$$\gamma : [0,1] \longrightarrow X$$

Topology Crash Course - Lecture 01

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Session Agenda

1. Crash Course Introduction
2. Background Poll Comments
3. Some Practical Tips

PART A

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1. Topological Spaces
2. Closed Sets
3. Subspace Topology
4. Neighbourhoods
5. Hausdorff Spaces
6. **Reference Study Material**
7. **Lecture Gap Poll**

PART B

PART A

Crash Course Motivation

Lie Groups, Lie Algebras & Representations

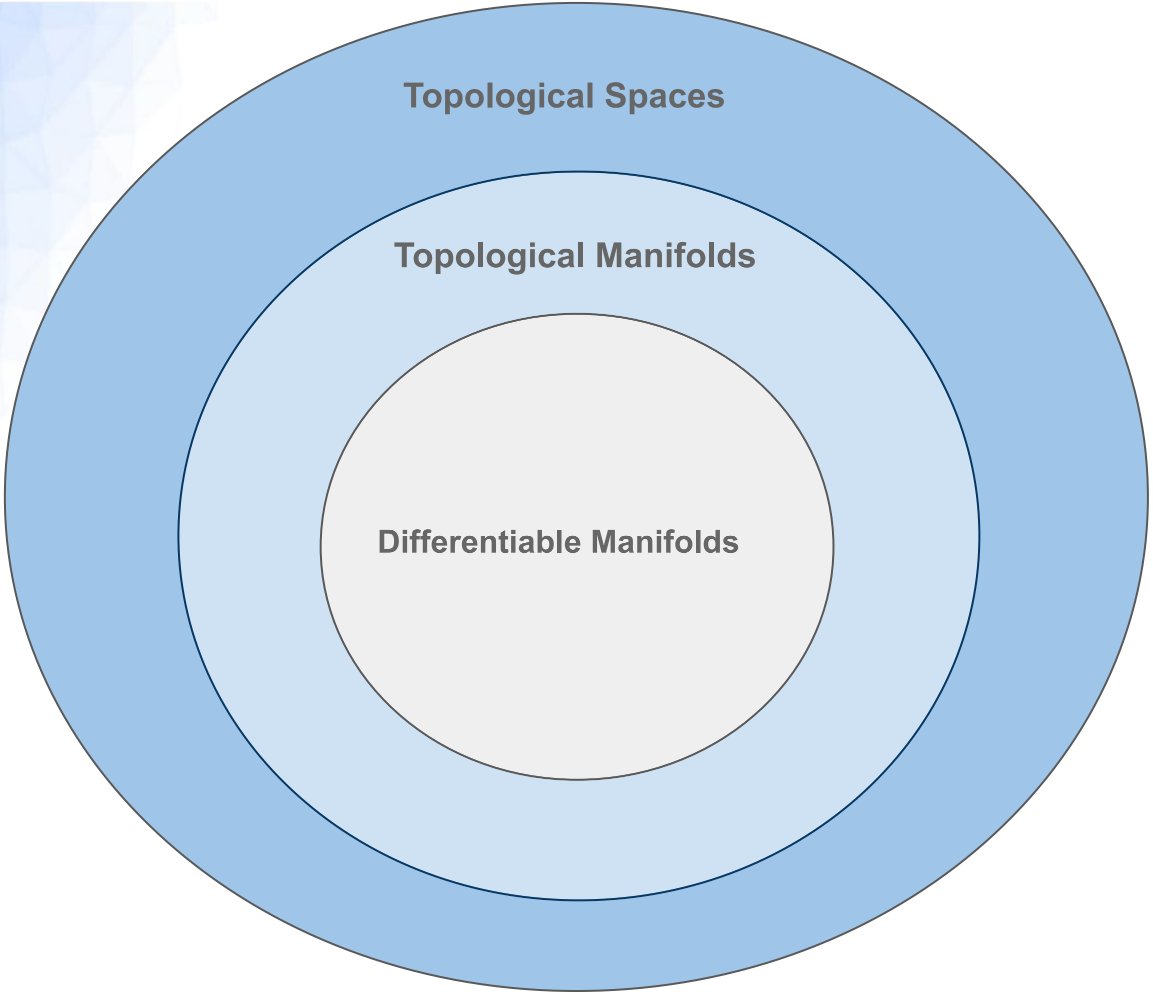
Module II (October - January?)

