$$\gamma : [0,1] \longrightarrow X$$

Topology Crash Course - Lecture 05

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Session Agenda

Pre-lecture
blah blah blah

PART A

1. Paths in Topological Spaces
2. Path-Connected Spaces
3. Path-Connectedness Challenge
4. Connectedness Comments

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PART B

1. Open Covers & Subcovers
2. Open Cover Challenge
3. Compact Spaces
4. Compact Subsets
5. Compactness Challenge
6. Why Compactness Matters?
7. **Lecture Gap Poll**

Lecture 5 Coverage

General Coverage

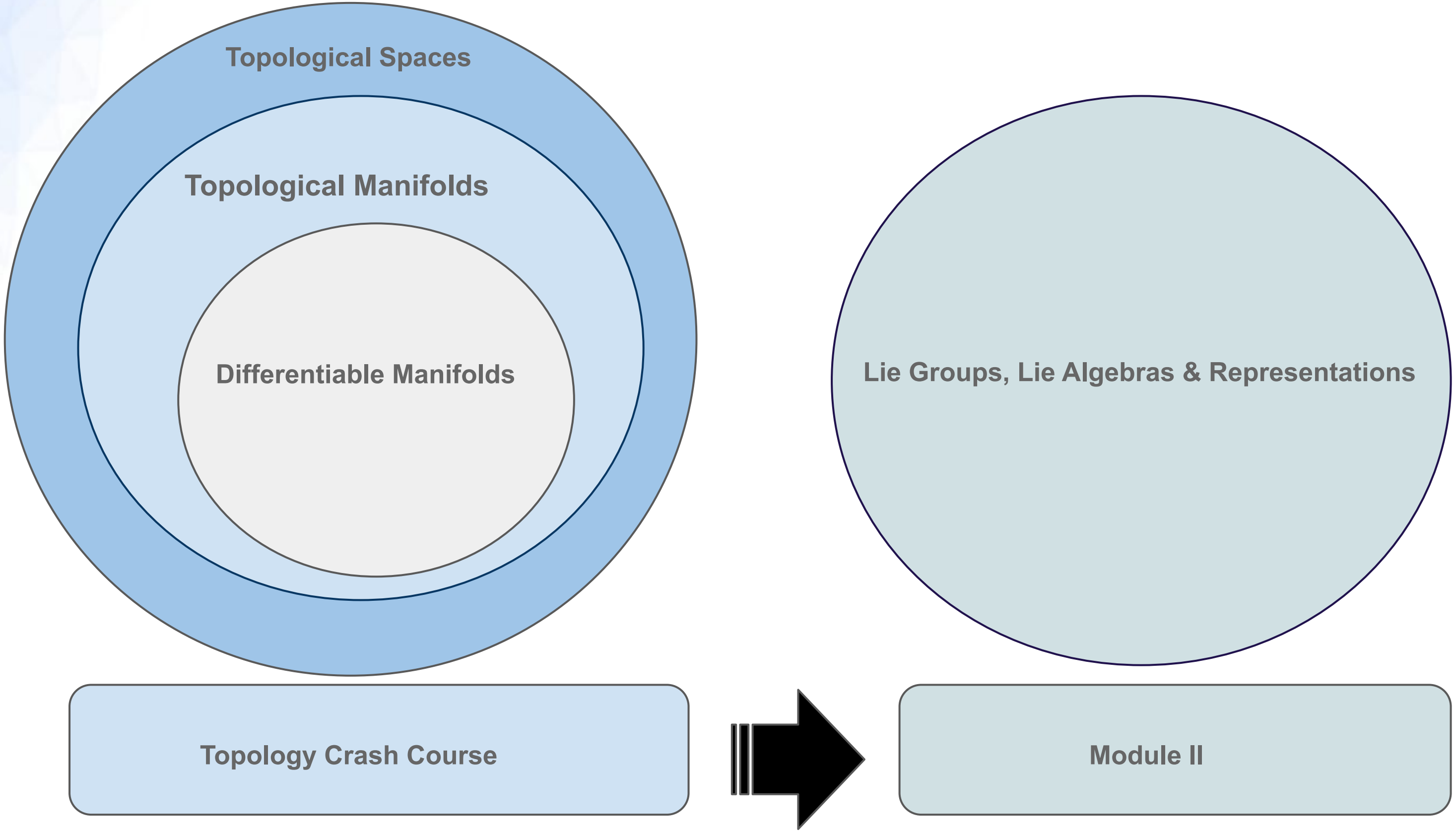
Including a brief intro to Fiber Bundles?

