Random Variables and Probability Models

Chapter 14

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Probability Models

- An insurance company pays 10,000 if you die or 5,000 for a disability.
- The amount the company pays is a **random variable**: a numeric value based on the outcome of a random event.
- It is a **discrete random variable**, since we can list all the possible outcomes
- A **continuous random variable** is a random variable that is not discrete.
- The collection of all possible values and their probabilities is called a **probability model**.

OUTCOME	PAYOUT	PROBABILITY
Death	10,000	1/1000
Disability	5,000	2/1000
Neither	0	997/1000

Expected Value

- The expected value is the average amount that is likely to occur if there are many trials.
- Expected Value Formula:

$$\mu = E(x) = \sum xP(x)$$

$$E(x) = (10,000)\,\frac{1}{1,000} + (5,000)\,\frac{2}{1,000} + 0\,\frac{997}{1,000} = 20$$

- The company expects to pay an average of about 20 per policy per year.

Standard Deviation of a Probability Model

- Consider data from many outcomes of a random variable.
- The variance and standard deviation of these outcomes will measure the spread of the data.

$$\sigma^2 = \sum (x - \mu)^2 P(x)$$

- The standard deviation is the square root of the variance.

$$SD(x) = \sigma = \sqrt{\sigma^2} = \sqrt{\sum (x - \mu)^2 P(x)}$$

About the standard deviation

- Not just the standard deviation of the X values
- A weighted average
- Measures how outcomes will likely be spread out if many are selected
- Will be large if there is a high probability of both small values and large values

Standard Deviation of Insurance Example

- Insurance policy costs 50.
- The expected payout is 20 and the standard deviation is 386.78.
- The insurance company's expected profit is 50 - 20 = 30.
- Would you buy this policy?
- The standard deviation gives an indication of the high risk to the insurer.

```
    Outcome Payout Probability
      Death  10000       0.001
 Disability   5000       0.002
    Neither      0       0.997
```

```
m <- sum(insurance$Payout *
            insurance$Probability)
m
```

```
[1] 20
```

```
sqrt(sum( (insurance$Payout - m)^2  *
            insurance$Probability))
```

```
[1] 386.8
```

Adding or Subtracting a Constant

- Adding or subtracting a constant to all the data values shifts the expected value by that constant.

$$\begin{aligned} E(X+c) &= E(X) + c \\ E(X-c) &= E(X) - c \end{aligned}$$

- Adding or subtracting a constant to all the data values has no effect on the standard deviation.

$$\begin{aligned} Var(X \pm c) &= Var(X) \\ SD(X \pm c) &= SD(X) \end{aligned}$$

A Coupon on Top of the Valentine’s Discount

The Valentine's Discount

- $E(X) = 5.83$, $SD(X) = 8.62$

If everybody brings a coupon for 5 off, what are the new expected value and standard deviation?

- $E(X + 5) = 5.83 + 5 = 10.83$
- $SD(X + 5) = 8.62$

Multiplying by a Constant

$$\begin{aligned} E(cX) &= cE(X) \\ Var(cX) &= c^2 Var(X) \\ SD(cX) &= |c| SD(X) \end{aligned}$$

- There is a double the rewards special Valentines Day discount. The rewards are now 40 and 20 instead of 20 and 10. What are the new expected value and standard deviation?
- $E(2X) = 2E(X) = (2)(5.83) = 11.66$
- $SD(2X) = 2SD(X) = (2)(8.62) = 17.74$
- With the double rewards special the restaurant expects an average discount of 11.66 and a standard deviation of 17.24.

The Addition Rule

$$\begin{aligned} E(X+Y) &= E(X) + E(Y) \\ Var(X+Y) &= Var(X) + Var(Y) \end{aligned}$$

- Two couples try the Valentines Day discount. For each: E(X) = 5.83 and SD(X) = 8.62.
- What is the combined expected value and SD?
- Let the total discount be: $T = X_1 + X_2$

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 5.83 + 5.83 = 11.66$$

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) = 8.62^2 + 8.62^2 = 148.6088$$

$$SD(X_1 + X_2) = \sqrt{148.6099} = 12.19$$

- Notice that since the variables are independent, the standard deviation, 12.19, is less than the standard deviation, 17.24, of the double discount.

The Subtraction Rule

- Roll two dice. Each die outcome will have the same SD.
- Subtracting the standard deviations gives 0, but the standard deviation of the differences is not 0.
- The range of the differences is –5 to 5, larger than the range for a single die: 1 to 6.

$$E(X - Y) = E(X) - E(Y)$$
$$Var(X - Y) = Var(X) + Var(Y)$$

Subtracting Discounts

- A competing restaurant also has a game style discount: E(X) = 10, SD = 15.
- How much more can you expect to save compared with the Valentines Day Discount: E(X) = 5.83, SD = 8.62?
- $E(W - X) = 10 - 5.83 = 4.17$
- What is the standard deviation of the differences?
- The competing restaurant's discount averages 4.17 more than the Valentines Day Discount. The standard deviation for the difference is 17.30.

The Binomial Model

Searching for Walt's Card (see class 9)

- 20% of the cereal boxes have Walt's card.
- What is the expected number of boxes to open to get a Walt card?

Bernoulli Trial

- Two outcomes: success or failure
- The probability of success, p, is the same for each trial.
- The trials are independent.

