Solutions 9 (8 3/4).

Example 1. Mark 58 points in such a way that the distance between two neighbours is equal to 1 cm. Our "pigeons" are these points and our "pigeon holes" are the 57 colours. By the Pigeon Hole Principle, we can find at least two pigeons in the same pigeon hole, which means that we can find at least two points with the same colour. The distance between them is an integer number in cm by our construction.
Example 2. Consider all possible results for the class test. You can get 16 different outcomes: $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,1,0)$, $(0,1,1,1)$, $(1,0,0,0)$, $(1,0,0,1)$, $(1,0,1,0)$, $(1,0,1,1)$, $(1,1,0,0)$, $(1,1,0,1)$, $(1,1,1,0)$, $(1,1,1,1)$. These results are the "pigeon holes", students are the "pigeons". Note that $33 = 2 \times 16 + 1$. Then by the General Pigeon Hole Principle, you can find $2 + 1 = 3$ students in the same pigeon hole, i.e. with the same results.
Example 3. There are 3 "pigeon holes" corresponding to 3 possible remainders modulo 3: 0, 1 and 2. The amounts of mushrooms correspond to 4 pigeons. By the Pigeon Hole Principle, two of the amounts have the same remainder, which means that they differ by a multiple of 3.
Example 4. Once again consider 3 "pigeon holes" corresponding to the remainders modulo 3. We have 7 natural numbers (our "pigeons"), so taking into account that $7 = 2 \times 3 + 1$, by the General Pigeon Hole Principle, there are at least $2 + 1 = 3$ numbers in the same "pigeon hole" , i.e. with the same remainder. By adding these 3 numbers up you get a number with a zero remainder, thus it is a multiple of 3.
Comment. This is just an example of how the General Pigeon Hole Principle can be applied. Of course, it is not hard to show that even if you take initially only 5 numbers you are still able to form such a triple.
Problem 8.1. (a) In this problem we have 6 "*pigeons*" corresponding to 6 random natural numbers. The number of "*pigeon holes*" is 5 and they correspond to all possible remainders modulo 5. By Pigeon Hole Principle, we can always find a "pigeon hole" with 2 "*pigeons*" in it. This means there always will be 2 natural numbers with the same remainder modulo 5, so their difference will be a multiple of 5.
(b) Divide the remainders modulo 5 in the following groups ("pigeon holes"): 0, 1 and 4, 2 and 3. Our 4 numbers are "pigeons" . Now by the Pigeon Hole Principle, two of the numbers will end up in the same "pigeon hole". But no matter what group you take they will either have the same remainders, so the difference is a multiple

of 5, or their remainders will sum up to a multiple of 5. This is exactly what we are asked to prove here.

Problem 8.2. There are only two "pigeon holes" corresponding to odd remainders modulo 4: 1 and 3. By the Pigeon Hole Principle, two out of three odd numbers a, b, c will have the same remainder, say a and b. Then $a \times b$ is 1 modulo 4. Therefore, $a \times b - 1$ is a multiple of 4.

Answer: No, it's not possible.

Problem 8.3. Divide 999999 by 60: $999999 = 999960 + 39 = 60 \times 16666 + 39$. Define 16667 *"pigeon holes"*: the first consists of numbers from 1 to 60, the second consists of numbers from 61 to 120, the third consists of numbers from 121 to 180, ..., the 16666-th consists of numbers from 999901 to 999960 and the 16667-th consists of numbers from 999961 to 999999. From here we can see that in this problem it is necessary to consider 16668 *"pigeons"*, i.e. to buy 16668 tickets. In that case due to Pigeon Hole Principle there will be 2 *"pigeons"* in the same *"pigeon hole"* meaning that there will be 2 numbers among the numbers of the form $60 \times n + 1$, ..., $60 \times (n+1)$ for some $n = 1, 2, ..., 16666$ or among the numbers 999961, ..., 999999. In both cases the difference of these numbers will be less or equal to 59, thus not more than 60. It is not enough to buy 16667 tickets since it is possible that we can get the following 16667 numbers: 1, 61, 121, ..., 999901, 999961. The difference between any two of these numbers is greater or equal to 60.

Answer: 16668 tickets.

Problem 8.4. First we notice that all possible values of the remainders of square numbers modulo 5 are 0, 1 and 4 (for example, look up our tables from Problem Session 7). Thus it is essential to define "*pigeon holes*" in correspondence with these three remainders. We have 7 randomly generated integer numbers (7 "*pigeons*"). We put every number into the "*pigeon hole*" corresponding to the remainder of its square modulo 5. By the General Pigeon Hole Principle, there will be 3 "*pigeons*" in one "*pigeon hole*", i.e. 3 numbers which have the same remainder of their squares modulo 5. So we are done.

Problem 8.5. Assume that it is possible. Denote the products of the numbers in these three groups by a, b and c. We have a,b,$c \leq 71$, and, consequently, $a \times b \times c \leq 71^3$(*). We can compute $a \times b \times c$ as it is just a product of 1, 2, ..., 9 taken in another order: $a \times b \times c = 9! = 362880$. And easily we get $71^3 = 357911$. Therefore, the inequality (*) doesn't hold meaning that one of the numbers a, b or c should be greater than 71. So, our assumption is incorrect.

