Homework 4

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $\mathcal{A} \subset P(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_σ be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure. Show that:

   a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

   b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$.

   Solution:

   (a) We recall that by the definition of the outer measure induced by μ_0,

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \mid A \subset \cup_{j=1}^{\infty} E_j, E_j \in \mathcal{A} \right\}.$$

Thus given $\epsilon > 0$, there is some collection of sets $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ such that $E \subset \cup_{j=1}^{\infty} A_j$ and $|\mu^*(E) - \sum_{j=1}^{\infty} \mu_0(A_j)| < \epsilon$. We know that since algebras are closed under compliments, we can pick the sets A_j to be disjoint (review the lesson on σ-algebras if you forgot this!) and since we defined $\mathcal{A}_\sigma$ to be the algebra that contains all countable unions of members of $\mathcal{A}$, we know that $A := \cup_{j=1}^{\infty} A_j \in \mathcal{A}_\sigma$, so by the definition of a premeasure on $\mathcal{A}_\sigma$, we have that

$$\mu_0(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

where the last equality follows from the fact that $\mu^* | \mathcal{A} = \mu_0$. Now by the definition of outer measure, we get that

$$\sum_{j=1}^{\infty} \mu_0(A_j) = \sum_{j=1}^{\infty} \mu^*(A_j) \geq \mu^*(A) \geq \mu^*(E),$$

whence we have that

$$\sum_{j=1}^{\infty} \mu_0(A_j) - \mu^*(E) < \epsilon \implies \sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \epsilon \implies \mu^*(A) \leq \mu^*(E) + \epsilon$$

as desired.

(b) We first show the forward direction, so suppose E is μ^*-measurable. This means by definition that for any $A \subset X$, we have that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. We also know that from part (a), for each n, there is some $A_n \in \mathcal{A}_\sigma$ such that $E \subset A_n$ and $\mu * (A_n) - \mu^*(E) < 1/n$. Let $B := \cap_{n=1}^\infty A_n$, so $B \in \mathcal{A}_{\sigma\delta}$. Since E is μ^*-measurable, we have that

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \backslash E) \implies \mu^*(B \backslash E) = \mu^*(B) - \mu^*(E).$$

Finally since we have that $B \subset A_n$ for each n, given any $\epsilon > 0$, there is some N such that $1/N < \epsilon$, and in particular we have that

$$\mu^*(B \backslash E) = \mu^*(B) - \mu^*(E) \leq mu^*(A_n) - \mu^*(E) < 1/N < \epsilon.$$

So for any $\epsilon > 0$, $\mu^*(B \backslash E) < \epsilon$, thus $\mu^*(B \backslash E) = 0$ as desired.

We now prove the other direction. Let $E \subset X$ and suppose such a B exists as in the statement. First we will show that B is μ^* measurable. To see this we simply note that B is in the σ-algebra generated by $\mathcal{A}$, and since every set in $\mathcal{A}$ is μ^* measurable, every set in the σ-algebra it generates will also be μ^* measurable (this comes from the final proposition and theorem in the outer measure lesson). So it follows that for any $A \subset X$

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \geq \mu^*(A \cap E) + \mu^*(A \cap B^c),$$

where the last inequality comes from the fact that $A \cap B \supset A \cap E$. We also observe that $A \cap E^c = (A \cap B^c) \cup (A \cap B \cap E^c) = A \cap (B^c \cup (B \backslash E))$. So it follows from the definition of an outer measure that

$$\mu^*(A \cap E^c) = \mu^*((A \cap B^c) \cup (A \cap B \cap E^c)) \leq \mu^*(A \cap B^c) + \mu^*(A \cap (B \backslash E)) = \mu^*(A \cap B^c),$$

where the final equality follows since the outer measure of a subset of a null set is zero. So now we have that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

meaning that E is indeed μ^*-measurable.

2. Let μ^* be an outer measure on X induced by a premeasure μ_0 where $\mu_0(X) < \infty$. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

   Solution: The forward direction is fairly easy to see - we start by supposing $E \subset X$ is μ^*-measurable. Since X is in the algebra on which μ_0 is defined, we know from the proposition in the lesson that $\mu^*(X) = \mu_0(X)$. Since E is μ^* measurable, we know that for any set $A \subset X$ we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$, and in particular taking $A = X$ we get

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(E) + \mu^*(E^c) \implies \mu^*(X) - \mu^*(E^c) = \mu^*(E),$$

thus we have

$$\mu_*(E) = \mu_0(X) - \mu^*(E^c) = \mu^*(X) - \mu^*(E^c) = \mu^*(E).$$

To see the other direction, we now suppose that $\mu^*(E) = \mu_*(E)$. Pick A_n and B in the previous problem. Since A_n is μ^*-measurable, we have that

$$\mu^*(E^c) = \mu^*(E^c \cap A_n) + \mu^*(E^c \cap A_n^c) = \mu^*(A_n \backslash E) + \mu^*(A_n^c).$$

Since A_n^c is μ^*-measurable, we know by what we proved above that $\mu^*(A_n^c) = \mu_*(A_n^c) = \mu^*(X) - \mu^*(A_n)$. Also by our hypothesis, we have that $\mu^*(E^c) = \mu^*(X) - \mu^*(E)$. Thus substituting in above we have

$$\begin{aligned}\mu^*(X) - \mu^*(E) = \mu^*(E^c) = \mu^*(A_n \backslash E) + \mu^*(A_n^c) &= \mu^*(A_n \backslash E) + \mu^*(X) - \mu^*(A_n) \\ \implies \mu^*(A_n) - \mu^*(E) &= \mu^*(A_n \backslash E).\end{aligned}$$

Finally, since $B \subset A_n$, we have that for all $n \in \mathbb{N}$,

$$\mu^*(B \backslash E) \leq \mu^*(A_n \backslash E) = \mu^*(A_n) - \mu^*(E) < \frac{1}{n},$$

thus $\mu^*(B \backslash E) = 0$, so we can apply part (b) from the first problem to get that E is indeed μ^*-measurable.

3. Come up with an example of a set X, an algebra $\mathcal{A}$, outer measure μ^* and a set in $E \subset X$ such that $\mu_*(E) \neq \mu^*(E)$.

   Solution: We will let $X = [0, 1]$, $\mathcal{A} = \{\emptyset, X, [0, 1/2), [1/2, 1]\}$, which is an algebra (actually a σ-algebra), and we will simply define a premeasure on $\mathcal{A}$ to be the length of the intervals, and taking the empty set to zero (one can verify this is a premeasure). We consider $E = (1/4, 3/4)$. The only cover of this is $[0, 1/2) \cup [1/2, 1] = X$, so $\mu^*(E) = 1$. Likewise one can show that $\mu^*(E^c) = 1$. However note that $\mu_*(E) = \mu_0(X) - \mu^*(E^c) = 1 - 1 = 0$ So $\mu^*(E) \neq \mu_*(E)$, so we have an example of a set that is not μ^*-measurable.

   Remark: This example also provides an example that, in general, if $E \subset A$, $\mu^*(A \backslash E) \neq \mu^*(A) - \mu^*(E)$. Often times people erroneously assume that this is true.