Problem Sheet 10.

Pi Day is celebrated on March 14th (3/14) around the world.
Pi (Greek letter "π") is the symbol used in mathematics to represent a constant – the ratio of the circumference of a circle to its diameter. Pi cannot be represented as a ratio of two natural numbers, therefore you can't express it as a finite decimal or as a repeating decimal. Here is some decimal approximation of Pi:

$$\pi \approx 3,14159265358979323846264338327 95$$

Unfortunately, we are not going to celebrate this day properly, but still we prepared a kind of a "π-pie" for you.

One of the most common mathematical objects is summation. It is used in a large variety of problems. So it is important to learn how you can operate with sums of various types. Today is the first session devoted to this notion. The idea behind the following examples is to use different summations for counting the same value.

Example 1. The Royal Ball was held at Buckingham Palace on the first day of Easter holidays. Every gentleman danced with three ladies that evening. And every lady danced with three gentlemen. Prove that the number of gentlemen and the number of ladies attending the ball were the same.

Example 2. In each box of 7×7 grid some numbers are written in such a way that every box contains only one number. Fred calculated the sum of the numbers in every column and in every row. He concluded that not depending on which row and which column you choose the first sum will always be positive and the second will always be negative. Don't you think Fred made a mistake in his calculations?

Example 3. All faces of a polyhedron are triangles. How many edges does this polyhedron have if the number of faces is 20?

Problem 10.1. A group of year 7 and year 8 students shook each others hands. It turns out that every year 7 student has shaken eight hands and every year 8 student has shaken seven hands. Compare the number of year 7 students to the number of year 8 students.

Problem 10.2. There are 2000 employees in the investment banking firm "Silverman Romans". Every time two co-workers meet each other they exchange one £10 note for two £5 notes. (A gives B one £10 note and B gives A two £5 notes.) Is it possible that during the day every employee gives exactly 10 notes?

Problem 10.3. Each Sunday afternoon, barkeeper Clint counts the total income from the past week. But he does it in a very unusual way. First he counts notes and coins with value not less than 1, then he counts notes and coins with value not less than 2, next he counts notes and coins with value not less than 3, etc. After that he sums up all these numbers. Why does such method give you the right answer for the total amount of money?

Problem 10.4. A red carpet covers a staircase which has 10 steps. This staircase is 2.5 metres high and 2 meters long. There is another staircase of the same height and width which has 15 steps. Do you need to enlarge the old carpet so it can cover the new staircase?

Problem 10.5. A soccer ball is sewn from 32 pieces. Those pieces are pentagon or hexagon form. The ball is sewn in such a way that edges of neighbouring polygons match exactly. Pentagons are blue, and hexagons are white. It is known that every blue piece has only white neighbours and every white piece has 3 white and 3 blue neighbours. Count the number of white pieces.

Problem 10.6. I choose 25 numbers in such a way that the following condition holds true: no matter which three numbers you pick from the chosen 25 numbers, I can find the fourth number among the remaining 22 numbers so that the sum of the chosen four will be positive. Is it true that the sum of all 25 numbers is positive?

Problem 10.7. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{199} - \frac{1}{200} = \frac{1}{101} + \frac{1}{102} + ... + \frac{1}{200}$$