Lie group

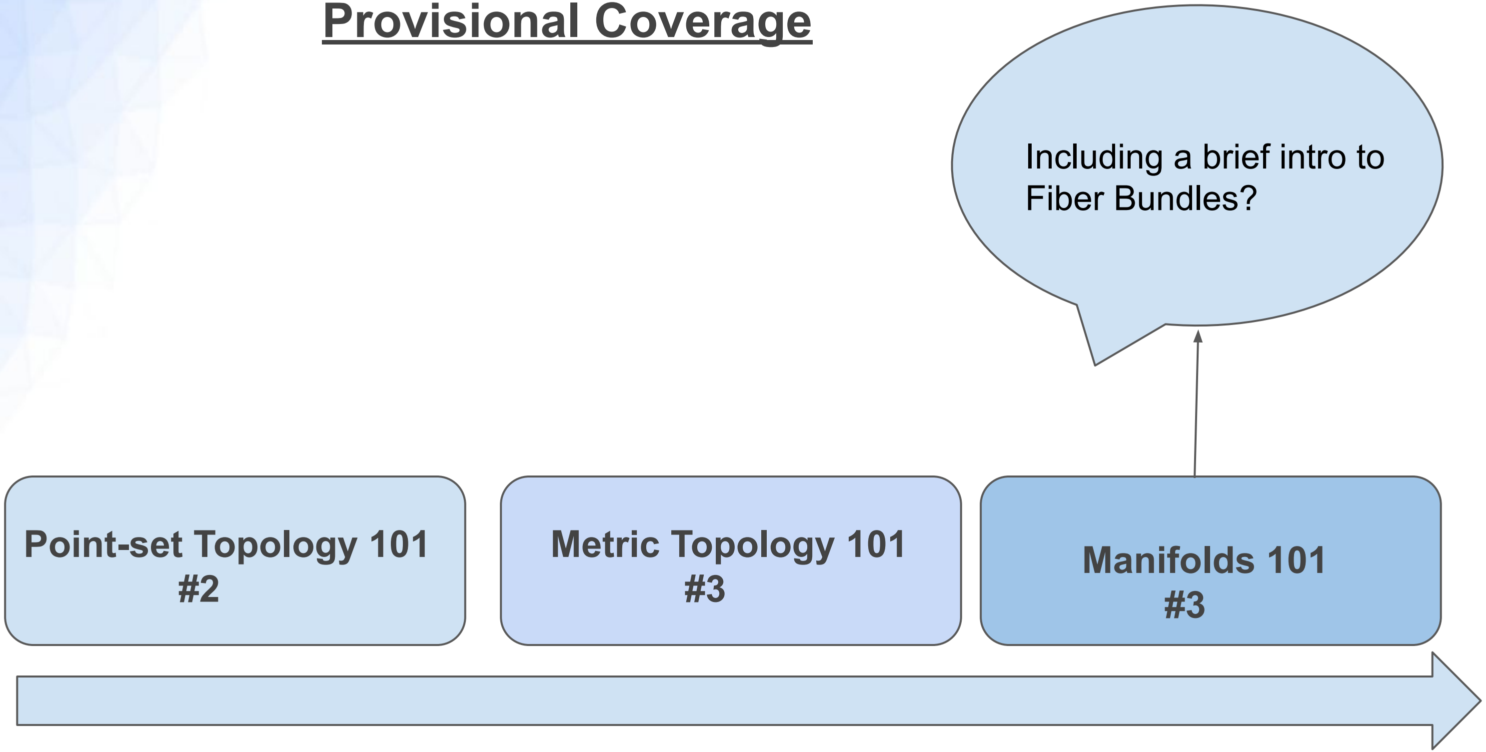
From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.


