QF Group Theory CC2022
By
Zaiku Group

Lecture 14

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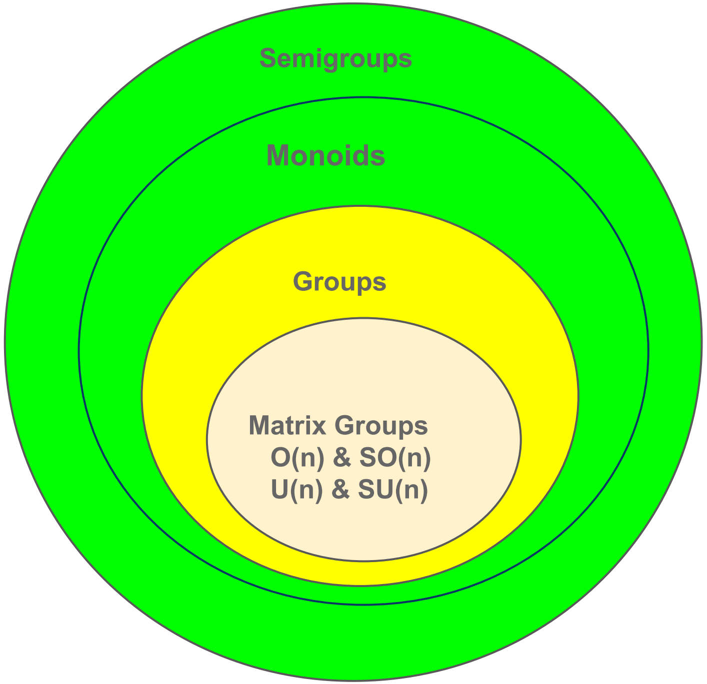
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
14/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed! We're here!

Left and Right Cosets

Definition 1.0

Let G be a group and H be a subgroup of G. Then for any element $g \in G$, we define the left $H-$ coset of generated by g as $gH = \{gh \mid h \in H\}$.

- It's easy to see that $g \in gH$ right?
- We write $|gH|$ to denote the cardinality of gH.

Definition 1.1

Let G be a group and H be a subgroup of G. Then for any element $g \in G$, we define the right $H-$ coset of generated by g as $Hg = \{hg \mid h \in H\}$.

- It's also easy to see that $g \in Hg$ right?
- We analogously write $|Hg|$ to denote the cardinality Hg.

Natural questions: Are gH and Hg necessarily subgroups of G? If no, under what circumstances are they subgroups of G?

Side note: The notion of coset was invented by Galois! But the terminology introduced by American group theorist George Abram Miller.

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Examples of Cosets (A)

Left Cosets

Suppose $G = S_3$ and $H = \{(1), (13)\}$. Now consider the cycles (1), (12) and (13). Then we'll get the following left cosets:

1. $(1)H = H$
2. $(12)H = \{(12), (132)\}$
3. $(13)H = \{(13), (1)\}$

Right Cosets (Challenge 1)

Suppose again $G = S_3$ and $H = \{(1), (13)\}$. Let consider the same cycles (1), (12) and (13). You're challenged to compute their corresponding right cosets below!

1. $H(1)$.
2. $H(12)$.
3. $H(13)$.

Examples of Cosets (B)

Left Cosets

Let $G = \mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under mod 9 addition and $H = \{0, 3, 6\}$. Now, because we're dealing with addition, it's more convenient to write a left coset as $g + H$ instead of gH! We'll then have the following left cosets:

1. $3 + H = H$.
2. $1 + H = \{1, 4, 7\}$.
3. $2 + H = \{2, 5, 8\}$.

Left Cosets (Challenge 2)

Let us again set $G = \mathbb{Z}_9$ and $H = \{0, 3, 6\}$ as above. You're encouraged to compute the following left cosets:

1. $4 + H$.
2. $5 + H$.
3. $6 + H$.
4. $7 + H$.
5. $8 + H$.

- After the examples above, do you notice anything interesting compared to the previous examples?

Facts about cosets

Theorem 1.0

Let G be a group and H a subgroup of G. Then the following properties hold:

1. $g \in gH$ for all $g \in G$.
2. $gH = H$ iff $g \in H$ for all $g \in G$.
3. $g_1H = g_2H$ iff $g_1^{-1}g_2 \in g_2H$ for all $g_1, g_2 \in G$.
4. Either $g_1H = g_2H$ or $g_1H \cap g_2H = \emptyset$ for all $g_1, g_2 \in G$.
5. $|g_1H| = |g_2H|$ for all $g_1, g_2 \in G$.
6. gH is a subgroup of G iff $g \in H$.

