmatrix} \rightarrow \begin{bmatrix}4 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}13\\3\end{bmatrix} \qquad \alpha_1 = \frac{13}{4} \quad \alpha_2 = \frac{3}{2}$$

$$X^* = \alpha_1 v^{(1)} + \alpha_2 v^{(2)} = \frac{13}{4}\begin{bmatrix}0\\0\\-2\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} \rightarrow \boxed{X^* = \begin{bmatrix}1.5\\1.5\\-5\end{bmatrix}}$$

$\text{Proj}_{V_3} X_3 \qquad \langle v^{(1)}, v^{(1)} \rangle = 3\cdot 0^2 + 1^2 + 2^2 = 5 \quad \langle v^{(2)}, v^{(2)} \rangle = 1^2 + 3^2 + 1^2 = 11 \quad \langle v^{(3)}, v^{(3)} \rangle = 1^2 + 5^2 + 1^2 = 27$

$\langle v^{(2)}, v^{(1)} \rangle = \langle v^{(1)}, v^{(2)} \rangle = \cancel{0 \cdot 1} + 1\cdot 3 + \cancel{2\cdot 0} + \cancel{0\cdot 1} + \cancel{1\cdot 0} = 3 \quad \langle v^{(3)}, v^{(1)} \rangle = \langle v^{(1)}, v^{(3)} \rangle = \cancel{0\cdot 0} + 1^2 + 2\cdot 5 + \cancel{0^2} + \cancel{0\cdot 1} = 11$

$\langle v^{(2)}, v^{(3)} \rangle = \langle v^{(3)}, v^{(2)} \rangle = \cancel{1\cdot 0} + 3\cdot 1 + \cancel{0\cdot 5} + \cancel{1\cdot 0} + \cancel{0\cdot 1} = 3 \quad \langle X, v^{(1)} \rangle = \cancel{3\cdot 0} + \cancel{0\cdot 1} + -1\cdot 2 + \cancel{0\cdot 2} + \cancel{0\cdot 2} = -2$

$\langle X, v^{(2)} \rangle = 3\cdot 1 + \cancel{0\cdot 3} + \cancel{-1\cdot 0} + 2\cdot 1 + \cancel{2\cdot 0} = 5 \quad \langle X, v^{(3)} \rangle = \cancel{3\cdot 0} + \cancel{0\cdot 1} + -1\cdot 5 + \cancel{0\cdot 2} + 2\cdot 1 = -3$

$$\begin{bmatrix}\langle v^{(1)}, v^{(1)} \rangle & \langle v^{(2)}, v^{(1)} \rangle & \langle v^{(3)}, v^{(1)} \rangle \\ \langle v^{(1)}, v^{(2)} \rangle & \langle v^{(2)}, v^{(2)} \rangle & \langle v^{(3)}, v^{(2)} \rangle \\ \langle v^{(1)}, v^{(3)} \rangle & \langle v^{(2)}, v^{(3)} \rangle & \langle v^{(3)}, v^{(3)} \rangle\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}\langle X, v^{(1)} \rangle \\ \langle X, v^{(2)} \rangle \\ \langle X, v^{(3)} \rangle\end{bmatrix} \rightarrow \begin{bmatrix}5 & 3 & 11\\3 & 11 & 3\\11 & 3 & 27\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}-2\\5\\-3\end{bmatrix}$$

$\det(V) = 5\cdot(11\cdot 27 - 9) - 3(3\cdot 27 - 3\cdot 11) + 11(9 - 11^2) \rightarrow \det(V) = 64$

$$\text{Adj}(V) = \begin{bmatrix}(-1)^2(11\cdot 27 - 9) & (-1)^3(3\cdot 27 - 3\cdot 11) & (-1)^4(9-121)\\ (-1)^5(3\cdot 27 - 3\cdot 11) & (-1)^6(5\cdot 27 - 121) & (-1)^7(15-33)\\ (-1)^8(9-121) & (-1)^9(15-33) & (-1)^{10}(5\cdot 11 - 9)\end{bmatrix}^T \rightarrow \text{Adj}(V) = \begin{bmatrix}288 & -48 & -112\\-48 & 14 & 18\\-112 & 18 & 46\end{bmatrix} \quad V^{-1} = \frac{1}{4}\begin{bmatrix}18 & -3 & -7\\-3 & 7/8 & 9/8\\-7 & 9/8 & \frac{23}{8}\end{bmatrix}$$

