From Randomness to Probability

Chapter 12

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Red Light, Green Light

Each day you drive through an intersection and check if the light is red, green, or yellow.

- Day 1: red
- Day 2: green
- Day 3: red

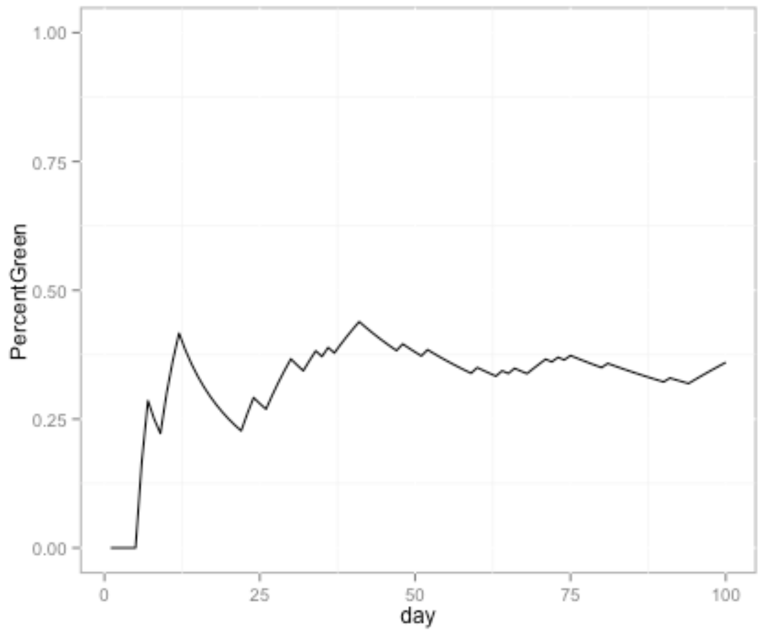
Before you begin, you know:

- The possible outcomes
- An outcome will occur.

After you finish, you know:

- The outcomes that occurred

```
flips <- sample(c(0,1,0), 100, replace=TRUE)
df <- data.frame(day=1:100,
    PercentGreen=cumsum(flips) / 1:100)
ggplot(df, aes(x=day, y=PercentGreen)) +
    geom_path() + ylim(c(0,1))
```



Random Phenomena Vocabulary

- Trial: Each occasion which we observe a random phenomena
- Outcome: The value of the trial for the random phenomena
- Event: The combination of the trial’s outcomes
- Sample Space: The collection of all possible outcomes

For example, flipping two coins:

- Trial: The flipping of the two coins
- Outcome: Heads or tails for each flip
- Event: HT, for example
- Sample Space: S = {HH, HT, TH, TT}

The Law of Large Numbers

- If you flip a coin once, you will either get 100% heads or 0% heads.
- If you flip a coin 1000 times, you will probably get close to 50% heads.

The Law of Large Numbers states that for many trials, the proportion of times an event occurs settles down to one number.

- This number is called the **empirical probability**.

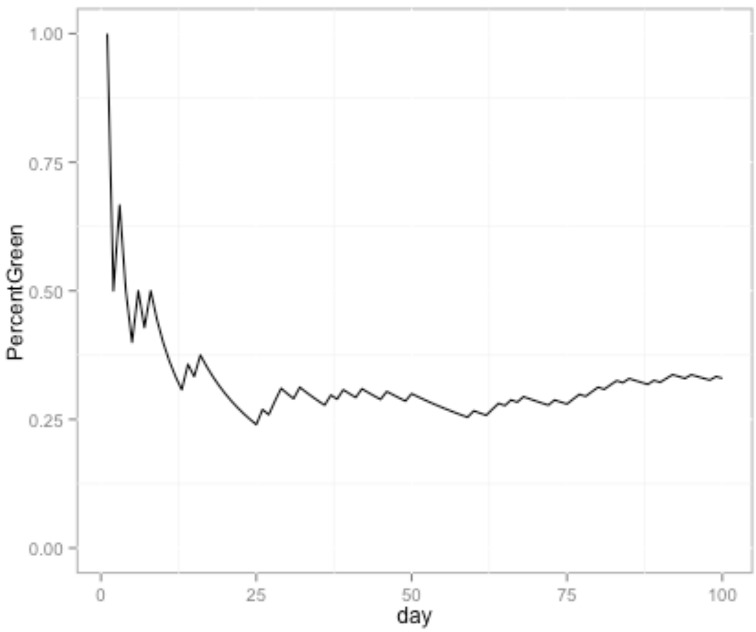
Requirements:

