Black Scholes and Martingales

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Outline

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Consider a market with N+1 securities. A trading strategy is a vector $\theta(t) = (\theta_1(t), .., \theta_N(t))$ that describes the units of stock held before time t.

Definition

The value process resulting from a trading strategy will be denoted with $V(t)$

$$V(t) = \sum \theta_i S^i(t) \tag{1}$$

where θ_t is F_{t-1} measurable.

Definition

A self-financing trading strategy is a strategy,θ_t , where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. In particular, a self-financing strategy satisfies

$$V(t+1) - V(t) = \sum \theta_i (S^i(t+1) - S^i(t)) \tag{2}$$

This condition states that the value of a self-financing portfolio just before trading or re-balancing is equal to the value of the portfolio just after trading, i.e., no additional funds have been deposited or withdrawn. In particular we see that (1) may also be taken as the definition of a self-financing trading strategy.

Definition

An arbitrage opportunity is a self financing trading strategy with $V_0 = 0$ and $V_T > 0$ (or $V_t \geq 0$ and $V_T > 0$ on with positive probability).

Definition

A contingent claim is a random variable whose value is known by time T i.e. F_T measurable.

Definition

A contingent claim is attainable if there exist a self financing trading strategy such that $V_T = C$. The value of the claim must equal the value of replicating portfolio if there is a unique trading strategy.

Definition

A numeraire is a tradable security with positive price.

Theorem

Given two assets, $S(t)$ the stock price and $B(t)$ the bank account. If there exists a unique measure Q such that $\frac{S(t)}{B(t)}$ is martingale, then the price of any contingent claim X is just the expectation under risk-neutral measure

$$B(t) \times \mathbb{E}^Q \left[\frac{X}{B(T)} | \mathcal{F}_t \right]$$

and there exists a replicating portfolio in stock and bank account. Furthermore, there is no arbitrage.

Remark Do not think of expectation in terms as of law of large numbers (i.e. after large number of i.i.d trials we converge to expected value). We will always be taking expectation with respect to risk neutral measure which is the price of a replicating self financing trading strategy which enforces the no arbitrage condition.

Outline

Filtration represents all the possible outcomes by time t.

Example

When flipping a coin we have the possible outcome $\mathcal{F}_3$ is the information we know by third coin toss.

Martingale is a fair process.

Definition

An adapted process X_t with respect to a filtration $\mathcal{F}_t$ is a martingale if and only if:

- for all $t \geq s$,$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$

As a consequence a martingale requires constant expectation.

Definition

A process W_t is called a Brownian Motion if:

- W_t is a Gaussian process with mean 0 and variance t
- W_t is an adapted process with respect to a filtration $\mathcal{F}_t = \bigcup_{0 \leq u \leq t} W_u$
- for all $t > s$, $W_t - W_s$ is independent of W_u for all $0 \leq u \leq s(\mathcal{F}_s = \bigcup_{0 \leq u \leq s} W_u)$

A diffusion is a process composed by a finite variation part and a stochastic integral. It is useful to define diffusions X_t as solution to the stochastic differential equations:

$$dX_t = \underbrace{\mu_t dt}_{\text{drift}} + \underbrace{\sigma_t dW_t}_{\text{diffusion}}, X(0) = x_0$$

The integral notation of the equation above is:

$$X_t = x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

Example

Brownian Motion and Martingales:

- the process W_t, has the property $\mathbb{E}[W_t] = 0$ for any t. (martingale)
- the process W_t^2, $\mathbb{E}[W_t^2] = t$ is not a martingale. (in fact, it is a super-martingale)

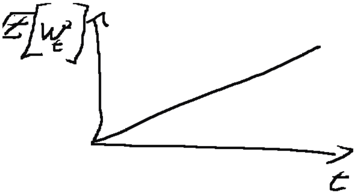

Figure: Illustration of $\mathbb{E}[W_t^2]$ v.s. t

Theorem (Ito's Lemma)

Given a diffusion process X_t

$$dX_t = \mu_t dt + \sigma_t dW_t$$

then $f(X_t)$ satisfies:

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\sigma_t^2 dt \tag{3}$$

Note that if $f(\cdot,\cdot)$ is a function of x, t Ito's Lemma becomes:

$$df(t, X(t)) = \frac{\partial f}{\partial x}(t, X(t))dX(t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt$$

Example (Solving a SDE)

