Problem Sheet 2.

Proof by contradiction. A form of indirect proof, that establishes the truth or validity of a proposition by showing that the proposition's being false would imply a contradiction.

Pigeon Hole Principle. If we must put $N + 1$ or more pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

Proof: Suppose no more than one pigeon were in each hole. Then there would be no more than N pigeons altogether, which contradicts the assumption that we have at least $N + 1$ pigeons. Thus, we proved it by contradiction.

King Arthur is a legendary British leader, according to medieval histories and romances, who led the defence of Britain against Saxon invaders in the late 5th and early 6th centuries AD. The details of Arthur's story are mainly composed of folklore and literary invention, and his historical existence is debated and disputed by modern historians.

Example 1. One million pine trees grow in a mythical forest of Brocéliande. It is known that no pine tree has more than 600000 pine needles on it. Show that two pine trees in the forest must have the same number of pine needles.

Example 2. Impressed by a knight's achievements, King Arthur has rewarded him with a piece of land measuring 4km×4km. Unfortunately, 15 apple trees grow on the land, and these trees all belong to the king. (The king loves apples.) The king's will is that the knight builds a 1km×1km castle within the lands granted, but he must not destroy any of the king's apple trees. Can the knight accept this prize from the king?

Problem 2.1. In Merlin's diary the **General Pigeon Hole Principle** was stated as "If we must put $Nk + 1$ or more pigeons into N pigeon holes, then some pigeon hole must contain at least $k + 1$ pigeons". Can you prove it?

Problem 2.2. King's Arthur daughter is a very whimsical child and she doesn't like the father's apple trees at all. While her father was seeking The Holy Grail in The Holy Land, she decided to build a nesting box in his favourite garden. It is known that the garden of 10m×10m consists of 51 apple trees. Show that she can find a place in the garden for a 2m×2m nesting box such that at least 3 apple trees will be demolished when the nesting box is constructed.

Problem 2.3. Seventeen kings are placed on a chess board. Show that there exist two which attack each other.

Problem 2.4. King Arthur's wife, Guinevere is in danger. Sir Lancelot, who is the king's chief knight, has a plan to save her. He needs to divide his battalion of 44 knights into 9 units with unequal number of knights per unit. Is it possible to divide the battalion or will it be the case that he has to join them?

Problem 2.5. One day, a tragedy has arrived upon the kingdom. King Arthur died in a great battle. As he does not have a next of kin, 11 knights are the contenders to become the new king. Some of the knights are friends with each other and others are not. Show that there are at least two knights who have equal number of friends.