$$\gamma : [0, 1] \longrightarrow X$$

Topology Crash Course - Lecture 04

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Session Agenda

Pre-lecture
blah blah blah

PART A

1. Topological Bases
2. Product Topology
3. Projection Maps

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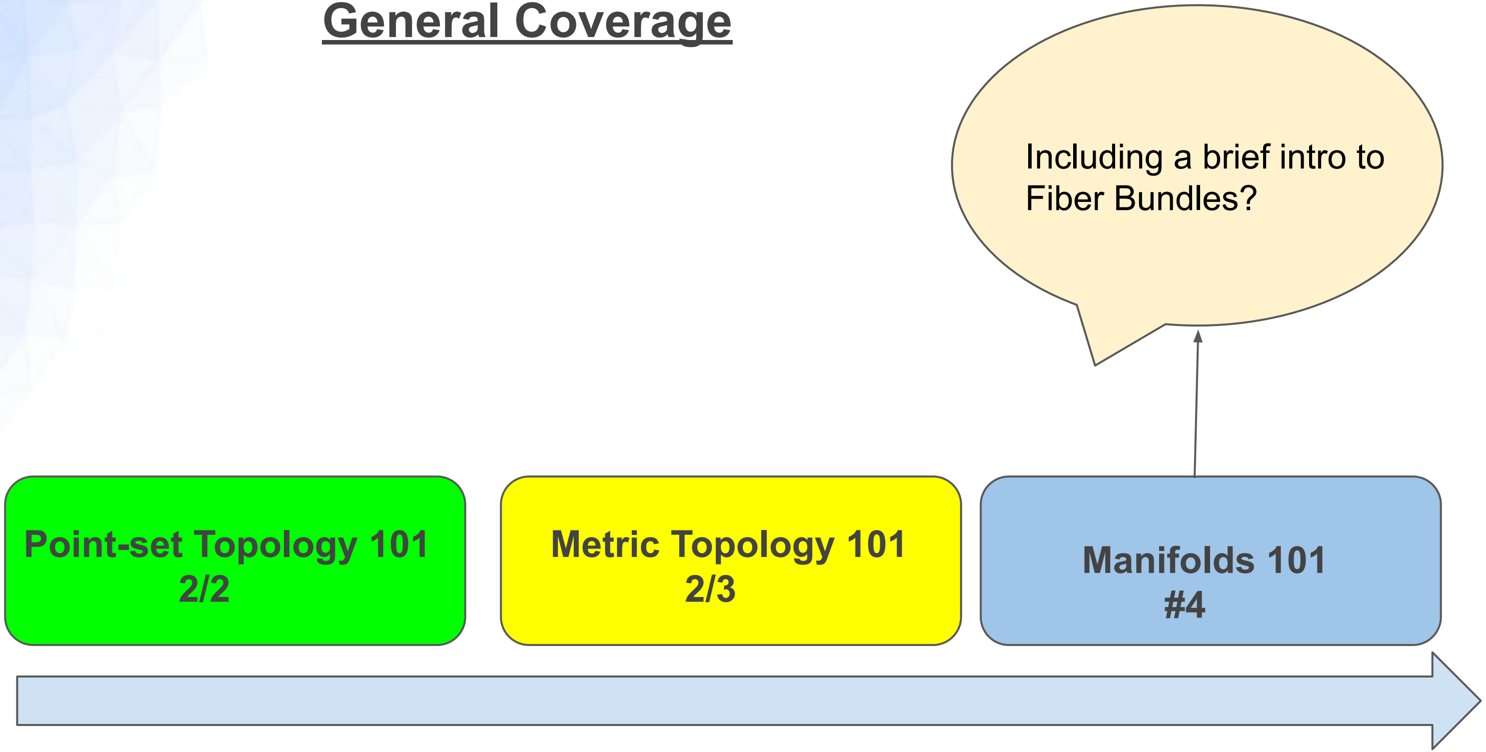
PART B

1. Topological Invariants
2. Connected Spaces
3. Connected subsets
4. Connectedness Challenge
5. **Lecture Gap Poll**

Lecture 4 Coverage

General Coverage



#n is the number of live lectures | Completed | Ongoing

PART A

Topological Bases

Definition (1.0)

Let $(X, \mathcal{T})$ be a topological space. A collection of open sets $\mathcal{B} \subseteq \mathcal{T}$ is called a basis (or just base) if for any open set $O \in \mathcal{T}$ there is a $\mathcal{B}' = \{B_i\}_{i \in I} \subseteq \mathcal{B}$ such that $O = \bigcup_{i \in I} B_i$.

