Homework 1

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**Directions:** Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Prove - without using the Fundamental Theorem of Calculus - that for any $b > 0$,

$$\int_0^b x^3 \, dx = \frac{b^4}{4}.$$

**Hint:** Follow along with the example we did in class. You will need the following summation formula:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let $A_1$ be a finite subset of $[0,1]$, that is, $A_1 = \{q_{11}, q_{12}, \ldots, q_{1n_1}\}$. Let $A_2$ be a finite subset of $[0,1]\backslash A_1$, so $A_2 = \{q_{21}, q_{22}, \ldots, q_{2n_2}\}$, and $A_1 \cap A_2 = \emptyset$. In general, let $A_k$ be a finite subset of $[0,1]\backslash \left(\bigcup_{i=1}^{k-1} A_i\right)$. Thus each $A_k$ contains finitely many points of $[0,1]$ and they are mutually disjoint, that is, $A_k \cap A_j = \emptyset$ whenever $k \neq j$. Now define a function $f(x)$ as follows:

$$f(x) = \begin{cases} \frac{1}{n} & x \in A_n \\ 0 & x \notin \bigcup_{i=1}^{\infty} A_i \end{cases}$$

Prove $f$ is Riemann integrable over $[0,1]$ and find its value.

**Hint:** This will be very similar to how we handled Thomae's Function. In fact, Thomae's Function is just a special case of the more general function defined in this problem!

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