Topological Spaces
Topological Manifolds
Differentiable Manifolds

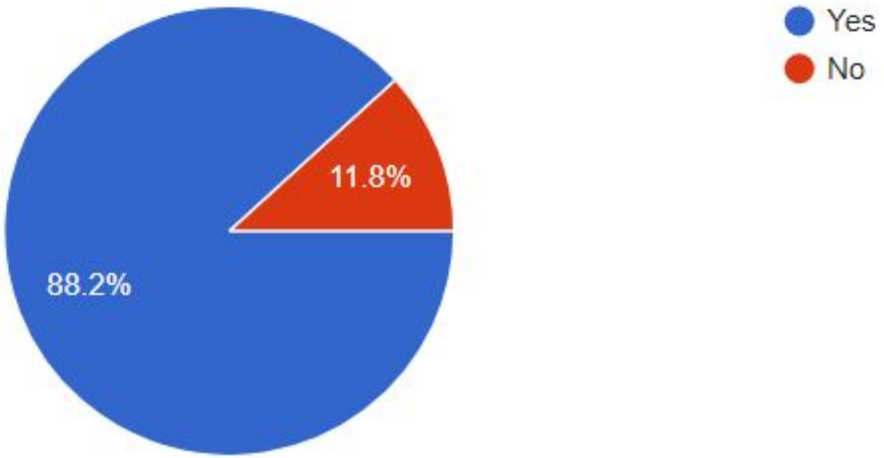
Provisional Coverage



#n is the number of live lectures.

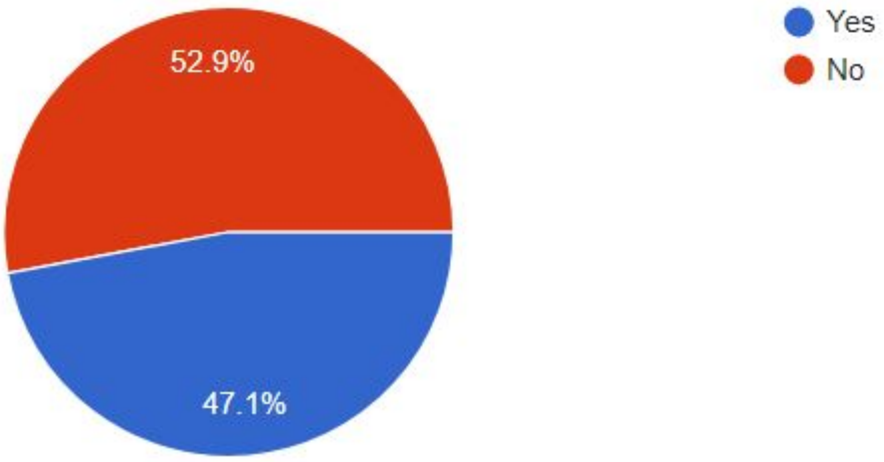
Background Poll

Have you been exposed to University level mathematics? For example, undergraduate level calculus and real analysis.