$$\alpha = V^{-1} \cdot X \rightarrow \alpha = \frac{1}{4}\begin{bmatrix}18 & -3 & -7\\-3 & 7/8 & 9/8\\-7 & 9/8 & \frac{23}{8}\end{bmatrix}\begin{bmatrix}-2\\5\\-3\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-2\cdot 18 + -3\cdot 5 + -7\cdot -3\\ -3\cdot -2 + \frac{7}{8}\cdot 5 + \frac{9}{8}\cdot -3\\ -7\cdot -2 + \frac{9}{8}\cdot 5 + \frac{23}{8}\cdot -3\end{bmatrix} \rightarrow \alpha = \begin{bmatrix}-\frac{15}{2}\\7/4\\11/4\end{bmatrix}$$

$$X^* = -\frac{15}{2}\cdot\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix} + \frac{7}{4}\begin{bmatrix}1\\3\\0\\1\\0\end{bmatrix} + \frac{11}{4}\begin{bmatrix}0\\1\\5\\0\\1\end{bmatrix} \rightarrow X^* = \frac{1}{4}\begin{bmatrix}7\\2\\-5\\7\\11\end{bmatrix}$$

b) $\mathrm{Proj}_{A_1}(X_1)$ $\quad X^* = \alpha V + b \rightarrow \langle X-b, V\rangle = \alpha\langle V,V\rangle \rightarrow \alpha = \frac{\langle X-b\rangle}{\langle V,V\rangle}$

$$X-b = \begin{bmatrix}3\\-1\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix} \qquad \alpha = \frac{4\cdot 1 + 2\cdot -2}{5} \rightarrow \alpha = 0 \qquad X^* = 0\cdot\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

$\mathrm{Proj}_{A_2}(X_2)$

$$X^* = \sum_{i=1}^{2} \alpha_i V_i + b \rightarrow \langle X-b, V_j\rangle = \left\langle \sum_{i=1}^{2}\alpha_i V_i, V_j\right\rangle \quad j\in\{1,2\} \qquad X-b = \begin{bmatrix}2\\4\\-4\end{bmatrix}$$

$$\begin{bmatrix}\langle V_1,V_1\rangle & \langle V_2,V_1\rangle\\ \langle V_2,V_1\rangle & \langle V_2,V_2\rangle\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}\langle X-b,V_1\rangle\\ \langle X-b,V_2\rangle\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}4 & -2 & 8\\ -2 & 3 & 2\end{array}\right] \rightarrow \left[\begin{array}{cc|c}4 & -2 & 8\\ 0 & 2 & 6\end{array}\right] \rightarrow \left[\begin{array}{cc|c}4 & 0 & 14\\ 0 & 2 & 6\end{array}\right] \quad \begin{array}{l}2\alpha_2 = 6 \rightarrow \alpha_2 = 3\\ 4\alpha_1 = 14 \rightarrow \alpha_1 = \frac{7}{2}\end{array}$$

$$X^* = \frac{7}{2}\begin{bmatrix}0\\0\\-2\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\-3\\-1\end{bmatrix} \rightarrow X^* = \begin{bmatrix}0+3+0\\0+3+-3\\-7+3+-1\end{bmatrix} \rightarrow X^* = \begin{bmatrix}3\\0\\-5\end{bmatrix}$$

