QUANTUM FORMALISM

Matrix Groups 101 - Part 2

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Matrix Conjugate
3. Hermitian Conjugate
4. Hermitian Conjugate Properties

Part A

1. The Circle Group: U(1)
2. Unitary Groups: U(2) & U(n)
3. Special Unitary Groups: SU(2) & SU(n)
4. Study Material Comments
5. Fireside Chat Update

Part B

December 2020 Agenda

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Lecture dates The next fireside chat

January 2021 Calendar

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26 Australia Day	27	28	29	30
31						

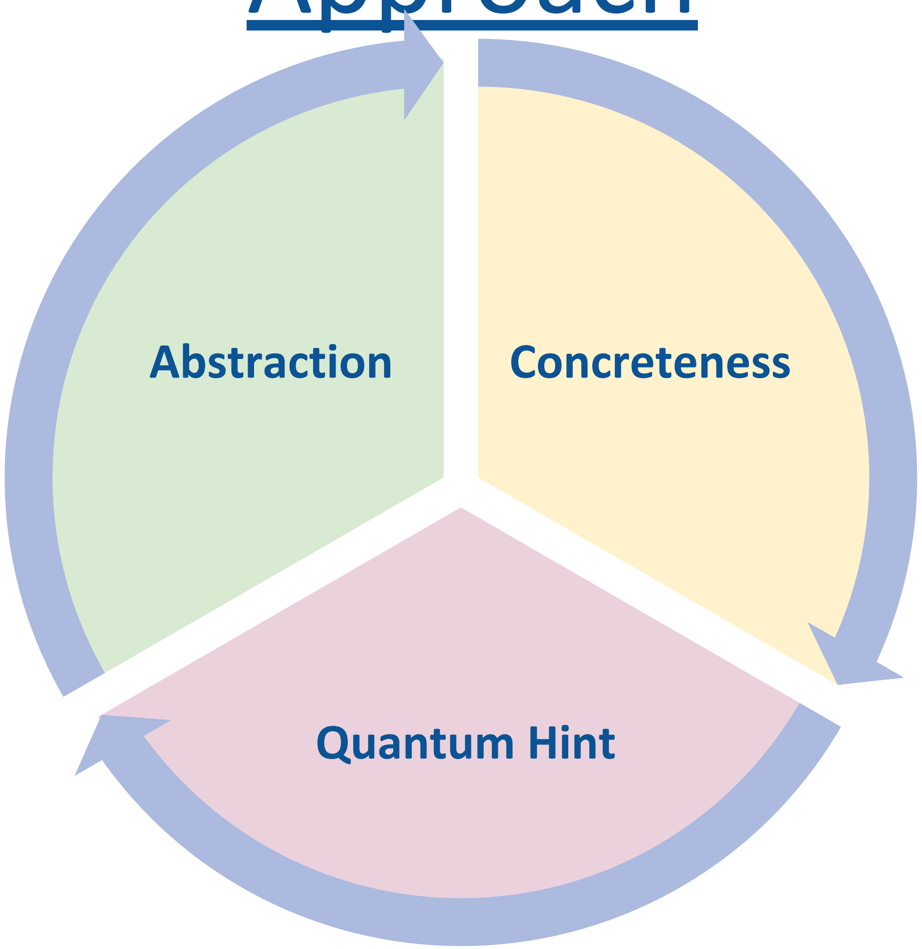
Lecture dates

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Operators + Composite Systems #3
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

Course Pedagogical Approach



PART A

Matrix Conjugate

Definition (1.0)

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$, we define $A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} \in M_2(\mathbb{C})$.

- Where a_{ij}^* is the conjugate of a_{ij} i.e. if $a_{ij} = x + yi$ then $a_{ij}^* = x - yi$.
- Mathematicians normally use the notation $\bar{a}_{ij}$ to denote the conjugate of a_{ij} and so use $\bar{A}$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$X^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, Z^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Conjugate Transpose

Definition (1.1)

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C}), A^\dagger = (A^*)^T = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \in M_2(\mathbb{C})$.