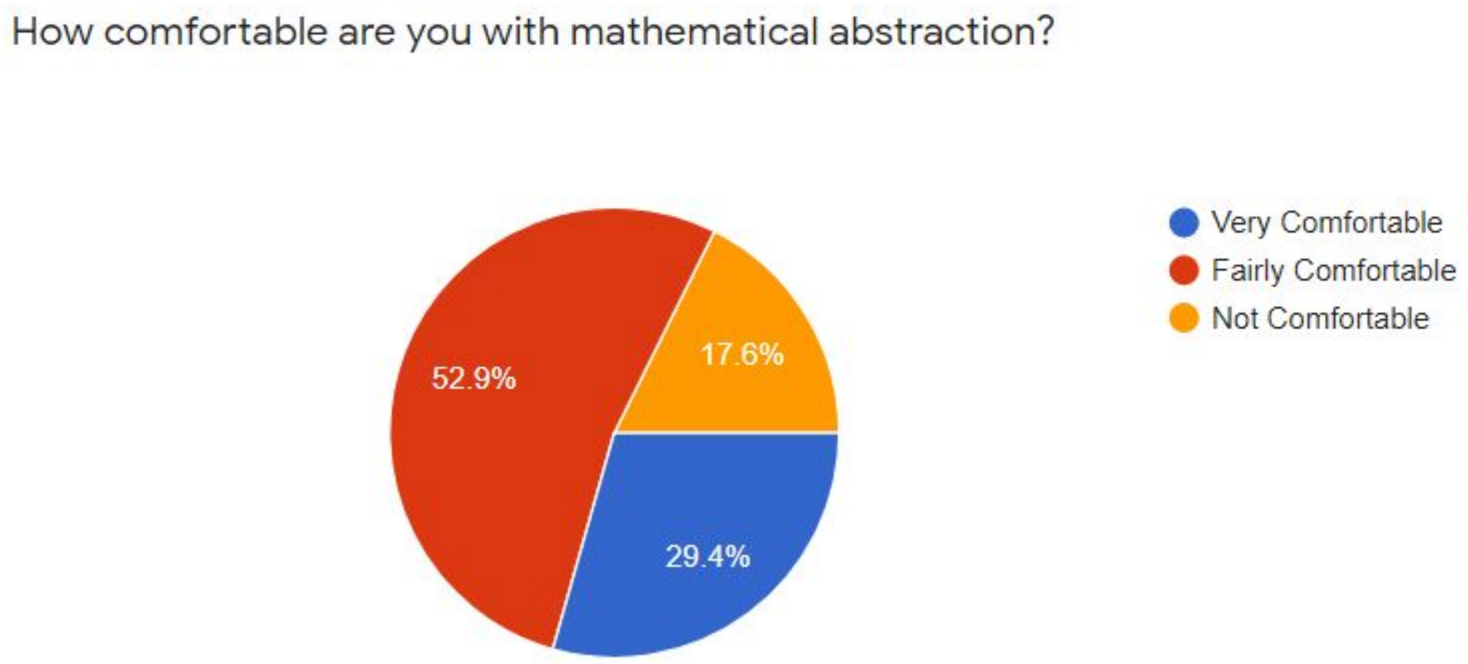


Background Poll

Have you been exposed to topology before?



Background Poll



Some Practical Tips

1. Embrace abstraction i.e. take the abstraction red pill!:)
2. One step at a time approach to the covered abstract concepts.
3. Try build your own intuition of the covered abstract concepts.
4. Try do proofs by yourself before checking other people's proofs.
5. If you struggle to understand a concept, cross reference different sources.
6. Setup a study group where you can present proofs to each other.

How to Overcome the Crash Course Challenges?

PART B

MacTutor

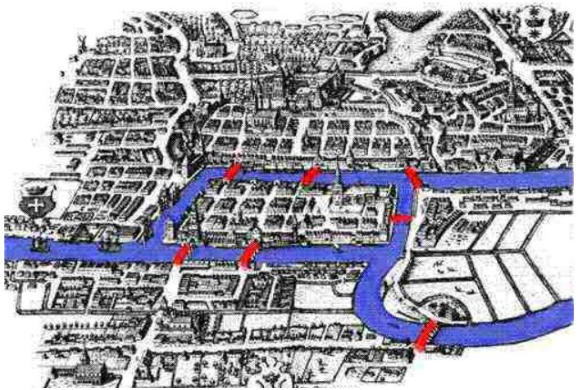
HOME BIOGRAPHIES HISTORY TOPICS MAP CURVES SEARCH

A history of Topology

Topological ideas are present in almost all areas of today's mathematics. The subject of topology itself consists of several different branches, such as point set topology, algebraic topology and differential topology, which have relatively little in common. We shall trace the rise of topological concepts in a number of different situations.

Perhaps the first work which deserves to be considered as the beginnings of topology is due to Euler. In 1736 Euler published a paper on the solution of the *Königsberg bridge problem* entitled *Solutio problematis ad geometriam situs pertinentis* Ⓣ. The title itself indicates that Euler was aware that he was dealing with a different type of geometry where distance was not relevant.

Here is **a diagram of the Königsberg bridges**



mathshistory.st-andrews.ac.uk/HistTopics/Topology_in_mathematics/

Topological Spaces

Definition (1.0)

Let X be a non-empty set. A topology on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. If $\{O_1, O_2, \ldots, O_k\} \subseteq \mathcal{T}$ then $O_1 \cap O_2 \cap \ldots \cap O_k \in \mathcal{T}$.
3. If $\{O_i\}_{i \in I} \subseteq \mathcal{T}$ then $\bigcup_{i \in I} O_i \in \mathcal{T}$.

- The pair $(X, \mathcal{T})$ is called a topological space. By convention, whenever the topology $\mathcal{T}$ is understood from the context, we'll just write X and call it a topological space. We'll also call the elements of set X as 'points'.
- The elements of $\mathcal{T}$ are called open sets.
- Is $\mathcal{T} = \mathcal{P}(X)$ a topology on X?
- What about $\mathcal{T} = \{\emptyset, X\}$?

