Solutions 10

Example 1. Denote by n the number of ladies and by m the number of gentlemen. Let's count the total number of dance couples which were formed during the Royal Ball. We can do it in two ways. On the one hand, since every lady danced with 3 gentlemen there were formed $3 \times n$ couples. On the other hand, due to the fact that every gentleman danced with 3 ladies the number of couples is $3 \times m$. Therefore, we equate $3 \times n = 3 \times m$ which is equivalent to $n = m$.
Example 2. Let's count the sum of the numbers in all the boxes of this grid. We can do it in two ways. Denote by x_1, x_2, x_3, x_4, x_5, x_6, x_7 the sums of all numbers in rows 1, 2, 3, 4, 5, 6, 7 respectively. Similarly define y_1, y_2, y_3, y_4, y_5 y_6, y_7 to be the sums of all numbers in columns 1, 2, 3, 4, 5, 6, 7. On one hand, the sum of all 49 numbers is equal to $x_1 + x_2 + ... + x_7$ and is negative since we know that all x_i's are negative. On the other hand, it equals $y_1 + y_2 + ... + y_7$ which has positive value because all y_i's are positive. It is impossible that a positive number equals a negative number. Thus, Fred made a mistake.
Underlying idea: if you want to compute the sum of the numbers written in boxes of $n \times m$ grid you can group them up according to their row or to their column.
Answer: He made a mistake.
Example 3. Every edge belongs to some two neighbouring faces. Since every face has triangular form, it contains 3 edges. Therefore, if we sum up the number of edges over all faces of the polyhedron we get 3×20. And by the first argument, this number is doubling the total number of edges (we counted every edge twice). So the number of edges is $\frac{3\times 20}{2} = 30$.
Comment: In case when all the edges has the same length such polyhedron is called an icosahedron.
Answer: 20 edges.
Problem 10.1. Let's count the total number of handshakes in two ways. Assume there were n year 7 students and m year 8 students. On one hand, the total number of handshakes is $8 \times n$ since every 7 year student shook 8 hands. On the other hand, this number is equal to $7 \times m$ since every year 8 student shook exactly 7 hands. From these observations we deduce that $8 \times n = 7 \times m$, so $n = \frac{7}{8}m$. Thus, the number of 7 year students is less than the number of 8 year students.
Answer: The number of year 7 students is less than the number of year 8 students.

Problem 10.2. One exchange "deal" requires 3 notes (one £10 note and two £5 notes). Denote by n the number of "deals" during the day and by m the number of employees. Then we can compute the number of notes used for all deals during the day in two ways. On one hand, it equals $10 \times m$ since every employee gave 10 notes. On the other hand, by the first argument, it is equal to $3 \times n$. We obtain the following equality $10 \times m = 3 \times n$. Substituting $m = 2000$ we arrive at $n = \frac{20000}{3}$ and the fraction on the right-hand side is not an integer number. Therefore, such situation is not possible.
Answer: No, it's impossible.

Problem 10.3. We need to check that every note or coin with fixed value n is counted exactly n times. And this is straightforward since Clint counts this note only when he counts notes and coins with the value at least 1, the value at least 2, the value at least 3, ..., the value at least n so n times in total.

Problem 10.4. The carpet covers the staircase only if it covers all the steps fully, thus, the length of this carpet should be equal to the sum of horizontal and vertical components of all the steps. We divide this sum into two parts: the vertical and the horizontal, and work out the sum within each part separately. The height (the vertical part) of a stairs' step can be projected on the height of the staircase. The sum of these projections equals to the height of the staircase which is 2.5 m in our case. Similarly we can project the width (the horizontal part) of each step on the width of the staircase and they will sum up to the width of the staircase, i.e. 2 m. These observations don't depend on the number of steps, therefore no matter what the number of steps is, the carpet should be of length $2.5 + 2 = 4.5$ metres in order to cover the staircase. Since the carpet covers the original staircase, it covers another staircase as well.
Answer: No, you don't need.

Problem 10.5. Denote by a the number of white pieces and by b the number of blue pieces. Let's count the number of white neighbours of all blue pieces. Since every blue piece has 5 white neighbours, this number is $5 \times b$. But, in this sum we counted every white part 3 times due to the fact that every white part is a neighbour of exactly 3 blue parts. So we deduce that $5 \times b = 3 \times a$. We know that there are all together 32 pieces. Thus $a + b = 32$, and by substituting $b = \frac{3}{5}a$, we obtain $\frac{8}{5}a = 32$. This gives us $a = 20$.
Answer: 20 white pieces.

Problem 10.6. Consider the following example: we have 24 numbers equal to -1 and one number 4. If you choose three -1, then the 4th number should be 4 in order

the sum to be equal to 1 which is positive. If you choose two -1's and one 4 you don't have much choice for the last number (it has to be -1) and the sum will be again equal to 1. Therefore, this example satisfies our condition. But the total sum is negative: $-1 \times 24 + 4 = -20$.
Answer: It is not true.
Comment: Instead of the existence of the 4th number, assume that every four numbers have positive sum. What will be the answer in that case?

Problem 10.7. Every natural number n can be uniquely written as a product of some power of 2 and an odd number (you just combine all odd factors of prime decomposition to get the second factor for this expression). Assume $n = 2^k \times a$ is the representation for n (a is odd). Then the following equalities hold:

$$\frac{1}{n} = \frac{1}{2^k \times a} = \frac{1}{2^{k-1} \times a} - \frac{1}{2^k \times a} = \frac{1}{2^{k-2} \times a} - \frac{1}{2^{k-1} \times a} - \frac{1}{2^k \times a} =$$

$$= \frac{1}{a} - \frac{1}{2 \times a} - \ldots - \frac{1}{2^{k-1} \times a} - \frac{1}{2^k \times a}$$

For every number $\frac{1}{n} = \frac{1}{2^k \times a}$ on the right-hand side of our sum the derived expression $\frac{1}{a} - \frac{1}{2 \times a} - \ldots - \frac{1}{2^{k-1} \times a} - \frac{1}{2^k \times a}$ is presented on the left-hand side (for odd numbers from 101 to 200 this expression coincides with the number since $k = 0$). Moreover, every number on the left-hand side is a part of some representation of that kind (if you multiply any odd number by 2 several times you will get a number between 101 and 200 at some point) and these representations don't intersect for different odd numbers a. Therefore, we proved the desired equality.