Problem Sheet 8.

Recall the Pigeon Hole Principle from Problem Set 2.

Pigeon Hole Principle. If we must put $N+1$ or more pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

General Pigeon Hole Principle. If we must put $Nk+1$ or more pigeons into N pigeon holes, then some pigeon hole must contain at least $k+1$ pigeons

Example 1. An infinite straight line is coloured in 57 colours. Prove that you can find two points with a distance between them being an integer number in cm.

Example 2. A class test consisted of four problems. For each problem you can either receive 1 point or 0 points. There are 33 students in a class who took this test. Prove that no matter what the results of the test are, you can find three students such that they have the same marks for each of the four problems.

Normally, it is not possible to solve a mathematical problem by using only one method or one technique. Most of the time you need to apply several ideas to get the full proof or the correct answer. Combining different methods and techniques allows you to solve a large variety of problems. Today we consider examples of how the Pigeon Hole Principle and the theory of remainders can work together.

Example 3. Frodo Baggins, Samwise Gamgee, Meriadoc Brandybuck and Peregrin Took were gathering mushrooms in the forest of Shire. When the hobbits got home after a long day under the baking sun, they compared the number of mushrooms each of them picked. Prove that you can choose two of them with the difference in their "results" being a multiple of 3.

Example 4. A computer generates seven random natural numbers. Show that you can always choose three numbers out of them such that their sum is a multiple of 3.

"Hobbits really are amazing creatures, as I have said before. You can learn all that there is to know about their ways in a month, and yet after a hundred years they can still surprise you at a pinch." (The Lord of the Rings, The Fellowship of the Ring, Book I, Chapter 2: "The Shadow of the Past")

Problem 8.1. (a) This time the computer produces 4 random natural numbers. Prove that you can find two numbers among them such that their sum or their difference is a multiple of 5. (b) You are given 10 integers. Prove that you can always choose such numbers among them, that the sum of the chosen numbers is divisible by 10.

Problem 8.2. Is it possible to find three odd numbers a, b and c, such that none of the numbers $a \times b - 1$, $b \times c - 1$, $c \times a - 1$ is a multiple of 4?

Problem 8.3. Out of the four hobbits, Pippin is the youngest. At first, he was not taking their mission seriously until he felt bored to the point that he came up with an argument. He was so adamant with his argument that he started to persuade other members of the fellowship. His argument stated that you can divide the digits 1, 2, ..., 9 into three groups in such a way that the product of the numbers within each group is not greater than 71. Is it liable to say that his argument is valid?

Problem 8.4. Before leaving the Middle-Earth, Frodo handed his Hobbit-Hole down to Sam in order for his friend to settle down for life. The apartment consisted of 11 rooms. The areas of all rooms (if counted in square meters) were different natural numbers, each of which was not greater than 20. Prove that there were two rooms with areas a and b, such that a divides b.

Problem 8.5. The strength of the orc army doubles every day. Initially there was only 1 orc. Saruman the White was making a report for Lord Sauron when suddenly he noticed that between two different (not necessarily neighbouring) days the number of orcs increased by a multiple of 2016. Was it a mistake?