Examples of Topology

- Let $X = \{\beta_1, \beta_2, \beta_3\}$. Is $\mathcal{T} = \{\emptyset, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$ a topology on X?
- What if $\mathcal{T} = \{\emptyset, X, \{\beta_2\}, \{\beta_3\}, \{\beta_1, \beta_2\}\}$?
- If $X = \{0, 1\}$, then $\mathcal{T} = \{\emptyset, X, \{1\}\}$ is a topology on X (Sierpinski topology).
- What if $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}\}$?
- For a finite set X, here is an interesting table about the number of possible topologies on X based on its cardinality:

Cardinality of X	Possible number of topologies
1	1
2	4
3	29
4	356
5	6942
6	209527

- Can you generalise the above table to any cardinality n i.e. can you find a general formula that tells how many topologies are there for X if its cardinality is n?

Side note: Let X be any non-empty set and consider $\mathcal{T} = \{\emptyset, Z \subseteq X \mid X \setminus Z \text{ is finite }\}$. $\mathcal{T}$ is a topology on X aka Zariski topology on X. This is a very important topology in Algebraic Geometry, it helped build the modern foundations of the subject!

Closed Sets

Definition (1.1)

A subset $A \subseteq X$ is said to be a closed set in respect to the topology $\mathcal{T}$ if $X \setminus A \in \mathcal{T}$ i.e. the complement of A in X is an open set.

- Is X a closed set in respect to the topology $\mathcal{T}$? What about the empty set $\emptyset$?
- Let $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\emptyset, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$. Is the subset $A = \{\beta_3\}$ closed in respect to $\mathcal{T}$? What if $A = \{\beta_2\}$?

Proposition (1.0)

Let $(X, \mathcal{T})$ be a topological space. Then the following is true:

1. If $A_1, A_2, \ldots, A_k$ are closed then $\bigcup_{i=1}^{k} A_i$ is closed.
2. If $\{A_i\}_{i \in I}$ is closed then $\bigcap_{i \in I} A_i$ is closed.

Proof : Homework challenge!

Important Notes

- In general, a subset $A \subseteq X$ can be the following in respect to a topology $\mathcal{T}$ on X:

1. Open
2. closed
3. Open and closed
4. Open and not closed
5. Not open and closed
6. Not open and not closed

- A subset $A \subseteq X$ is called 'clopen' if it is both open and closed!
- Later you'll see that in a 'connected' topological space, X and $\emptyset$ are the only clopen subsets!
- Is it true that if $\mathcal{T} = \mathcal{P}(X)$, then every open set is clopen?

Subspace Topology

Definition (1.2)

Let $(X, \mathcal{T})$ be a topological space and consider a subset $A \subseteq X$. We define the topology $\mathcal{T}_A$ on A as $\mathcal{T}_A = \{A \cap O \mid O \in \mathcal{T}\}$.

- It's clear that $\mathcal{T}_A$ is a topology on A right? For example, $\emptyset, A \in \mathcal{T}_A$ because $A = A \cap X$ and $\emptyset = A \cap \emptyset$. Likewise, the rest of the conditions for topology hold in $\mathcal{T}_A$?
- The pair $(A, \mathcal{T}_A)$ is called the subspace (or subset) topology in respect to the topological space $(X, \mathcal{T})$.

Proposition (1.1)

Let $(X, \mathcal{T})$ be a topological space and $(A, \mathcal{T}_A)$ be a subspace. If $B \in \mathcal{T}_A$ and $A \in \mathcal{T}$ then $B \in \mathcal{T}$.

Proof : Homework challenge!

- Hence, there is an interesting link between open sets of a subspace topology if the underlying set happens to be an open set in the parent topological space!

- Is it true that, if A is open in X then $\mathcal{T}_A$ consists of all open sets of X contained in A? What if A happens to be closed?

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Subspace Challenge

- Let $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\emptyset, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$. Now consider the subset $A = \{\beta_1, \beta_3\}$. Which of the following is the true subspace topology for A:
  1. $\mathcal{T}_A = \{\emptyset, A, \{\beta_2\}\}$
  2. $\mathcal{T}_A = \{\emptyset, A, \{\beta_1\}\}$
  3. $\mathcal{T}_A = \{\emptyset, A, \{\beta_3\}\}$
- What is the subspace topology if:
  1. $A = \{\beta_2, \beta_3\}$
  2. $A = \{\beta_1, \beta_2\}$
  3. $A = \{\beta_1\}$
  4. $A = \{\beta_2\}$
  5. $A = \{\beta_3\}$
- The cardinality of X is 3, hence there are 28 other topologies apart from the one considered above. As a homework challenge, you're encouraged to:
  1. Construct at least 3 other topologies different from the discrete and indiscrete topologies.
  2. Construct at least 3 subspaces from the 3 topologies above.