Point-set Topology 101
2/2

Metric Topology 101
3/3

Manifolds 101
#4

#n is the number of live lectures | Completed | Ongoing


Topological Spaces
Topological Manifolds
Differentiable Manifolds
Lie Groups, Lie Algebras & Representations
Topology Crash Course
Module II

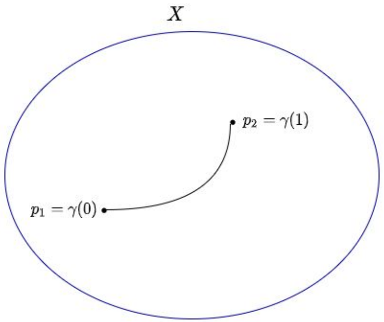
PART A

Paths in Topological Spaces

Definition (1.0)

Let $(X, \mathcal{T})$ be a topological space and $p_1, p_2 \in X$. We say the points p_1 and p_2 are connected by a path in X if there exists a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$.

- We say p_1 and p_2 are the endpoints of the path. Intuitively, a path is just an abstraction of being able to walk from point p_1 to the point p_2.



- If γ is a homeomorphism, then it's called an 'arc-path' (or just arc) from p_1 to p_2.

- Obviously, if p_1 and p_2 are connected by a path and p_2 is connected to p_3 by a path then p_1 and p_3 are also connected by a path right? In fact, the notion of path in X defines an equivalence relation $\sim$ in X i.e. $p_1 \sim p_2$ iff the two points are connected by a path.

- Wouldn't it be nice that any two points in X can be connected by a path?

Path-Connected Spaces

Definition (1.1)

A topological space $(X, \mathcal{T})$ is path-connected if every two points $p_1, p_2 \in X$ can be connected by some path $\gamma : [0, 1] \longrightarrow X$.

- Intuitively, X being path-connected means we can walk between any two points in X. From a modelling point of view, this is great for our friends in physics right?
- X is called 'arcwise-connected' if there exists an arc-path between any two points. Interestingly, it can be proven that a Hausdorff space is path connected iff it is arcwise- connected!

Side note: Some authors use 'arcwise-connected' to mean the same thing as 'path connected' when dealing with Hausdorff spaces.

Path-Connectedness Is an Invariant

Proposition (1.0)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If X is path-connected and $f : X \longrightarrow Y$ is a continuous map, then $f(X)$ is path-connected.

Proof : Homework challenge!

- Hence, path-connectedness is a topological invariant! Therefore it can be used to check whether two topological spaces are homeomorphic.

Theorem (1.0)

If a topological space $(X, \mathcal{T})$ is path-connected, then it is connected.

Proof : Textbook or try prove it yourself!

- Please note the opposite is not true i.e. not every connected space is path-connected! A famous counter-example is the so-called 'Topologist's Sine Curve' which connected but not path connected.

- Hence, path-connectedness is a stronger notion than connectedness. Very often it is easier and more intuitive to try to prove that a space is path-connected first before trying to prove if it's connected!

Path-Connectedness Challenge

- Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two path-connected topological spaces. Is it true that the product space $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ is also path-connected?
- Which of the following topological spaces are path-connected (therefore connected):

1. The closed interval $[0, 1]$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
2. The open interval $(0, 1)$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
3. The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^2$.
4. The torus (aka doughnut) $\mathbb{T}^2 = S^1 \times S^1$.
5. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^3$.
6. $\mathbb{R}$ equipped with the standard topology.
7. $\mathbb{R} \setminus \{0\}$ equipped with the standard topology.
8. $\mathbb{R}^2 \setminus \{(0, 0)\}$ equipped with the standard topology.

- Which of the topological spaces above are arc-connected?

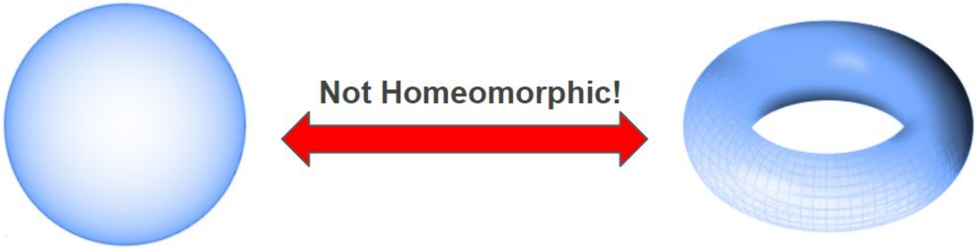
Connectedness Comments

- We left out some other important notions associated with connectedness and path-connectedness:

  1. **Local Connectedness and local-path connectedness:**
     Connectedness and path-connectedness are what topologists call 'global properties' of a space X i.e. they say something about X as a whole. Whereas local connectedness and local path-connectedness are 'local properties' of a space X i.e. they say what happens locally at arbitralily small neighbourhoods of points.

