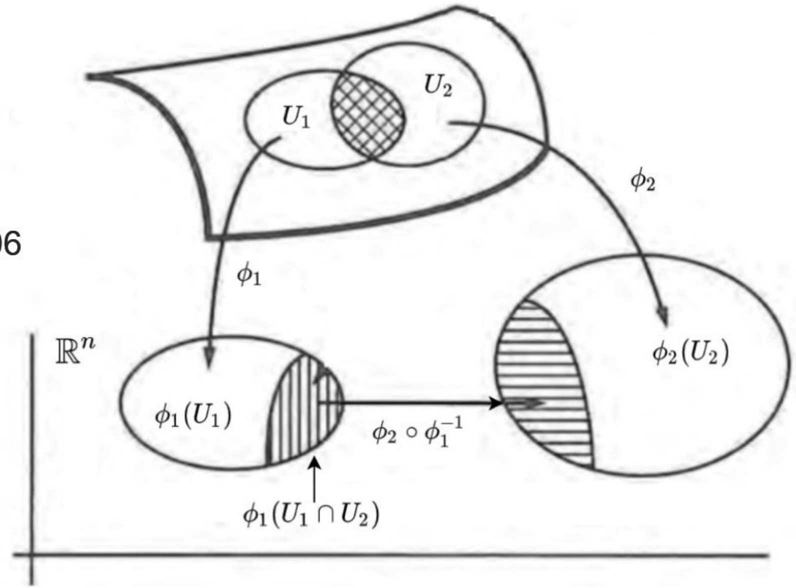
Topology Crash Course: Lecture 06



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Session Agenda

Pre-lecture blah blah blah

1. Paracompactness Hack/Shortcut
2. Abstraction Comments
3. N-dimensional Charts
4. Simple Chart Example
5. Chart Challenge
6. Charts Around Points
7. Topological Manifolds
8. Topological Manifolds Challenge

PART A

+

1. Refresh Challenge (Map Differentiability Classes)
2. **Study Materials**
3. **Lecture Gap Poll**
4. **Informal Manifold Joke**

PART B

Lecture 6 Coverage

General Coverage

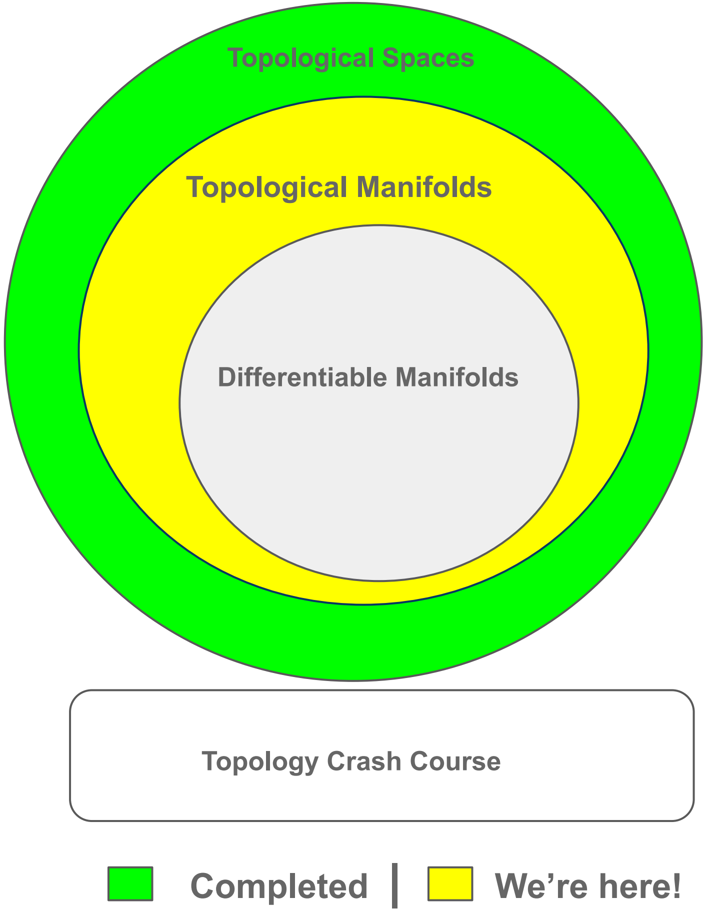
Including a brief intro to Fiber Bundles?

