

Inferences About Means

Chapter 18

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The Central Limit Theorem (CLT)

When a random sample is drawn from a population with mean m and standard deviation s , the sampling distribution has:

- Mean: μ
- Standard deviation: $\frac{\sigma}{\sqrt{n}}$
- Approximately Normal distribution as long as the sample size is large.
- The larger the distribution, the closer to Normal.

Weight of Angus Cows

The weight of Angus cows is Normally distributed with $\mu = 1309$ pounds and $\sigma = 157\text{lbs}$. What does the CLT say about the mean weight of a random sample of 100 Angus cows?

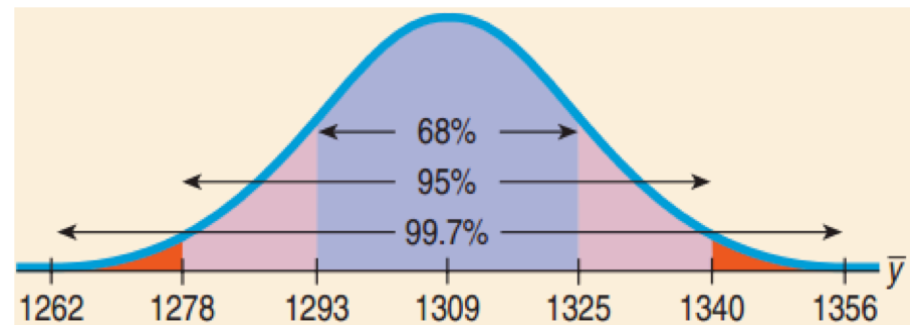
- Sample means \bar{y} will average 1309 pounds.

- $SD(\bar{Y}) = \frac{\sigma}{\sqrt{n}} = \frac{157}{\sqrt{100}} = 15.7\text{lbs}$

The CLT says the sampling distribution will be approximately Normal: $N(1309, 15.7)$.

For the means of all random samples,

- 68% will be between 1293.3 and 1324.7 lb.
- 95% will be between 1277.6 and 1340.4 lb.
- 99.7% will be between 1261.9 and 1356.1 lb.

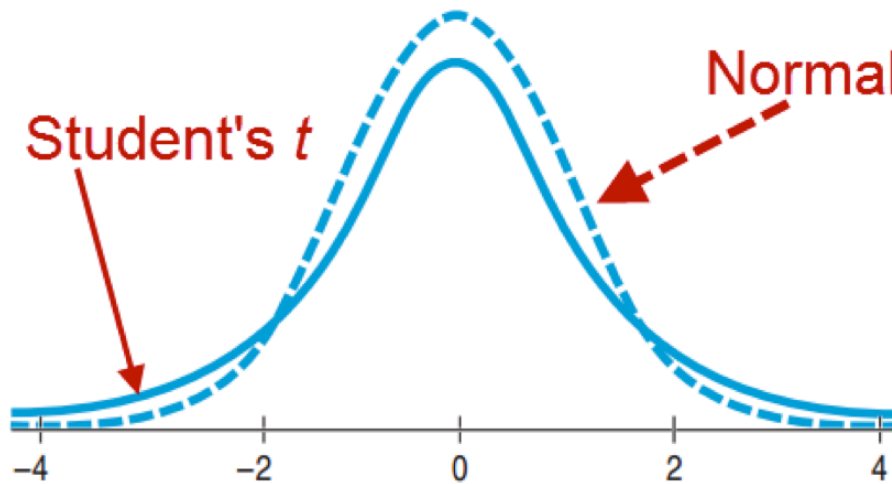


The Challenge of the CLT

- CLT tells us $SD(\bar{y}) = \frac{\sigma}{\sqrt{n}}$
- We would like to use this for Confidence Intervals and Hypothesis Testing.
- Unfortunately, we almost never know σ .
- Using s almost works: $SE(\bar{y}) = \frac{s}{\sqrt{n}}$, but not quite.
- When using s , the Normal model has some error.
- William Gosset came up with new models, one for each n that works better.

