Numerical Methods in Finance : Pricing in Discrete time

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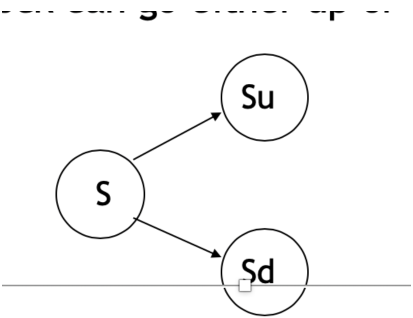
June 13, 2023

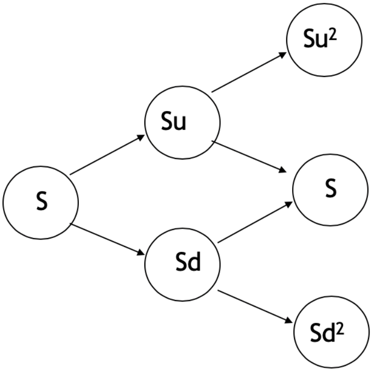
SDE for stock in the BS model, risk-neutral measure

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t) \tag{2.1}$$

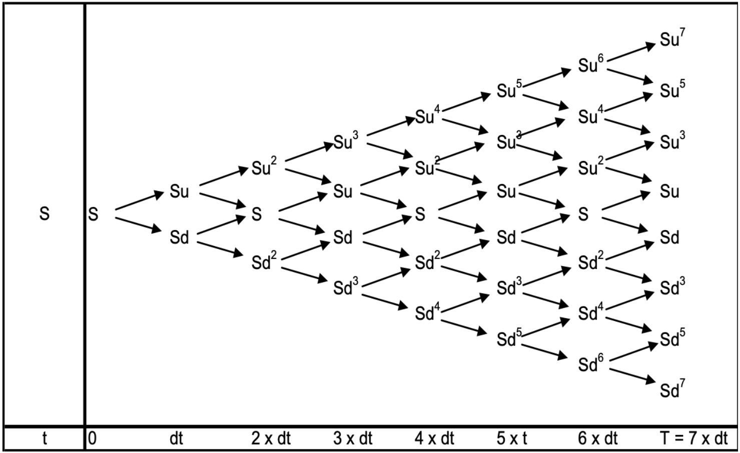
Approximation of this process: after a small time step the stock can go either up or down



- Take u and d such that u*d = 1
- After two steps the tree recombines

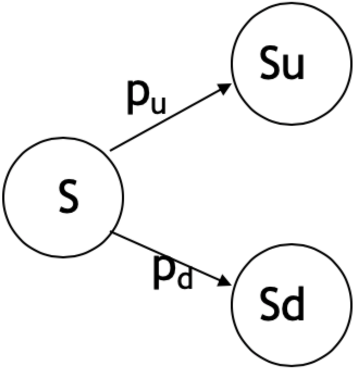


- After more steps



- In risk neutral measure all assets are martingale

$$\boldsymbol{E}_t \left(S_{t+\Delta} e^{-r\Delta t} \right) = S_t$$



- In discrete setting this just reduces of solving

$$S = p_u u S e^{-r\Delta t} + p_d d S e^{-r\Delta t}$$

Solving for solution shows:

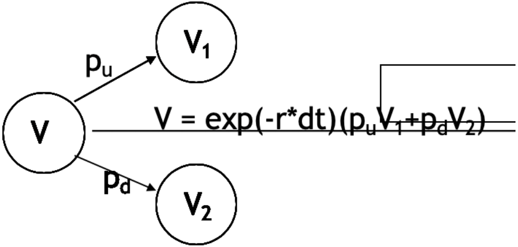
- the solution is :

$$p_u = \frac{e^{rt} - d}{u - d}$$
$$p_d = \frac{u - e^{rt}}{u - d}$$

- Up/Down u, d are chosen such that are chosen to ensure that the volatility of the stock is σ
$$u = e^{\sigma\sqrt{t}}, d = e^{-\sigma\sqrt{t}}$$
- What happens if there is no solution ? (no martingale measure)

Option pricing

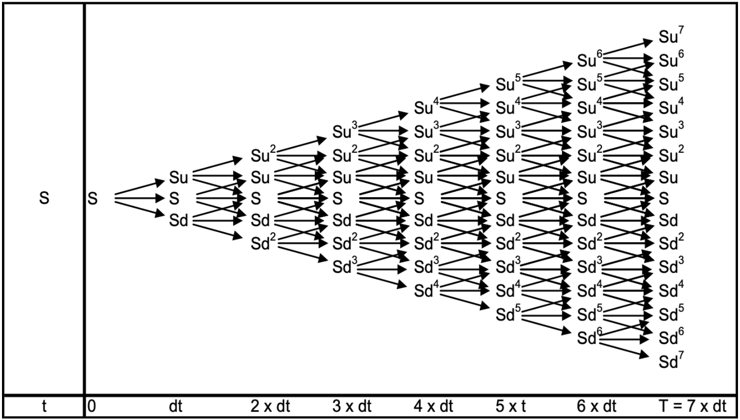
- Option pricing is done by backward induction
- Start at the last time step: the price of the option is known, it's simply the payoff
- Then proceed backward: at each node of the tree the price of the option is the calculated from the price of the option at the two nodes branching from it



