Regla de Simpson

$I = \int_a^b f(x)\,dx$ Este metodo cambia el integrando por un polinomio interpolador de grado dos.

$\Omega = \{(a, f(a)), (x_m, f(x_m)), (b, f(b))\}$ donde $x_m = \frac{a+b}{2}$

$$f(x) = p_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)$$

$$I = \int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f b \; dx$$

Usando la aditividad de la integral e integrando por partes se tiene:

$$I_1 = \int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)\,dx = \frac{f(a)}{(a-b)(a-x_m)} \int_a^b (x-b)(x-x_m)\,dx$$

$x_m = a + h$ entonces $a - x_m = -h$ $\qquad h = \frac{(b-a)}{2}$

$b - a = 2h$ entonces $a - b = -2h$

$x_m - b = -h$

Tomando $x - b = dv$ $\quad x - x_m = u$

$$\int_a^b (x-b)(x-x_m)dx = (x-x_m)\frac{(x-b)^2}{2}\Big|_a^b - \frac{1}{2}\int_a^b (x-b)^2 dx$$

$$= -(a-x_m)\frac{(a-b)^2}{2} - \frac{1}{2}\cdot\frac{(x-b)^3}{3}\Big|_a^b$$

$$= -(a-x_m)\frac{(a-b)^2}{2} + \frac{1}{2}\frac{(a-b)^3}{3}$$

$$= -(-h)\frac{(-2h)^2}{2} + \frac{1}{2}\frac{(-2h)^3}{3}$$

$$= 2h^3 - \frac{8h^3}{6} = \frac{2h^3}{3}$$

$$I1 = \frac{f(a)}{(a-b)(a-x_m)}\cdot\frac{2h^3}{3} = \frac{f(a)}{2h^2}\cdot\frac{2h^3}{3} = \frac{h}{3}f(a)$$

$$I2 = \int_a^b \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) = \frac{f(x_m)}{(x_m-a)(x_m-b)}\int_a^b (x-a)(x-b)\,dx$$

$$\int_a^b (x-a)(x-b)dx = (x-a)\frac{(x-b)^2}{2}\Big|_a^b - \frac{1}{2}\int_a^b (x-b)^2 dx$$

$$= -\frac{1}{2}\frac{(x-b)^3}{3}\Big|_a^b$$

$$= \frac{1}{2}\frac{(a-b)^3}{3} = \frac{1}{6}(a-b)^3 = -\frac{8h^3}{6} = -\frac{4}{3}h^3$$

$$I2 = \frac{f(x_m)}{(x_m-a)(x_m-b)}\cdot -\frac{4}{3}h^3 = \frac{f(x_m)}{-h^2}\cdot -\frac{4h^3}{3} = \frac{4}{3}h\,f(x_m)$$

$$I_3 = \int_a^b \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)\,dx = \frac{f(b)}{(b-a)(b-x_m)} \int_a^b (x-a)(x-x_m)\,dx$$

$$\int_a^b (x-a)(x-x_m)\,dx = (x-x_m)\frac{(x-a)^2}{2}\Big|_a^b - \frac{1}{2}\int_a^b (x-a)^2\,dx$$

$$= (b-x_m)\frac{(b-a)^2}{2} - \left(\frac{1}{2}\,\frac{(x-a)^3}{3}\right)\Big|_a^b$$

$$= (b-x_m)\frac{(b-a)^2}{2} - \frac{1}{2}\,\frac{(b-a)^3}{3}$$

$$= h \cdot \frac{(2h)^2}{2} - \frac{1}{2}\,\frac{(2h)^3}{3} = 2h^3 - \frac{4}{3}h^3 = \frac{2h^3}{3}$$

$$I_3 = \frac{f(b)}{(b-a)(b-x_m)} \cdot \frac{2h^3}{3} = \frac{f(b)}{2h^2} \cdot \frac{2h^3}{3} = \frac{h}{3} f(b)$$

$$I = I_1 + I_2 + I_3$$

$$I = \frac{h}{3} f(a) + \frac{4}{3} h f(x_m) + \frac{h}{3} f(b)$$

$$I = \frac{h}{3}\left(f(a) + 4f(x_m) + f(b)\right)$$