Homework 1

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Prove - without using the Fundamental Theorem of Calculus - that for any $b > 0$,

$$\int_0^b x^3 \, dx = \frac{b^4}{4}.$$

Hint: Follow along with the example we did in class. You will need the following summation formula:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: We will follow along similarly to the examples done in class. We will show that for all $\epsilon > 0$ we can find some partition P_n such that we can make $U(f, P_n) - L(f, P_n) < \epsilon$. We will divide $[0, b]$ into n equal subintervals of length b/n, and let $x_0 = 0$, $x_1 = b/n$, and in general, $x_i = ib/n$ for $i = 0, 1, 2, \ldots, n$. Then

$$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i)^3 \frac{b}{n} = \frac{b^4}{n^4} \sum_{i=1}^{n} i^3 = \frac{b^4}{n^4} \frac{n^2(n+1)^2}{4} = \frac{b^4}{4} \left(\frac{n+1}{n} \right)^2.$$

We have that, for all n, $U(f, P_n) > b^4/4$. Computing now the lower sum we have

$$L(f, P_n) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_{i-1})^3 \frac{b}{n} = \frac{b^4}{n^4} \sum_{i=1}^{n} (i-1)^3 = \frac{b^4}{n^4} \left(\frac{n^2(n+1)^2}{4} - n^3 \right)$$

and for all n, we have

$$L(f, P_n) = \frac{b^4}{4} \left(\frac{n-1}{n} \right)^2 < \frac{b^4}{4}.$$

We will now prove that we can make $U(f, P_n) - L(f, P_n)$ small. Let $M > b^4/\epsilon$. Then for all $n \geq M$, we have

$$U(f, P_n) - L(f, P_n) = \frac{b^4}{4}\left(\frac{(n+1)^2}{n^2} - \frac{(n-1)^2}{n^2}\right) = \frac{b^4}{n} < b^4\frac{\epsilon}{b^4} = \epsilon.$$

So by the theorem from class, we know x^3 is integrable on $[0, b]$, and the value must be the only value between all the upper and lower sums, which is $b^4/4$, i.e.,

$$\int_0^b x^3\, dx = \frac{b^4}{4}.$$

2. Let A_1 be a finite subset of $[0, 1]$, that is, $A_1 = \{q_{11}, q_{12}, \ldots, q_{1n_1}\}$. Let A_2 be a finite subset of $[0, 1]\backslash A_1$, so $A_2 = \{q_{21}, q_{22}, \ldots, q_{2n_2}\}$, and $A_1 \cap A_2 = \emptyset$. In general, let A_k be a finite subset of $[0, 1]\backslash \left(\bigcup_{i=1}^{k-1} A_i\right)$. Thus each A_k contains finitely many points of $[0, 1]$ and they are mutually disjoint, that is, $A_k \cap A_j = \emptyset$ whenever $k \neq j$. Now define a function $f(x)$ as follows:

$$f(x) = \begin{cases} \frac{1}{n} & x \in A_n \\ 0 & x \notin \bigcup_{i=1}^{\infty} A_i \end{cases}$$

Prove f is Riemann integrable over $[0, 1]$ and find its value.

Hint: This will be very similar to how we handled Thomae's Function. In fact, Thomae's Function is just a special case of the more general function defined in this problem!

Solution: We first let $\epsilon > 0$, and then there is some $N \in \mathbb{N}$ such that $1/N < \epsilon$. Notice the set of points such that $f(x) \geq 1/N$ is $A_1 \cup A_2 \cup \ldots \cup A_N$, which is a finite union of finite sets, hence there are only finitely many points, say $q_1 < q_2 < \ldots < q_r$ such that $f(q_j) \geq 1/N$. We pick $x_0 = 0 < x_1 < q_1 < x_2 < x_3 < q_2 < x_4 < x_5 \ldots < x_{2r-1} < q_r < x_{2r} < x_{2r+1} = 1$, and choose them such that $x_{2k} - x_{2k-1} < \epsilon/r$. Now let $P = \{x_0, \ldots, x_{2r+1}\}$.

Note that now

$$U(f, P) = \sum_{i=1}^{2r+1} M_i(x_i - x_{i-1}) = \sum_{i \text{ odd}} M_i(x_i - x_{i-1}) + \sum_{i \text{ even}} M_i(x_i - x_{i-1}).$$

Since $f(x) \leq 1$, on the even intervals (that contain one of the q_i) we have that $M_i(x_i - x_{i-1}) < 1\epsilon$. Over the intervals where i is odd, we have that $f(x) < \epsilon$. Thus

$$U(f, P) = \sum_{i \text{ odd}} M_i(x_i - x_{i-1}) + \sum_{i \text{ even}} M_i(x_i - x_{i-1}) < \sum_{i \text{ odd}} \epsilon(x_i - x_{i-1}) + \sum_{i \text{ even}} \epsilon/r < \epsilon + \epsilon = 2\epsilon,$$

where K is some constant. The inequality follows because the sums of the lengths of some subset of the intervals will be at most 1.

Thus we can make $U(f, P) < \epsilon$, thus

$$\overline{\int_0^1} f = \inf_P\{U(f, P)\} = 0.$$

Since the collection $\bigcup_{i=1}^{\infty} A_i$ is countable, then every interval in our partition must contain some element x such that $f(x) = 0$. Thus $m_i = 0$ for all i for any partition P, meaning the lower intergral is also zero. Thus since the lower and upper integral are both the same value, by definition

$$\int_0^1 f = 0.$$

Remark: Alternatively one can avoid a denseness argument and simply note that since $f(x) \geq 0$,

$$0 \leq \underline{\int_0^1} f \leq \overline{\int_0^1} f = 0 \implies \underline{\int_0^1} f = 0.$$