$\mathrm{Proj}_{A_3}(X_3)$

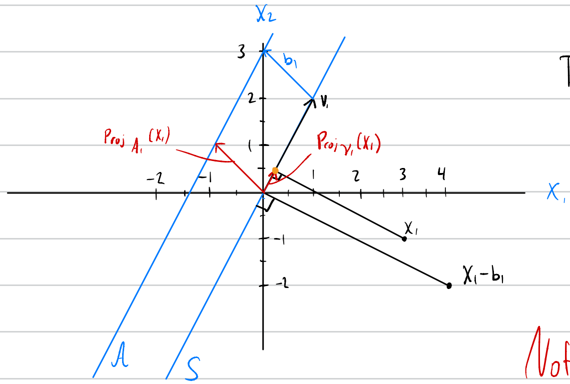
$$X-b = [4,0,-2,4,1]^T \quad \langle X-b, V^{(1)}\rangle = -4 \quad \langle X-b, V^{(2)}\rangle = 4+4 = 8 \quad \langle X-b, V^{(3)}\rangle = -10+1 = -9$$

$$\begin{bmatrix}\langle V^{(1)},V^{(1)}\rangle & \langle V^{(2)},V^{(1)}\rangle & \langle V^{(3)},V^{(1)}\rangle\\ \langle V^{(1)},V^{(2)}\rangle & \langle V^{(2)},V^{(2)}\rangle & \langle V^{(3)},V^{(2)}\rangle\\ \langle V^{(1)},V^{(3)}\rangle & \langle V^{(2)},V^{(3)}\rangle & \langle V^{(3)},V^{(3)}\rangle\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}\langle X-b,V^{(1)}\rangle\\ \langle X-b,V^{(2)}\rangle\\ \langle X-b,V^{(3)}\rangle\end{bmatrix} \rightarrow \begin{bmatrix}5 & 3 & 11\\ 3 & 11 & 3\\ 11 & 3 & 27\end{bmatrix}\alpha = \begin{bmatrix}-4\\8\\-9\end{bmatrix} \qquad \det(V) = 64$$

$$V^{-1} = \frac{1}{4}\begin{bmatrix}18 & -3 & -7\\ -3 & 7/8 & 9/8\\ -7 & 9/8 & \frac{23}{8}\end{bmatrix} \qquad \alpha = V^{-1}X = \frac{1}{4}\begin{bmatrix}18 & -3 & -7\\ -3 & 7/8 & 9/8\\ -7 & 9/8 & \frac{23}{8}\end{bmatrix}\begin{bmatrix}0\\8\\1\end{bmatrix} \rightarrow \alpha = \begin{bmatrix}0 + \frac{-24}{4} + \frac{-7}{4}\\ 7/4 + 9/32\\ 9 + \frac{23}{8}\end{bmatrix} \rightarrow \alpha = \begin{bmatrix}-\frac{33}{4}\\ 71/32\\ 89/32\end{bmatrix}$$

$$X^* = -\frac{33}{4}\cdot\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix} + \frac{71}{32}\begin{bmatrix}1\\3\\0\\1\\0\end{bmatrix} + \frac{89}{32}\begin{bmatrix}0\\1\\5\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\1\\-2\\1\end{bmatrix} \rightarrow X^* = \frac{1}{32}\begin{bmatrix}39\\38\\-51\\7\\121\end{bmatrix} = \begin{bmatrix}39/32\\19/16\\-51/32\\7/32\\121/32\end{bmatrix}$$

c)



The point on γ_1 closest to X_1 is

$\mathrm{Proj}_{\gamma_1}(X_1) = \begin{bmatrix}0.2\\0.4\end{bmatrix}$ Orange on plot

Note that $\mathrm{Proj}_S(X_1 - b_1) = 0$

So it is not shown on plot.

d)

$$z^{(k)} = a^{(k)} - \sum_{i=1}^{k-1} \langle a^{(k)}, q^{(i)} \rangle q^{(i)}$$

$$z^{(1)} = v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad p^{(1)} = \frac{z^{(1)}}{\|z^{(1)}\|} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix}$$

$$z^{(2)} = v^{(2)} - \langle v^{(2)}, q^{(1)} \rangle q^{(1)} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{12}{5} \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$$

$$\|z^{(2)}\| = \sqrt{1^2 \cdot 2 + \left(\tfrac{12}{5}\right)^2 + \left(\tfrac{6}{5}\right)^2} = \frac{\sqrt{230}}{5} \qquad q^{(2)} = \begin{bmatrix} 5/\sqrt{230} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ 5/\sqrt{230} \\ 0 \end{bmatrix}$$