- $A^\dagger$ is also known as the Hermitian transpose of A and physicists often call it 'the A dagger'. Mathematicians use A^* instead of $A^\dagger$!
- The definition can be extended to matrices in $M_n(\mathbb{C})$.
- Just as a side note for now, A is called Hermitian if $A = A^\dagger$.

Proposition (1.0)

Let $A, B \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$. The following identities hold:

1. $(A^\dagger)^\dagger = A$.
2. $(\lambda A)^\dagger = \lambda^* A$.
3. $(A + B)^\dagger = A^\dagger + B^\dagger$.
4. $(AB)^\dagger = B^\dagger A^\dagger$.
5. $\det(A^\dagger) = det(A)^*$.
6. If A is invertible then $A^\dagger$ is also invertible.

Proof : Homework challenge?

- Of course, the properties above also apply to matrices in $M_n(\mathbb{C})$.

PART B

The Circle Group

Definition (1.0)

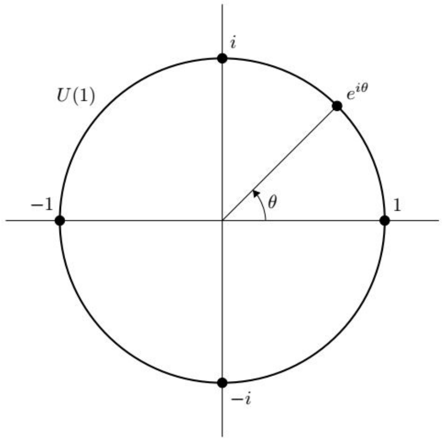
$\mathbb{C}^* = \left\{ z \in \mathbb{C} \mid z \neq 0 \right\}$ i.e. the set of all non-zero complex numbers.

- $\mathbb{C}^*$ is an abelian group under the multiplication in $\mathbb{C}$.

Definition (1.1)

We can now define this special set $U(1) = \left\{ \lambda \in \mathbb{C}^* \mid |\lambda| = 1 \right\}$.

- It's easy to see that $U(1)$ forms a subgroup of $\mathbb{C}^*$ right?
- $U(1)$ is called the circle group because it can be identified with the unit circle on the complex plane.



- The elements of the circle group are of the form $e^{i\theta}$.
- The group multiplication in $U(1)$ can then be defined for two elements $A = e^{i\theta_1}$ and $B = e^{i\theta_2}$ as $AB = e^{i(\theta_1 + \theta_2)}$.

Quantum Hint

- $U(1)$ is the simplest example of a Lie Group that we'll cover in the advanced module.
- $U(1)$ is an important group in particle physics because it's an example of what theoretical physicists call 'gauge symmetry':
  - Quantum Electrodynamics (QED) is called 'abelian gauge theory' because $U(1)$ is its 'gauge group' and as you know $U(1)$ is abelian!
  - $U(1)$ is indeed an important piece of the gauge group for the so-called Standard Model of particle physics.
  - For those interested in quantum hardware, I believe QED provides the theoretical basis for the so-called 'Cavity Quantum Electrodynamics' (CQED)? There is also something called 'Circuit Quantum Electrodynamics' (cQED)!

The Unitary Group

Definition (1.2)

We can define $U(2) = \left\{ A \in GL(2, \mathbb{C}) \mid AA^\dagger = A^\dagger A = \mathbb{I} \text{ i.e. } A^{-1} = A^\dagger \right\}$.

- The following matrices are of course elements of $U(2)$:
  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- We can indeed generalise the definition above to any $n > 0$:
  $U(n) = \left\{ A \in GL(n, \mathbb{C}) \mid AA^\dagger = A^\dagger A = \mathbb{I} \right\}$.
- Elements of $U(n)$ are called unitary matrices or unitary operators when they act on the complex vector space $\mathbb{C}^n$ which is generally equipped with a Hilbert space structure.
- Is $U(2)$ or $U(n)$ in general form a group?

Proposition (1.0)

$U(2)$ is a subgroup of $GL(2, \mathbb{C})$.

Proof : Homework challenge!

