$$\gamma : [0,1] \longrightarrow X$$

Topology Crash Course - Lecture 02

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Session Agenda

1. Pre-lecture comments
2. QF Industry Fellowship
3. **Lecture 01 Poll**

PART A

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1. Subset Interior
2. Subset Closure
3. Subset Exterior & Boundary
4. Continuous Maps & Homeomorphisms
5. **Reference Study Material**
6. **Lecture Gap Poll**

PART B

+

Measurable Spaces

PART C

PART A

General Coverage

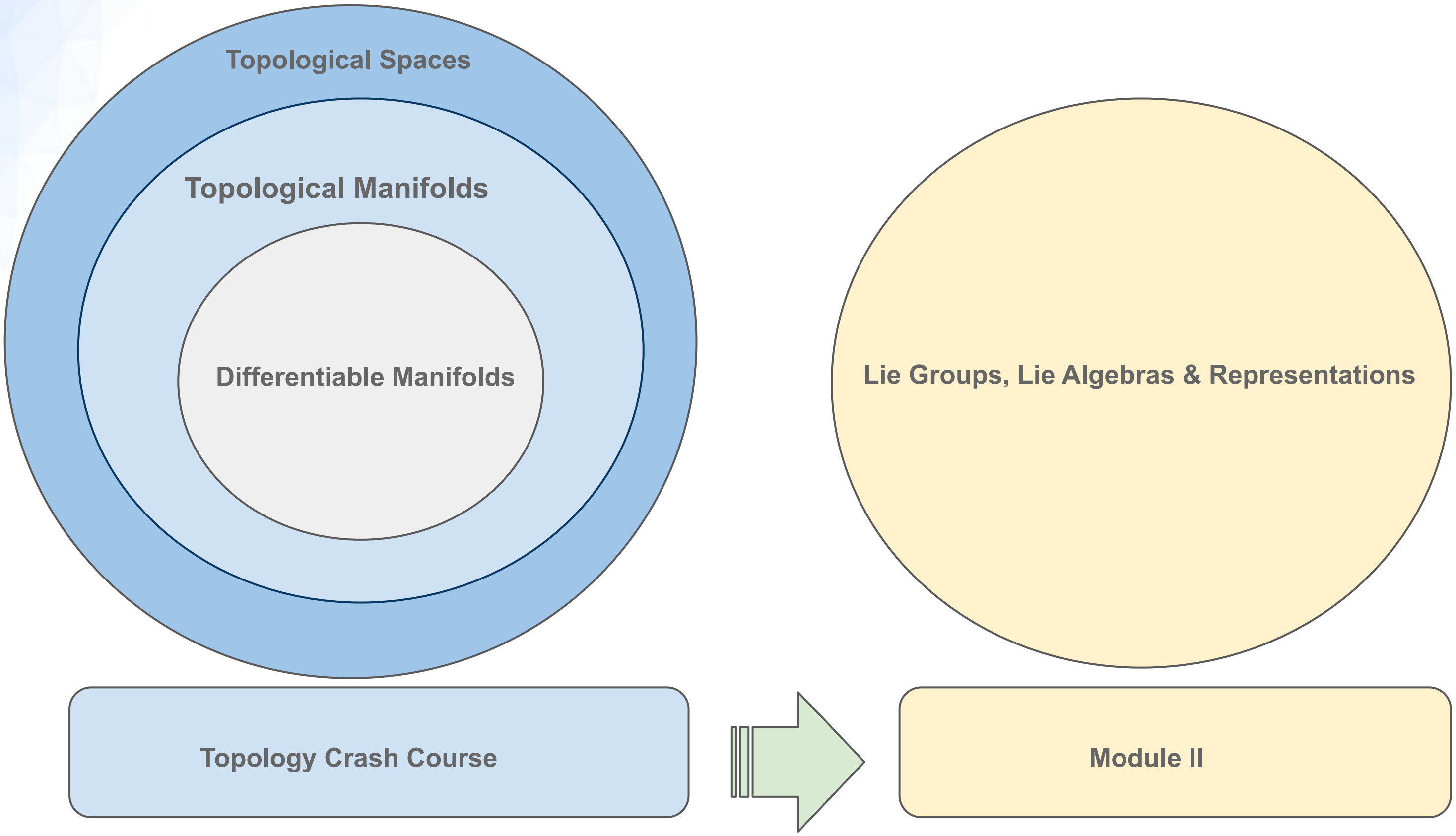
Including a brief intro to Fiber Bundles?

