QUANTUM FORMALISM

Matrix Algebra - Part 2

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Lecture 08 Recap
3. Matrix Transpose
4. Symmetric Matrices
5. Skew Symmetric Matrices
6. Matrix Trace
7. Study Materials Comments
8. Double session poll

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Matrix Operators #2
Group Theory 101 #1	Linear Operators 101 #2	Complex Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Mat. Groups: GL(2, C) & U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

Lecture 08 Recap

Invertible Matrices

Definition (1.1)

A matrix $A \in M_n(\mathbb{C})$ is invertible if there exists $A^{-1} \in M_n(\mathbb{C})$ such that $AA^{-1} = A^{-1}A = \mathbb{I}$.

- In mathematics literature $A \in M_n(\mathbb{C})$ is called singular matrix if it's not invertible and non-singular matrix if it's invertible!

Proposition (1.0)

If $A, B \in M_n(\mathbb{C})$ are invertible, then the following statements are true:

1. AB is also invertible.
2. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : Homework challenge?

- Hence, the set of all invertible matrices in $M_n(\mathbb{C})$ denoted $GL(n, \mathbb{C})$ is a group!
- Please note that some authors use the notation $GL_n(\mathbb{C})$.

Diagonal Matrices

Definition (1.2)

$A \in M_n(\mathbb{C})$ is called a diagonal matrix if it has the following form:

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \text{ such that } a_{ij} = 0 \text{ if } i \neq j.$$

- The following elements of $M_2(\mathbb{C})$ are examples of diagonal matrices: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

Proposition (1.1)

If $A, B \in M_n(\mathbb{C})$ are diagonal, then AB is also diagonal and $AB = BA$.

Proof : Homework challenge?

Invertible Diagonal Matrices

Proposition (1.2)

Let $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. The following statements are true:

1. D is invertible iff $\lambda_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$ i.e if all the diagonal elements are nonzero.
2. $$D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{pmatrix}.$$

Proof : Homework challenge?

Lecture 09

Matrix Transpose

Definition (1.0)

For $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C})$, A^T is defined as:

$$A^T = [a_{ji}] = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

- $M_2(\mathbb{C})$ examples:

For $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$$X^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, Z^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Transpose Properties

Proposition (1.0)

Let $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then the following properties hold:

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$.
3. $(\lambda A)^T = \lambda A^T$.
4. $(AB)^T = B^T A^T$.

Proof : Homework challenge?

- Recall that the commutator of two matrices A and B is defined as $[A, B] = AB - BA$ or more algebraically correct $[A, B] = AB + (-BA)$.

- As hands on home challenge, try play with following:
  1. Compute $([X, Y])^T$, $[X, Y^T]$, $[Y^T, X]$, $[H, Y^T]$ and other combinations of your choice.
  2. Apply the results of the above computation to $|0\rangle$ and $|1\rangle$.

Symmetric Matrices

Definition (1.1)

We say a matrix $A \in M_n(\mathbb{C})$ is symmetric if $A^T = A$.

- Hence, it's easy to see that every diagonal matrix is symmetric.

Theorem (1.0)

Let $A, B \in M_n(\mathbb{C})$ be symmetric and $\lambda \in \mathbb{C}$. Then the following is true:

1. If A is invertible then A^{-1} is symmetric.
2. $A + B$ is symmetric.
3. λA is symmetric.
4. If AB is symmetric then $[A, B] = 0$ i.e. $AB = BA$.

Proof : Homework challenge?

- Which of the following $M_2(\mathbb{C})$ matrices are symmetric?

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Skew Symmetric Matrices

Definition (1.2)

$A \in M_n(\mathbb{C})$ is skew symmetric (or antisymmetric) if $A^T = -A$.

Theorem (1.1)

Let $A, B \in M_n(\mathbb{C})$ be skew symmetric and $\lambda \in \mathbb{C}$. Then the following is true:

1. If A is invertible then A^{-1} is skew symmetric.
2. $A + B$ and $[A, B]$ are skew symmetric.
3. A^T and λA are skew symmetric.

- Is any of the famous $M_2(\mathbb{C})$ matrices below skew symmetric?

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

- In the advanced module we'll see that skew symmetric matrices form a Lie Algebra!

Matrix Trace

Definition (1.3)

For $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C})$, the trace of A is defined as:

$Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$.

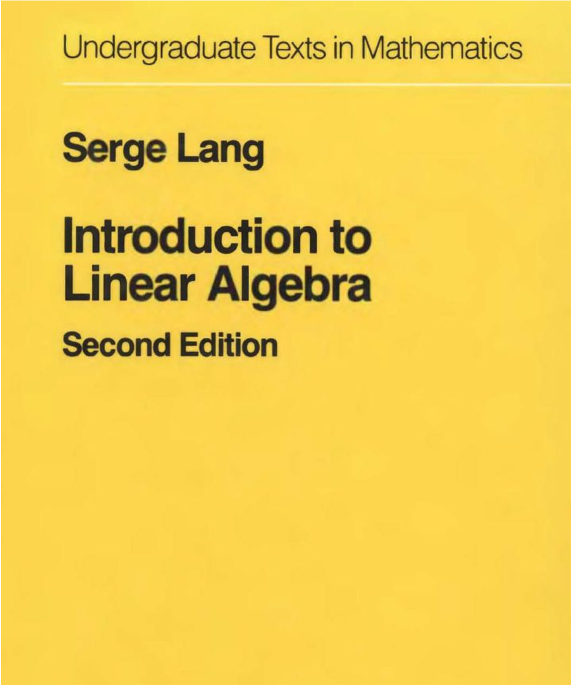
Proposition (1.1)

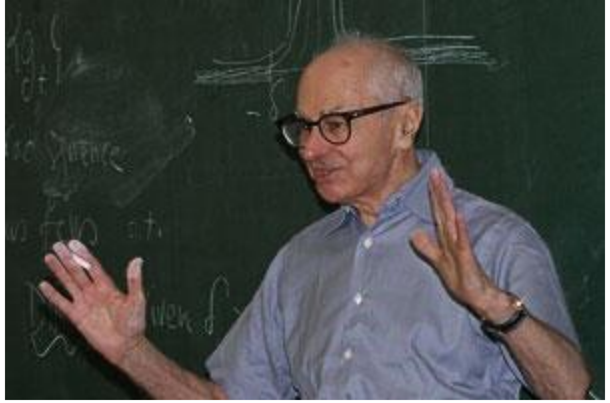
Let $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then the following properties hold:

1. $Tr(A^T) = Tr(A)$.
2. $Tr(A + B) = Tr(A) + Tr(B)$.
3. $Tr(\lambda A) = \lambda Tr(A)$.
4. $Tr(AB) = Tr(BA)$.

- In respect to the trace, can you notice anything interesting in common with the following matrices?

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.





Serge Lang (RIP)

Where should you focus?

Chapter II Matrices and Linear Equations *(Pages 42 - 85)*

QUANTUM FORMALISM

- **GitHub (Curated study materials):** *github.com/quantumformalism*
- **YouTube:** youtube.com/zaikugroup
- **Twitter:** *@ZaikuGroup*
- ***Gitter:*** *gitter.im/quantumformalism/community*