$$z^{(3)} = v^{(3)} - \langle v^{(3)}, q^{(1)} \rangle q^{(1)} - \langle v^{(3)}, q^{(2)} \rangle q^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} - \frac{11}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} + \frac{18}{\sqrt{230}} \begin{bmatrix} 5/\sqrt{230} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ 5/\sqrt{230} \\ 0 \end{bmatrix} = \begin{bmatrix} 9/23 \\ -6/23 \\ 3/23 \\ 9/23 \\ 1 \end{bmatrix}$$

$$\|z^{(3)}\| = \sqrt{\left(\tfrac{9}{23}\right)^2 \cdot 2 + \tfrac{6}{23}^2 + \tfrac{9}{23}^2 + 1^2} = \frac{4\sqrt{46}}{23}$$

$$q^{(3)} = \begin{bmatrix} 9/4\sqrt{46} \\ -6/4\sqrt{46} \\ 3/4\sqrt{46} \\ 9/4\sqrt{46} \\ 23/4\sqrt{46} \end{bmatrix}$$

Orthonormal basis for V_3: $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/\sqrt{230} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ 5/\sqrt{230} \\ 0 \end{bmatrix}, \begin{bmatrix} 9/4\sqrt{46} \\ -6/4\sqrt{46} \\ 3/4\sqrt{46} \\ 9/4\sqrt{46} \\ 23/4\sqrt{46} \end{bmatrix} \right\}$ 3 0 −1 2 2

$\mathrm{Proj}_{V_3}(X_3)$

$\langle q^{(1)}, q^{(1)} \rangle = 1 \quad \langle q^{(2)}, q^{(2)} \rangle = 1 \quad \langle q^{(3)}, q^{(3)} \rangle = 1 \quad \langle q^{(1)}, q^{(2)} \rangle = 0 \quad \langle q^{(2)}, q^{(3)} \rangle = 0 \quad \langle q^{(1)}, q^{(3)} \rangle = 0$

$\langle X, q^{(1)} \rangle = -\frac{2}{\sqrt{5}} \quad \langle X, q^{(2)} \rangle = \frac{1}{\sqrt{230}} \cdot (3 \cdot 5 + -1 \cdot -6 + 2 \cdot 5) = \frac{31}{\sqrt{230}} \quad \langle X, q^{(3)} \rangle = \frac{1}{4\sqrt{46}} (-6 + 5 \cdot 3 + 23 \cdot 1) = \frac{32}{4\sqrt{46}}$

$$\begin{bmatrix} \langle q^{(1)}, q^{(1)} \rangle & \langle q^{(2)}, q^{(1)} \rangle & \langle q^{(3)}, q^{(1)} \rangle \\ \langle q^{(1)}, q^{(2)} \rangle & \langle q^{(2)}, q^{(2)} \rangle & \langle q^{(3)}, q^{(2)} \rangle \\ \langle q^{(1)}, q^{(3)} \rangle & \langle q^{(2)}, q^{(3)} \rangle & \langle q^{(3)}, q^{(3)} \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \langle X, q^{(1)} \rangle \\ \langle X, q^{(2)} \rangle \\ \langle X, q^{(3)} \rangle \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \alpha = \begin{bmatrix} -2/\sqrt{5} \\ 31/\sqrt{230} \\ \frac{22}{4\sqrt{46}} \end{bmatrix}$$

$\alpha_1 = -2/\sqrt{5}$
$\alpha_2 = 31/\sqrt{230}$
$\alpha_3 = \frac{22}{\sqrt{46}}$