Point-set Topology 101
2/2

Metric Topology 101
3/3

Manifolds 101
1/4

#n is the number of live lectures | Completed | We're here!


Topological Spaces
Topological Manifolds
Differentiable Manifolds
Topology Crash Course
Completed
We're here!

Crash Course Motivation

Lie Groups, Lie Algebras & Representations

Module II (January 2022 start date?)

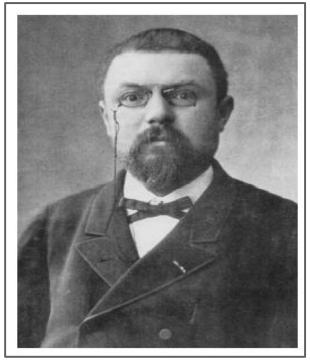
Lie group

From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.

Bernhard Riemann



Henri Poincaré

The Genesis of Manifolds

Preview

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 17 October 15, 1931 Number 10

A SET OF AXIOMS FOR DIFFERENTIAL GEOMETRY

By O. Veblen and J. H. C. Whitehead

Department of Mathematics, Princeton University

Communicated September 9, 1931

1. *Introductory.*—The axioms set forth in this note are intended to describe the class of manifolds of n dimensions to which the theories nowadays grouped together under the heading of differential geometry are applicable. The manifolds are classes of elements called points, having a structure which is characterized by means of coördinate systems. A *coördinate system* is a (1–1) correspondence, $P \longrightarrow x$, between a set of points, $[P]$, of the manifold, and a set, $[x]$, of ordered sets of n real numbers, $x = (x^1, \ldots, x^n)$. For convenience we call any ordered set of n real numbers an arithmetic point and the totality of arithmetic points, for a fixed n, the arithmetic space of n dimensions. Each point P which corresponds in a coördinate system, $P \longrightarrow x$, to an arithmetic point x, is said to be represented by x. The set, $[P]$, of all points represented in a given coördinate system is called the *domain* of the coördinate system, and the set, $[x]$, of the arithmetic points which represent them is called its *arithmetic domain*.

If $P \longrightarrow x$ and $P \longrightarrow y$ are two coördinate systems having the same

JOURNAL ARTICLE

A Set of Axioms for Differential Geometry

O. Veblen and J. H. C. Whitehead

Proceedings of the National Academy of Sciences of the United States of America
Vol. 17, No. 10 (Oct. 15, 1931), pp. 551-561 (11 pages)
Published By: National Academy of Sciences

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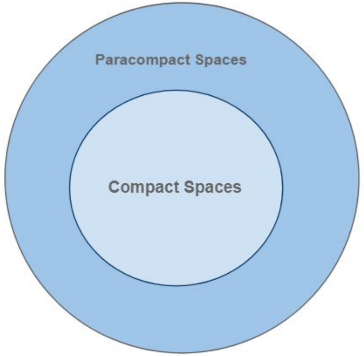
First Rigorous Abstract Definition of Manifolds

PART A

Paracompactness Hack

- It is very often a convention to build the notion of topological manifold on top of paracompact topological spaces i.e. to assume that the underlying topological space $(X, \mathcal{T})$ is paracompact as prerequisite for the definition of manifold. However, paracompactness is a weaker notion than compactness i.e. every compact space is paracompact.



- For our purposes (Module II), we can get away without directly defining paracompactness by assuming our topological spaces to be:
  1. Hausdorff
  2. Second-countable i.e. having countable basis
- The two conditions above essentially guarantee us metrizability on X i.e. X can be equipped with a metric (distance).
- Hence, from now on, automatically assume our topological spaces to satisfy the above.

