QUANTUM FORMALISM

Finite dimensional Hilbert spaces - Part 2

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Axiom #1 Recap
3. Normalised vectors
4. Orthogonal Vectors
5. Orthonormal Bases
6. Separable Hilbert space

Part A

1. LFs Induced by the Inner Product (Bras!)
2. The Dual Space
3. Riesz Representation Theorem
4. Study Material Comments

Part B

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Operators + Composite Systems #3
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

January 2021 Calendar

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26 Australia Day	27	28	29	30
31						

Lecture dates

PART A

Complex Hilbert space

Definition (1.0)

A complex Hilbert space is vector space over $\mathbb{C}$ denoted $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ that induces a complete norm $\| \cdot \|$.

- Hence, by definition a Hilbert space is a Banach space in respect to the induced norm.
- $\mathbb{C}^n$ is obvious examples of complex Hilbert spaces! Indeed any finite dimensional complex Hilbert space $\mathcal{H}$ of dimension n is unitari ly equivalent (isomorphic) to $\mathbb{C}^n$!
- Hilbert spaces are often presented as the rock stars of quantum formalism. In reality, Banach spaces also play a very important role in the formalism e.g. the $\mathbb{C}^*$-algebra of bounded operators is a Banach space.
- Is a Banach space necessarily a Hilbert space?

The Axioms of Quantum Mechanics

Axiom 1: The states of quantum systems are modelled by **normalised vectors** on **separable complex Hilbert spaces**.

- So to understand what the axiom is telling us mathematically, we need to know the meaning of the following Jargons:

1. Normalised vectors
2. Separable Hilbert space

- As previously mentioned, the axiom is actually referring to what physicists call 'pure states'. There are also the so-called mixed states!
- These normalised vectors representing the states of physical systems are called 'state vectors'.

Normalised Vectors

Definition (1.1)

A vector $\psi \in \mathcal{H}$ is said to be normalised if $\|\psi\| = 1$ where the norm $\|\cdot\|$ is of course induced by the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$.

- Hence, definition demystifies one of the jargons in axiom 1!
- Why the requirement for normalised state vectors? As course break challenge, try research or ask a physicist why the normalised state vectors matter!
- $\mathcal{H} = \mathbb{C}^2$ then $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are both normalised?
- Which of the following vectors are normalised? Are they all normalised?

$$\psi_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \psi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \psi_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ and } \psi_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}.$$

Orthogonal Vectors

Definition (1.2)

Two vectors $\psi_1, \psi_2 \in \mathcal{H}$ are said to be orthogonal if $\langle \psi_1, \psi_2 \rangle = 0$.

- Authors often write $\psi_1 \perp \psi_2$ to denote that ψ_1 and ψ_2 are orthogonal.
- If $\mathcal{H} = \mathbb{C}^2$ then $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are orthogonal?

Definition (1.3)

Let W be a linear subspace of $\mathcal{H}$. The orthogonal compliment of W is defined as $W^{\perp} = \{\psi \in \mathcal{H} \mid \langle \psi, \Phi \rangle = 0 \text{ for all } \Phi \in W\}$.

Proposition (1.0)

Let W be a linear subspace of $\mathcal{H}$. Then the following is true:

1. $W^{\perp}$ is a linear subspace of $\mathcal{H}$.
2. $(W^{\perp})^{\perp} = W$.

- Why do physicists want state vectors that are orthogonal? As course break challenge, try research or ask a physicist why orthogonal state vectors matter!

Orthonormal Bases

Definition (1.4)

A basis $B = \{e_1, e_2, \ldots, e_n\}$ of $\mathcal{H}$ is said to be an orthonornal basis in $\mathcal{H}$ if $\|e_i\| = 1$ for all $i \in \{1, \ldots, n\}$ and $\langle e_i, e_j \rangle = 0$ for all $i \neq j$ i.e. any two distinct elements of B are orthogonal.

- Often authors write 'ON' as abbreviation for the word 'orthonormal' and so you may hear the term 'ON basis'!
- Let $\mathcal{H} = \mathbb{C}^2$ and $B = \left\{ |0\rangle, |1\rangle \right\}$. Then B is an orthonormal basis in $\mathbb{C}^2$ right?
- Are there more orthonormal bases in $\mathbb{C}^2$ other than B ? As homework challenge, you are encouraged to find them!
- Since a Hilbert space has more than just the linear structure, the actual underlying bases are technically known as 'Schauder' bases as oppose to 'Hamel' bases!

Separable Hilbert Space

Definition (1.5)

A Hilbert space $\mathcal{H}$ is said to be separable if it has a countable ON basis $B = \{e_1, e_2, \ldots, e_n\}$.

- Hence, it's trivial to prove that $\mathbb{C}^n$ is a separable Hilbert space!
- It can be proved that if $B = \{e_1, e_2, \ldots, e_n\}$ is an ON basis in $\mathcal{H}$, then the following are equivalent:

1. $\psi = \sum_{i=1}^{n} \langle e_i, \psi \rangle e_i$ for all $\psi \in \mathcal{H}$.
2. $\langle e_i, \psi \rangle = 0$ for all i iff $\psi = 0_{\mathcal{H}}$.
3. $\|\psi\|^2 = \sum_{i=1}^{n} \|\langle e_i, \psi \rangle\|^2$ for all $\psi \in \mathcal{H}$.