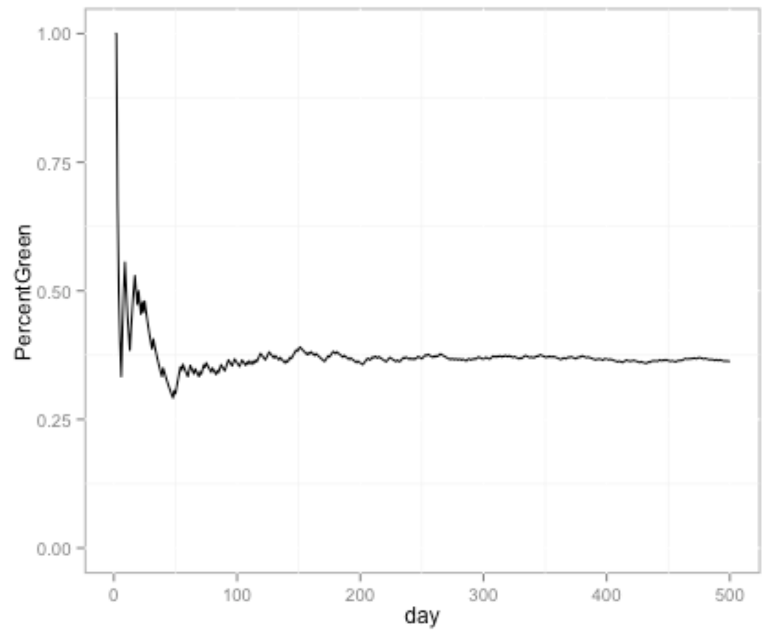
- Identical Probabilities: The probabilities for each event must remain the same for each trial.
- Independence: The outcome of a trial is not influenced by the outcomes of the previous trials.

Empirical probability: $P(A) = \frac{num \quad times \quad A \quad occurs}{num \quad of \quad trials}$ (in the long run)

Red Light, Green Light: The Law of Large Numbers

- After many days, the proportion of green lights encountered is approximately 0.34.
- P(green) = 0.34.
- If we recorded more days, the probability would still be about 0.34.





The Nonexistent Law of Averages

Wrong

- If you flip a coin 5 times and get five tails, then you are due for a head on the next flip.
- You put 10 quarters in the slot machine and lose each time. You are just a bad luck person, so you have a smaller chance of winning on the 11th try.
- There is no such thing as the Law of Averages for short runs.

Theoretical Probability

American Roulette

- 18 Red, 18 Black, 2 Green
- If you bet on Red, what is the probability of winning?

Theoretical Probability

$$P(A) = \frac{num \quad times \quad A \quad occurs}{num \quad of \quad trials}$$

$$P(red) = \frac{18}{38}$$

Heads or Tails

Flip 2 coins. Find P(HH)

- List the sample space:
  S = {HH, HT, TH, TT}
- $P(HH) = \frac{1}{4}$

Flip 100 coins. Find the probability of all heads. The sample space would involve 1,267,650,600,228,229,401,496,703,205,376 different outcomes. Later, we will see a better way.

Equally Likely?

What's wrong with this logic?

- Randomly pick two people.
- Find the probability that both are left-handed.
- Sample Space
  S = {LL, LR, RL, RR}
- $P(LL) = \frac{1}{4}$

Since left-handed and right-handed are not equally likely, this method does not work.

Personal Probability

What's your chance of getting an A in statistics?

- You cannot base this on your long-run experience.
- There is no sample space of events with equal probabilities to list.
- You can only base your answer on personal experience and guesswork.
- Probabilities based on personal experience rather than long-run relative frequencies or equally likely events are called personal probabilities.

Rules 1 and 2

Rule 1: $0 \leq P(A) \leq 1$

- You can’t have a –25% chance of winning.
- A 120% chance also makes no sense.
- Note: Probabilities are (generally) written in decimals.
    - 45% chance → $P(A) = 0.45$

Rule 2: $P(S) = 1$

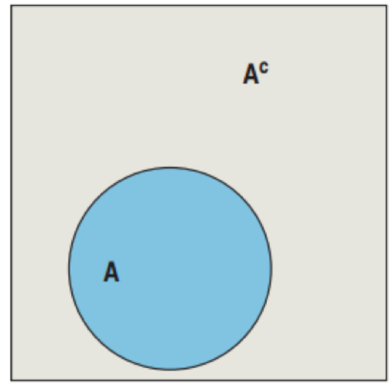
- The set of all possible outcomes has probability 1.
- There is a 100% chance that you will get a head or a tail.