Preview

A NOTE ON PARACOMPACT SPACES

ERNEST MICHAEL[1]

1. Introduction. The purpose of this paper is to prove the following theorem, which asserts that for regular topological spaces paracompactness is equivalent to an apparently weaker property, and derive some of its consequences.

THEOREM 1. *Let X be a regular topological space. Then X is paracompact if and only if*

(*) *every open covering of X has an open refinement $\mathcal{V}=\bigcup_{i=1}^{\infty}\mathcal{V}_i$, where each $\mathcal{V}_i$ is a locally finite collection of open subsets of X.*

Let us quickly recall the definitions of the terms which are used in the statement of Theorem 1, and which will be used throughout this paper. Let X be a topological space. A collection $\mathcal{R}$ of subsets of X is called *open* (resp. *closed*) if every element of $\mathcal{R}$ is open (resp. closed) in X. A *covering* of X is a collection of subsets of X whose union is X; observe that in this paper a covering need *not* be open. If $\mathcal{R}$ is a covering of X, then by a *refinement* of $\mathcal{R}$ we mean a covering $\mathcal{V}$ of X such that every element of $\mathcal{V}$ is a subset of some element of $\mathcal{R}$. A collection $\mathcal{R}$ of subsets of X is *locally finite* if every $x\in X$ has a neighborhood which intersects only finitely many elements of $\mathcal{R}$. Finally, X is *paracompact* [**3**, p. 66] if it is Hausdorff, and if every open covering of X has an open, locally finite refinement. (Metric spaces and compact Hausdorff spaces are paracompact (cf. [**11**] and [**3**]),

JOURNAL ARTICLE

A Note on Paracompact Spaces

Ernest Michael

Proceedings of the American Mathematical Society
Vol. 4, No. 5 (Oct., 1953), pp. 831-838 (8 pages)
Published By: American Mathematical Society

https://doi.org/10.2307/2032419
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Abstraction Comments

‘’Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.’’

Johann Wolfgang von Goethe

N-dimensional Charts

Definition (1.0)

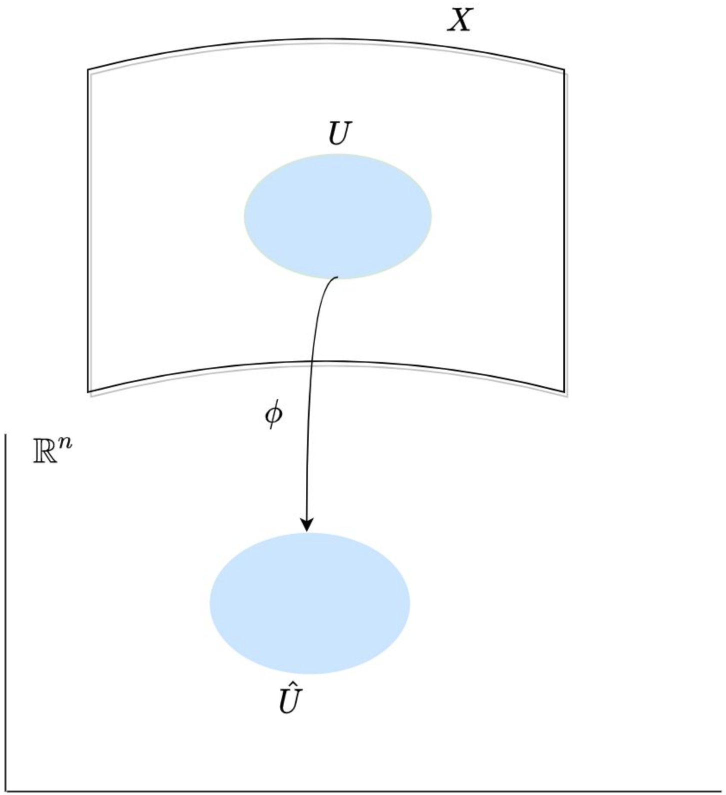
Let $(X, \mathcal{T})$ be a topological space. An $n-$ dimensional coordinate chart on X is a triple $(U, \hat{U}, \phi)$ satisfying the following conditions:

