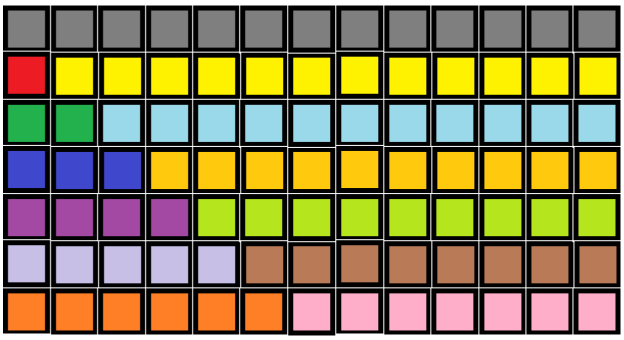
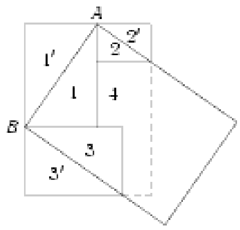
Solutions for December Maths Battle 2015. The Royal Grammar School, High Wycombe.

Problem 1. Octopuses give four different answers, and therefore we have at least 3 liars among them. Suppose all octopuses are liars. A liar octopus has 7 legs, so four liars have 28 legs, but in this case the blue one tells the truth. This is a contradiction. Now suppose exactly three octopuses are liars. There must be $3 \times 7 + 6 = 27$ legs in total or $3 \times 7 + 8 = 29$ legs. This is only possible if the green octopus tells the truth and has 6 legs. Consequently, the blue, the red and the yellow are lying, and, subsequently, have 7 legs each.
Answer: 7 legs - the red, the yellow and the blue; 6 legs - the green.
Problem 2. See the picture below.



Answer: Yes.
Problem 3. Construct a rectangular cuboid $1 \times 2 \times 3$ from 2 "corners". Make a rectangular cuboid $3 \times 2 \times 3$ from three rectangular cuboids $1 \times 2 \times 3$ by putting one on another. You still have one "corner" and 6 small cubes from which you can make a rectangular cuboid $3 \times 1 \times 3$. Putting together two constructed rectangular cuboids completes the cube $3 \times 3 \times 3$.
Problem 4. How can we find the sum of 8 initial numbers? It is easy to see that this sum is equal to $11 + 13 + 15 + 17$ on one hand and to $12 + 14 + 16 + 18$ on another. But these numbers are not the same which means that the described situation is not possible.
Answer: it is not possible.
Problem 5. The sheets are equal in size. The area covered by triangles 1', 2', 3'

is equal to the area covered by triangles 1, 2, 3 (see the picture below). Thus, the covered part of the bottom sheet is larger than the uncovered part. The difference in area between the covered and the uncovered part is equal to the area covered by 4.



Answer: the covered part.
Problem 6. Assume that it is possible. The sum of six numbers from the set $-1, 0, 1$ can take 13 possible values: $-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$. These values are our "pigeon holes". The number of "pigeons" corresponds to the number of all rows(6), all columns(6) and all main diagonals(2), so it is equal to 14(=13+1). By the pigeon hole principle, we can find a pigeon hole with at least two pigeons in it. This contradicts our assumption.
Answer: it is not possible.
Problem 7. Assume the contrary. Let's call a group of same type creatures sitting next to each other and flanked by creatures of another type a "clan". Note that there 1) can't be a clan of more than 3 elves and 2) can't be a clan consisting of only one dwarf. From 1) there are at least 13 clans of elves (use the pigeon hole principle taking in mind that a clan of elves should contain only 1 or 2 elves). From 2) we get that there are no more than 12 clans of dwarves (use again the pigeon hole principle). But these clans should alternate, thus, their numbers should be equal. But they are not equal. This is a contradiction.

Problem 8. In any 100 consecutive five-digit numbers you can find a number ending in "00" which means that it can be written as $100 \times \overline{abc} = \overline{abc00}$, so it is not "sustainable". Consider 99 five-digit numbers: $10001, 10002, ..., 10099$. All of these numbers are greater than $10000 = 100 \times 100$ and less than $10100 = 101 \times 100$. The numbers 10000 and 10100 are "non-sustainable" , and they are the least possible "non-sustainable" numbers. Thus you cannot find a "non-sustainable" number between them. Therefore, $10001, 10002, ..., 10099$ are 99 consecutive "sustainable" numbers.
Answer: 99.