$$x^* = -\frac{2}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} + \frac{31}{\sqrt{230}} \begin{bmatrix} 5/\sqrt{230} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ 5/\sqrt{230} \\ 0 \end{bmatrix} + \frac{22}{\sqrt{46}} \begin{bmatrix} 9/4\sqrt{46} \\ -6/4\sqrt{46} \\ 3/4\sqrt{46} \\ 9/4\sqrt{46} \\ 23/4\sqrt{46} \end{bmatrix} = \begin{bmatrix} 0 + \frac{31 \cdot 5}{230} + \frac{22 \cdot 9}{4 \cdot 46} \\ -\frac{2}{5} + \frac{31 \cdot 12}{230} + \frac{22 \cdot -6}{4 \cdot 46} \\ -\frac{4}{5} + \frac{31 \cdot -6}{230} + \frac{22 \cdot 3}{4 \cdot 46} \\ 0 + \frac{31 \cdot 5}{230} + \frac{22 \cdot 9}{4 \cdot 46} \\ 0 + 0 + \frac{23 \cdot 22}{4 \cdot 46} \end{bmatrix} = \begin{bmatrix} 7/4 \\ 1/2 \\ -5/4 \\ 7/4 \\ 11/4 \end{bmatrix} = \text{Same as before}$$

$\mathrm{Proj}_{A_3}(X_2) \quad X - b = [4, 0, -2, 4, 1]^T$

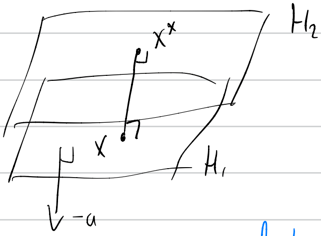
$\langle X-b, q^{(1)} \rangle = -\frac{4}{\sqrt{5}} \quad \langle X-b, q^{(2)} \rangle = \frac{1}{\sqrt{230}} (4 \cdot 5 + -2 \cdot -6 + 4 \cdot 5) = \frac{52}{\sqrt{230}} \quad \langle X-b, q^{(3)} \rangle = \frac{1}{4\sqrt{46}} (4 \cdot 9 + -2 \cdot 3 + 4 \cdot 9 + 1 \cdot 23) = \frac{89}{4\sqrt{46}}$

$$I \alpha = \begin{bmatrix} -4/\sqrt{5} \\ 28/\sqrt{230} \\ 89/4\sqrt{46} \end{bmatrix} \to \begin{matrix} \alpha_1 = -\frac{4}{\sqrt{5}} \\ \alpha_2 = 28/\sqrt{230} \\ \alpha_3 = 89/4\sqrt{46} \end{matrix}$$

$$x^* = -\frac{4}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} + \frac{52}{\sqrt{230}} \begin{bmatrix} 5/\sqrt{230} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ 5/\sqrt{230} \\ 0 \end{bmatrix} + \frac{89}{4\sqrt{46}} \begin{bmatrix} 9/4\sqrt{46} \\ -6/4\sqrt{46} \\ 3/4\sqrt{46} \\ 9/4\sqrt{46} \\ 23/4\sqrt{46} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + \frac{52 \cdot 5}{230} + \frac{89 \cdot 9}{16 \cdot 46} - 1 \\ -\frac{4}{5} + \frac{52 \cdot 12}{230} + \frac{89 \cdot -6}{16 \cdot 46} + 0 \\ -\frac{8}{5} + \frac{52 \cdot -6}{230} + \frac{89 \cdot 3}{16 \cdot 46} + 1 \\ 0 + \frac{52 \cdot 5}{230} + \frac{89 \cdot 9}{16 \cdot 46} - 2 \\ 0 + 0 + \frac{89 \cdot 23}{16 \cdot 46} + 1 \end{bmatrix} = \begin{bmatrix} 39/32 \\ 19/16 \\ -51/32 \\ 7/32 \\ 121/32 \end{bmatrix} = \text{Same as before}$$

Same results but easier computation since Id matrix.