The 10% Rule

- There are 10 cereal boxes and you sample 4 of them.
- Not independent, since the probability of success changes for the second if you have success on the first
- If the sample is more than 10% of the population, then the trials are far from being independent.

Probability of Getting 2 Walt in 5 Trials

- Bernoulli trials: Millions of boxes, sample size 5.
- P(X = 2) from Binom(n, p), n = 5, p = 0.2, q = 0.8.
- 2 successes, 3 failures. No quite 0.2^2 × 0.8^3.
- Must consider all orders of 2 successes and 3 failures.
- Number of ways of picking k items from n:

$$ {}_nC_k = \frac{n!}{k!(n-k)!} $$
$$ {}_5C_2 = \frac{5!}{2!(5-2)!} = 10 $$

- P(X = 2) = 10 × 0.22 × 0.83 = 0.2048

- n = Number of trials
- p = Probability of success
- q = 1 – p = Probability of failure
- X = Number of successes
- $P(X = x) = {}_nC_x p^x q^{n-x}$
- Mean = np
- Standard Deviation = $\sqrt{nqp}$

Binomial Models Using R

```
dbinom(x=2, size=5, prob=0.2)
```

```
[1] 0.2048
```

```
cards <- data.frame(Card=c('Walt',
        'Jesse','Hank'),
        Prob=c(.2,.3,.5))
dbinom(2, 5, prob=cards$Prob)
```

```
[1] 0.2048 0.3087 0.3125
```

Spam and the Binomial Model

91% of all email is spam. Your inbox has 25 emails. Find the mean, standard deviation, and the probability that 1 or 2 of the emails are not spam.

- n = 25, p = 1 – 0.91 = 0.09, q = 0.91
- Mean: np = (25)(0.09) = 2.25
- Standard Deviation = $\sqrt{nqp} = \sqrt{(25)(0.09)(0.91)} \approx 1.43$
- P(X = 1 or X = 2) = P(X=1) + P(X=2) = 0.2340 + 0.2777 = 0.5117

There is about a 51% chance of 1 or 2 emails that are not spam.

```
dbinom(1, 25, 0.09) + dbinom(2, 25, 0.09)
```

```
[1] 0.5117
```

The Trouble with Large Sample Sizes

The Red Cross has 32,000 donors and needs at least 1850 that are O−. Will they run out?

- The computations involve ridiculously large numbers.
- “At least” requires P(X = 1850), P(X = 1851), all the way up to P(X = 32,000).
- Mean = np = 1920
- $SD = \sqrt{npq} \approx 42.48$

The Solution for Large Sample Sizes

The Red Cross has 32,000 donors and needs at least 1850 that are O−. Will they run out (less than)?

- Mean = np = 1920
- $SD = \sqrt{npq} \approx 42.48$
- The normal model with the same mean and standard deviation is a very good approximation.

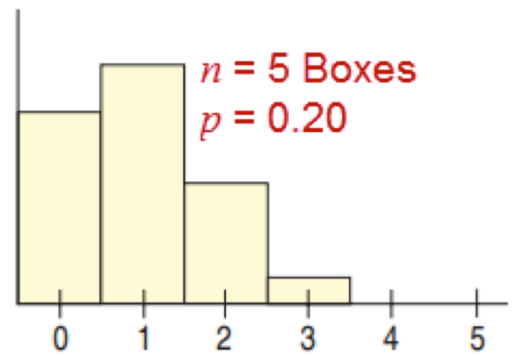
$$P(X < 1850) \approx P\left(z < \frac{1850 - 1920}{42.48}\right) \approx P(z < -1.65) \approx 0.05$$

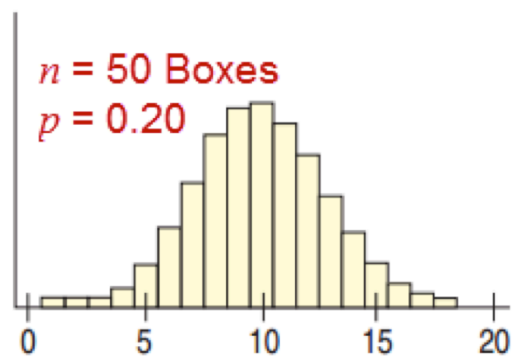
There is about a 5% chance that they will run out.

How Large is "Large Enough"

The Success/Failure Condition

- A Binomial is approximately Normal if we expect at least 10 successes and 10 failures.
    - $np \geq 10$
    - $nq \geq 10$
- This comes from the binomial being skewed for a small number of successes or failures expected.





Example: Spam and the Normal Approximation to the Binomial

Only 151 of 1422 emails got through your spam filter. Might the filter be too aggressive?

- What is the probability that no more than 151 of the emails are real messages?
- These emails represent less than 10% of all emails.
- $np = (1422)(0.09) = 127.98 \geq 10$
- $nq = (1422)(0.91) = 1294.02 \geq 10$
- Yes, the Normal model is a good approximation.

Example: Spam and the Normal Approximation to the Binomial

What is the probability that no more than 151 of the emails are real messages?

- $\mu = np = 127.98$
- $\sigma = \sqrt{npq} \approx 10.79$
- $P(X < 151) \approx P\left(z < \frac{151-127.98}{10.79}\right) \approx P(z < 2.13) \approx 0.9834$

There is over a 98% chance that no more than 151 of them were real messages. The filter may be working.