Adventures in Bayesian Structural Time Series

Part 3: Structural Time Series in Bayesian Context

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Outline

- Without Regression:

Outline

- Without Regression:
  - Prior

Outline

- Without Regression:
  - Prior
  - Posterior

Outline

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  - Prior
  - Posterior
- With Regression:

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Prior and Posterior Without Regression

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  - Specify distribution on parameters (variances)

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  - Kalman filter and Kalman smoother

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  - Markov Chain Monte Carlo

Prior and Posterior Without Regression

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- Posterior
  - Kalman filter and Kalman smoother
  - Markov Chain Monte Carlo
  - See Durbin and Koopman: Time Series Analysis by State Space Methods for details

Spike and Slab Prior

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- **Spike**: prior for probability of regression coefficients being zero
- **Slab**: prior on nonzero regression coefficients and variance

Spike and Slab Prior

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- β_γ: subset of βs with $\beta_k \neq 0$
- Independent Bernoulli prior:

$$\gamma \sim \prod_{i=1}^{K} \pi_k^{\gamma_k} (1 - \pi_k)^{1-\gamma_k}$$

Spike and Slab Prior

Slab

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  - Ω_{γ}^{-1}: submatrix corresponding to $\gamma_k = 1$

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- Conjugate prior:

$$\beta_\gamma | \sigma_\varepsilon^2, \gamma \sim N(b_\gamma, \sigma_\varepsilon^2 \Omega_\gamma) \qquad \frac{1}{\sigma_\varepsilon^2} | \gamma \sim \text{Gamma}\left(\frac{v}{2}, \frac{ss}{2}\right)$$

Spike and Slab Prior

Full Prior

$$p(\beta, \gamma, \sigma_\varepsilon^2) = p(\beta_\gamma | \gamma, \sigma_\varepsilon^2) p(\sigma_\varepsilon^2 | \gamma) f(\gamma)$$

Specifying Spike and Slab Prior

- ss, v: prior sum of squares and prior sample size

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  - $\Omega^{-1} = \frac{k}{n} X^T X$

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Posterior of β and σ_ε^2 given γ, y^*

- Z_t^*: observation matrix Z_t with $\beta^T x_t = 0$

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- $$\beta_\gamma | \sigma_\varepsilon, \gamma, y^* \sim N(\tilde{\beta}_\gamma, \sigma_\varepsilon^2 V_\gamma) \quad \frac{1}{\sigma_\varepsilon^2} | \gamma, y^* \sim \text{Gamma}\left(\frac{N}{2}, \frac{SS_\gamma}{2}\right)$$

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- $\tilde{\beta}_\gamma = V_\gamma (X_\gamma^T y^* + \Omega_\gamma^{-1} b_\gamma)$

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Posterior of β and σ^2_ε given γ, y^*

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- Z_t^*: observation matrix Z_t with $\beta^T x_t = 0$
- $y_t^* = y_t - {Z_t^*}^T \alpha_t$
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- $V_\gamma^{-1} = (X^T X)_\gamma + \Omega_\gamma^{-1}$
- $\tilde{\beta}_\gamma = V_\gamma (X_\gamma^T y^* + \Omega_\gamma^{-1} b_\gamma)$
- $N = v + n$
- $SS_\gamma = ss + {y^*}^T y^* + b_\gamma^T \Omega_\gamma^{-1} b_\gamma - \tilde{\beta}_\gamma^T V_\gamma^{-1} \tilde{\beta}_\gamma$

Spike and Slab Posterior

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Posterior of γ given y^*:

$$\gamma | y^* \sim C(y^*) \frac{|\Omega_\gamma^{-1}|^{1/2}}{|V_\gamma^{-1}|^{1/2}} \frac{p(\gamma)}{SS_\gamma^{N/2-1}}$$

Posterior Sampling

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