QF Group Theory CC2022
By
Zaiku Group

Lecture 20 (PART A)

Delivered by Bambordé Baldé

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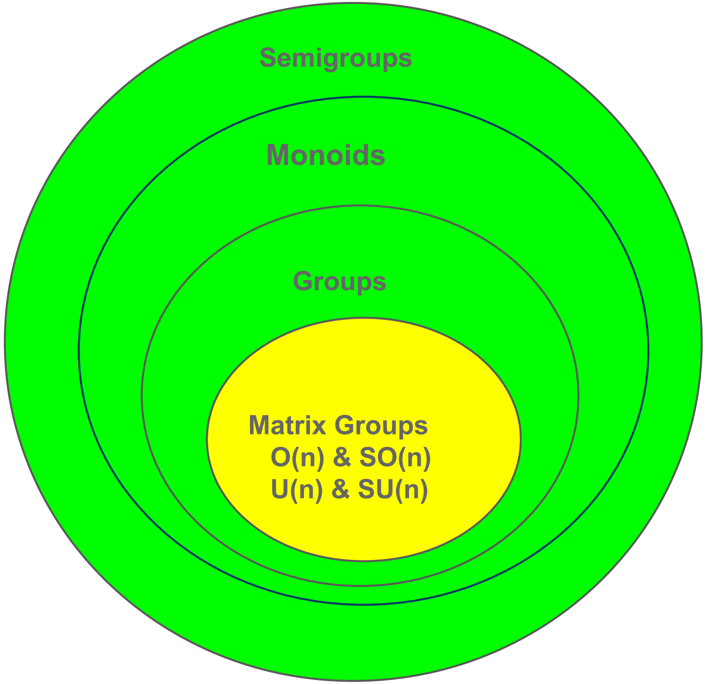
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
20A

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

quantumformalism.com

Group Representation Intuition

Group representation is essentially a way of getting elements of an abstract group G represented as matrices! Why would we want to do this?

1. During the crash course we've come across groups such as permutation groups where it becomes a hassle to do computations on them e.g computing with cycles!
2. Matrices are much easier and less abstract to perform computations with e.g. multiplying two matrices is less hassle than multiplying two cycles?

Linear Representations

Definition 1.0

Let G be a group. An $n-$ dimension linear presentation of G is a pair (V, ρ) satisfying the following conditions:

1. V is an $n-$ dimensional vector space over a field $\mathbb{F}$.
2. $\rho : G \longrightarrow GL(V)$ is a group homomorphism.

- For practical applications, the field $\mathbb{F}$ is usually either be the field of the reals $\mathbb{R}$ or complex numbers $\mathbb{C}$. We'll mostly stick with $\mathbb{C}$!
- Since $GL(V) \simeq GL(n, \mathbb{F})$, then we can instead consider $\rho : G \longrightarrow GL(n, \mathbb{F})$!
- We can use the group homomorphism $\rho : G \longrightarrow GL(V)$ to construct a left action of G on V via $(g, v) \in G \times V \longrightarrow \rho(g)v \in V$!

Definition 1.1

A representation (V, ρ) of G is said to be faithful if ρ is injective i.e. if $\rho(g_1) = \rho(g_2)$ iff $g_1 = g_2$ for all $g_1, g_2 \in G$.

- If (V, ρ) is faithful then $\rho(g) = I_n$ iff $g = 1_G$ i.e. $Ker(\rho) = \{1_G\}$?

Notation awareness:

- Some authors use the notation π for the homomorphism instead of ρ!

- Also, very often a representation (V, ρ) is referred to by its homomorphism ρ whenever the vector space V is understood from the context i.e. V is often not mentioned which may give an impression that a representation is just the homomorphism ρ!

Example (Complex Representations)

1. Let G be any group. Then we can construct an $n-$ dimensional representation $(\mathbb{C}^n, \rho)$ with $\rho : G \longrightarrow GL(n, \mathbb{C})$ defined as $\rho(g) = I_n$ for all $g \in G$.
2. Consider the permutation group $G = \{(1), (123), (132)\} \subset S_3$. We can construct a $3-$ dimensional representation $(\mathbb{C}^3, \rho)$ of G with $\rho : G \longrightarrow GL(3, \mathbb{C})$ defined as follows:

$$\rho((1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Is the representation faithful?

Challenge 1

Let $G = S_3$. You're encouraged to try construct the following:

1. A faithful 3−dimensional representation $(\mathbb{C}^3, \rho)$ of G.
2. A non-faithful 3−dimensional representation $(\mathbb{C}^3, \rho)$.

Subgroup Representations

Proposition 1.0

Let G be a group and H a subgroup of G. If (V, ρ) is a representation of G then $(V, \rho_{|H})$ is a representation of H where $\rho_{|H} : H \longrightarrow GL(V)$ is the restriction of ρ to H.

Proof : Homework challenge (almost trivial)!

- Hence, if we have a representation of a group G, then we can use it used to construct representations of any subgroup of G by using the restriction!

Proposition 1.1

Let G_1, G_2 be groups and $\phi : G_2 \longrightarrow G_1$ be a homomorphism. If (V, ρ) is a representation of G_1 then $(V, \rho \circ \phi)$ is a representation of G_2 i.e. $\rho \circ \phi : G_2 \longrightarrow GL(V)$ is a group homomorphism.

Proof : Homework challenge (trivial)!

Equivalent Representations

Definition 1.2

Two representations (V_1, ρ_1) and (V_2, ρ_2) of G are equivalent if there is an isomomorphism $T : V_1 \longrightarrow V_2$ such that $\rho_1(g) \circ T = T \circ \rho_2(g)$ for all $g \in G$.