Point-set Topology 101
2/2

Metric Topology 101
#3

Manifolds 101
#3

#n is the number of live lectures.



Topological Spaces
Topological Manifolds
Differentiable Manifolds
Lie Groups, Lie Algebras & Representations
Topology Crash Course
Module II

Quantum Industry Fellowship

$$\langle \Omega \rangle = \int_X \Omega \cdot \mu_P$$

1. Independent Thinkers
2. Mission Driven
3. Team Players
4. Humble

Zaiku Group + Partner Companies

Lecture 01 Poll

PART B

Subset Interior

Definition (1.0)

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The interior of A denoted as $\text{Int}(A)$ is the union of all open sets that are contained in A i.e. $\text{Int}(A) = \bigcup\{O \subseteq A \mid O \in \mathcal{T}\}$.

- Hence, $\text{Int}(A)$ is the largest open set contained in A.
- Some authors use the notation A° instead of $\text{Int}(A)$.

Proposition (1.0)

Given any topological space $(X, \mathcal{T})$ and $A \subseteq X$, the following hold:

1. A point $p \in \text{Int}(A)$ iff there exists a neighbourhood N_p such that $N_p \subseteq A$.
2. $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$ for all $B \subseteq X$.
3. If $A \subseteq B$ then $\text{Int}(A) \subseteq \text{Int}(B)$.

Proof : Homework challenge!

- To help build your own intuition, I challenge you to try draw the concept of interior of A!

Subset Closure

Definition (1.1)

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The closure of A denoted $\overline{A}$ is the intersection of all closed sets containing A i.e. $\overline{A} = \bigcap\{K \subseteq X \mid A \subseteq K\}$ with K being closed.

- Hence, $\overline{A}$ is the smallest closed set containing A?
- Some authors use the notation $\mathrm{Cl}(A)$ instead of $\overline{A}$.

Proposition (1.1)

Given any topological space $(X, \mathcal{T})$ and $A \subseteq X$, the following hold:

1. $X \setminus \overline{A} = \mathrm{Int}(X \setminus A)$ and $X \setminus \mathrm{Int}(A) = (\overline{X \setminus A})$.
2. A point $p \in \overline{A}$ iff $N_p \cap A \neq \emptyset$ for any neighbourhood N_p of p.
3. $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ for all $B \subseteq X$.
4. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$ for all $B \subseteq X$.

- I challenge you to try draw the concept of closure of A!

Subset Exterior & Boundary

Definition (1.2)

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The exterior of A is defined as $\text{Ext}(A) = X \setminus \overline{A}$.

- An alternative notation is to just use $\text{Ext } A$ when convenient.

Definition (1.3)

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The boundary of A is defined as $\partial A = X \setminus \text{Int}(A) \cup \text{Ext}(A)$.

- Is $\partial A = \overline{A} \setminus \text{Int}(A)$ equivalent to the definition above?
- Is it true that $\text{Int}(A) \subseteq \overline{A}$? What about $A \subseteq \overline{A}$?

Proposition (1.2)

For any topological space $(X, \mathcal{T})$ and $A \subseteq X$, the following hold:

1. A point $p \in \partial A$ iff for any of its neighbourhood N_p, we have that $N_p \cap A \neq \emptyset$ and $N_p \cap (X \setminus A) \neq \emptyset$.

2. $X = \text{Int}(A) \cup \partial A \cup (X \setminus \overline{A})$.

Proof : Homework challenge!

Continuous Maps

Definition (1.4)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if for any open set $O_Y \in \mathcal{T}_2$, the pre-image $\text{Preim}_f(O_Y) \in \mathcal{T}_1$. Recall that $\text{Preim}_f(O_Y) = \{p \in X \mid f(p) \in O_Y\}$.

- Hence, under a continuous map f, the pre-image of any open set in the target topological space $(Y, \mathcal{T}_2)$ is an open set in the domain space X in respect to the chosen topology $\mathcal{T}_1$.
- Be warned that some authors use $f^{-1}(O_Y)$ instead of $\text{Preim}_f(O_Y)$. This can be confusing because the notation f^{-1} is also used to denote the inverse of f! Hence, it may give the wrong impression that all continuous maps are invertible!
- Is the preimage of any closed set in Y also closed in X?
- For those with real analysis experience, do you agree this definition of continuity is far simpler than the one you encounter in analysis with ϵ and δ?!
- Can we alternatively define the notion of continuity at a point $p \in X$ first and then generalise to all the points of X?