- $\mathcal{T}$ itself a basis right? For any open set $O \in \mathcal{T}$ we can make $\mathcal{B}' = \{\emptyset, O\}$ and so $O = \emptyset \cup O$?
- If $X = \mathbb{R}^2$ with the standard Topology $\mathcal{T}_d$ given by the Euclidean metric d. Then $\mathcal{B} = \{B_r(p) \mid p \in \mathbb{R}^2, r > 0\}$ forms a basis.
- In general, if X is a space induced by a metric topology $\mathcal{T}_d$. Then the collection of all the open balls $\mathcal{B} = \{B_r(p) \mid p \in X, r > 0\}$ indeed forms a basis for the metric topology. Another reminder that open balls are very important when it comes to the metric topology!

Comments

- In principle, a basis $\mathcal{B}$ can be an uncountable set. But when $\mathcal{B}$ is countable, then the topological space $(X, \mathcal{T})$ is said to be 'second-countable'. This is a very important assumption that is normally used with Hausdorfness in order to build additional structures such as manifolds on top of topological spaces.

- With vector spaces, the notion of basis leads us to the notion of 'dimension'. But can we similarly define the notion of dimension for topological spaces?

Basis Challenge

- If $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\emptyset, X, \{\beta_1\}, \{\beta_2\}, \{\beta_1, \beta_2\}\}$. Which of the following (if any) forms a basis:
  1. $\mathcal{B}_1 = \{X, \{\beta_1\}, \{\beta_2\}\}$
  2. $\mathcal{B}_2 = \{\emptyset, X, \{\beta_1\}, \{\beta_2\}\}$
  3. $\mathcal{B}_3 = \{\emptyset, X, \{\beta_1\}\}$
  4. $\mathcal{B}_3 = \{\emptyset, X, \{\beta_2\}\}$
- Consider the standard topology on $\mathbb{R}$. Is it true that the product topology $\mathbb{R} \times \mathbb{R}$ is the same as the standard topology on $\mathbb{R}^2$?
- If the above is true, does it also generalise to $\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}$ and $\mathbb{R}^n$ for all $n > 2$?
- Let $\mathcal{B}$ be a basis for the space $(X, \mathcal{T})$ and let us define the collection $\mathcal{T}_{\mathcal{B}} = \{\bigcup O_i \mid O_i \in \mathcal{B}\}$. A mini challenge for you:
  1. Verify that the collection $\mathcal{T}_{\mathcal{B}}$ is indeed a topology on X!
  2. Are the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}_{\mathcal{B}})$ necessarily homeomorphic?

Note: $\mathcal{T}_{\mathcal{B}}$ is called the topology generated by $\mathcal{B}$ or basis topology.

Remark

- Given two topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$, we'll write $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ to denote the Cartesian product of the two underlying sets.

Product Topology

Definition (1.1)

A subset $P \subseteq X \times Y$ is open if for any $(x, y) \in P$ there are two open sets $O_1 \in \mathcal{T}_1$ and $O_2 \in \mathcal{T}_2$ such that:

1. $x \in O_1$ and $y \in O_2$.
2. $O_1 \times O_2 \subseteq P$.

- As a challenge, try generalise the above definition for an arbitrary number of topological spaces e.g. for $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \ldots, (X_k, \mathcal{T}_k)$ for $k > 2$.

Proposition (1.0)

Given two topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$, the collection of subsets $\mathcal{T}_1 \times \mathcal{T}_2 = \{P \subseteq X \times Y \mid P \text{ is open in } X \times Y\}$ is a topology on $X \times Y$.