- What can we price using trees? Calls/ American exercise / Mild Path dependent
- Pricing Black Scholes in tree vs discretization steps (see spreadsheet)

Trinomial trees

- In a trinomial tree there are 3 nodes branching from each node



- We can construct a trinomial tree by simply concatenating steps of the binomial tree

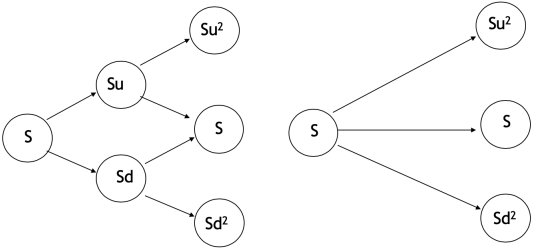


Figure: Trinomial Tree Construction from Binomial

- The convergence of the trinomial tree to Black Scholes:

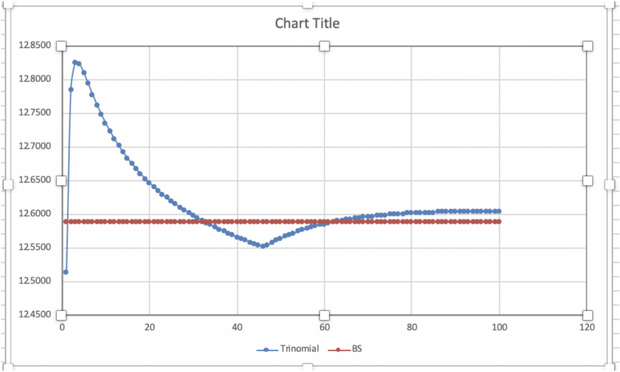


Figure: Tree Price vs Number of Steps Vs Black Scholes price

Binomial vs Trinomial trees comparison

- The price given by the trinomial tree does not exhibit the zig-zag pattern
- the trinomial tree simply picks every second price given by the binomial tree Because this trinomial tree with N time steps gives the same price as a binomial with 2N time steps
- The real advantage is speed: A binomial tree with 2N steps takes four times longer to run than one with N steps But the same result can be obtained from a trinomial tree with N steps, which takes only 33% longer to run

American Options: American put

- An American option on a security gives the holder of the option the right to *exercise* it at any day before the expiration date T. An exercise strategy τ is a rule specifying when the option should be exercised. In general τ is a random variable that can depend of the evolution of the model. However is not possible to base the decision to exercise on information that is not yet available.

- Mathematical setup:

$$V(t,s) = \sup_{\tau \in [t,T]} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-\tau)} \Phi(S_\tau) \middle| S_t = s \right]$$

where $\Phi(\cdot)$ is the payoff, also called *intrinsic value*.

- This formula resembles the formula of a European option, except that in this case the option can be exercised at any time. But... at which time? The problem is to find the best exercise time τ that maximizes the expectation.

With trees or Finite Differences (PDE) it is almost trivial to modify the scheme to allow for early exercise

- Notice that the time τ is a random variable! It can be different for different paths of the stock process.

- The solution of the problem provides a strategy, that at each time assesses if it is optimal to exercise the option or not, depending on the current value of the underlying stock.

- The American put does not have an analytical formula

- American Call: positive interest rates and no dividents the optimal exercise is at maturity (the payoff at maturity dominates the instrinc value)

- In a tree the conditional expectation comes for free

- For example the conditional expectation at the circled point here is the value calculated on the subtree that starts from the point

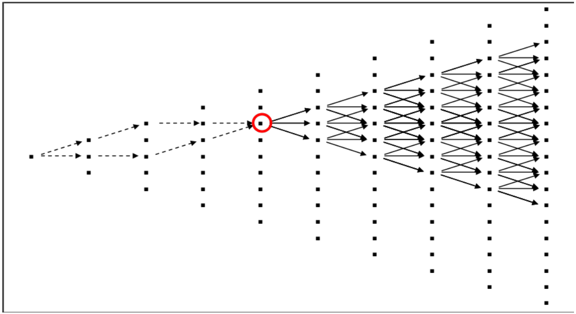


Figure: Conditional Expectation in Tree

Implementation of American Options in the Tree

- You need a numerical scheme to calculate its value, such as trees or Finite Differences
- At each step you can decide if you want to exercise the option by computing the continuation value vs the intrinsic value
- You do the backward induction in the following way:
- At the last time step the option is equal to the payoff
- At each other timestep you calculate the continuation value, as you do for the European option
- And then you replace this with the larger of the continuation value and the immediate exercise
- Not more difficult vs an European option