Equivalent Def. of Continuity

Definition (1.5)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \longrightarrow Y$ is continuous at a point $p \in X$ if for any open set $O_Y \in \mathcal{T}_2$ such that $f(p) \in O_Y$ there exists an open set $O_X \in \mathcal{T}_1$ such that $p \in O_X$ and $f(O_X) \subseteq O_Y$.

- Hence, equivalently, we can define a map $f : X \longrightarrow Y$ to be continuous if it is continuous at every point $p \in X$ as per the definition above.
- We can bridge the abstract/general definition in the previous slide with the above definition via:

Proposition (1.3)

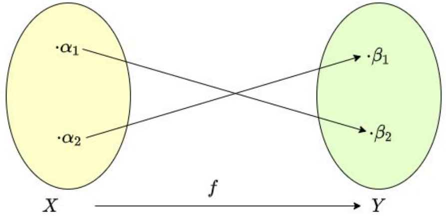
Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \longrightarrow Y$ is continuous as per the abstract/general definition in the previous slide iff it is continuous as per the definition above.

Proof : Homework challenge!

Simple Challenge

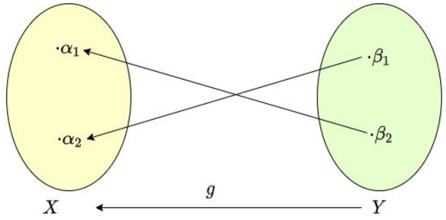
- Let $(X, \mathcal{T})$ be a topological space. Is the identity map $id_X : X \longrightarrow X$ continuous?
- Consider a topological space $(X, \mathcal{T}_1)$ with $\mathcal{T}_1 = \mathcal{P}(X)$ and $(Y, \mathcal{T}_2)$ be another topological space with $\mathcal{T}_2$ being any topology. Is true that any map $f : X \longrightarrow Y$ is continuous?
- Let $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2\}$ with the respective topologies $\mathcal{T}_1 = \{\emptyset, X, \{\alpha_1\}, \{\alpha_2\}\}$ and $\mathcal{T}_2 = \{\emptyset, Y\}$.

  Now let $f : X \longrightarrow Y$ be defined as follows:

  Is the map f continuous?

- Now let us consider the map $g : Y \longrightarrow X$ defined as follows:



Is the map g continuous?

Properties of Continuous Maps

Proposition (1.4)

Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous maps then the composition $g \circ f$ is a continuous map.

Proof : Homework challenge!

- As a side note, the space of complex valued continuous maps form a very interesting structure called $\mathbb{C}^*-$ Algebra!

Proposition (1.5)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$, then the map $f|_A : A \longrightarrow Y$ is also continuous.

Proof : Homework challenge!

- $f|_A$ is called the restriction of f to the subset $A \subseteq X$. Also, note that we're implicitly invoking the subspace topology on A.

Homeomorphisms

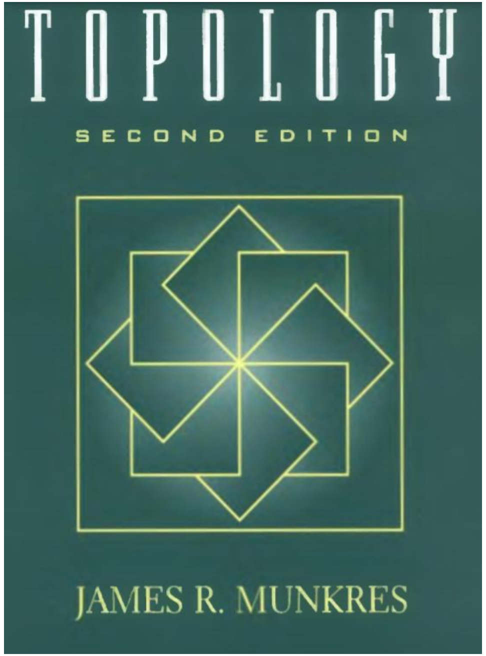
Definition (1.6)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A continuous map $f : X \longrightarrow Y$ is a homeomorphism if the following conditions hold:

1. f is bijective i.e. f is surjective & injective.
2. Its inverse $f^{-1} : Y \longrightarrow X$ is also continuous.

