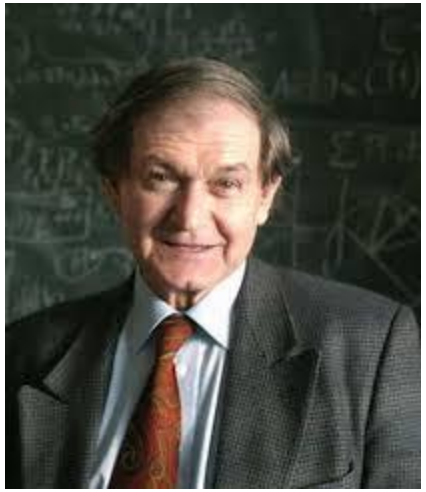
QUANTUM FORMALISM

Vector Spaces 101 - Part I

Bambordé Baldé | Co-Founder at Zaiku Group | Twitter: @zaikubalde • **zaikugroup.com** • October 9, 2020

Physics Nobel Prize 2020



Sir Roger Penrose

While at Cambridge working towards his doctorate he began to publish articles on semigroups, and on rings of matrices. In 1955 he published *A generalized inverse for matrices* in the *Proceedings of the Cambridge Philosophical Society*. In this paper Penrose defined a generalized inverse X of a complex rectangular (or possibly square and singular) matrix A to be the unique solution to the equations $AXA = A, XAX = X, (AX)^T = AX, (XA)^T = XA$. He used this generalized inverse for problems such as solving systems of matrix equations, and finding a new type of spectral decomposition. His second publication of 1955 was *A note on inverse semigroups* published in the same journal and co-authored with Douglas Munn. An inverse semigroup is a generalisation of a group and continues to be the subject of many research papers. This early paper gave several alternative definitions. In the following year Penrose published *On best approximation solutions of linear matrix equations* which used the generalized inverse of a matrix to find the best approximate solution X to $AX = B$ where A is rectangular and non-square or square and singular.

Refined Foundation Module

Rings and Fields 101 #1	Matrix Analysis #1	Quantum Matrix Operators #2
Group Theory 101 #1	Linear Operators 101 #2	Complex Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups: GL(n, C) & U(n) #3

