QF Group Theory CC2022
By
Zaiku Group

Lecture 19

Delivered by Bambordé Baldé

Friday, 25/11/2022

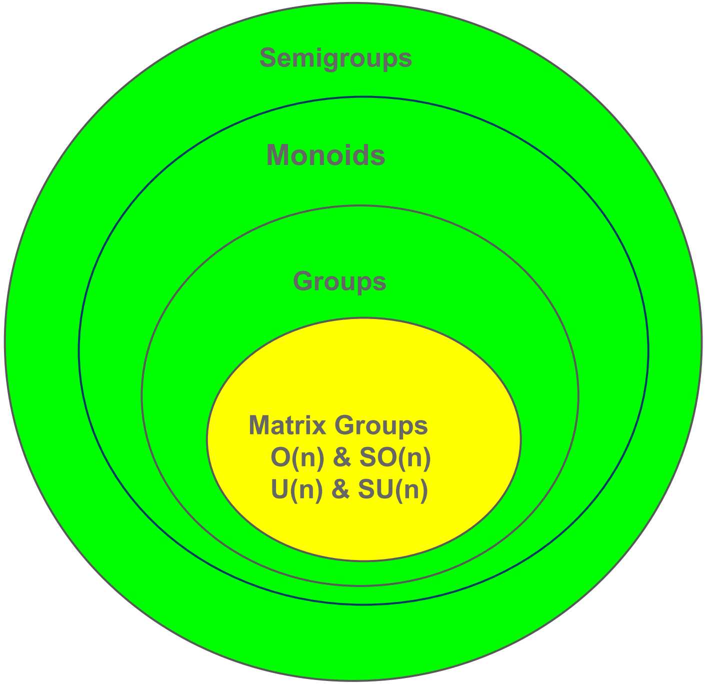
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
19/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

A Brief Linear Algebra Recap

Definition 1.0

We'll write $M_n(\mathbb{C})$ to denote the set of all $n \times n$ matrices over ~~the reals~~ $\mathbb{C}$.

- Some authors use the notation $M^{n\times n}(\mathbb{C})$ instead of $M_n(\mathbb{C})$.
- I'll assume everyone knows about the basics of $n \times n$ matrices over the reals $\mathbb{C}$ including; how to compute the transpose, perform addition and multiplication of $n \times n$ matrices.
- When equipped with the ordinary matrix addition or multiplication, which of the following is true?

1. $M_n(\mathbb{C})$ forms an abelian group structure under addition.
2. $M_n(\mathbb{C})$ forms a nonabelian group structure under multiplication.

Important: From linear algebra 101 an element $A \in M_n(\mathbb{C})$ induces a linear map $L_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, with $\mathbb{C}^n$ equipped with the canonical vector space structure over $\mathbb{C}$. Likewise, any linear map $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ induces an element $A_L \in M_n(\mathbb{C})$ i.e. linear maps on $\mathbb{C}^n \equiv n \times n$ matrices over $\mathbb{C}$.

Complex Matrix Groups

Definition 1.1

A subset $G \subset M_n(\mathbb{C})$ is a complex matrix group if it's a group under the ordinary matrix multiplication. This obviously implies the following:

1. If $A, B \in G$ then $AB \in G$ i.e. matrix multiplication is a closed binary operation in G.
2. If $A, B, C \in G$ then $A(BC) = (AB)C$ i.e. matrix multiplication is associative in G. This is trivial to show because it is associative in $M_n(\mathbb{C})$!
3. The identity matrix $I_n \in G$.
4. For any $A \in G$ there exists an inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

- Since G is a group, then all the abstract group-theoretic properties and constructions we've made so far also applies to it! Hence, we can ask about subgroups of G, left group actions, left cosets, orbits, stabilisers and so on.

The General Linear Group over $\mathbb{C}$

Proposition 1.0

Let us consider the subset of $M_n(\mathbb{C})$ defined as $GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$. Then $GL(n, \mathbb{C})$ is a complex matrix group under the ordinary matrix multiplication.

Proof : Homework challenge!

- As a hint to help you prove the above: Recall from kindergarten linear algebra that if $A \in M_n(\mathbb{C})$ and $det(A) \neq 0$, then A is invertible! In fact A is invertible iff $det(A) \neq 0$!
- $GL(n, \mathbb{C})$ is known in the literature as the general linear group of order n over $\mathbb{C}$. Also, some authors use the notation $GL_n(\mathbb{C})$!

Side note: Observe the following subtle facts about $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ as Lie groups:

1. $GL(n, \mathbb{C})$ is a noncompact connected Lie group of complex dimension n^2 and real dimension $2n^2$.
2. $GL(n, \mathbb{R})$ is a noncompact disconnected Lie group of dimension n^2.

The Complex Special Linear Group

Proposition 1.1

The set $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid det(A) = 1\}$ is a subgroup of $GL(n, \mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- $SL(n, \mathbb{C})$ is known in the literature as the complex special linear group.

Side note: Observe the following subtle facts about $SL(n, \mathbb{C})$ and $SL(n, \mathbb{R})$ as Lie groups:

1. $SL(n, \mathbb{C})$ is a noncompact connected Lie group of complex dimension $n^2 - 1$ and real dimension $2(n^2 - 1)$.
2. $SL(n, \mathbb{R})$ is a noncompact connected Lie group of dimension $n^2 - 1$.

