QF Group Theory CC2022
By
Zaiku Group

Lecture 01

Delivered by Bambordé Baldé

Friday, 25/02/2022

Session Agenda

1. Crash Course Motivation
2. Learning Journey Timeline
3. Course Approach Overview
4. Pre-course Survey Results

Pre-session Comments

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1. Binary Operations
2. Semigroup Structure
3. Semigroup Examples
4. Semigroups Homework

Main Session

Crash Course Motivation

Lie Groups, Lie Algebras & Representations

Module II (Summer 2022)

Lie group

From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.

Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
8 - 12?

Lie Groups & Rep.
10 - 16?

Completed | Starting today! | TBC (summer) | n is the number of live lectures |

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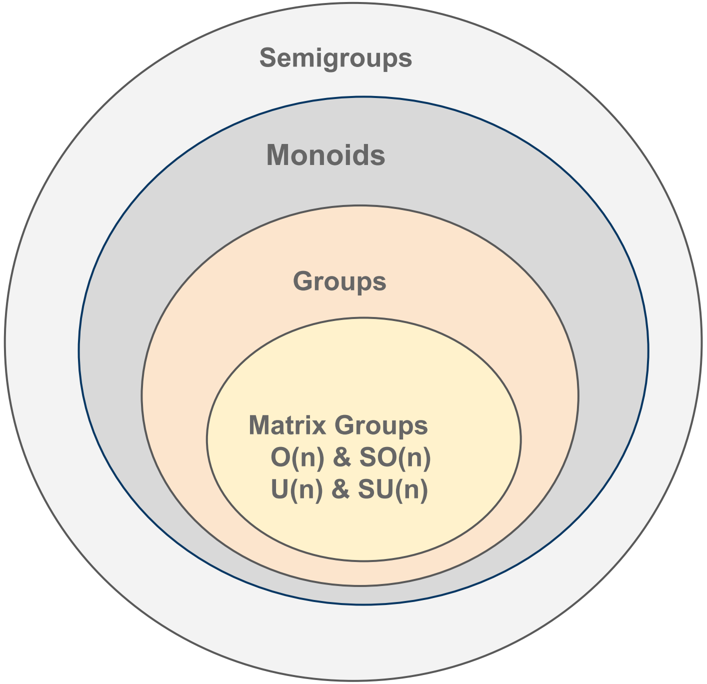
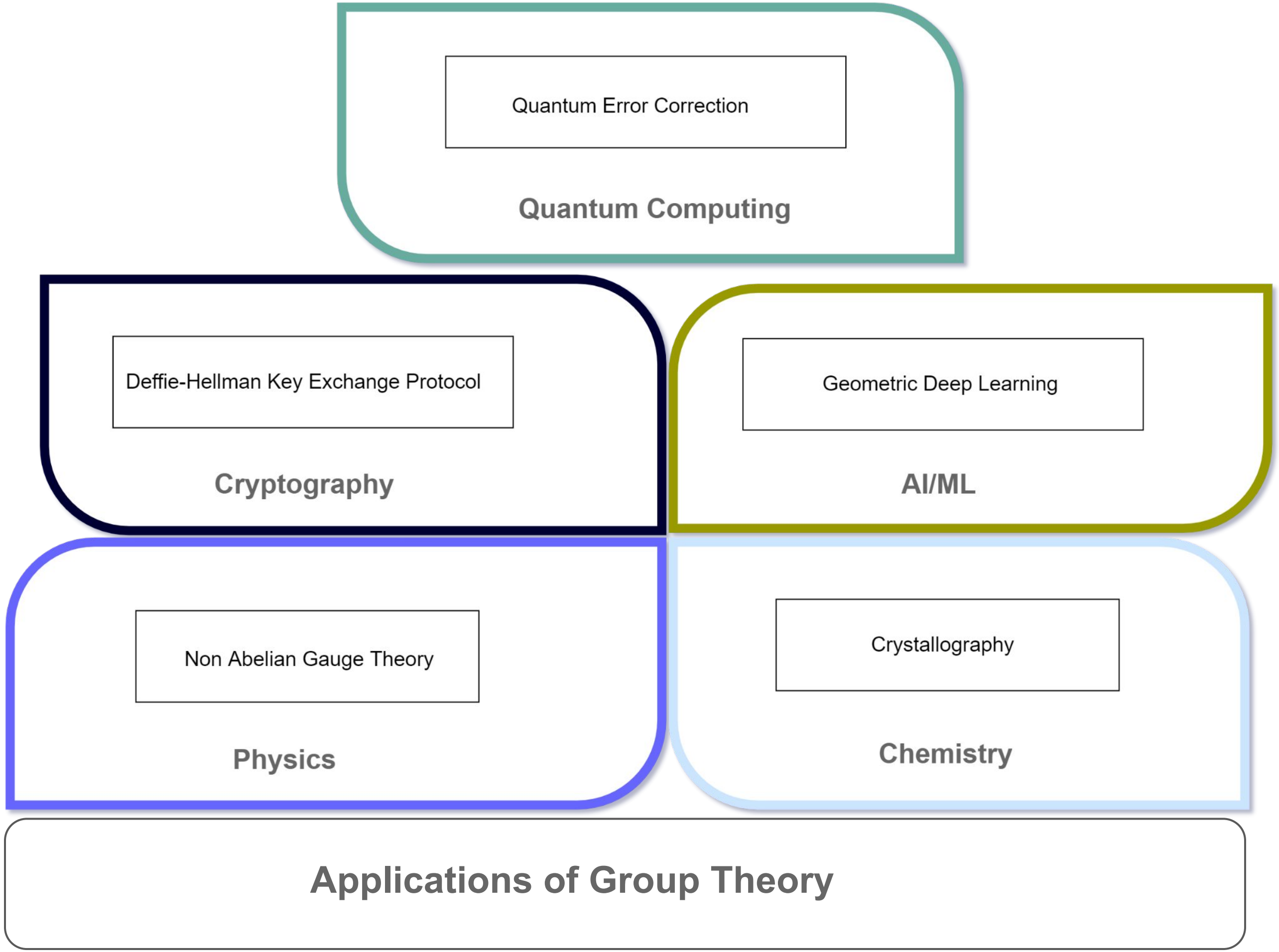