1. $U \subseteq X$ is a non empty open set i.e $U \in \mathcal{T}$ and $U \neq \emptyset$.
2. $\hat{U} \subseteq \mathbb{R}^n$ is an open set in respect to the standard topology on $\mathbb{R}^n$.
3. $\phi : U \longrightarrow \hat{U}$ is a homeomorphism i.e. a homeomorphism in respect to the subspace topologies structures of U in X and $\hat{U}$ in $\mathbb{R}^n$ respectively.

- The open set U is called the coordinate domain of the chart and ϕ is called the chart map. Guess what $\hat{U}$ is normally called? Or even better, what intuitive name would you give it?

Simple Diagram

- Here is a diagrammatic picture of an $n-$ dimensional chart on X:



A Simple Chart Example

- Let $X = \mathbb{R}^n$ be equipped with the standard topology and $\hat{U} = \mathbb{R}^n$ also obviously equipped with the standard topology. Then $(\mathbb{R}^n, \mathbb{R}^n, id_{\mathbb{R}^n})$ is an $n-$ dimensional chart where $id_{\mathbb{R}^n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map on $\mathbb{R}^n$ i.e. $id_{\mathbb{R}^n}(p) = p$ for all $p \in \mathbb{R}^n$.

Note: In the literature, $(\mathbb{R}^n, \mathbb{R}^n, id_{\mathbb{R}^n})$ is known as 'the standard chart on $\mathbb{R}^n$'.

- Let U be an open subset of $\mathbb{R}^n$ and $\hat{U} = U$. (U, U, id_U) is an $n-$ dimensional chart on the subspace U where $id_U : U \longrightarrow U$ is the identity map i.e. $id_U(p) = p$ for all $p \in U$.

Some Remarks

- For convenience/laziness, we'll often just write the pair (U, ϕ) instead of the triple $(U, \hat{U}, \phi)$!
- Since ϕ is a homeomorphism, then obviously $\hat{U} = \phi(U)$. Hence, we'll also often write $\phi(U)$ instead of $\hat{U}$.
- Often the notation x is used to denote the homeomorphism ϕ and so a chart would be denoted (U, x).

Chart Challenge

- Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^2$. Now let $U_1 = S^1 \setminus (0, -1)$ and $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_1$ let us define the map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = \frac{x}{y+1}$.

  Then (U_1, ϕ_1) forms a $1-$ dimensional chart on S^1.

- Consider the circle S^1 again and let $U_2 = S^1 \setminus (0, 1)$ with $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_2$ let us define the map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = \frac{x}{1-y}$. Is (U_2, ϕ_2) a $1-$ dimensional chart on S^1?

- Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $U_1 = S^2 \setminus (0, 0, -1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x, y, z) \in U_1$ let us define the map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = (\frac{x}{z+1}, \frac{y}{z+1})$.

  Then (U_1, ϕ_1) is a $2-$ dimensional chart on S^2.

- Consider the unit sphere S^2 again. Let $U_2 = S^2 \setminus (0, 0, 1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x, y, z) \in U_2$ let us define the map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = (\frac{x}{1-z}, \frac{y}{1-z})$. Is (U_2, ϕ_2) a $2-$ dimensional chart on S^2?

Charts Around Points

Definition (1.1)

Let $(X, \mathcal{T})$ be a topological space. A chart around a point $p \in X$ is an $n-$ dimensional chart (U, ϕ) such that $p \in U$.

- $\phi(U) \subseteq \mathbb{R}^n$ is called a local coordinate system around p.
- Wouldn't it be nice to have a local coordinate system for every point $p \in X$?