Completed | #n is the number of live lectures

Linear Algebra Section

- More Abstract and Algebraic Approach
- More tailored to QM/QC

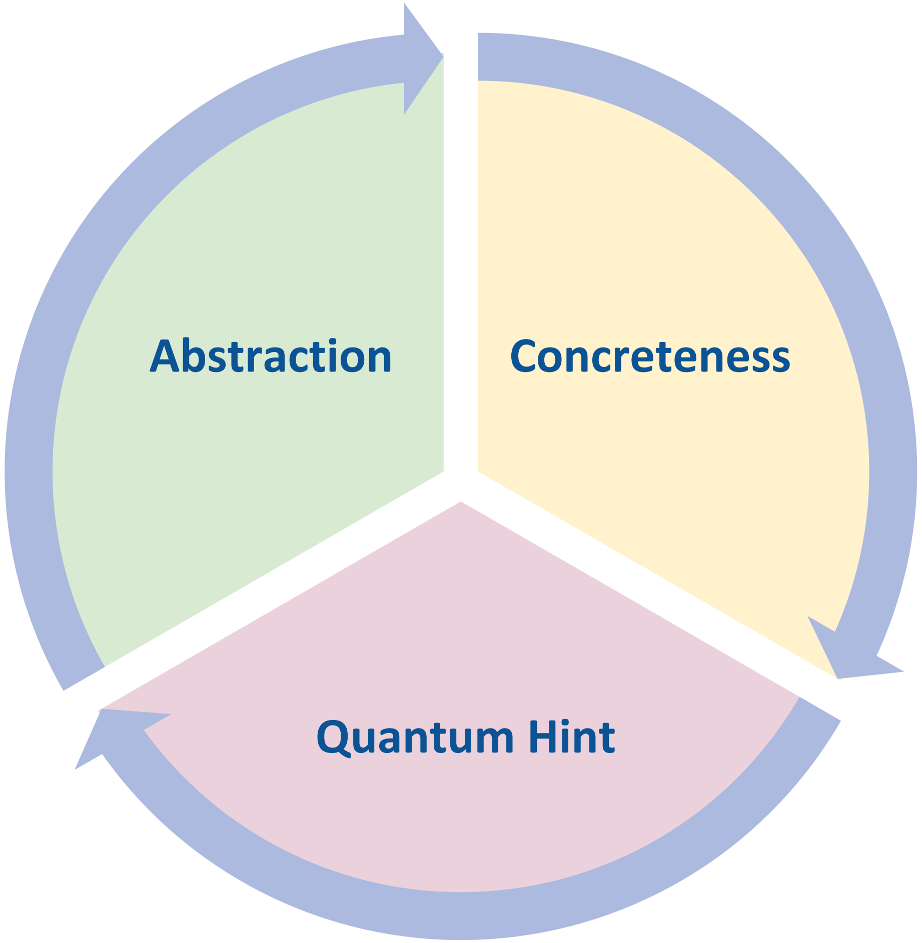
Live Sessions

Versus

- More Aligned to undergrad LA textbooks
- Not tailored to QM/QC

Study Materials

Live Session Approach



Lecture Agenda Summary

1. Additive Groups Axioms
2. Field Axioms
3. Complex Vector Space Axioms
4. Complex Vector Space Examples
5. Linear Subspaces
6. Linear Combinations
7. Linear Spans
8. Dirac Notation (Ket only)
9. Study Material Comments

Additive Groups Recap

Definition

Recall that an additive group is an abelian group $(G, +)$ i.e. a group satisfying the following conditions:

1. $g_1 + g_2 = g_2 + g_1$ for all $g_1, g_2 \in G$
2. There exists an element $0 \in G$ such that for all $g \in G$, $0 + g = g + 0 = g$ (identity or zero element)
3. $g_1 + (g_2 + g_3) = (g_1 + g_2) + g_3$ for all g_1, g_2, g_3 in G (associative)
4. For all $g \in G$ there exists $-g \in G$ such that $g + -g = 0$ (inverse)

- Examples of additive groups:
    1. $(\mathbb{Z}, +)$
    2. $(\mathbb{R}, +)$
    3. $(\mathbb{C}, +)$
    4. $(\mathbb{C}^2, +)$

What Is a Field?

Definition (1.9)

A field is a triple $(F, +, \times)$ satisfying the following axioms:

1. $(F, +, \times)$ is an integral domain
2. Every non-zero element $a \in F$ has a multiplicative inverse a^{-1}
   i.e. $aa^{-1} = a^{-1}a = 1$

Which of the following integral domains are fields?

1. $\cancel{\mathbb{Z}}$
2. $\mathbb{Q}$
3. $\mathbb{R}$
4. $\mathbb{C}$ *The field that we'll be working with!*
5. $\mathbb{F}_2$

Complex Vector Spaces 101

Definition (1.0)

A vector space space over $\mathbb{C}$ is a triple $(V, +, \cdot)$, where $(V, +)$ is an additive group and $\cdot$ is called scalar multiplication in V such that the following axioms hold:

1. $\alpha \cdot \psi \in V$ for all $\psi \in V$ and $\alpha \in \mathbb{C}$
2. $\alpha \cdot (\psi_1 + \psi_2) = \alpha \cdot \psi_1 + \alpha \cdot \psi_2$ for all $\alpha \in \mathbb{C}$ and $\psi_1, \psi_2 \in V$
3. $(\alpha + \beta) \cdot \psi = \alpha \cdot \psi + \beta \cdot \psi$ for all $\alpha, \beta \in \mathbb{C}$ and $\psi \in V$
4. $\alpha \cdot (\beta \cdot \psi) = \alpha\beta \cdot \psi$ for all $\alpha, \beta \in \mathbb{C}$ and $\psi \in V$
5. $1 \cdot \psi = \psi$ for all $\psi \in V$, where 1 is the multiplicative identity in $\mathbb{C}$

   Convention: Going forward we'll just write V to denote an abstract vector space instead of $(V, +, \cdot)$. Also, we'll omit the dot and just write $\alpha\psi$ instead of $\alpha \cdot \psi$!

Proposition (1.0)

If V is a vector space over $\mathbb{C}$, then the following is true:

1. For all $\psi \in V$ there is a unique additive inverse $-\psi \in V$ such that $\psi + -\psi = 0_V$
2. $0\psi = 0_V$ for all $\psi \in V$
3. $\alpha 0_V = 0_V$ for all $\alpha \in \mathbb{C}$
4. $-1\psi = -\psi$ for all $\psi \in V$

Proof : Homework exercise!

Vector Space Example 1

Definition

Recall that the set of all 2×1 matrices with entries in $\mathbb{C}$ is defined as follow $\mathbb{C}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$.