Consider the stock price SDE as follows:

$$dx_t = rx_t dt + \sigma x_t dW_t, x(0) = x_0$$

To apply Ito's lemma one must first compute:

$$f(x) = \log(x), \qquad f'(x) = \frac{1}{x}, \qquad f''(x) = -\frac{1}{x^2}$$

Then by Ito's lemma:

$$d(\log(x_t)) = \frac{1}{x_t}(rx_t dt + \sigma x_t dW_t) + \frac{1}{2}\left(-\frac{1}{x_t^2}\right)\sigma^2 x_t^2 dt$$

$$d(\log(x_t)) = rdt + \sigma dW_t - \frac{1}{2}\frac{1}{x_t^2}\sigma^2 x_t^2 dt$$

$$d(\log(x_t)) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

$$\log(x_t) = \log(x_0) + \int_0^t \left(r - \frac{1}{2}\sigma^2\right) ds + \sigma dW_t$$

$$\log(x_t) = \log(x_0) + \left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

$$x_t = x_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

Note that $W_t =$ is distributed$N(0, t)$.

Example (Change Density/Measure)

What is the change of measure factor such that a random variable X will be distributed:

$$\begin{cases} X \sim N(\mu, t), & \text{under a measure } P \\ X \sim N(\mu - rt, t), & \text{under a measure } Q \end{cases}$$

The change of measure (Radon-Nikodym derivative) under Q, is defined as the likelihood ratio

$$\text{measure } Q = \frac{\text{density } N(\mu - rt, t)}{\text{density } N(\mu, t)} = \frac{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-(\mu-rt))^2}{2t}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-\mu)^2}{2t}}}$$

Theorem (Girsanov (light version))

Let W_t be Brownian motion under a measure P. Then $W_t + at$ is a Brownian motion under a measure Q. The Radon-Nikodym derivative from P to Q is defined as:

$$exp(-aW_t - a^2/2 \cdot t)$$

Outline

For continuous time models, under the historical measure we have

$$\text{real world measure} = \begin{cases} dS_t = \mu S_t dt + \sigma X_t dW_t \\ dB_t = rB_t dt \end{cases}$$

Under the risk-neutral measure (martingale measure) Q the stock price satisfies:

$$\text{risk-neutral measure} = \begin{cases} dS_t = rS_t dt + \sigma X_t d\tilde{W}_t \\ dB_t = rB_t dt \end{cases}$$

where $\tilde{W}_t$ is a Brownian motion under Q and $S_t/B_t = e^{-rt}S_t$ is a martingale under Q. The measure Q makes $\tilde{W}(t) = W(t) - \frac{\mu - r}{\sigma}t$ a Brownian motion.

Example

Change of measure for a specific example: Consider IBM stock price, with volatility $\sigma = 20\%$, historical average return $\mu = 10\%$. How do we price a the call price for 3 month option on IBM $(S_T - K)^+$? The interest rate is assumed constant $r = 0.03$.

Currently, we calculate the average IBM return as 10% thus we can model:

$$dS_t = 0.1 S_t dt + 0.2 S_t dW_t \tag{4}$$

Under the risk-neutral pricing measure

$$dS_t = 0.03 S_t dt + 0.2 S_t d\tilde{W}_t \tag{5}$$

How to get from (4) to (5)? Consider the following

$$dS_t = 0.03 S_t dt + 0.2 S_t \underbrace{\left[dW_t + \frac{0.1 - 0.03}{0.2} dt \right]}_{= \tilde{W}_t} \leftarrow \text{Sharpe ratio}$$

Example (BS formula)

By the fundamental theorem of finance, the call price equals:

$$\mathbb{E}[e^{-rT}(S_T - K)^+]$$

One can break the expectation into two parts:

$$e^{-rT}\mathbb{E}[(S_T - K)^+] = e^{-rT}\mathbb{E}\left[S_T \mathbf{1}_{\{\mathbf{S_T > K}\}} - \mathbf{K}\mathbf{1}_{\{\mathbf{S_T > K}\}}\right]$$

The second term can be computed as follows:

$$\begin{aligned}\mathbb{E}\left[KI_{\{S_T > K\}}\right] &= K\mathbb{E}\left[I_{\{S_T > K\}}\right] \\ &= KP(S_T > K)\end{aligned}$$

Apply Ito's formula,

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

Given S_T one can compute the probability :

$$\begin{aligned}
P(S_T > K) \quad &= P\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma dW_T} > K\right) \\
&= P\left(\left(r-\tfrac{1}{2}\sigma^2\right)T + \sigma W_T > \log\tfrac{K}{S_0}\right) \\
&= P\left(\sigma W_T > \log\tfrac{K}{S_0} - \left(r-\tfrac{1}{2}\sigma^2\right)T\right) \\
&= P\left(W_T > \tfrac{1}{\sigma}\left[\log\tfrac{K}{S_0} - \left(r-\tfrac{1}{2}\sigma^2\right)T\right]\right)
\end{aligned}$$