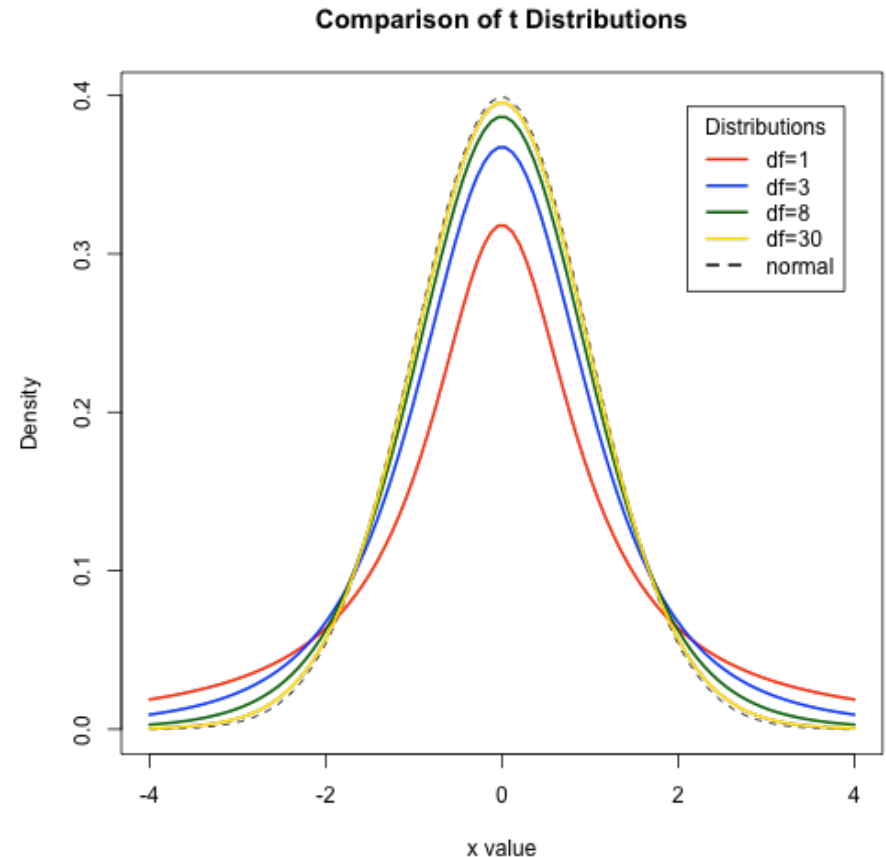
Gosset the Brewer

- At Guinness, Gosset experimented with beer.
- The Normal Model was not right, especially for small samples.
- Still bell shaped, but details differed, depending on n
- Came up with the “Student’s t Distribution” as the correct model



Degrees of Freedom

- For every sample size n there is a different Student's t distribution.
- Degrees of freedom: $df = n - 1$.
- Similar to the " $n - 1$ " in the formula for sample standard deviation
- It is the number of independent quantities left after we've estimated the parameters.



Confidence Interval for Means

Sampling Distribution Model for Means

- With certain conditions (seen later), the standardized sample mean follows the Student's t model with $n - 1$ degrees of freedom.

$$t = \frac{\bar{y} - \mu}{SE(\bar{y})}$$

- We estimate the standard deviation with

$$SE(\bar{y}) = \frac{s}{\sqrt{n}}$$

One Sample t-Interval for the Mean

When the assumptions are met (seen later), the confidence interval for the mean is

$$\bar{y} \pm t_{n-1}^* \times SE(\bar{y})$$

The critical value t_{n-1}^* depends on the confidence level, C, and the degrees of freedom $n - 1$.

Contaminated Salmon

A study of mirex concentrations in salmon found

- $n = 150$, $\bar{y} = 0.0913$ ppm, $s = 0.0495$ ppm
- Find a 95% confidence interval for mirex concentrations in salmon.
- $df = 150 - 1 = 149$
- $SE(\bar{y}) = \frac{0.0485}{\sqrt{150}} \approx 0.0040$
- $t_{149}^* = 1.976$