Proposition 1.2

Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. Then the determinant map $det : GL(n, \mathbb{C}) \longrightarrow \mathbb{C}^*$ taking $A \in GL(n, \mathbb{C})$ to $det(A) \in \mathbb{C}^*$ is a group homomorphism and $Ker(det) = SL(n, \mathbb{C})$.

A Special Complex Matrix Group in Disguise

Complex Numbers 101

Given a complex number $a = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$, the complex conjugate of a is defined as $\bar{a} = x - iy$.

Attention: Physicists often use the notation a^* instead of $\bar{a}$!

Proposition 1.3

The set $G = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$ is a subgroup of $GL(2, \mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- The group G above is a very special type of group in disguise! Can anyone unmask it? Can the quantum folks unmask it?

Side note: You'll learn in the next course that as a smooth manifold, G is diffeomorphic to the $3 - \textit{sphere}\ S^3$!

Matrix Conjugate Refresh

Definition 1.2

Given $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C})$, we define the conjugate as:

$\bar{A} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{pmatrix}$ where $\bar{a}_{ij} = x - iy$ for all $a_{ij} = x + iy$.

- Physicists often use the notation A^* instead of $\bar{A}$!

Conjugate Transpose Refresh

Definition 1.2 (using the mathematician's notation)

Given $A \in M_n(\mathbb{C})$, we define the conjugate transpose of A as $A^* = (\bar{A})^T$.

- Physicists use the notation $A^\dagger$ instead of A^*!
- We'll adopt the physicist notation for the conjugate transpose of a matrix and adopt the mathematician's notation for the conjugate of complex numbers!

Proposition 1.4

Let $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then the following identities hold:

1. $(A^\dagger)^\dagger = A$.
2. $(\lambda A)^\dagger = \bar{\lambda} A^\dagger$.
3. $(A + B)^\dagger = A^\dagger + B^\dagger$.
4. $(AB)^\dagger = B^\dagger A^\dagger$.
5. $det(A^\dagger) = \overline{det(A)}$.
6. If A is invertible then $A^\dagger$ is also invertible.

Proof : Homework challenge!

Side note: A matrix $A \in M_n(\mathbb{C})$ is said to be Hermitian if $A = A^\dagger$!

The Unitary Matrix Group

Proposition 1.5

The set $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^\dagger A = AA^\dagger = I_n\}$ is a subgroup of $GL(n, \mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- The group $U(n)$ is known in the literature as the unitary group.
- The group elements of $U(n)$ are indeed linear isometries in $\mathbb{C}^n$ i.e. they preserve the inner product in $\mathbb{C}^n$ and so the norm.
- So $U(n)$ is the complex version of the real orthogonal group $O(n)$!
- $U(n)$ is a very important group with applications in many topics such as theoretical physics and quantum information science.

Side note: Observe the following subtle facts about $U(n)$ and $O(n)$ as Lie groups:

1. $U(n)$ is compact and connected Lie group with 'real' dimension n^2.
2. $O(n)$ is compact and disconnected Lie group with dimension $\frac{n(n-1)}{2}$.

The Special Unitary Group

Proposition 1.6

The set $SU(n) = \{A \in U(n) \mid det(A) = 1\}$ is a subgroup of $U(n)$.

Proof : Homework challenge!

- $SU(n)$ is known in the literature as the special unitary group.
- So $SU(n)$ is the complex version of the special orthogonal group $SO(n)$!
- It's clear that $SU(n) = U(n) \cap SL(n, \mathbb{C})$ right?
- The group G in disguise we were playing with is indeed $SU(2)$!

  $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$

Side note: Observe the following subtle facts about $SU(n)$ and $SO(n)$ as Lie groups:

1. $SU(n)$ is compact and connected Lie group with 'real' dimension $n^2 - 1$.
2. $SO(n)$ is compact and connected Lie group with dimension $\frac{n(n-1)}{2}$.

SU(n) homework challenge

Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. Then the determinant map $det : U(n) \longrightarrow \mathbb{C}^*$ taking $A \in U(n)$ to $det(A) \in \mathbb{C}^*$ is a group homomorphism. What is $Ker(det)$?

- Also, Is it true $SU(n)$ is a normal subgroup of $U(n)$?

Side note tables

G	$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{O}(n,\mathbb{R})$	$\mathrm{SO}(n,\mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(2n,\mathbb{R})$
$\mathfrak{g}$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathrm{tr}\,x=0$	$x+x^t=0$	$x+x^t=0$	$x+x^*=0$	$x+x^*=0,\ \mathrm{tr}\,x=0$	$x+Jx^tJ^{-1}=0$
$\dim G$	n^2	n^2-1	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	n^2-1	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2\ (n\geq 3)$	$\mathbb{Z}_2\ (n\geq 3)$	$\mathbb{Z}_2\ (n\geq 3)$	$\mathbb{Z}_2\ (n\geq 3)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}$

G	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$

Credits for the tables: Prof Alexander Kirillov, Math. Department of State Univ. of New York at Stony Brook.

QUANTUM FORMALISM

GitHub: github.com/quantumformalism

YouTube: youtube.com/ZaikuGroup

Discord: discord.gg/SPcmcsXMD2

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