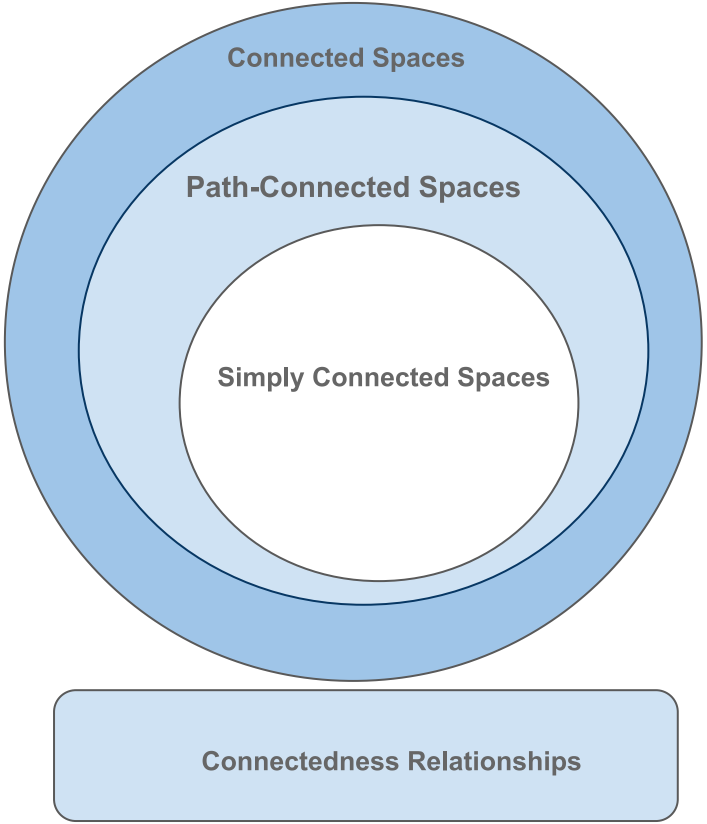
  Note: In general, global properties do not imply local properties and vice-versa. For example, X can be locally path-connected but fails to be path-connected!

  2. **Simply Connected Spaces:** How can we prove (formally) that the sphere is not homeomorphic to the torus?



In a nusthell, simply connected spaces are spaces that have no holes!

To formalise this, we need the notion of 'homotopy', which essentially helps detect holes in a space!


Connected Spaces
Path-Connected Spaces
Simply Connected Spaces
Connectedness Relationships

PART B

Open Covers

Definition (1.2)

Let $(X, \mathcal{T})$ be a topological space. A collection of open sets $\mathcal{C} = \{C_i\}_{i \in I} \subseteq \mathcal{T}$ is said to form an open cover if $\bigcup_{i \in I} C_i = X$.

- The indexing set I can be infinite or even uncountable!
- We could have started more abstractly with the notion 'cover' as being a subcollection of the power-set i.e. $\mathcal{C} = \{C_i\}_{i \in I} \subseteq \mathcal{P}(X)$.

Definition (1.3)

Let $\mathcal{C} = \{C_i\}_{i \in I} \subseteq \mathcal{T}$ be an open cover of $(X, \mathcal{T})$. A subcollection $\mathcal{C}' = \{C_j'\}_{j \in J} \subseteq \mathcal{C}$ where $J \subseteq I$ is called a subcover of $\mathcal{C}$ if it's also an open cover of $(X, \mathcal{T})$ i.e. $\bigcup_{j \in J} C_j' = X$.

- In principle, a subcover can be finite i.e. $J \subseteq I$ can be a finite set.

Open Cover Challenge

- Let $X = (0,1)$ be equipped with subspace topology of the standard topology on $\mathbb{R}$.
  Is $\mathcal{C} = \{(0,1), (0, \frac{1}{2}), (\frac{1}{2}, 1)\}$ an open cover of X i.e. is $X = (0,1) \cup (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$?

- Let now $X = (0,1]$ be equipped with subspace topology of the standard topology on $\mathbb{R}$ and let $C_n = (\frac{1}{n}, 1)$ for all $n \in \mathbb{N}$.

  So for example, $C_1 = (1,1)$, $C_2 = (\frac{1}{2}, 1)$, $C_3 = (\frac{1}{3}, 1)$ etc.

  Is $\mathcal{C} = \{C_n \mid n \in \mathbb{N}\}$ an open cover of X i.e. is $\bigcup_{n \in \mathbb{N}} C_n = X$?

- Consider $X = \mathbb{R}$ equipped with the standard topology. Let us construct the open intervals $C_n = (-n, n)$ for all $n \in \mathbb{N}$.