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qt(0.975, 149)
```

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[1] 1.976
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$$\bar{y} \pm t_{149}^* \times SE(\bar{y}) = 0.0913 \pm 1.976(0.0040)$$

$$\bar{y} \pm t_{149}^* \times SE(\bar{y}) = (0.0834, 0.0992)$$

- I'm 95% confident that the mean level of mirex concentration in farm-raised salmon is between 0.0834 and 0.0992 parts per million.

Notes about z and t

The Student's t distribution:

- Is unimodal.
- Is symmetric about its mean.
- Has higher tails than Normal.
- Is very close to Normal for large df.
- Is needed because we are using s as an estimate for σ .

If you happen to know σ , which almost never happens, use the Normal model and not Student's t.

Assumptions and Conditions

Independence Condition

- Randomization Condition: The data should arise from a suitably randomized experiment.
- Sample size $< 10\%$ of the population size.

Nearly Normal

- For large sample sizes ($n > 40$), not severely skewed.
- ($15 \leq n \leq 40$): Need unimodal and symmetric.
- ($n < 15$): Need almost perfectly normal.
- Check with a histogram.

How Much Sleep do College Students Get?

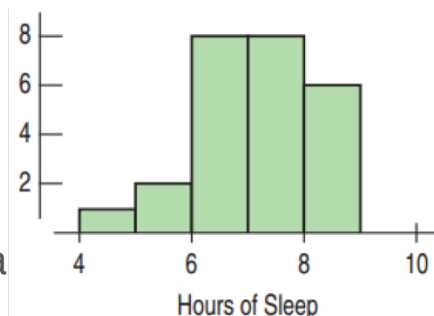
Build a 90% Confidence Interval for the Mean.

Plan: Data on 25 Students

Model

- Randomization Condition

The data are from a random survey.



- Nearly Normal Condition
Unimodal and slightly skewed, so OK
- Use Student's t-Model with $df = 25 - 1 = 24$.
- One-sample t-interval for the mean

Mechanics

- $n = 25; \bar{y} = 6.64; s = 1.075$
- $SE(\bar{y}) = \frac{s}{\sqrt{n}} = \frac{1.075}{\sqrt{25}} = 0.215 \text{ hours}$

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qt(0.95, 24)
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[1] 1.711
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- $t_{24}^* = 1.711$
- $ME = t_{24}^* \times SE\bar{y} = 1.711 \times 0.215 = 0.368 \text{ hours}$

90% $CI = 6.64 \pm 0.368$

How Much Sleep do College Students Get?

Conclusion: I'm 90 percent confident that the interval from 6.272 and 7.008 hours contains the true population mean number of hours that college students sleep.

What Not to Say

“90% of all students sleep between 6.272 and 7.008 hours each night.”

- The CI is for the mean sleep, not individual students.

“We are 90% confident that a randomly selected student will sleep between 6.272 and 7.008 hours per night.”

- We are 90% confident about the mean sleep, not an individual's sleep.

“The mean amount of sleep is 6.64 hours 90% of the time.”

- The population mean never changes. Only sample means vary from sample to sample.

“90% of all samples will have a mean sleep between 6.272 and 7.008 hours per night.”

- This interval does not set the standard for all other intervals. This interval is no more likely to be correct than any other.

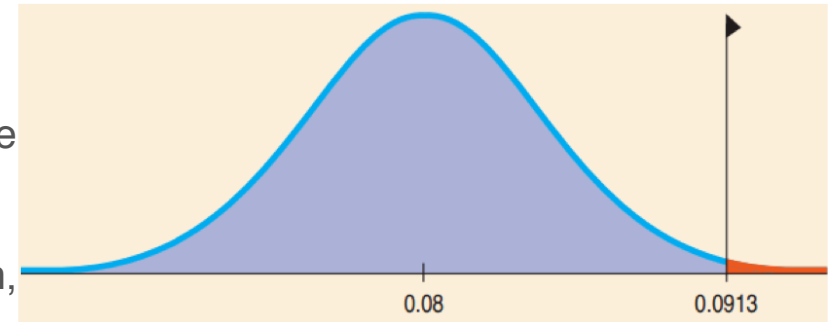
One-Sample t-Test for the Mean

- Assumptions are the same.
- $H_0 : \mu = \mu_0$
- $t_{n-1}^* = \frac{\bar{y} - \mu_0}{SE(\bar{y})}$
- Standard Error of \bar{y} : $SE(\bar{y}) = \frac{\sigma}{\sqrt{n}}$
- When the conditions are met and H_0 is true, the statistic follows the Student's t Model.
- Use this model to find the P-value.

Are the Salmon Unsafe?

EPA recommended mirex screening is 0.08 ppm.