Definition (1.2)

Let $(X, \mathcal{T})$ be a topological space. We say that a chart (U, ϕ) around $p \in X$ is centered at p if $0 \in \phi(U)$ and $\phi(p) = 0$.

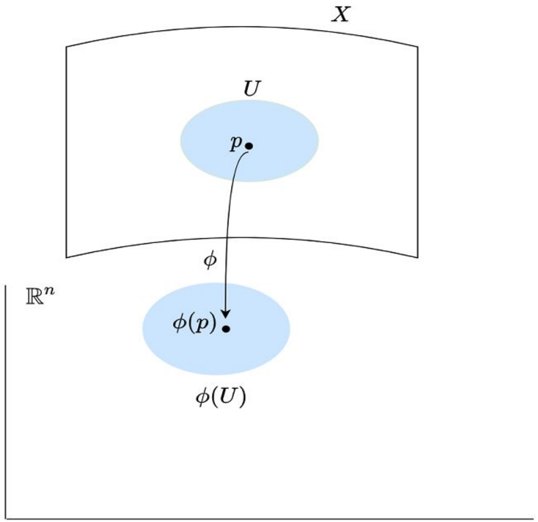
- Where is the standard chart $(\mathbb{R}^n, id_{\mathbb{R}^n})$ centered i.e. which $p \in \mathbb{R}^n$ satisfy $id_{\mathbb{R}^n}(p) = 0$?

Topological Manifolds

Definition (1.3)

A topological space $(X, \mathcal{T})$ is called an $n-$ dimensional manifold if every point $p \in X$ has an $n-$ dimensional coordinate chart around it i.e. for every $p \in X$ there exists an $n-$ dimensional coordinate chart (U, ϕ) such that $p \in U$.

- Hence, for any point $p \in X$ we can find an open set U containing p such that U is homeomorphic to some open set in $\mathbb{R}^n$.



- Therefore, we can naively say that an $n-$ dimensional manifold X looks locally like $\mathbb{R}^n$. So the flat Earth theorists may have a valid point 'locally'!

Some Properties of Manifolds

Proposition (1.0)

Let $(X, \mathcal{T})$ be an $n-$ dimensional manifold. If $U \subset X$ is an open subset of X, then as subspace of X, U is also an $n-$ dimensional manifold.

Proof : Homework challenge!

- Interestingly, if $n \neq m$, then a manifold cannot be both $n-$ dimensional and $m-$ dimensional manifolds i.e. the dimension of a manifold is invariant! Hence, the dimension of manifold is a well-defined notion.
- When $n = 2$, then X is also known as a 'topological surface'.
- Often the term '$n-$ manifold' is used instead '$n-$ dimensional manifold'.

Convention: From now on, we'll just write X instead of $(X, \mathcal{T})$.

Manifold Challenge

- Which of the following topological spaces are manifolds:
  1. The closed interval $[0, 1]$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
  2. The open interval $(0, 1)$ equipped with the subspace topology of the standard topology on $\mathbb{R}$.
  3. The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^2$.
  4. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ equipped with subspace topology of the standard topology on $\mathbb{R}^3$.
  5. $\mathbb{R}$ equipped with the standard topology.
  6. $\mathbb{R} \setminus \{0\}$ equipped with the standard topology.
  7. $\mathbb{R}^2 \setminus \{(0, 0)\}$ equipped with the standard topology.