  For example, $C_1 = (-1,1)$, $C_2 = (-2,2)$, $C_3 = (-3,3)$ etc.
  Is $\mathcal{C} = \{C_n \mid n \in \mathbb{N}\}$ an open cover of $\mathbb{R}$ i.e. is $\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}$?

- Consider $X = \mathbb{R}$ again equipped with the standard topology. But now let $C_k = (k, k+2)$ for all $k \in \mathbb{Z}$.

  So for example, $C_0 = (0,2)$, $C_1 = (1,3)$, $C_{-1} = (-1,1)$ etc.

  Is $\mathcal{C} = \{C_k \mid k \in \mathbb{Z}\}$ an open cover of $\mathbb{R}$ i.e. $\bigcup_{k \in \mathbb{Z}} C_k = \mathbb{R}$?

- Try find at least a subcover for the open covers above. Are there finite subcovers?

Compact Spaces

Definition (1.4)

A topological space $(X, \mathcal{T})$ is compact if every open cover $\mathcal{C}$ has a finite subcover $\mathcal{C}' = \{C_j'\} \subseteq \mathcal{C}$ where $j \in \{1, \ldots, k\}$ for some $k \in \mathbb{N}$.

- This means that X can be written as finite union of the subcover i.e. $\bigcup_{j=1}^{k} C_j' = C_1' \cup \ldots \cup C_k' = X$.
- Hence, if X is a finite set, then $(X, \mathcal{T})$ is always compact regardless of our choice for the the topology $\mathcal{T}$.
- Likewise, let X be any set (finite or infinite) and equipped with the indiscrete topology i.e. $\mathcal{T} = \{\emptyset, X\}$. Then $(X, \mathcal{T})$ is obviously compact!
- Suppose now that X is infinite and equipped with the discrete topology i.e. $\mathcal{T} = P(X)$. Is $(X, \mathcal{T})$ compact?

Compact Subsets

Definition (1.5)

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is compact if the induced subspace topology $(A, \mathcal{T}_A)$ is compact.

- Does a proper subset $A \subset X$ being compact necessarily means the parent space $(X, \mathcal{T})$ is compact?

Proposition (1.1)

If $(X, \mathcal{T})$ is compact, then a closed subset $A \subseteq X$ is compact.

Proof : Homework challenge?

- Recall that $A \subseteq X$ being closed means its complement in X is open i.e. $X \setminus A \in \mathcal{T}$.

- Worth noting that in general compactness does not imply closed. However, if $(X, \mathcal{T})$ is Hausdorff and $A \subseteq X$ is compact, then A is closed in X!

- Another interesting property is that if $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are compact Hausdorff spaces, then any bijective continuous map $f : X \longrightarrow Y$ is a homeomorphism!

Compactness Is an Invariant

Proposition (1.2)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If X is compact and $f : X \longrightarrow Y$ is a continuous map, then $f(X)$ is compact.

Proof : Homework challenge!

- Hence, as you might have already suspected, compactness is a topological invariant! Therefore it can be used to check whether two topological spaces are homeomorphic.

Compactness Challenge

- Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two compact spaces. Is it true that the product space $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ is also compact?
- Which of the following topological spaces are compact:
  1. The closed interval $[0, 1]$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
  2. The open interval $(0, 1)$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
  3. The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^2$.
  4. The torus (aka doughnut) $\mathbb{T}^2 = S^1 \times S^1$.
  5. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^3$.
  6. $\mathbb{R}$ equipped with the standard topology.
  7. $\mathbb{R} \setminus \{0\}$ equipped with the standard topology.
  8. $\mathbb{R}^2 \setminus \{(0, 0)\}$ equipped with the standard topology.

Why Compactness Matters?

- In Module II we'll learn about 'Compact Lie Groups'. These are very important class of Lie Groups with applications in Physics and Quantum Computation. Famous examples are:

  1. The unitary group $U(n)$ and the special unitary group $SU(n)$.
  2. The orthogonal group $O(n)$.
  3. The special orthogonal group $SO(n)$.

Lie Theory for Quantum Control

Abstract

One of the main theoretical challenges in quantum computing is the design of explicit schemes that enable one to effectively factorize a given final unitary operator into a product of basic unitary operators. As this is equivalent to a constructive controllability task on a Lie group of special unitary operators, one faces interesting classes of bilinear optimal control problems for which efficient numerical solution algorithms are sought for. In this paper we give a review on recent Lie-theoretical developments in finite-dimensional quantum control that play a key role for solving such factorization problems on a compact Lie group.

Q-CTRL

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Hermann Weyl

‘’If a city is compact, it can be guarded by a finite number of arbitrarily short-sighted policemen.’’

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