Course Approach Overview



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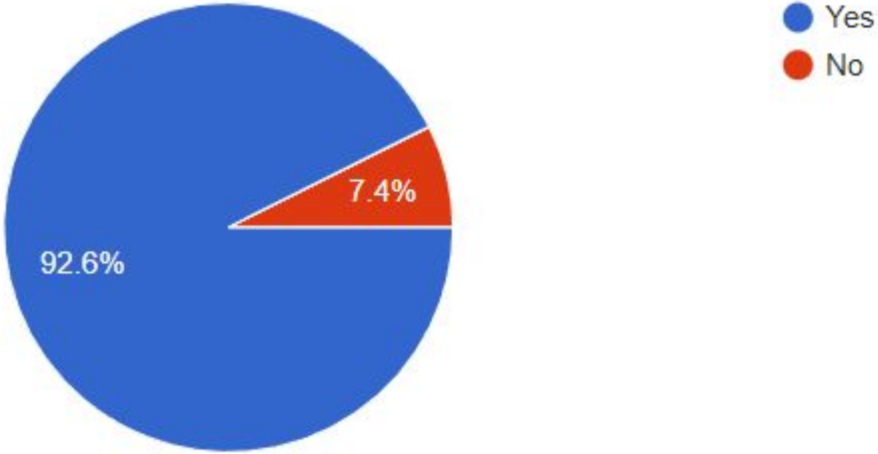
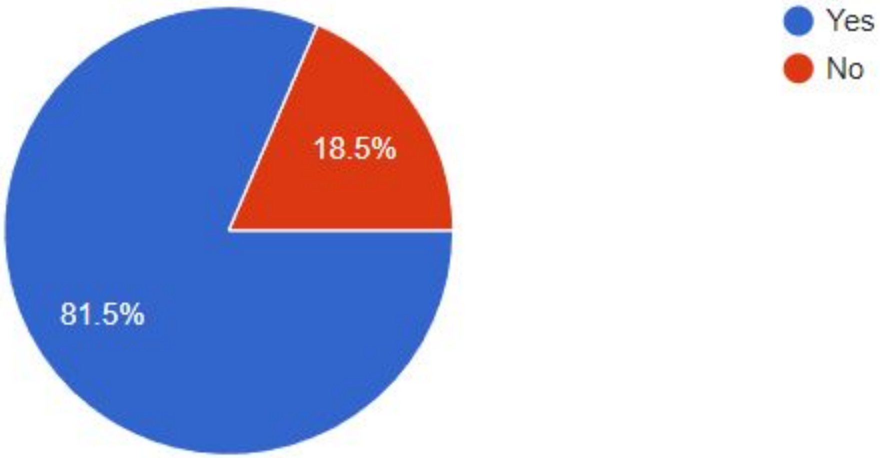
Have you been exposed to University level mathematics? For example, undergraduate level calculus and real analysis.

27 responses



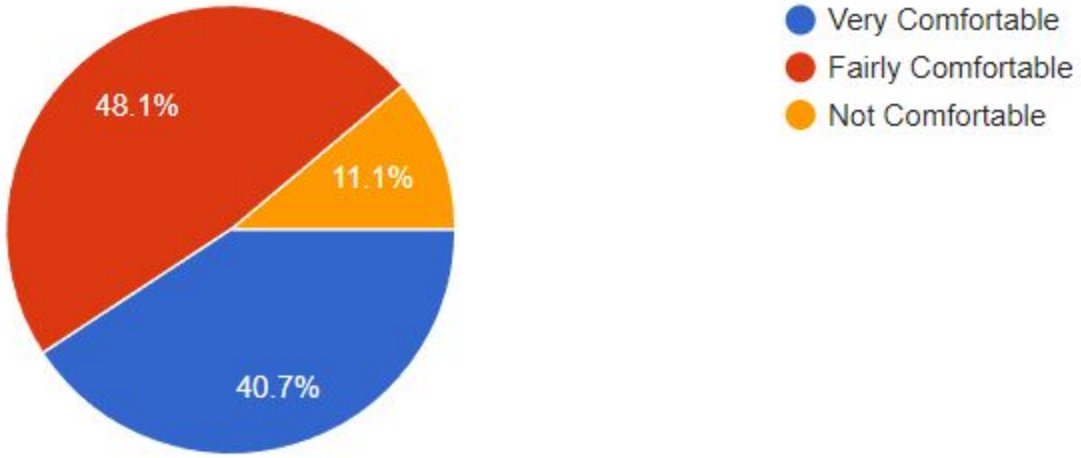
Have you been exposed to group theory before?