We already know that $\mathbb{C}^2$ forms an additive group as follows:

1. For $\psi_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in $\mathbb{C}^2$, the addition is defined as $\psi_1 + \psi_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$.
2. For any $\psi = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{C}^2$ and $\alpha \in \mathbb{C}$, we can define the scalar product as $\alpha\psi = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$. Hence, $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.

Famous Vectors in $\mathbb{C}^2$

$$\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \psi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \psi_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \psi_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \psi_5 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}.$$

Vector Space Example 2

Definition

For any natural number $n \geq 1$, we can indeed define the set all of $n \times 1$ matrices with entries in $\mathbb{C}$ as $\mathbb{C}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{C} \right\}$.

It's not hard to see that $\mathbb{C}^n$ also forms a vector space over $\mathbb{C}$ in a very natural way right?

- In quantum computing, you normally want $n = 2^k$ where $k \geq 1$ is the number of qubits under consideration. You'll understand this better when you learn about tensor product of Hilbert spaces! ☺

Linear Subspaces

Definition (1.1)

Let V be a vector space over $\mathbb{C}$. A subset $L \subseteq V$ is a linear subspace of V if L is also a vector space over $\mathbb{C}$ under the same scalar multiplication.

- Let $V = \mathbb{C}^2$. Is $L = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \right\}$ a linear subspace of $\mathbb{C}^2$? What about $W = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \right\}$?

- Is there a minimum and necessary condition for a subset of V to be a linear subspace?

Proposition (1.1)

A subset $L \subseteq V$ is a linear subspace of V if only if the following conditions hold:

1. $\psi_1 + \psi_2 \in L$ for all $\psi_1, \psi_2 \in L$ i.e. addition is closed in L.
2. $\alpha\psi \in L$ for all $\psi \in L$ and $\alpha \in \mathbb{C}$ i.e. scalar multiplication in L is also closed.

Proof : Homework!

Proposition (1.2)

If L and W are linear subspaces of V then $L \cap W$ is also a subspace of V.

Proof : Homework! Prop. 1.1 means that all you need to show is that:

1. If $\psi_1, \psi_2 \in L \cap W$ then $\psi_1 + \psi_2 \in L \cap W$
2. If $\psi \in L \cap W$ then $\alpha\psi \in L \cap W \; \forall \alpha \in \mathbb{C}$

Linear Combinations

Definition (1.2)

Let $\psi_1, \psi_2, ..., \psi_n \in V$. Their linear combination is defined as the sum $\sum_{i=1}^{n} \alpha_i \psi_i = \alpha_1\psi_1 + \alpha_2\psi_2 + ... + \alpha_n\psi_n$ where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$.

- Let $V = \mathbb{C}^2$ and consider the example of the two famous vectors $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If we make $\alpha_1 = \frac{1}{\sqrt{2}}$ and $\alpha_2 = \frac{i}{\sqrt{2}}$, then we can create the following linear combination: $\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$
- At some point you'll figure it out that what physicists and quantum computing people call 'superposition' is just a special type of linear combination!☺

Definition (1.3)

Let $\psi_1, \psi_2, ..., \psi_n \in V$. Their linear span is defined as $\text{Span}(\psi_1, \psi_2, ..., \psi_n) = \left\{ \sum_{i=1}^{n} \alpha_i \psi_i \mid \alpha_i \in \mathbb{C} \right\}$ i.e. $\text{Span}(\psi_1, ..., \psi_n)$ is the set of all possible linear combinations of the vectors $\psi_1, \psi_2, ..., \psi_n$.

- So a vector $\psi \in \text{Span}(\psi_1, \psi_2, ..., \psi_n)$ if there exist some $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ such that $\psi = \sum_{i=1}^{n} \alpha_i \psi_i$.
- Does $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$? The answer is yes because $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Is $\text{Span}(\psi_1, \psi_2 ......, \psi_n)$ a linear subspace of V?

Proposition (1.3)

Span($\psi_1, \psi_2, ..., \psi_n$) is a linear subspace of V.