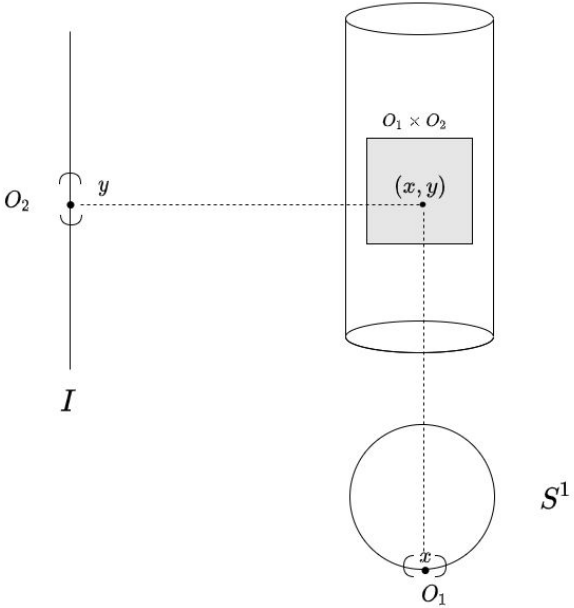
- Is it true that if X and Y are Hausdorff spaces then $X \times Y$ is also Hausdorff?

- Let $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ be a product topological space. Does the collection $\mathcal{B} = \{O_1 \times O_2 \mid O_1 \in \mathcal{T}_1 \text{ and } O_2 \in \mathcal{T}_2\}$ form a basis?

Product Examples

- Consider the Cylinder $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ with $0 \leq z \leq 1$. C can become topological space if we equip it with the subspace topology from the standard topology on $\mathbb{R}^3$.

  We can also realise C as product topology between the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and the unit interval $I = [0, 1]$ i.e. $C = S^1 \times I$.



- Indeed many important topological spaces are constructed via the product topology. Some other familiar examples include:
  1. **Unit Square**: $[0, 1] \times [0, 1]$
  2. **Torus**: $S^1 \times S^1$
- As a challenge, you are encouraged to play around the products of some subspaces of the standard topologies on $\mathbb{R}^n$ for $n \in \{2, 3, 4\}$.

Projection Maps

Definition (1.2)

Let $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ be a product topological space. We can define the following maps:

1. $\pi_X : X \times Y \longrightarrow X$ as $\pi_X(x, y) = x$ for all $(x, y) \in X \times Y$.
2. $\pi_Y : X \times Y \longrightarrow Y$ as $\pi_Y(x, y) = y$ for all $(x, y) \in X \times Y$.

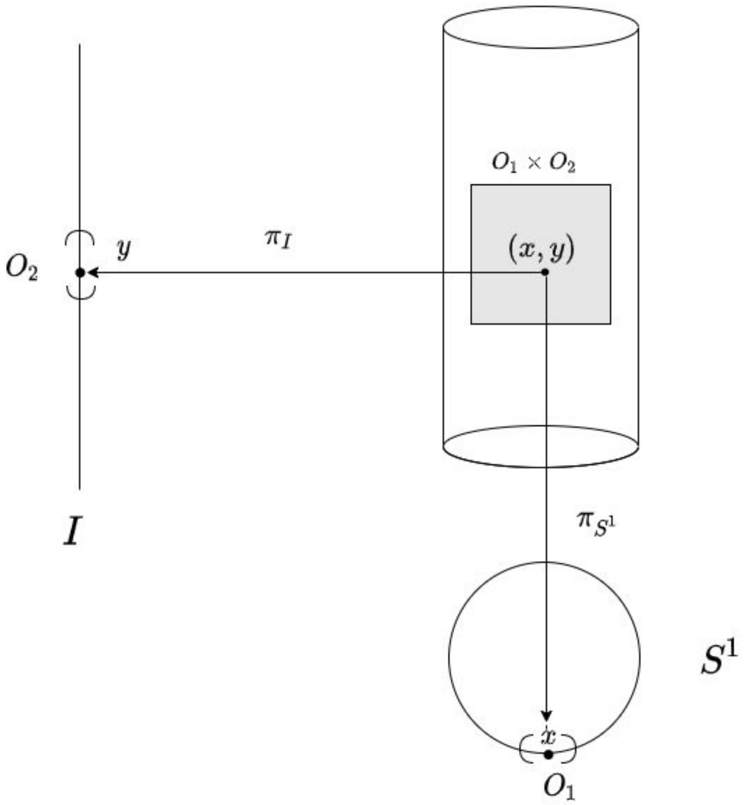
- π_X and π_Y are called projection maps. You really need to feel comfortable with product topology and projection maps in order to not struggle with the basic abstraction of Fibre Bundles!