PART B

Differentiability Classes on $\mathbb{R}^n$

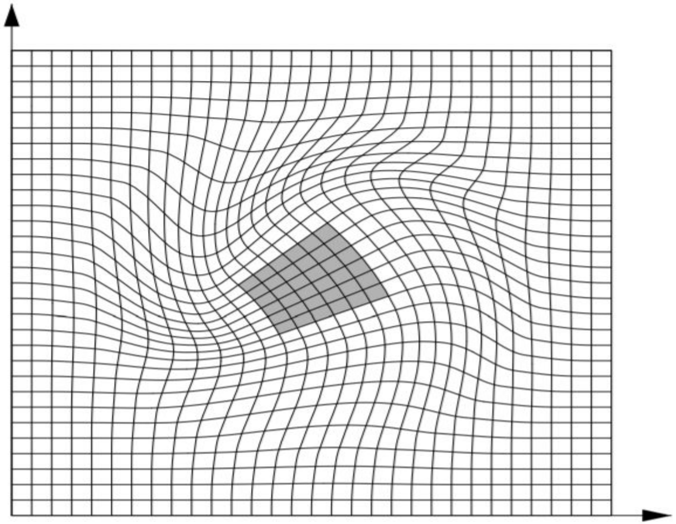
Definition

Recall from undergrad that given a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we can put f in the following categories:

1. C^0 class if f is continuous.
2. C^1 class if f is continuous and when differentiated once, the result of the differentiation is still continuous.
3. C^k class if f is continuous and when differentiated $k-$ times, the result of the differentiation is still continuous for some $k > 1$.
4. C^∞ class if f is continuous and when differentiated infinitely many times, the result of the differentiation is still continuous. Another name for a C^∞ class map f is 'smooth map'.

- C^∞ maps are very important to us because they will lead us to 'smooth manifolds' and their structure preserving maps i.e. 'diffeomorphisms'.

Multivariable Calculus



4.6 Higher Order Derivatives

Partial differentiation can be carried out more than once on nice enough functions. For example if

$$f(x, y) = e^{x \sin y}$$

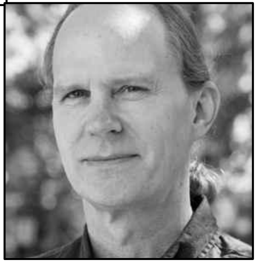
then

$$D_1 f(x, y) = \sin y e^{x \sin y}, \quad D_2 f(x, y) = x \cos y e^{x \sin y}.$$

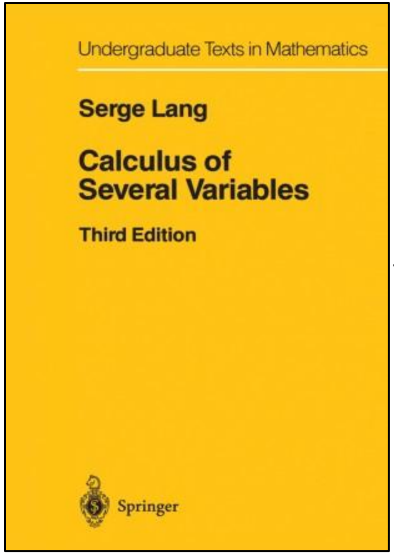
Taking partial derivatives again yields

$$\begin{aligned}
D_1 D_1 f(x, y) &= \sin^2 y e^{x \sin y}, \\
D_1 D_2 f(x, y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y}, \\
D_2 D_1 f(x, y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y} = D_1 D_2 f(x, y), \\
D_2 D_2 f(x, y) &= -x \sin y e^{x \sin y} + x^2 \cos^2 y e^{x \sin y},
\end{aligned}$$

and some partial derivatives of these in turn are,



Prof. Jerry Shurman



CHAPTER VI

Higher Derivatives

In this chapter, we discuss two things which are of independent interest. First, we define partial differential operators (with constant coefficients). It is very useful to have facility in working with these formally.

Secondly, we apply them to the derivation of Taylor's formula for functions of several variables, which will be very similar to the formula for one variable. The formula, as before, tells us how to approximate a function by means of polynomials. In the present theory, these polynomials involve several variables, of course. We shall see that they are hardly more difficult to handle than polynomials in one variable in the matters under consideration.

The proof that the partial derivatives commute is tricky. It can be omitted without harm in a class allergic to theory, because the technique involved never reappears in the rest of this book.