Neighbourhoods

Definition (1.3)

Let $(X, \mathcal{T})$ be a topological space. A subset $N \subseteq X$ is said to a neighbourhood of a point $p \in X$ if there is an open set $O \subseteq N$ such that $p \in O$.

- Obviously, the fact that $O \in \mathcal{T}$ doesn't mean $N \in \mathcal{T}$. However, if $N \in \mathcal{T}$, then N is called an open neighbourhood of p.
- Be aware that, often some authors assume $N \in \mathcal{T}$ in their definition of neighbourhood!

Proposition (1.2)

Let $(X, \mathcal{T})$ be a topological space. Then a subset $A \subseteq X$ is open iff A is a neighborhood of all its points $p \in A$.

Proof : Homework challenge!

- Hence, X is obviously an open neighbourhood of any point $p \in X$.

Proposition (1.3)

Let $(X, \mathcal{T})$ be a topological space and $p \in X$. If $N_1 \subseteq N_2$ is a neighbourhood of p then N_2 is also a neighbourhood of p.

Proof : Homework challenge!

Hausdorff Spaces

Definition (1.4)

A topological space $(X, \mathcal{T})$ is called a Hausdorff space if for any two distinct points $p_1, p_2 \in X$, there are neighbourhoods N_{p1} and N_{p2} of the respective points such that $N_{p1} \cap N_{p2} = \emptyset$.

- In the early days of Topology, the definition above was part of the definition of a topological space. It was called the 'Hausdorff Axiom'.
- Alternatively, mathematicians use the terms 'separated spaces' or 'T2 spaces'.
- Many important examples topological spaces are Hausdorff spaces. Very often, constructions made on topological spaces (e.g. manifolds) require the spaces to be Hausdorff.
- Is any of the examples of topological spaces encountered Hausdorff? Can you construct some examples?

Hausdorffness Properties

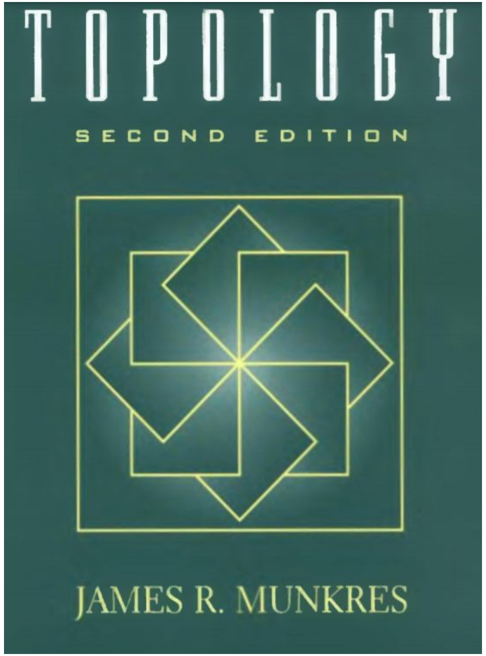
Proposition (1.4)

If a topological space $(X, \mathcal{T})$ is Hausdorff and $A \subseteq X$. Then the subspace topology of $(A, \mathcal{T}_A)$ is also Hausdorff.

Proof : Homework challenge!

- The proposition above is already hinting that Hausdorff spaces have nice structural properties i.e. they are well-behaved.
- Interestingly, in a Hausdorff space X, every singleton $\{p\} \subset X$ is closed i.e. $X \setminus \{p\}$ is open. This property will become handy when constructing certain type of $n-$ dimensional topological manifolds such as the $n-$ Sphere.
- Another interesting fact that we'll see later, is that topological spaces induced by the metric topology are Hausdorff!

Side note: As a side note, when X is infinite and $\mathcal{T}$ is the Zariski topology then $(X, \mathcal{T})$ cannot be Hausdorff!



Point-set Topology

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Lecture Gap Poll

1 Week Gap **versus** 2 Weeks Gap