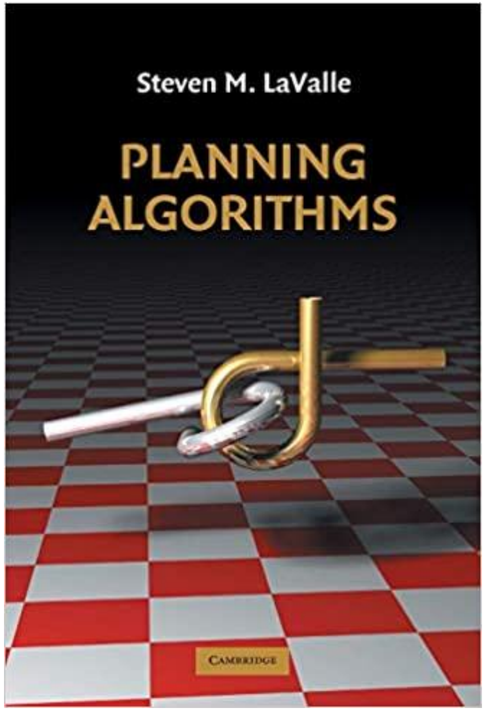
- We write $X \simeq Y$ and say the two topological spaces are homeomorphic if an homeomorphism between the two exists.
- Homeomorphisms are indeed structure preserving maps in topology. In particular, if $X \simeq Y$ and $Y \simeq Z$ then $X \simeq Z$ right?
- In Linear Algebra, when two finite dimensional vector spaces have the same dimension, it follows that they must be isomorphic.
- Are there any intrinsic topological properties such that when two spaces satisfy these properties, it follows that the two spaces are homeomorphic?

- Homeomorphisms are the reason why for Topologists, donuts and cups are the same thing!
- In a nutshell, two spaces (think shapes) are homeomorphic if one can be deformed into the other without cutting or gluing.
- Taking into account what is said above, can the sphere be homeomorphic to a donut?



Reference Material

quantumformalism.com



Extra Reference Material

quantumformalism.com

QUANTUM FORMALISM

GitHub: github.com/quantumformalism

YouTube: youtube.com/ZaikuGroup

Discord: discord.gg/SPcmcsXMD2

Twitter: twitter.com/ZaikuGroup

LinkedIn: linkedin.com/company/zaikugroup

Lecture Gap Poll

1 Week Gap **versus** 2 Weeks Gap

PART C

Measurable Spaces

Definition

Let X be a non-empty. A $\sigma-$ algebra on X is a collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

1. $\emptyset \in \mathcal{M}$.
2. If $A \in \mathcal{M}$ then its complement $X \setminus A \in \mathcal{M}$.
3. $\{A_1, A_2, \ldots, A_k\} \subseteq \mathcal{M}$ then $A_1 \cup A_2 \cup \ldots \cup A_k \in \mathcal{M}$.

- The pair $(X, \mathcal{M})$ is called a 'measurable space'. The elements of the collection $\mathcal{M}$ are called measurable sets.
- Is X measurable i.e. does $X \in \mathcal{M}$?
- Let X to be any set.
  1. Is $\mathcal{M} = \mathcal{P}(X)$ a $\sigma-$ algebra on X?
  2. What about $\mathcal{M} = \{\emptyset, X\}$?
- Given a topological space of $(X, \mathcal{T})$ of your choice. Can you construct a $\sigma-$ algebra $(X, \mathcal{M})$ from the topology?

Note: Please do not use the discrete or in discrete topologies on X!

Why Measurable Spaces?

- Measurable spaces are important because they are the fundamental building blocks of 'Measure Theory'. The subject is a very important branch of mathematics for many reasons including:
  1. It is the basis of the axiomatic foundation for 'Probability Theory'. As some of you may remember, Andrey Kolmogorov used the language of Measure Theory to give an abstract definition of probability i.e. the notion of 'probability measure'.
  2. Borel sets are special type of measurable sets built from the standard topology. For example, the spectrum of a Self-Adjoint Operator on an infinite dimensional Hilbert space lies in a Borel sets.
  3. It makes possible the proper definition of integration i.e. Lebesgue integration.