27 responses



How comfortable are you with mathematical abstraction?

27 responses



Binary Operation on a Set

Definition 1.0

Let S be a nonempty set. Informally, a binary operation $*$ on S is a rule that takes any two elements $a, b \in S$ to generate another element $a * b \in S$.

- More formally, a binary operation $*$ on S is a map $* : S \times S \longrightarrow S$.
- Hence, given $(a, b) \in S \times S$, $a * b$ is just an abbreviation for $*((a, b))$ i.e. $a * b$ is an abuse of notation!
- It is possible to equip a set S with more than one binary operation! For example, the algebraic structures of rings and fields are obtained that way.

Definition 1.1

Let S be a nonempty set. A binary operation $*$ on S is said to be commutative (or abelian) if $a * b = b * a$ for any pairs $a, b \in S$. Otherwise, whenever we have $a * b \neq b * a$ for some $a, b \in S$, we say that $*$ is a noncommutative (or non-abelian) binary operation on S.

Binary Operation Examples (Part A)

Example 1

Let S be the set of natural numbers $\mathbb{N}$ and let the operation $*$ be the ordinary addition of natural numbers $+$.

- $+$ defines a binary operation on $\mathbb{N}$ right?

Example 2

Let us consider $S = \{a \in \mathbb{N} \mid a \text{ is odd } \}$ and $*$ be the ordinary multiplication of natural numbers $\times$.

- Does $\times$ define a binary operation on S?

Example 3

Let consider again $S = \{a \in \mathbb{N} \mid a \text{ is odd } \}$ and let now $*$ be the ordinary addition of natural numbers $+$.

- Does $+$ also define a binary operation on S?

Binary Operation Examples (Part B)

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection } \}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is $\circ$ a binary operation on S? If yes, is it abelian or non-abelian?

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $*$ also a binary operation on $M_n(\mathbb{C})$? Is it abelian or non-abelian?

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $*$ also a binary operation on $GL(n, \mathbb{C})$? Is it abelian or non-abelian?
- What if $*$ is now the ordinary addition of matrices?

Semigroup Structure

Definition 1.2

A semigroup is a pair $(S, *)$ where S is a nonempty set and $*$ is a binary operation on S such that $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.

- The condition $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$ is called the 'associativity law' and we say that the operation $*$ is associative.
- Whenever the operation $*$ is understood from the context and fixed, we just say S is a semigroup and we omit writing the pair $(S, *)$.
- A semigroup $(S, *)$ is said to be abelian or non-abelian if $*$ is a abelian or non-abelian binary operation respectively.

Definition 1.3

Let $(S, *)$ be a semigroup and $S' \subseteq S$. Then S' is said to be subsemigroup of $(S, *)$ if $(S', *)$ is also a semigroup.

- Obviously, $(S, *)$ is trivially a subsemigroup of itself!

Semigroup Examples

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection }\}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is S a semigroup under $\circ$? If yes, is it abelian or non-abelian?

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $M_n(\mathbb{C})$ a semigroup under matrix multiplication? Is it abelian or non-abelian? What about under matrix addition?

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $GL(n, \mathbb{C})$ a semigroup under matrix multiplication?

What about under matrix addition?

Semigroups Structure Challenge

1. Let $(S, *)$ be a semigroup and let $S' = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}$. Is it true that $(S', *)$ is a subsemigroup of $(S, *)$?
2. Let $(S_1, *_1)$ and $(S_2, *_2)$ be two semigroups. Construct a semigroup structure on the Cartesian product $S_1 \times S_2$ using the respective semigroup structure. Can you generalise your construction to $(S_1, *_1), (S_2, *_2), \ldots, (S_n, *_n)$?
3. Assuming that $(S_1, *_1)$ is abelian and $(S_2, *_2)$ is non-abelian, is your constructed semigroup structure on $S_1 \times S_2$ abelian or non-abelian?
4. Identify at least a nontrivial subsemigroup structure for the constructed semigroup structure on $S_1 \times S_2$ above.
5. Let $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{Z}_4 = \{0, 1, 3\}$. Identify at least a semigroup structure for $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.
6. Identify at least a subsemigroup structure (if any) from the identified semigroup structures on $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ above.

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