Adventures in Bayesian Structural Time Series

Part 3: Structural Time Series in Bayesian Context

Andrew Bates, Josh Gloyd, Tyler Tucker

Outline

- Without Regression:

Outline

- Without Regression:
  - Prior

Outline

- Without Regression:
  - Prior
  - Posterior

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- With Regression:

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Without Regression

- Prior
  - Specify distribution on parameters (variances)

Without Regression

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Without Regression

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- Prior
  - Specify distribution on parameters (variances)
- Posterior
  - Kalman filter and Kalman smoother

Without Regression

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  - Markov Chain Monte Carlo

Spike and Slab Prior

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- **Spike**: prior for probability of regression coefficients being zero
- **Slab**: prior on nonzero regression coefficients and variance

Spike and Slab Prior

Spike

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Spike and Slab Prior

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- β_γ: subset of βs with $\beta_k \neq 0$
- Independent Bernoulli prior:

$$\gamma \sim \prod_{i=1}^{K} \pi_k^{\gamma_k}(1-\pi_k)^{1-\gamma_k}$$

Spike and Slab Prior

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Slab

- Ω^{-1}: symmetric matrix

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Slab

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  - Ω_{γ}^{-1}: submatrix corresponding to $\gamma_k = 1$

Slab

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- Conjugate prior:

$$\beta_\gamma | \sigma_\varepsilon^2, \gamma \sim N(b_\gamma, \sigma_\varepsilon^2 \Omega_\gamma) \qquad \frac{1}{\sigma_\varepsilon^2} | \gamma \sim \text{Gamma}\left(\frac{v}{2}, \frac{ss}{2}\right)$$

Spike and Slab Prior

Full Prior

$$p(\beta, \gamma, \sigma_\varepsilon^2) = p(\beta_\gamma|\gamma, \sigma_\varepsilon^2)p(\sigma_\varepsilon^2|\gamma)p(\gamma)$$

Specifying Spike and Slab Prior

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  - $\Omega^{-1} = \frac{k}{n} X^T X$

Spike and Slab Posterior

Posterior of β and σ_ε^2 given γ, y^*

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- $$\beta_\gamma | \sigma_\varepsilon, \gamma, y^* \sim N(\tilde{\beta}_\gamma, \sigma_\varepsilon^2 V_\gamma) \quad \frac{1}{\sigma_\varepsilon^2} | \gamma, y^* \sim \text{Gamma}\left(\frac{N}{2}, \frac{SS_\gamma}{2}\right)$$

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- $\tilde{\beta}_\gamma = V_\gamma (X_\gamma^T y^* + \Omega_\gamma^{-1} b_\gamma)$
- $N = v + n$
- $SS_\gamma = ss + {y^*}^T y^* + b_\gamma^T \Omega_\gamma^{-1} b_\gamma - \tilde{\beta}_\gamma^T V_\gamma^{-1} \tilde{\beta}_\gamma$

Spike and Slab Posterior

Posterior of γ given y^*:

$$\gamma | y^* \sim C(y^*) \frac{|\Omega_\gamma^{-1}|^{1/2}}{|V_\gamma^{-1}|^{1/2}} \frac{p(\gamma)}{SS_\gamma^{N/2-1}}$$

Posterior Sampling

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