- An equivalent way to state the above is to say the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\rho_2(g)"] & V_2 \end{tikzcd}$$

- We write $(V_1, \rho_1) \sim (V_2, \rho_2)$ or alternatively just $\rho_1 \sim \rho_2$!

Challenge 2

Is $\sim$ as defined above an equivalence relationship?

Example (Equivalent Representations)

- Consider the additive group $\mathbb{Z}_n$ (often also written as $\mathbb{Z}/n\mathbb{Z}$). Then we can construct two equivalent $2-$ dimensional representations of $\mathbb{Z}_n$, $(\mathbb{Z}_n, \rho_1)$ and $(\mathbb{Z}_n, \rho_2)$ as follows:

1. We define the map $\rho_1 : \mathbb{Z}_n \longrightarrow GL(2, \mathbb{C})$ as $\rho_1(x) = \begin{pmatrix} cos(\frac{2\pi x}{n}) & -sin(\frac{2\pi x}{n}) \\ sin(\frac{2\pi x}{n}) & cos(\frac{2\pi x}{n}) \end{pmatrix}$ for all $x \in \mathbb{Z}_n$.

2. We define the map $\rho_2 : \mathbb{Z}_n \longrightarrow GL(2, \mathbb{C})$ as $\rho_2(x) = \begin{pmatrix} e^{\frac{2\pi xi}{n}}) & 0 \\ 0 & e^{\frac{-2\pi xi}{n}} \end{pmatrix}$ for all $x \in \mathbb{Z}_n$.

Homomorphism of Representations

Definition 1.3

Let (V_1, ρ_1) and (V_2, ρ_2) be representations of a group G. A linear map $L : V_1 \longrightarrow V_2$ is a homomorphism of the two representations if $L \circ \rho_1(g) = \rho_2(g) \circ L$ for all G.

- The definition is equivalent to the commutative of the following diagram:

$$\begin{tikzcd}
V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "L"'] & V_1 \arrow[d, "L"] \\
V_2 \arrow[r, "\rho_2(g)"] & V_2
\end{tikzcd}$$

- L is also known as a $G-$ linear map in the literature!
- L happens to be an isomorphism as well, we say the two representations are isomorphic and write $(V_1, \rho_1) \simeq (V_2, \rho_2)$.

Proposition 1.2

Let (V_1, ρ_1) and (V_2, ρ_2) be representations of a group G. If a $G-$ linear map $L : V_1 \longrightarrow V_2$ is an isomorphism, then its inverse $L^{-1} : V_2 \longrightarrow V_1$ is also $G-$ linear.

Proof : Homework challenge!

Group Subrepresentations

Definition 1.4

Let (V, ρ) be a representation of G and $W \subseteq V$ be a linear subspace. Then we say W is $G-$ invariant if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$.

- Another way to state the above is to say W is $G-$ invariant under the underlying group action.
- Trivially, V and the zero space $\{0_V\}$ are $G-$ invariant right?

Definition 1.5

Let (V, ρ) be a representation of G and $W \subseteq V$ be G-invariant subspace. Then we call the representation $(W, \rho_{|W})$ a subrepresentation of (V, ρ).

- It's clear that $\rho_{|W} : G \longrightarrow GL(W)$ is a homomorphism right?

Irreducible Representations

Definition 1.6

A representation (V, ρ) of G is said to be irreducible if the only $G-$ invariant subspaces are V and the zero subspace $\{0_V\}$.

- If the representation admits a nontrivial $G-$ invariant subspace, then we say it is reducible.

Proposition 1.3

Any $1-$ dimensional representation (V, ρ) of a group G is irreducible.

Proof : Homework (trivial)!

- A hint to see the above is to recall that being $1-$ *dimensional* representation means $V \simeq \mathbb{C}$!

Curiosity: Given an $n-$ dimensional representation (V, ρ) of a group G, is there a trick to verify if (V, ρ) is irreducible?

Theorem 1.0 (Eigenvector trick)

Let (V, ρ) be a $2-$ dimensional representation of a group G. Then (V, ρ) is irreducible iff there is no eigenvector $\psi \in V$ common to all $\rho(g)$ with $g \in G$.

- There is a similar trick for $3-$ dimensional representations, but this requires extra condition such as G being finite!

Challenge 3

Consider the additive group $\mathbb{Z}_n$ (often also written as $\mathbb{Z}/n\mathbb{Z}$). Are the following 2− dimensional representations of $\mathbb{Z}_n$, $(\mathbb{Z}_n, \rho_1)$ and $(\mathbb{Z}_n, \rho_2)$ defined below irreducible?

1. We define the map $\rho_1 : \mathbb{Z}_n \longrightarrow GL(2, \mathbb{C})$ as $\rho_1(x) = \begin{pmatrix} cos(\frac{2\pi x}{n}) & -sin(\frac{2\pi x}{n}) \\ sin(\frac{2\pi x}{n}) & cos(\frac{2\pi x}{n}) \end{pmatrix}$ for all $x \in \mathbb{Z}_n$.

2. We define the map $\rho_2 : \mathbb{Z}_n \longrightarrow GL(2, \mathbb{C})$ as $\rho_2(x) = \begin{pmatrix} e^{\frac{2\pi xi}{n}}) & 0 \\ 0 & e^{\frac{-2\pi xi}{n}} \end{pmatrix}$ for all $x \in \mathbb{Z}_n$.

Challenge 4

Can you build a 2− dimensional complex representation of the permutation group S_3?

GitHub: github.com/quantumformalism

YouTube: youtube.com/ZaikuGroup

Discord: discord.gg/SPcmcsXMD2

LinkedIn: linkedin.com/showcase/quantum-formalism

Twitter: twitter.com/ZaikuGroup