Convergence of binomial tree to American put vs European put

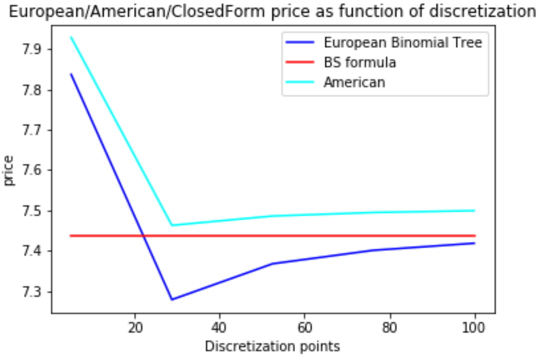


Figure: American Put vs European Put vs Black Scholes vs discretization : T=1, S =100,K=100

Discretizing the SDE

- Remember that the (standard) Brownian motion $\{W_t\}_{t \geqslant 0}$ is a continuous time stochastic process, that satisfies the following properties:
  - $W_0 = 0$.
  - The increments are stationary and independent
  - It is a martingale.
  - It has continuous paths, but nowhere differentiable.
  - $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $t \geqslant s \geqslant 0$.
- For more info see here [wiki](https://en.wikipedia.org/wiki/Brownian$_m otion$).

$$W_{t_i + \Delta t} - W_{t_i} = \Delta W_i \sim \mathcal{N}(0, \sigma^2 \Delta t).$$

- The process at time T is given by $W_T = \sum_i \Delta W_i$

$$W_T \sim \mathcal{N}(0, \sigma^2 T).$$

Euler Maruyama method

- Let us divide the time interval $[0, T]$ in $0 = t_0, t_1, ..., t_{n-1}, t_n = T$.
- We can choose equally spaced points t_i such that $\Delta t = t_{i+1} - t_i = \frac{T}{N}$ for each $1 \leqslant i \leqslant N$.
- A generic Itô-Diffusion SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, X_0 = 0$$

- can be approximated by:

$$X_{i+1} = X_i + \mu(t_i, X_i)\Delta t + \sigma(t_i, X_i)\Delta W_i$$
$$X(t_i) = X_i, W(t_i) = W_i, X_0 = 0$$

- The quantity to simulate is $W_{i+1} - W_i = \Delta W_i \sim \mathcal{N}(0, \sqrt{\Delta t})$.

Black Scholes SDE (log normal)

- Simulate Brownian motion paths
- Take exponential based on Ito's formula in Risk Neutral measure

$$dS_t = rS_t dt + \sigma S_t dW_t$$

- By applying the Itô lemma on $\log S_t$ it is possible to solve this equation

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

- Price Options by taking the average i.e. value of the call option:

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \middle| S_0\right] \approx \frac{1}{N}\sum_{i=1}^{N}(S_T^i - K)^+$$

For a put option I use this payoff $(K - S_T)^+$ inside the expectation.

Convergence of Monte Carlo for European Call Option S=100,K=100,vol=23%

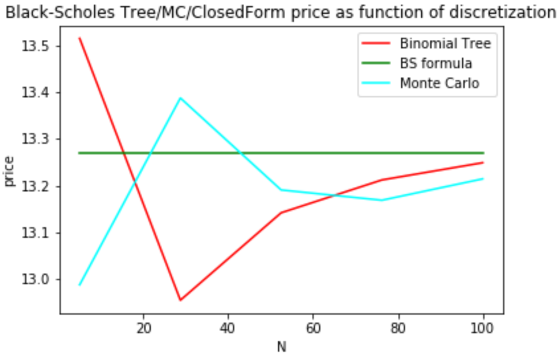


Figure: Binomial Tree Vs Monte Carlo vs Closed Form vs Discretization

Variance reduction techniques for Monte Carlo

- Antithetic Variables i.e take X and -X have the same distribution for standard normal
- For each N simulations with X also add -X which results in 2N paths with lower variance
- Moment matching : for example the standard mean and standard deviation of Normal variable is 0 and 1
- after generating a sequence of random variables make sure to correct the mean and standard deviation by adding the mean variance correction
- Always start the sequence of random numbers from the same seed in order to ensure stable greeks