Rule 3: The Complement Rule

Complements

- Define A^C as the complement of A.
- A^C is the event of A not happening.
- If A is the event of rolling a 5 on a six sided die, then AC is the event of not rolling a 5: {1, 2, 3, 4, 6}
- $P(A) = 1/6$; $P(A^C) = ?$
- $P(A^C) = 5/6 = 1 - 1/6$

The Rule of Complements: $P(A^C) = 1 - P(A)$



Red Light Green Light and Complements

We know that P(green) = 0.34. Find P(not green):

- Not green is the complement of green.
- Use the rule of complements:
  $P(notgreen) = P(green^C)$
  P(not green) = 1 – P(green)
  P(not green) = 1 – 0.34
  P(not green) = 0.66

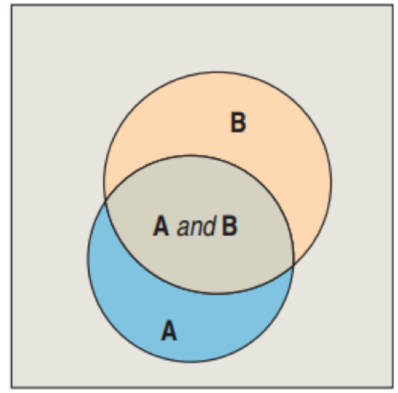
The probability of the light not being green is 0.66.

Rule 4: The Addition Rule

Suppose
P(sophomore) = 0.2 and P(junior) = 0.3

Find P(sophomore OR junior)
Solution: 0.2 + 0.3 = 0.5

This works because sophomore and junior are disjoint events. They have no outcomes in common.



The Addition Rule

- If A and B are disjoint events, then

$$P(A \quad OR \quad B) = P(A) + P(B)$$

Red Light, Green Light, Yellow Light

Given that P(green) = 0.35 and P(yellow) = 0.04

- Find P(red).

Solution: Use the Rule of Complements and the Addition Rule.

$$P(red) = 1 - P(red^C)$$
$$P(red) = 1 - P(greenORyellow)$$
$$P(red) = 1 - [P(green) + P(yellow)]$$
$$P(red) = 1 - [0.35 + 0.04]$$
$$P(red) = 1 - 0.39$$
$$P(red) = 0.61$$

The Sum of Probabilities

The sum of all the probabilities of every disjoint event must equal 1.

What's wrong with the following statement?

- Probabilities for freshmen, sophomore, junior, senior are: 0.25, 0.23, 0.22, 0.20.
  0.25 + 0.23 + 0.22 + 0.20 = 0.90
- Since they do not add to 1, something is wrong.

How about the following?

- P(owning a smartphone) = 0.5 and
- P(owning a computer) = 0.9
- This is fine, since they are not disjoint.

Rule 5: The Multiplication Rule

The probability that an Atlanta to Houston flight is on time is 0.85.

- If you have to fly every Monday, find the probability that your first two Monday flights will be on time.

Multiplication Rule: For independent events A and B:

$$P(A \quad AND \quad B) = P(A) \times P(B)$$

P(1st on time AND 2nd on time) = P(1st on time) × P(2nd on time)
P(1st on time AND 2nd on time) = 0.85 × 0.85
P(1st on time AND 2nd on time) = 0.7225

Red Light AND Green Light AND Yellow Light

Find the probability that the light will be red on Monday, green on Tuesday, and yellow on Wednesday.

- The multiplication rule works for more than 2 events.
- P(red Mon. AND green Tues. AND yellow Wed.) =
  P(red Mon.) × P(green Tues.) × P(yellow Wed.) =
  $0.61 \times 0.35 \times 0.04 =$
  0.00854

At Least One Red Light

Find the probability that the light will be red at least one time during the week.

Use the Complement Rule

P(at least 1 red) = 1 – P(no reds)
$= 1 - (0.39 \times 0.39 \times 0.39 \times 0.39 \times 0.39 \times 0.39 \times 0.39)$
≈ 0.9986

M&Ms: 38% Pink, 36% Teal, 16% Purple

- First notice that $0.38 + 0.36 + 0.16 = 0.9 \neq 1$
- There must be other colors.

Question: Find the probability that a Japanese survey respondent will want either pink or teal. P(pink OR teal)

- Pink and teal are disjoint.
- Apply the Addition Rule.

Question: P(1st purple AND 2nd purple)
Find the probability that two Japanese survey respondents will want purple.

- The choice made by the first respondent does not affect the choice of the other. The events are independent.
- Use the Multiplication Rule.

M&Ms: 38% Pink, 36% Teal, 16% Purple

Question: P(pink OR teal)

Mechanics:

- P(pink OR teal) = P(pink) + P(teal)
  P(pink OR teal) = 0.38 + 0.36
  P(pink OR teal) = 0.74

Conclusion: The probability that the respondent chose pink or teal is 0.74.

Question: P(1st purple AND 2nd purple)

- Mechanics:
  P(1st purple AND 2nd purple) = P(1st purple) × P(2nd purple)
  P(1st purple AND 2nd purple) = 0.16 × 0.16
  P(1st purple AND 2nd purple) = 0.0256

Conclusion: The probability that both choose purple is 0.0256.