- Indeed for any $n > 0$, $U(n)$ is a subgroup of $GL(n, \mathbb{C})$.
  - For QC purposes, you'll probably want $n = 2^k$ where k is the number of qubits so that the elements of $U(n)$ can act as unitary operators on the Hilbert space of the $k-$ qubit system i.e. $\mathbb{C}^{2^k}$.
- In the final section of this module you'll learn more about the interesting properties of $U(n)$ elements acting as operators on $\mathbb{C}^n$ state vectors.
- We'll also explore interesting relationships between unitary operators and hermitian operators!

Unitary Group Properties

Proposition (1.1)

The centre of $U(2)$ is $Z(U(2)) = \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in U(1) \right\}$.

Proof : Homework challenge? Which qubit gates are in $Z(U(2))$?

- The proposition above is indeed true for any $n > 0$:

$$Z(U(n)) = \left\{ \lambda \mathbb{I}_n \mid \lambda \in U(1) \right\} \text{ where } \mathbb{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- Is it true that $Z(U(n)) \simeq U(1)$?

Proposition (1.2)

$\det : U(2) \longrightarrow \mathbb{C}^*$ is a group homomorphism and $Ker(\det) \subseteq SL(2, \mathbb{C})$.

Proof : Homework challenge!

- $\det : U(n) \longrightarrow \mathbb{C}^*$ is also a homomorphism and of course $Ker(\det) \subseteq SL(n, \mathbb{C})$.

The Special Unitary Group

Definition (1.3)

We can now define $SU(2) = \left\{ A \in U(2) \mid \det(A) = 1 \right\}$.

- We can indeed generalise the definition above to any $n > 0$:

$$SU(n) = \left\{ A \in U(n) \mid \det(A) = 1 \right\}.$$

- It's easy to see that $SU(n) = U(n) \cap SL(n, \mathbb{C})$ right?

Proposition (1.3)

$SU(2)$ is a subgroup of $U(2)$.

Proof : Homework challenge!

- Indeed for any $n > 0$, $SU(n)$ is a subgroup of $U(n)$.

Special Unitary Group Properties

Proposition (1.4)

$$Z(SU(2)) = \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in U(1) : \lambda^2 = 1 \right\}.$$

Proof : Homework challenge? Which qubit gates are in $Z(SU(2))$?

- Can the propositon above be generalised to any $n > 0$?

$$Z(SU(n)) = \left\{ \lambda \mathbb{I}_n \mid \lambda \in U(1) : \lambda^n = 1 \right\} \text{ where } \mathbb{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Proposition (1.5)

$Ker(\det) \subseteq SU(2)$ where $\det : U(2) \longrightarrow \mathbb{C}^*$.

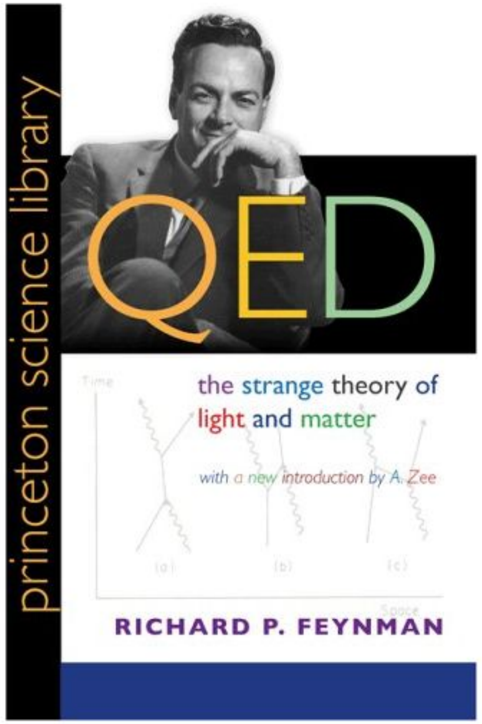
Proof : Homework challenge!

