

QF Group Theory CC2022

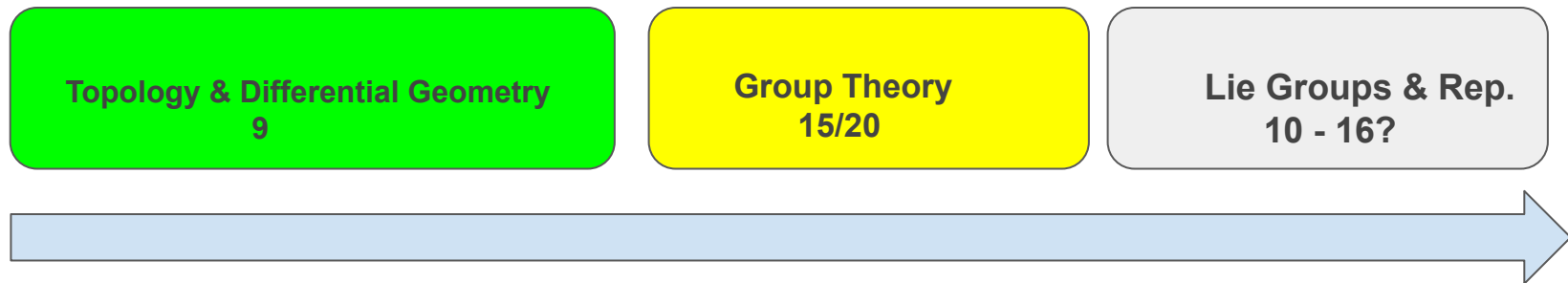
By Zaiku Group

Lecture 15

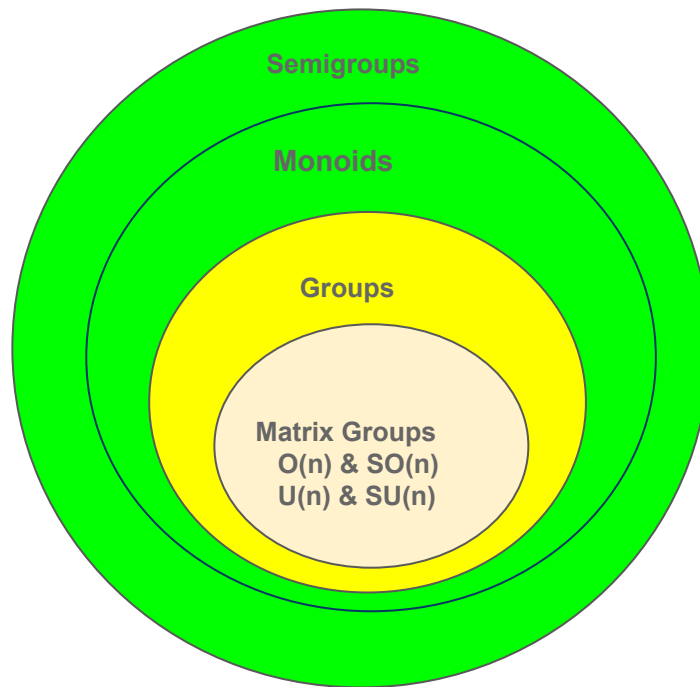
Delivered by Bambordé Baldé

Friday, 23/09/2022

Learning Journey Timeline



■ Completed | ■ Ongoing | ■ TBC (summer) | n is the number of live lectures |



Course Approach Overview



Completed!



We're here!

Quick Summary of Lecture 14 Concepts

- In lecture 14 we covered the following concepts:
 - ① Left and right cosets.
 - ② The index of a subgroup.
 - ③ Normal subgroups.
 - ④ Quotient groups.
- Today we'll layout some important results related to the concepts above before jumping to group actions.

Product of subgroups

Definition 1.0

Let G be a group, H and K be subgroups of G . The (internal) product HK is the set defined as $HK = \{hk \mid h \in H, k \in K\}$.

- HK is not necessarily a subgroup of G !

Theorem 1.0

Let G be a group, H and K be subgroups of G . Then the following hold:

- ① HK is a subgroup iff $HK = KH$.
- ② If either H or K are normal in G then HK is a subgroup of G .