Since $W_T \sim N(0, T)$, we have $\frac{W_T}{\sqrt{T}} \sim N(0, 1)$. Also note that $\Phi(x)$ is the c.d.f. of normal distribution, i.e., $P(N(0,1) < x) = \Phi(x)$
In addition, $x \sim N(0,1)$, $-x \sim N(0,1)$

$$
\begin{aligned}
\dots \qquad &= P\left(\frac{W_T}{\sqrt{T}} > \frac{1}{\sigma\sqrt{T}}\left[\log\frac{K}{S_0} - \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \\
&= P\left(\underbrace{-\frac{W_T}{\sqrt{T}}}_{\sim N(0,1)} < \underbrace{\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]}_{=''d2''}\right) \\
&= \underbrace{\Phi\left(\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right)}_{\text{easy part}}
\end{aligned}
$$

Let us compute the first term :

$$\begin{aligned} e^{-rT}\mathbb{E}\left[S_T \mathbf{1}_{\{\mathbf{S_T}>\mathbf{K}\}}\right] &= e^{-rT}\mathbb{E}\left[S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T}\mathbf{1}_{\{\mathbf{S_T}>\mathbf{K}\}}\right] \\ &= S_0\mathbb{E}^Q\left[\mathbf{1}_{\{\mathbf{S_T}>\mathbf{K}\}}\right] \\ &= S_0 P^Q(S_T > K) \end{aligned}$$

Where under the Q measure $\tilde{W}_t = W_t - \sigma t$ is Brownian Motion. By the same arguments as before:

$$S_0 P^Q(S_T > K) = S(0)P^Q\left(\underbrace{-\frac{W_T}{\sqrt{T}}} < \underbrace{\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]}\right)$$

$$S(0)P^Q\left(\underbrace{-\frac{W_T - \sigma T}{\sqrt{T}}}_{\sim N(0,1)} < \underbrace{\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]}_{=\text{"}d1\text{"}}\right)$$

Finding the replicating portfolio using martingales. By the fundamental theorem of finance we have the value of the call at time t:

$$V(t, S_t) = \mathbb{E}[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t]$$

where

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W(t))}$$

with $W_T - W_t$ is independent of W_t. We can now write:

$$\begin{aligned} V(t, S_t) &= \mathbb{E}[e^{-r(T-t)}(S_T - K)^+ | S_t] \\ &= \mathbb{E}[e^{-r(T-t)}(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W(t))} - K)^+ | S_t] \\ &= S_t * N(d1) - Ke^{-r(T-t)} N(d2) \end{aligned}$$

Example (Black-Scholes by replication)

$$\text{real world measure} = \begin{cases} dS_t = \mu S_t dt + \sigma X_t dW_t \\ dB_t = rB_t dt \end{cases}$$

Our goal is to find a replicating portfolio ie. $\Delta_t S_t$ such that the sum of the option value $V(S_t, t)$ and hedged portfolio is riskless.

By applying Ito's lemma we have:

$$\begin{aligned} dV(S_t, t) = &\quad (\frac{\partial V(S_t, t)}{dS_t}\mu S_t + \frac{\partial V(S_t, t)}{dt} + 1/2\frac{\partial^2 P(S_t, t)}{dS_t^2}\sigma^2 S_t^2)dt \\ + &\quad \frac{\partial V(S_t, t)}{dS_t}\sigma S_t dW_t \end{aligned}$$

The hedged portfolio $\Pi_t = \Delta_t S_t - V(t, S_t)$ satisfies the following equation:

$$d\Pi_t = (...)dt + (\Delta_t \sigma S_t - \frac{\partial V(S_t,t)}{dS_t}\sigma S_t)dW_t$$

It is clear now that $\Delta_t = \frac{\partial V(S_t,t)}{dS_t}$, and by doing this Π_t becomes a riskless portfolio.

$$d\Pi_t = (-\frac{\partial V(S_t,t)}{dt} - 1/2\frac{\partial^2 V(S_t,t)}{dS_t^2}\sigma^2 S_t^2)dt$$

Clearly, the riskless portfolio grows at money market rate:

$$d\Pi_t = r\left(\frac{\partial V(S_t,t)}{dS_t}S_t - V(S_t,t)\right)dt$$

which in turn yields the Black-Scholes PDE:

$$rS_t\frac{\partial V(S_t,t)}{dS_t} + \frac{\partial V(S_t,t)}{dt} + 1/2\frac{\partial^2 V(S_t,t)}{dS_t^2}\sigma^2 S_t^2 = rV(S_t,t)$$