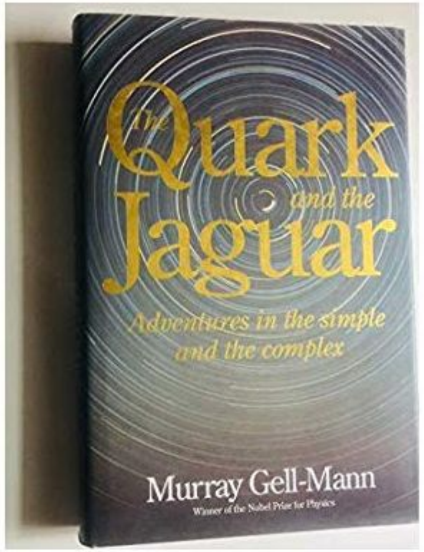
- For $\det : U(n) \longrightarrow \mathbb{C}^*$ we also have $Ker(\det) \subseteq SU(n)$?
- As extra home challenge, I encourage you to identify the famous two-level qubit gates that are elements of $SU(4)$ i.e gates with determinant 1. How many such gates are out there?

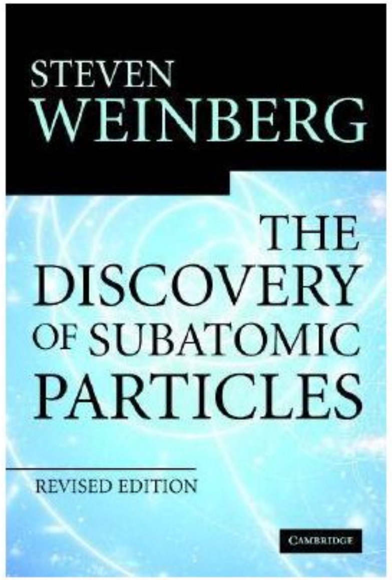
QHint (Ask a particle Physicist to confirm!)

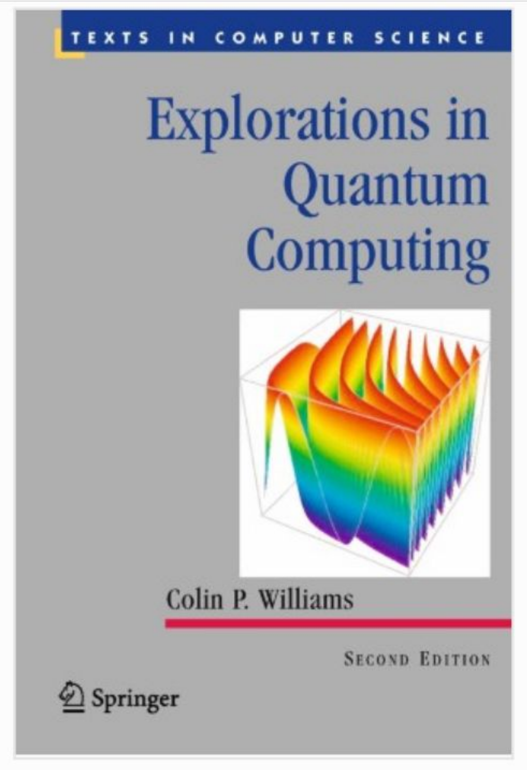
- $SU(n)$ groups are important in topics such as gauge theories:
  - The bible of particle physics called the Standard Model (SM) is a 'non-abelian gauge theory' because its gauge group is $SU(3) \times SU(2) \times U(1)$ where:
    1. $SU(3)$ is responsible for the quark colour changes (red, green, blue) i.e. responsible for the strong nuclear force or in other words Quantum Chromodynamics (QCD) stuff.
    2. $SU(2)$ is responsible for convertion between up and down quarks i.e. responsible for the weak nuclear force.
    3. $U(1)$ is responsible for the QED force.
  - What about gravity? Why is it not included in the SM?

Particle Physics Books









Colin P. Williams

Where should you focus?

Chapter 2: Quantum Gates

--------*(Pages 51 - 122)*----------

DEC
15
05:10pm

Fireside Chat with Laur Nita

By Zaiku Group · Unlisted

11 DAYS | 1 HRS | 50 MIN | 1 SEC

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Guest Bio

Laurentiu Nita is the founder of Quarks Interactive: a company with the mission of making quantum computing fun (through gamification) and accessible (by removing the need-to-know mathematics for understanding computation). He managed to secure funding to create Quantum Odyssey, a video game where the

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