- Are farmed salmon contaminated beyond the permitted EPA level?
- Recap: Sampled 150 salmon. Mean 0.0913 ppm, Standard Deviation 0.0495 ppm.
- $H_0 : \mu = 0.08$
- $H_A : \mu > 0.08$



Are the Salmon Unsafe?

One-Sample t-Test for the Mean

- $n=150$; $df=149$; $\bar{y} = 0.0913$; $s=0.0495$
- $SE(\bar{y}) = \frac{0.0495}{\sqrt{150}} \approx 0.0040$
- $t_{149}^* = \frac{0.0913-0.08}{0.0040} = 2.825$
- $P(t_{149} > 2.825) = 0.0027$

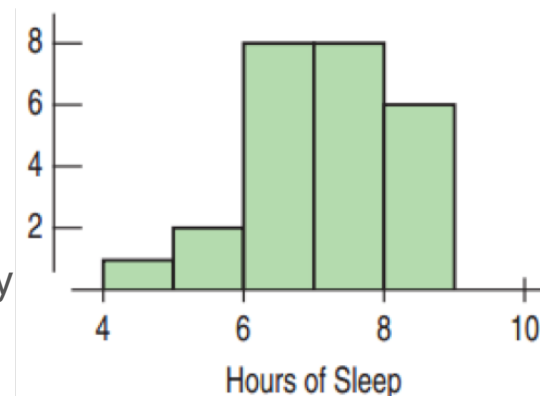
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1 - pt(2.825, 149)
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[1] 0.002688
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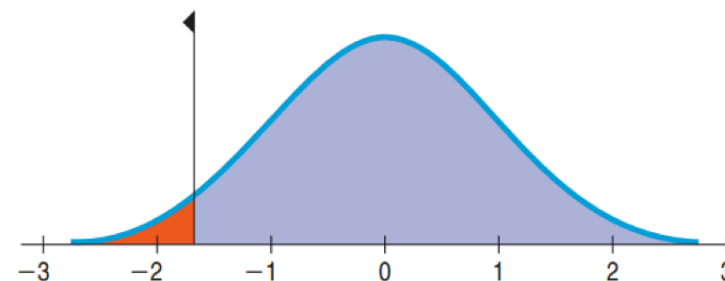
- Since the P-value is so low, reject H_0 and conclude that the population mean mirex level does exceed the EPA screening value.

Do Students Get at Least Seven Hours of Sleep? 25 Surveyed.

- Plan: Does the mean amount of sleep exceed 7 hours?
- Hypotheses: $H_0 : \mu = 7$; $H_A : \mu > 7$
- Model
 - Randomization Condition: The students were randomly and independently selected .
 - Nearly Normal Condition: unimodal and symmetric
- Use the Student's t-model, $df = 24$
- One-sample t-test for the mean



Do Students Get at Least Seven Hours of Sleep? 25 Surveyed.



Mechanics

- $n = 25; \bar{y} = 6.64; s = 1.075$
- $SE(\bar{y}) = \frac{s}{\sqrt{n}} = \frac{1.075}{\sqrt{25}} = 0.215 \text{ hours}$
- $t = \frac{\bar{y} - \mu_0}{SE(\bar{y})} = \frac{6.64 - 7}{0.215} \approx -1.67$
- P-value = $P(t_{25} < -1.67) \approx 0.054$

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pt(-1.67, 24)
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[1] 0.05396
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Conclusion: P-value = 0.054 says that if students do sleep an average of 7 hrs., samples of 25 students can be expected to have an observed mean of 6.64 hrs. or less about 54 times in 1000.

- With 0.05 cutoff, there is not quite enough evidence to conclude that the mean sleep is less than 7.
- The 90% CI: (6.272, 7.008) contains 7.
- Collecting a larger sample would reduce the ME and give us a better handle on the true mean hours of sleep.

Intervals and Tests

Confidence Intervals

- Start with data and find plausible values for the parameter.
- Always 2-sided

Hypothesis Tests

- Start with a proposed parameter value and then use the data to see if that value is not plausible.
- 2-sided test: Within the confidence interval means fail to reject H_0 . P-value = $1 - C$ is the cutoff.
- 1-sided test: P-value = $(1 - C)/2$ is the cutoff.

Sleep, Confidence Intervals and Hypothesis Tests

90% Confidence interval: (6.272, 7.008)

- For a 2-tailed test with a 10% cutoff, any $6.272 \leq \mu_0 \leq 7.008$ would result in failing to reject H_0 .
- For a 1-tailed test (“ $<$ ”) with a 5% cutoff, any $\mu_0 \leq 7.008$ would result in failing to reject H_0 .
- For a 1-tailed test (“ $>$ ”) with a 5% cutoff, any $6.272 \leq \mu_0$ would result in failing to reject H_0 .

The Challenge of Finding the Sample Size

$$ME = t_{n-1}^* \times \frac{s}{\sqrt{n}}$$

To find the necessary sample size in order to have a small enough margin of error:

- Decide on acceptable ME.
- Determine s: Use a pilot to estimate s.
- Determine t_{n-1}^* : Use z^* as an estimate. By the 68-95-99.7 Rule, use 2 for 95% confidence.