Proposition (1.1)

The projection maps $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are continuous maps.

Proof : Homework challenge!

Projection Examples



PART B

Topological Invariants

Definition (1.3)

A property P of a topological space $(X, \mathcal{T})$ is said to be a topological invariant (or topological property) if whenever $(X, \mathcal{T})$ is homeomorphic to another topological space $(Y, \mathcal{T}')$, then the topological space $(Y, \mathcal{T}')$ also has property P.

- In Linear Algebra, the dimension n of a finite dimensional vector space V is an invariant for V i.e. if $V \simeq V'$ then $\dim(V') = n$. Hence, the notion of dimension helps classify finite-dimensional vector spaces.
- Topological invariants also help classify topological spaces. This is important because very often it is much easier to prove existence of a topological invariant than checking whether there is exists a homemeomorphism between two topological spaces X and Y!

- Is Hausdorffness a topological property? What about metrizability?

Naked Eye Invariant Test

- In lecture 2, we concluded that the donut and the cup must be Homeomorphic by noticing with our naked eyes that the two shapes can be continuosly deformed from one into another without cutting gluing. Hence, for Topologists, donuts and cups are the same thing!

- Taking into the above point, can you see a potential topological invariant present in each object with your eyes?

Connected Spaces

Definition (1.4)

A topological space $(X, \mathcal{T})$ is connected if X cannot be written as the union $X = O_1 \cup O_2$ where $O_1, O_2 \in \mathcal{T}$ such that $O_1 \cap O_2 = \emptyset$ and $O_1, O_2 \neq \emptyset$.

- Another way to say $(X, \mathcal{T})$ is connected, is that X and $\emptyset$ are the only subsets of X that are both open and closed i.e. clopen!
- As you can guess, $(X, \mathcal{T})$ is not connected (disconnected) if X can be written as the union $X = O_1 \cup O_2$ where $O_1, O_2 \in \mathcal{T}$ such that $O_1 \cap O_2 = \emptyset$ and $O_1, O_2 \neq \emptyset$.
- Connectedness is a very important notion in topology as many examples of topological spaces that we encounter in applied topics such as physics are connected!
- Is connectedness a topological invariant i.e. if X is Connected and X is homeomorphic to Y then Y is also connected?

Connected Subsets

Definition (1.5)

Given a topological space $(X, \mathcal{T})$, we say that a subset $A \subseteq X$ is connected in X if its induced subspace topology $(A, \mathcal{T}_A)$ is connected.

- Is $[0, \frac{1}{2}]$ as subset of $\mathbb{R}$ with the standard topology connected?

Proposition (1.2)

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If X is connected and $f : X \longrightarrow Y$ is a continuous map, then $f(X)$ is connected in Y.

Proof : Homework challenge!

- Did the proposition above convince that connectedness is indeed a topological invariant?

Proposition (1.3)

If A and B are two non-disjoint (i.e. $A \cap B \neq \emptyset$) connected subsets of X, then their union $A \cup B$ is connected.

Proof : Homework challenge!

- Can you generalise the proposition above to an arbitrary number of non-disjoint connected subsets e.g. $A_1, A_2, \ldots A_k$?

- Under what condition(s) can we have $A \cap B$ is connected?

Connectedness Challenge

- Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two connected topological spaces. Prove that the product space $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ is connected.
- Let $X = (0, 1)$ with the subspace topology from the standard topology on $\mathbb{R}$. Is X a connected topological space?
- What if $X = [0, 1]$ with the subspace topology from the standard topology on $\mathbb{R}$?
- Which of the following real line intervals equipped with the subspace topology from the standard topology on $\mathbb{R}$ are connected?
  1. $[a, b]$
  2. $[a, b)$
  3. (a, b)
- Is the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ equipped with the standard subspace topology on $\mathbb{R}^3$ connected?
- Is $\mathbb{R}$ with the standard topology connected? Or even better, $\mathbb{R}^n$ for $n > 1$?

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