Maths Battle. February. Solutions.

Problem 1. The first product can be written as

$$3 \times 111111111111 \times 8 \times 1111111111111 = 24 \times 111111111111 \times 1111111111111 =$$
$$= 4 \times 6 \times 111111111111 \times 1111111111111 = 444444444444 \times 6666666666666.$$

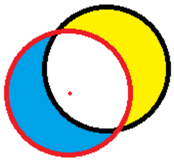
And this is definitely less than the second product. The difference between these products is

$$444444444444 \times (6666666666667 - 6666666666666) = 444444444444.$$

Problem 2. If we have three distinct numbers a, b and c satisfying $a > b > c$ and $a+b+c = 100$, then $a \geq 35$ and $c \leq 32$. If $a \leq 34$ the sum $a+b+c$ is not more than $34+33+32 = 99$ and if $c \geq 33$ the sum $a+b+c$ is not less than $35+34+33 = 102$ - in both cases we have a contradiction with $a+b+c = 100$. From this observation we deduce that at the first look there were at least 35 red balls (and at most 32 green balls) and at the second look there were at most 32 red balls (and at least 35 green balls). It is possible only if initially we had 3 magic balls coloured in red and after these magic balls became green. From this we get that the number of yellow balls was equal $100 - 35 - 32 = 33$ during the first and the second observation.
Answer: 33 yellow balls.
Problem 3. *Solution 1:* There are two categories into which we can fit the three sizes: those sizes for which there are more right boots than left boots, and those sizes for which there are more left boots than right boots (if there happens to be an equal number of right and left boots in one size, we put that size in the second category). It follows that two sizes lie in the same category. Let us say that sizes 41 and 42 have more right boots than left boots (an analogous argument will hold if two sizes have more left boots than right boots).
Now there are 300 left boots in all, and at most 200 left boots in any one size. Therefore, the sum of the left boots in any two sizes is at least 100. We have shown that there are at least 100 left boots in sizes 41 and 42 (taken together), and that each of these sizes contains more right boots than left boots. Hence each left boot has a match, and there are at least 100 good pairs in the warehouse.
Solution 2: Consider two "pigeon holes" for this problem. The first corresponds to

sizes for which there are more right boots than left boots. The second corresponds to sizes for which the number of right boots is less or equal to the number of left boots. We have three sizes, so by the Pigeon Hole Principle, there will be two sizes in one category. We can assume this is the first category and the sizes are 41 and 42. Denote by x, y and z the numbers of usable pairs if sizes 41, 42 and 43 respectively. Assume the contrary, so $x+y+z<100$. From our assumptions we know that the number of right boots of sizes 41 and 42 are $200-x$ and $200-y$ respectively. The number of right boots of size 43 is at least z since we can form z suitable pairs. Then the total number of right boots is at least $(200-x)+(200-y)+z=400-(x+y+z)+2z>300+2z$, but this contradicts the fact that there are 300 right boots. If two sizes belong to the second "pigeon hole" we consider the total number of left boots.

Problem 4. Draw a circle with the same radius around the marked dot. The yellow part of the initial circle and the blue part of the new circle are equal by the symmetry argument. So you can make the new circle from the given one by putting yellow part instead of blue. See picture.



Problem 5. Consider the remainders of 1, 2, ..., 1000 by the division algorithm by 25. There will be 40 numbers with the remainder 1, 40 numbers with the remainder 2, ..., 40 numbers with remainder 24 and 40 numbers with zero remainder. So we have 25 groups of 40 numbers in each. The first move of Alex can be arbitrary, say, he takes a card from the group x (corresponding to the remainder x). Then Bob makes his move and takes a card from some group y. Our strategy (as Alex) from this point is to take a card from the same group y. Then again Bob makes his move taking a card from the group z and we again "copy" him by taking the card from the group z. This will continue until we can't copy his move. That can happen only when he takes the last card from group x. In this situation we take an arbitrary card and follow our strategy again. When the game ends the total sum of the numbers on the cards of Alex will have the same remainder modulo 25 (denote it by a) as the total sum of the numbers on Bob's cards (why?). But it is possible only in case of zero remainder, since $1+2+3+...+1000=(1+1000)+(2+999)+(3+998)+...+(500+501)=$ $=1001\times 500$ is a multiple of 25 and $2\times a$ is divisible by 25 only if a is a multiple of 25.

Answer: No, he can't.

Problem 6. Consider the following numbers:

$$2^2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13^2 \times 17 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17^2 \times 19 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19^2 \times 23 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23^2 \times 29$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29^2$$

From their prime decompositions we can see that they satisfy the required conditions and instead of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 we can use any other 10 distinct prime numbers.