Solutions 7.

Example 1.

n=3,+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

n=3,×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

n=5,+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

n=5,×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Example 2. Since 7 has the remainder 2, 7^2 has the remainder 4, 7^3 has the remainder 3, 7^4 has the remainder 1, 7^5 has the remainder 2 (according to the table from Example 1). From this moment the values will repeat: 2, 4, 3, 1, 2, 4, 3, 1, ... From this observation we conclude that the remainder of 7^m depends only on the remainder of m by the division algorithm by 4, or, in other words, 7^m and 7^{m+4} have the same remainder. Therefore, from $57 = 4 \times 14 + 1$ we deduce that 7^{57} has the remainder 2 by the division algorithm by 5. Or, in other words, $7^{57} \equiv 2(mod\, 5)$.
Answer: 2.

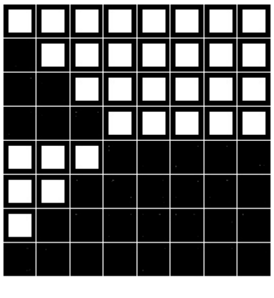
Example 3. The only possible remainders from division by 3 are 0, 1 or 2. Thus, their squares can only have 0 or 1 as remainders by the table from Example 1.
Answer: 0 or 1.

Problem 7.1.

n=7,+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

n=7,×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Problem 7.2. Note that $2013 = 11 \times 183$. Then 2014, 2015, 2016, 2017, 2018 has remainders 1, 2, 3, 4, 5 respectively. Thus our expression has the same remainder as $2\times3\times4+1^2\times5^3 = 24+25\times5$, which has the same remainder as $2+3\times5 = 17 = 6+11$. Therefore, the remainder of the initial expression is 6.
Answer: 6.
Problem 7.3. Assume it is possible. Then all possible numbers of coins inside the bags have 1 or 2 as their remainders modulo 3. But each of the operations (adding 1, adding 2, multiplying by 2) will change the remainder if it is nonzero. Therefore the remainders should alternate. But it is not possible as 99 is an odd number. Thus, we come to contradiction. It means that there has to be at least one bag containing such number of coins that it is a multiple of 3.
Problem 7.4. Consider $n = 4$. Odd numbers can have 1 or 3 as a remainder. It means that a square of an odd number can have only 1 as a remainder ($3 \times 3 = 2 \times 4 + 1$). If you sum up three odd squares we will get a number with remainder 3, and it cannot be a square of an odd natural number.
Answer: it is not possible.
Problem 7.5. The numbers $2222 = 7\times317+3$ and $5555 = 7\times793+4$ has remainders 3 and 4 respectively. Similar to Example 2 the remainder 3 has "period" 6 (3^m and 3^{m+6} have the same remainders modulo 7) and the remainder 4 has "period" 3 (4^m and 4^{m+3} have the same remainders modulo 7). Since $5555 = 6 \times 925 + 5$ and $2222 = 3\times74+2$, the number $2222^{5555}+5555^{2222}$ has the same remainder as 3^5+4^2, which has the same remainder as $5+2 = 7$. The last equality means that the initial number is divisible by 7.
Problem 7.6. Since all the columns should contain the same amount of beans, the total number of beans should be divisible by 8. We know that all the rows contain different number of beans, therefore we have 9 possible values, i.e. 0, 1, 2, 3, 4, 5, 6, 7, 8. We need to choose 8 out of them with the sum being a multiple of 8. The total sum is $0+1+2+3+4+5+6+7+8 = 36$ and it is 4 modulo 8, so we need to fit 0, 1, 2, 3, 5, 6, 7, 8 beans into different rows in such a way that every column has 4 beans. See the picture below.



Problem 7.7. (a) Label the books by 1, 2, and 3. At first they are in their

original positions of 1, 2, 3, on the shelf. We can place book 1 in position 2, and then book 3 must go to 1, so the only possible result is 3, 1, 2. If we place book 1 in position 3, then book 2 must go in position 1, so the only possible result is 2, 3, 1. Thus, there are two ways to arrange the books with none in its original position.
(b) Label the books 1, 2, 3, and 4. At first they are in the original positions of 1, 2, 3, 4 on the shelf. We can place book 1 in position 2, 3, or 4. With each of these moves, we will find the same number of options for arranging the books. Therefore let's count the number of arrangements when book 1 ends up in position 4, and then multiply that number by 3 to get the total number of arrangements. Place book 4 temporarily in position 1, so we have the books in the order 4, 2, 3, 1. The two rearrangements of the first three books, now 4, 2, 3, that leave none in its current place, will give an arrangement of all four books that leave none in its original place. In addition, the arrangement 4, 3, 2 gives an arrangement of 4, 2, 3 that leaves none of the four books in its original place. There are three arrangements of 4, 2, 3 that lead to arrangements of all four with none in its original place with book 1 in the fourth place. Multiply this number by 3 to get a total of nine rearrangements that leave no book in its original position.
(c) Label the books in order 1, 2, 3, 4, 5. Book 1 can be placed in any one of four places: 2, 3, 4, or 5, and the number of possible rearrangements in each of these cases will be the same, so we can calculate the number of outcomes with book 1 in position 5 and multiply by 4. Place the book 5 temporarily in position 1. Then the books 5, 2, 3, 4 should be rearranged either so that book 5 remains in position 1 while the books 2, 3, and 4 are rearranged amongst themselves, or so that no one of these four books remains in its position. From (a) we know there are two arrangements with book 5 occupying the first position, and from (b) we know there are 9 arrangements when we move book 5. This gives 11 rearrangements with book 1 in position five, so there are $4 \times 11 = 44$ arrangements of five books with none occupying its original position.
Answer: (a) 2; (b) 9; (c) 44.