Proof : Homework! Recall that Proposition 1.1 means that we only need to prove the following two things:

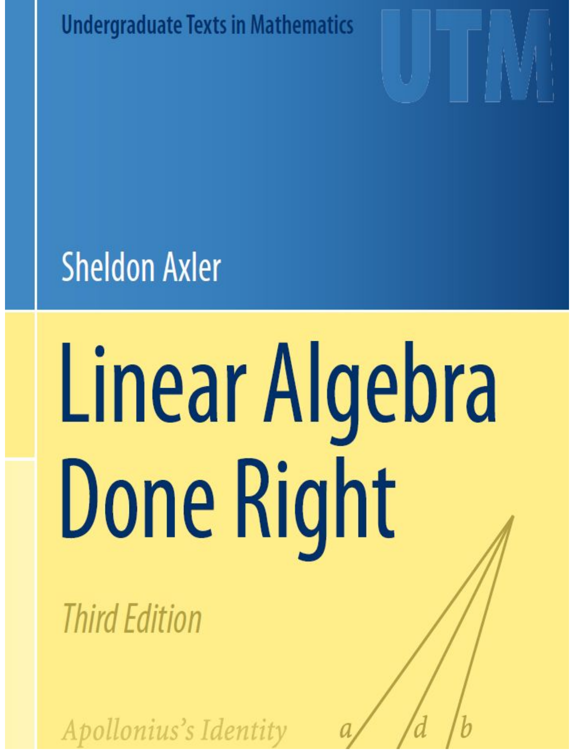
1. $+$ is closed in Span($\psi_1, \psi_2, ..., \psi_n$).
2. The scalar product is closed in Span($\psi_1, \psi_2, ..., \psi_n$).

Dirac Notation (Ket)

- Given a vector space V over $\mathbb{C}$. We'll write $|\psi\rangle$ to denote an element of V instead ψ i.e. we'll put the elements of V inside these strange objects $|\rangle$ called kets!
- As such linear combinations will now look like $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |\psi_i\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle .... + \alpha_n|\psi_n\rangle$.
- In quantum computing there is a special convention of denoting the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as follows:

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Always be alert to not mistake $|0\rangle$ with the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\mathbb{C}^2$!

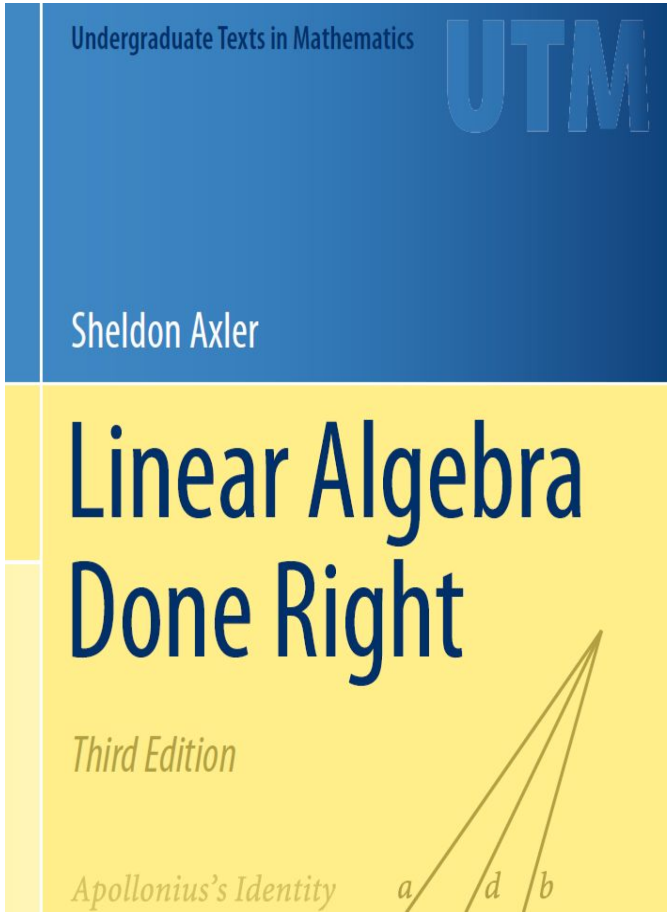
- Later we'll see that for each $|\psi\rangle \in V$ we can construct a strange object $\langle\psi|$ called bra!



Prof. Sheldon Axler

Where should you focus?

Vector spaces (*Pages 1 - 20)*



1.12 Definition *addition in $\mathbf{F}^n$*

Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

Often the mathematics of $\mathbf{F}^n$ becomes cleaner if we use a single letter to denote a list of n numbers, without explicitly writing the coordinates. For example, the result below is stated with x and y in $\mathbf{F}^n$ even though the proof requires the more cumbersome notation of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$.

1.13 Commutativity of addition in $\mathbf{F}^n$

If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

Proof Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= (y_1, \dots, y_n) + (x_1, \dots, x_n) \\ &= y + x, \end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in $\mathbf{F}^n$ and the third equality holds because of the usual commutativity of addition in $\mathbf{F}$. ■

If a single letter is used to denote an element of $\mathbf{F}^n$, then the same letter with appropriate subscripts is often used

The symbol ■ means "end of the proof".

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