Moment matching

- Assume that we want to simulate random normals with mean 0 and standard deviation σ
- Let X_i i.i.d (independent and identical distributed), for $1 \leqslant i \leqslant n$, with $\mathbb{E}[X_i] = 0$ and $Var[X_i] = \sigma^2$.
- Sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- Sample variance $Var[\bar{X}] = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$.
- Add a correction term for mean and variance

$$Y_i = (X_i - \bar{X}) \frac{\sigma}{\sqrt{Var[X_i]}}$$

American Monte Carlo

- It's difficult to price an option with early exercise using Monte Carlo
  - But some options require Monte Carlo to price
  - Path-dependent options
  - Options on baskets
  - Options priced using models with many factors

- Options with early exercise are called American if you can exercise at any time, or Bermudan if you can exercise only at certain times
- Hence a Monte Carlo scheme that allows you to price options with early exercise is called American Monte Carlo, or AMC for short

What is the issue with Monte Carlo

- But Monte Carlo doesn't use backward induction
- the problem is: How can we compute the continuation value?
- The continuation value is the value of the option if we continue to hold it
- As such it is the risk-neutral expectation of the payoff of the option
- So we need to be able to calculate conditional expectations in Monte Carlo

Optimal exercise in Monte Carlo
There is no sub-Monte Carlo starting from the circled point

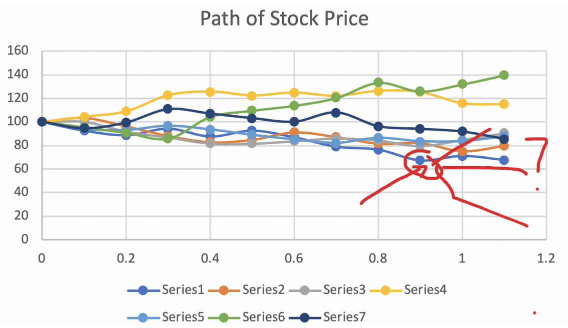


Figure: Path stock price in Monte Carlo What is the conditional expectation of payoff given at point ?

To calculate the continuation value, we could start a sub-Monte Carlo there

- And that was how people did this for a while (this scheme is called the Broadie-Glasserman scheme)
- But this is very expensive computationally
- We need to do a sub-Monte Carlo for each path
- To reduce from the computational burden we can start a sub-Monte Carlo at a discrete number of points and interpolate elsewhere

Conditional Expectation by Regression

- The first American Monte Carlo algorithm was devised by Longstaff-Schwartz
- used linear regression to find the conditional expectation of the payoff given the current stock price
- The Algorithm works using a forward pass and computes the linear regression coefficients
- at one step before Maturity $T - \Delta t$ for a put compute the Linear regression coefficients of $(K - S_T)^+ e^{-r\Delta t}$ vs $S_{T-\Delta t}, S^2_{T-\Delta t}, S^3_{T-\Delta t}, S^4_{T-\Delta t}$

Exercise decision

- approximate $\boldsymbol{E}(K - S_T)^+ e^{-r\Delta t} | S_{T-\Delta t} = a_0 + a_1 S_{T-\Delta t} + a_2 S^2_{T-\Delta t} + a_3 S^3_{T-\Delta t} + a_4 S^4_{T-\Delta t}$
- repeat the simulation to avoid looking into the future
- at $T - \Delta t$ on each path i compare the intrinsic value $(K - S^i_{T-\Delta t})^+$ with $a_0 + a_1 S^i_{T-\Delta t} + a_2 {S^i_{T-\Delta t}}^2 + a_3 {S^i_{T-\Delta t}}^3 + a_4 {S^i_{T-\Delta t}}^4$
- if the intrinsic value greater vs continuation value exercise the option and set the option value to the payoff
- else do not exercise and go backwards one more step

Other types of regression

- Kernel-based, or non-parametric regression
- Non-linear regression
- Generalized linear model (GLM)
- Generalized additive model (GAM)

In general as long as the value that we're regressing is a good predictor of the optimal exercise decision there is not a lot of value in going over least squares

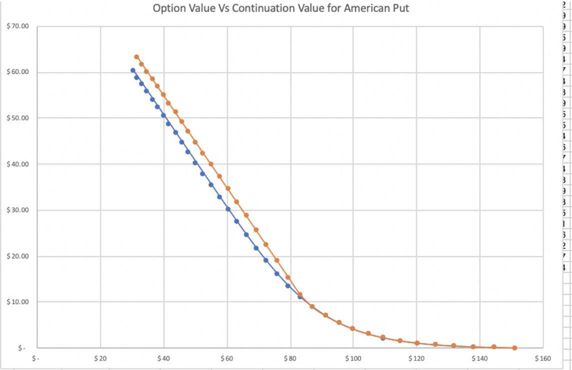


Figure: Option Value vs Continuation Value for American Put

Steven Shreve *Stochastic Calculus for Finance I: Continuous-Time Models*
ISBN-13: 978-1441923110

Steven Shreve *Stochastic Calculus for Finance II: Continuous-Time Models*
ISBN-13: 978-1441923110