Challenge 1

Let $G = S_3$, $H = \{1, (12)\}$ and $K = \{1, (13)\}$. You're encouraged to:

- ① Compute the product HK .
- ② Verify whether or not, HK is a subgroup of G . What about the product KH ?

Quotient map

Definition 1.1

Let G be a group and $N \triangleleft G$. The quotient map $\pi : G \longrightarrow G/N$ is defined as $\pi(x) = xN$ for all $x \in G$.

- The quotient map π is a homomorphism right?
- The map π is also known as 'canonical projection'.

Challenge 2

Let G be a group, $N \triangleleft G$ and $\pi : G \longrightarrow G/N$ be the quotient map. Is it true that $\text{Ker}(\pi) = N$? If not, what is $\text{Ker}(\pi)$?

- A gentle reminder that $\text{Ker}(\pi) = \{x \in G \mid \pi(x) = 1_{G/N}\}$.

Isomorphism theorems (A)

Theorem 1.1 (The first isomomorphism theorem for groups)

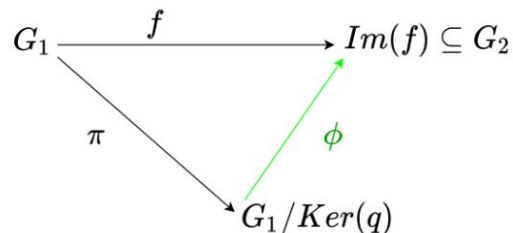
If a map $f : G_1 \longrightarrow G_2$ is a group homomorphism, then

$G_1/\text{Ker}(f) \simeq \text{Im}(f)$ i.e there exists an isomorphism

$\phi : G_1/\text{Ker}(f) \longrightarrow \text{Im}(f)$.

Proof : Homework challenge (not very hard to prove)!

- The theorem is equivalent to saying there exists an isomorphism $\phi : G_1/\text{Ker}(f) \longrightarrow \text{Im}(f)$ such that the following diagram commutes:



Challenge 3

Let $G_1 = \mathbb{Z}$ under ordinary addition and $G_2 = \mathbb{Z}_n$ under mod n addition. Construct a homomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ and verify that Theorem 1.1.

Isomorphism theorems (B)

Theorem 1.2 (The second isomorphism theorem for groups)

Let G be a group, $N \triangleleft G$ and H a subgroup of G . Then the following hold:

- 1 HN is a subgroup of G .
- 2 $N \cap H \triangleleft G$.
- 3 $H/(N \cap H) \simeq HN/N$.

Proof : Do you fancy having a go? It's not technically very hard to prove!

Note: The third isomorphism theorem will be included in the final slide after the session!

Group Actions on Sets

Definition 1.2

Let G be a group and X a set. A left action of G on the set X is a 'rule' that takes a pair $(g, x) \in G \times X$ and produces an element $gx \in X$ such that the following conditions hold:

- ① $1_G x = x$ for all $x \in X$.
- ② $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1, g_2 \in G$ and $x \in X$.
- Observe that for $g \in G$ we can define a map (left translation)
 $g_L : X \longrightarrow X$ as $g_L(x) = gx$ for all $x \in X$. Also note that:
 - ① g_L has an inverse map $(g^{-1})_L$ and so g_L is a bijection in X i.e. $g_L \in \text{Sym}(X)$!
 - ② We can define a map $\sigma : G \longrightarrow \text{Sym}(X)$ as $\sigma(g) = g_L$ for all $g \in G$.
The map σ is then a homomorphism right?

Important conclusion: A left G -action on X gives us a homomorphism from G to $\text{Sym}(X)$ and conversely, every such homomorphism yields an action. Hence, a G -action on $X \equiv$ group homomorphisms from G to $\text{Sym}(X)$!

Group Actions (Examples)

- 1 Let G be a group, we can define a left action on G that takes $(g, x) \in G \times G$ to $gx \in G$.
- 2 Let G be a group and H a subgroup of G . We can define a left H -action that takes $(h, g) \in H \times G$ to $hx \in G$.
- 3 Let G be a group and N a normal subgroup of G . We can define an action on the set of left cosets of N that takes $(g, C) \in G \times G/N$ to $gC \in G/N$.



**QUANTUM
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