§1. *Repeated partial derivatives*

Let f be a function of two variables, defined on an open set U in 2-space. Assume that its first partial derivative exists. Then $D_1 f$ (which we also write $\partial f/\partial x$ if x is the first variable) is a function defined on U. We may then ask for its first or second partial derivatives, i.e. we may form $D_2 D_1 f$ or $D_1 D_1 f$ if these exist. Similarly, if $D_2 f$ exists, and if the first partial derivative of $D_2 f$ exists, we may form $D_1 D_2 f$.

Suppose that we write f in terms of the two variables (x, y). Then we can write

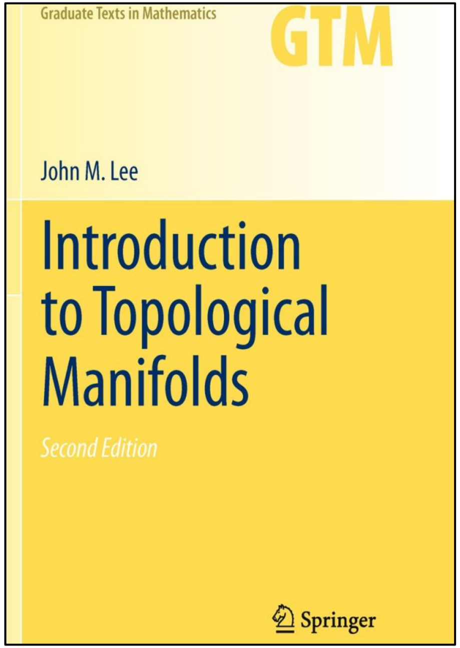
$$D_1 D_2 f(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (D_1(D_2 f))(x, y),$$

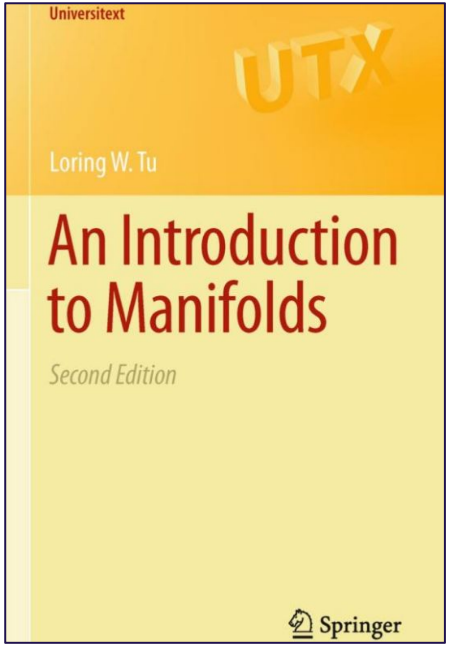
and

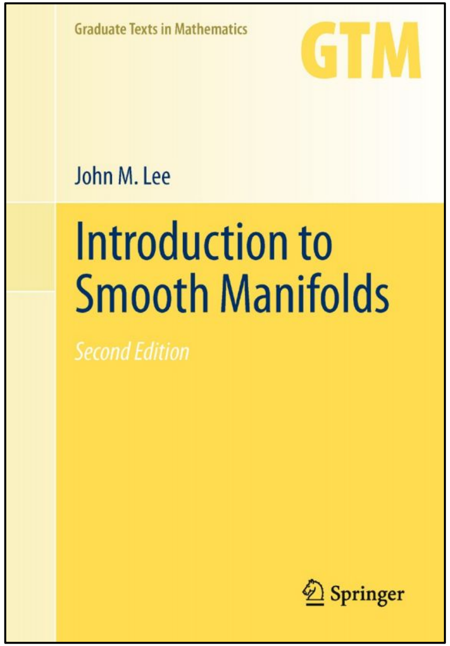
$$D_2 D_1 f(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (D_2(D_1 f))(x, y).$$

For example, let $f(x, y) = \sin(xy)$. Then

$$\frac{\partial f}{\partial x} = y\cos(xy) \quad \text{and} \quad \frac{\partial f}{\partial y} = x\cos(xy).$$







Topological & Diff. Manifolds

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