3. $\mathcal{H}_i = \{x \in \mathbb{R}^n \mid a^T x = b_i\},\ i \in [2]$



$x - x^* = ca$

$a^T(x - x^*) = c a^T a$

$a^T x - a^T x^* = c\,\|a\|_2^2$

$b_1 - b_2 = c\,\|a\|_2^2 \qquad c = \dfrac{b_1 - b_2}{\|a\|_2^2}$

distance $= |c|\,\|a\|_2 = \dfrac{b_1 - b_2}{\|a\|_2^2}\|a\|_2 = \dfrac{b_1 - b_2}{\|a\|_2}$

4. $f(x) = -\sum_{l=1}^{m} \log(b_l - a_l^T x) \quad x, a_l \in \mathbb{R}^n \quad b_l \in \mathbb{R}$

$$f = -\sum_{l=1}^{m} \log\left(b_l - \sum_{i=1}^{n} a_i x_i\right)$$

$n \times 1 \cdot 1 \times n$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \partial f / \partial x_n \end{bmatrix} = -\sum_{l=1}^{m} \begin{bmatrix} \frac{a_l^{(1)}}{b_l - a_l^T x} \\ \vdots \\ \frac{a_l^{(n)}}{b_l - a_l^T x_n} \end{bmatrix} = -\sum_{l=1}^{m} \frac{1}{b_l - a_l^T x} \begin{bmatrix} a_l^{(1)} \\ a_l^{(2)} \\ \vdots \\ a_l^{n} \end{bmatrix} = -\sum_{l=1}^{m} \frac{a_l}{b_l - a_l^T x}$$

$$\nabla^2 f(x) = \nabla(\nabla f(x))^T = \nabla - \sum_{l=1}^{m} \frac{a_l^T}{b_l - a_l^T x} = -\sum_{l=1}^{m} a_l^T (b_l - a_l^T x)^{-2} \cdot \nabla(b_l - a_l^T x)$$

$$= -\sum_{l=1}^{m} a_l^T (b_l - a_l^T x)^{-2} \cdot -\begin{bmatrix} a_l^{(1)} \\ a_l^{(2)} \\ \vdots \\ a_l^{(n)} \end{bmatrix} \to \nabla^2 f(x) = \sum_{l=1}^{m} a_l a_l^T (b_l - a_l^T x)^{-2}$$

$$f(x) \approx f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

$$= -\sum_{l=1}^{m} \log(b_l - a_l^T x_0) - \sum_{l=1}^{m} \frac{a_l^T}{b_l - a_l^T x_0}(x - x_0) + \frac{1}{2}(x - x_0)^T \cdot \sum_{l=1}^{m} \frac{a_l a_l^T}{(b_l - a_l^T x_0)^2}(x - x_0)$$

5a) $x(t+1) = Ax(t) + Bu(t) \quad y(t) = Cx(t)$

$x(0) = 0$

$x(1) = Ax(0) + Bu(0) = Bu(0)$

$x(2) = Ax(1) + Bu(1) = ABu(0) + Bu(1)$

$x(3) = Ax(2) + Bu(2) = A^2Bu(0) + ABu(1) + Bu(2)$

$$x(t) = A^{T-1}Bu(0) + A^{T-2}Bu(1) + \cdots + Bu(T-1)$$
$$= \sum_{i=1}^{T-1} A^{T-1-i}Bu(t)$$

$$y(t) = C\sum_{i=1}^{T-1} A^{T-1-i}Bu(t) = \sum_{i=1}^{T-1} CA^{T-1-i}Bu(t)$$

$$Y = \begin{bmatrix} CA^{T-1}B & CA^{T-2}B & \cdots & CAB & CB \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T-1) \end{bmatrix}$$

$$H = \begin{bmatrix} CA^{T-1}B & CA^{T-2}B & \cdots & CAB & CB \end{bmatrix}$$

b) The range of H is the set of all $\underbrace{\text{transformations of the input sequence } U(T) \text{ through } H}_{Y(T)}$, with the initial condition $x(0) = 0$

Application Questions:

1a) $\nabla f_1 = [2\ 3]^T$ $\nabla f_2 = [2x-y \quad 2y-x]^T$ $\nabla f_3 = [\cos(y-5) - (x-5)\cos(x-5) \quad (5-x)(\sin(y-5) - \sin(x-5))]^T$

2a) $\nabla^2 f_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \nabla^2 f_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\nabla^2 f_3 = \begin{bmatrix} -(y-5)\sin(5-x) & \sin(5-y) - \cos(5-x) \\ \sin(5-y) - \cos(5-x) & -(x-5)\cos(5-y) \end{bmatrix}$$