- This marks the end of the axiom 1 jargons that we needed to decode right:

1. Normalised vectors
2. Separable Hilbert space

- As homework challenge, you are encouraged to research why separable Hilbert spaces are required for the quantum formalism.
- Interestingly, just like any finite dimensional Hilbert space is isomorphic to $\mathbb{C}^n$. It turns out that any infinite dimensional separable complex Hilbert space is unitarily equivalent to $l^2(\mathbb{N})$!

PART B

Linear Functionals Induced by the Inner Product

Definition (1.6)

For any $\psi \in \mathcal{H}$ we can construct a map $L_\psi : \mathcal{H} \longrightarrow \mathbb{C}$ induced by the inner product as $L_\psi \Phi = \langle \psi, \Phi \rangle$ for all $\Phi \in \mathcal{H}$.

Proposition (1.1)

For all $\psi \in \mathcal{H}$, $L_\psi : \mathcal{H} \longrightarrow \mathbb{C}$ is a linear map (aka linear functional).

Proof : Homework challenge?

- Physicists normally write $\langle \psi |$ to denote L_ψ and call it 'bra'!
- Physicists also use the Dirac notation e.g. place Φ inside the ket to get $|\Phi\rangle$. Then the action of the bra $\langle \psi |$ on the ket $|\Phi\rangle$ becomes $\langle \psi || \Phi \rangle$. This is the standard inner product notation for physicists i.e. they replace $\langle \psi, \Phi \rangle$ with $\langle \psi || \Phi \rangle$ which they further abbreviate as $\langle \psi | \Phi \rangle$!

- We know that given an ON basis $B = \{e_1, e_2, \ldots, e_n\}$ in $\mathcal{H}$, then $\psi = \sum_{i=1}^{n} \langle e_i, \psi \rangle e_i$ for all $\psi \in \mathcal{H}$. Using the Dirac notation, the expression becomes: $|\psi\rangle = \sum_{i=1}^{n} \langle e_i | \psi \rangle |e_i\rangle$
- Which of the above expressions is cleaner for you?

- What is the Kernel of $\langle \psi |$? It's a linear subspace of $\mathcal{H}$ of course. But do you notice anything interesting about Ker($\langle \psi |$)?

Concrete Representation of Bras

- If $\mathcal{H} = \mathbb{C}^n$ and $|\psi\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then the concrete representation of its bra becomes the row matrix $\langle\psi| = \begin{pmatrix} x_1^* & x_2^* & \dots & x_n^* \end{pmatrix}$.
- The action $\langle\psi||\psi\rangle$ is then just the matrix multiplication i.e. multiplication of the row matrix (bra) with the column matrix (ket).

The Dual Space

Definition (1.7)

The set of all linear functionals $L : \mathcal{H} \to \mathbb{C}$ is called the dual space of $\mathcal{H}$ and it's usually denoted $\mathcal{H}^*$.

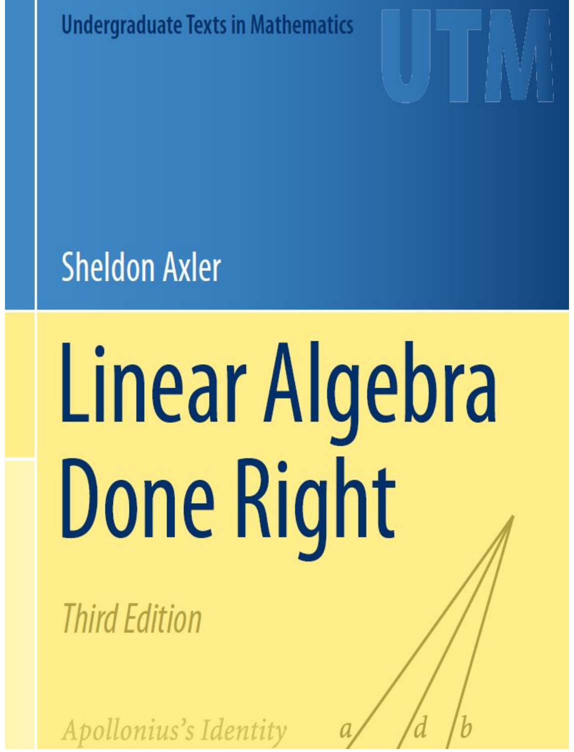
Check out the previous session's study material recommendation on dual spaces: **3F: Duality (pages 101 - 113)**.

Theorem (Riesz Representation)

Let $\mathcal{H}$ be a Hilbert space and $L \in \mathcal{H}^*$. Then there exists a unique $\psi \in \mathcal{H}$ such that $L = \langle \psi, \cdot \rangle$ i.e. $L\Phi = \langle \psi, \Phi \rangle$ for all $\Phi \in \mathcal{H}$.

Proof : See study material (188)!

- The theorem above is very handy as it establishes a linear isomorphism $\mathcal{H}^* \simeq \mathcal{H}$ and so $\dim(\mathcal{H}^*) = \dim(\mathcal{H})$!



Prof. Sheldon Axler

Where should you focus?

6.B: Orthonormal Bases (180 - 192)

6.C: Orthogonal Complements (*193 - 202)*

QF

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- **GitHub (Curated study materials):** *github.com/quantumformalism*
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QF

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