Proof : As a challenge, try to prove some of the properties by yourself!
Natural question 3: Do the properties above hold for right cosets too?

The index of a subgroup

Definition 1.2

Let G be a group and H be a subgroup of G. Then the index of H in G denoted $[G : H]$ is the number of left cosets i.e. $[G : H]$ is the cardinality of the $\{gH \mid g \in G\}$.

- As you'll see in the next couple slides, $\{gH \mid g \in G\}$ is a very special set for certain subgroups!
- Alternative notation for the index is $(G : H)$ or sometimes $|G : H|$!
- It's obvious that if $H = \{1_G\}$, where 1_G is the group identity, then $[G : H] = |G|$ i.e. the index of the trivial subgroup coincides with the order of the group G!

Theorem 1.1 (Lagrange)

Let G be a finite group and H a subgroup of G. Then $|G| = [G : H]|H|$.

Curiosity question (homework): Does the the theorem above holds if G is infinite?

Examples of subgroup index

- Consider $G = \mathbb{Z}$ the group of the integers under ordinary addition. If $H = 2\mathbb{Z}$ then $[\mathbb{Z} : 2\mathbb{Z}] = 2$ right? Also, does $[\mathbb{Z} : n\mathbb{Z}] = n$ hold for any positive integer n?
- Let now $G = \mathbb{R}$ the group of the reals under ordinary addition. If we make If $H = 2\mathbb{Z}$, what is $[\mathbb{R} : n\mathbb{Z}]$? What if $H = \mathbb{Z}$?

Normal subgroups

Definition 1.3

Let G be a group and H be a subgroup of G. We say H is a normal subgroup if for any $h \in H$ we have $ghg^{-1} \in H$ for all $g \in G$.

- We write $H \lhd G$ to denote H is a normal subgroup of G.
- It's obvious that if G is abelian, then all the subgroups of G are normal. So things are more interesting when G is nonabelian!

Theorem 1.2

Let $\phi : G_1 \longrightarrow G_2$ be a group homomorphism. Then $Ker(\phi) \lhd G_1$.

Proof : Homework challenge!

- Hence, group homomorphisms are a great source of normal subgroups even if the underlying groups are nonabelian!

Challenge 3

Is it true $H \lhd G$ iff $gH = Hg$ for all $g \in G$?

Examples of Normal Subgroups

- Let A_n be the subgroup of even permutations of S_n (aka Alternating group) encountered in lecture 12. A_n is a normal subgroup of S_n!
  An easy way to see A_n is normal is to recall that $A_n = Ker(sign)$ where $sign : S_n \longrightarrow \{1, -1\}$ is a group homomorphism. The group operation in $\{1, -1\}$ is the ordinary multiplication.
  Also, recall that for any permutation $\sigma \in S_n$, the homomorphism $sign$ is defined as follows:

$$sign(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

- Let G be a nonabelian group and $Z(G)$ be the center of G i.e. $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Then $Z(G)$ is normal!

Normal subgroups challenge

Challenge 5

Let G be a group with normal subgroups $H_1 \lhd G$ and $H_2 \lhd G$. Is it true $H_1 \cap H_2 \lhd G$ i.e. is the intersection of two normal subgroups normal?

- What about $H_1 \cup H_2$?

Challenge 6

Let $G = S_3$ and $H = \{(1), (123), (132)\}$. Is $H \lhd G$? Also, what is the index $[G : H]$?

Quotient groups

Definition 1.4

Let G be a group and H a normal subgroup of G. We define the set $G/H = \{gH \mid g \in G\}$ i.e. G/H is the set of all $H-$ left cosets.

- Now, we can use the group operation in G to define a binary operation in G/H as follows, for $g_1 H, g_2 H$ we define $(g_1 H)(g_2 H) = (g_1 g_2)H$ where obviously $g_1 g_2$ is the operation in G.

Proposition 1.0

Under the binary operation above, G/H is a group with identity $1_G H$ and for each coset $gH \in G/H$, the group inverse is $g^{-1}H$.

Proof : Homework challenge!

- As you can guess, G/H is called quotient group! Another alternative name is factor group.

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