Answer: No, we can't.

Problem 8.6. From the formulation of the problem it is clear that the strength of the orc army grows as the powers of 2. Now the question is about whether it is possible to find two different powers of 2 with the difference being a multiple of 2016. There are 2016 possible remainders modulo 2016: 0, 1, ..., 2015. They are our

"pigeon holes". Our pigeons are $2^0 = 1$, $2^1 = 2$,... 2^{2015}, 2^{2016} - 2017 numbers in total. By the Pigeon Hole Principle, two of them will have the same remainder modulo 2016. Therefore, their difference is a multiple of 2016. So, it is possible.
Answer: No, it wasn't a mistake.
Problem 8.7. The idea is similar to the one we used in the problem about orcs. By construction, the total number of points on the screen at any time is a power of 3. We consider the following 98 numbers: 3^1, 3^2, ..., 3^{98}. We have 97 "*pigeon holes*" corresponding to the remainders modulo 97. Thus, we deduce by the Pigeon Hole Principle that we have two powers of 3 which differ by a multiple of 97, say 3^k and 3^l with $k < l$. If $k = 2$, then we are done since then $3^l - 3^k = 3^l - 9$ is a multiple of 97, so 3^l is 9 modulo 97. Assume $k \neq 2$. Then $3^l - 3^k = 3^k \times (3^{l-k} - 1)$ is divisible by 97. And since 3^k and 97 don't have any common divisors, $3^{l-k} - 1$ should be divisible by 97. From this we can say that $3^{l-k+2} - 9$ (we just multiply previous expression by $3^2 = 9$) should be a multiple of 97. Thus, 3^{l-k+2} is 9 modulo 97.
Answer: Yes, there will be a moment.
Problem 8.8. By the Main Theorem of Arithmetics, every number can be expressed as a product of an odd number and a power of 2. By using this observation form 10 "pigeon holes" corresponding to different odd factors of such decomposition: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. We have 11 "pigeons" to fit these "pigeon holes" , so by the Pigeon Hole Principle, there will be at least two numbers x and y in the hole a. We can express these numbers in the following way: $x = a \times 2^n$ and $y = a \times 2^m$. Clearly, one with the greater power of 2 is divisible by the other.
We can describe "pigeon holes" more precisely:
1) 1, 2, 4, 8, 16;
2) 3, 6, 12;
3) 5, 10, 20;
4) 7, 14;
5) 9, 18;
6) 11;
7) 13;
8) 15;
9) 17;
10) 19.
Problem 8.9. Denote these numbers by x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}. Consider the following sums: x_1, $x_1 + x_2$, $x_1 + x_2 + x_3$, ..., $x_1 + x_2 + ... + x_{10}$. If one of the sums is a multiple of 10 we are done. If not, then by the Pigeon Hole Principle, there will be two with the same remainders modulo 10. Therefore, their difference is a multiple of 10, and also is a sum of the elements of some subset.
Problem 8.10. Denote by x kg the weight of one cow. The total weight of a herd of 101 cows is $x + 2S$ kg, where S kg is the total weight of the first group of 50

remaining cows (also it is the total weight of the second group of 50 cows). It means that x has the same parity as the total weight of 101 cows. Since it is true for an arbitrary cow, then either each cow's weight is an even number, or each cow's weight is an odd number.

In the first case, we can divide the weight of each cow by 2, and the assumptions of the problem do not change because we can still remove each cow and divide the remaining ones into two groups of 50, each of which weighs the same.

In the second case, we can add 1 kg to the weight of each cow and get even numbers as weights. Again, the assumptions of the problem do not change because we can still remove each cow and divide the remaining ones into two groups of 50, each of which weighs the same. Subsequently, we repeat the argument from the first case, and divide the new weights by 2. The newly obtained weights satisfy the assumptions of the problem again.

In the both cases none of the weights has increased. Continuing such manipulations (dividing the weight of each cow by 2 if the weight is even, or adding 1 and subsequently dividing by 2 if the wieght is odd) we get 1 kg as some cow's weight at some point. After this moment we will only apply the second case which will be decreasing all the weights not equal to 1. So, we end up with all 1's.

To finish the proof we reverse the transformations we did, i.e. we multiply by 2 or add 1. By doing these reverse manipulations we get that all the initial numbers are the same.

Problem 8.11. The last digit of E^3 is E itself, because $E \times E$ has last digit F and $E \times F$ has last digit E. Thus E must be 4 or 9 (the cases 0, 1, 5, 6 are excluded because they lead to $F = E$).

Next, 9 is excluded because it would lead to $F = 1$, and the last line above the second stroke would only have 4 digits instead of 5. So $E = 4$ and $F = 6$.

Since E is even, both V and I have to be even (it can be seen from the second and the third lines). If one of them equals to 0, so does the other. Therefore either $I = 2$ and $V = 8$, or vice versa. It is easy to see that the both cases work out, so the number is either 6284 or 6824.