QUANTUM FORMALISM

Finite dimensional Hilbert spaces - Part 1

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Complex Vector Space Review
3. The Axioms of QM (#1)
4. Why Hilbert spaces

Part A

1. Complex Inner Product Space
2. Complex Normed Space (Induced)
3. Complex Hilbert space
4. Study Material Comments

Part B

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Operators + Composite Systems #3
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

PART A

Linear Independence and Bases

Definition (1.0)

Let V be a vector space over $\mathbb{C}$. $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle \in V$ are linearly independent if $\sum_{i=1}^{n} \alpha_i |\psi_i\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle .... + \alpha_n|\psi_n\rangle = 0$ if only if $\alpha_i = 0 \; \forall i \in \{1, ..., n\}$ i.e. $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent in $\mathbb{C}^2$?

Definition (1.1)

A subset $B = \{|e_1\rangle, |e_2\rangle ......, |e_n\rangle\}$ of V forms a basis (Hamel) in V if:

1. $|e_1\rangle, |e_2\rangle ......, |e_n\rangle$ are linearly independent i.e. $\sum_{i=1}^{n} \alpha_i |e_i\rangle = 0$ if only if $\alpha_i = 0 \; \forall i \in \{1, ..., n\}$.
2. Span$(|e_1\rangle, |e_2\rangle, ..., |e_n\rangle) = V$ i.e. any $|\psi\rangle \in V$ can be written (uniquely) as linear combination of $|e_1\rangle, |e_2\rangle, ..., |e_n\rangle$.

- Let $B = \left\{ |0\rangle, |1\rangle \right\}$. It's clear that B forms a basis in $\mathbb{C}^2$ right?
- Are there more bases in $\mathbb{C}^2$ other than B ?

Dimension Invariance

Definition (1.2)

If $B = \{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ forms a basis in V then its cardinality is called the dimension of V and denoted $\dim(V)$ or just $\dim V$.

Proposition (1.0)

If $B = \{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ and $B' = \{|e'_1\rangle, |e'_2\rangle, ..., |e'_m\rangle\}$ are bases in V then $|B| = |B`|$ i.e. $n = m$.

Proof: Homework challenge!

- As you already know, in general, $\dim(\mathbb{C}^n) = n$.
- For quantum computing purposes, you'll want $n = 2^k$ i.e. $\mathbb{C}^{2^k}$ where k is the number of qubits under consideration.
- We'll next see that if V_1 and V_2 are two vector spaces over $\mathbb{C}$. Then, $dim(V_1) = dim(V_2)$ iff $V_1 \simeq V_2$ vise versa.

Isomorphism Theorems

Theorem (1.0)

Let V and W be vector spaces over $\mathbb{C}$. Then the following are equivalent:

1. $V \simeq W$.
2. $\dim(V) = \dim(W)$.

Proof : Homework challenge!

Theorem (1.1)

If V is a vector space over $\mathbb{C}$ and $\dim(V) = n$. Then $V \simeq \mathbb{C}^n$.

- So for example, take $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. Because $\dim(\mathbb{C}^2 \otimes \mathbb{C}^2) = 4$ it then follows that $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.
- The theorem above makes our life very easy in terms of building a theory on a $n-$ dimensional complex vector space because we can just use $\mathbb{C}^n$ even if we start with a very abstract V.

The Axioms of Quantum Mechanics

Axiom 1: The states of quantum systems are modelled by **normalised vectors** on **separable complex Hilbert spaces**.

- Introductory quantum mechanics textbooks/courses often hide the technical details that the Hilbert space has to be separable.
- Technically, physicists call the states mentioned in the axiom as 'pure states'. There are also the so-called 'mixed states'! This is where the so-called density matrix formalism is useful because it can deal with both pure and mixed states.
- In the advanced module, we may rephrase the axioms in this module in terms of the density matrix formalism. But we'll need $\mathbb{C}^*$-algebras in order to do so in a clean mathematical fashion!

Extra comments

- In classical mechanics (Hamilton formalism): The states of a system are elements of the cotangent bundle T^*M of some symplectic manifold M of dimension n.
- There is a field of mathematical physics called 'Geometric Quantisation' which essentially tries to construct a Hilbert space from the symplectic manifold. This leads to the idea of using category theory e.g. quantisation could then be conceptually thought of as functor from the category of symplectic manifolds to the category of Hilbert spaces.

Why Hilbert Spaces?

In the early days of quantum mechanics, there seemed to be two competing theories :

1. **The Schrodinger wave mechanics:** This version of quantum mechanics gave emphasis on the so-called wave function $\Psi(x)$ which encoded the notion of state. Although physicists did not realise at the time, Schrodinger's wave functions were actually elements of a very special complex vector space $L^2(\mathbb{R}^3)$ i.e. the space of square integrable functions (Lebesgue sense).
2. **The Heisenberg matrix mechanics:** This version of quantum mechanics gave emphasis on observables and lacked the notion of state. The observables were encoded as 'matrices' built from $l^2(\mathbb{N})$. This is indeed a special complex vector space called the space of square summable sequences.

- Some years later, John von Neumann realised that since $L^2(\mathbb{R}^3)$ and $l^2(\mathbb{N})$ have an abstract mathematical structure in common i.e. they are both complex Hilbert spaces. He proposed that the Heisenberg observables be encoded as self-adjoint operators on a Hilbert space whilst the Schrodinger wave functions as unit vectors on the same Hilbert space!
- It was also proved that $L^2(\mathbb{R}^3)$ and $l^2(\mathbb{N})$ are unitarily equivalent i.e. a more sophisticated way of saying the two spaces are isomorphic!

PART B

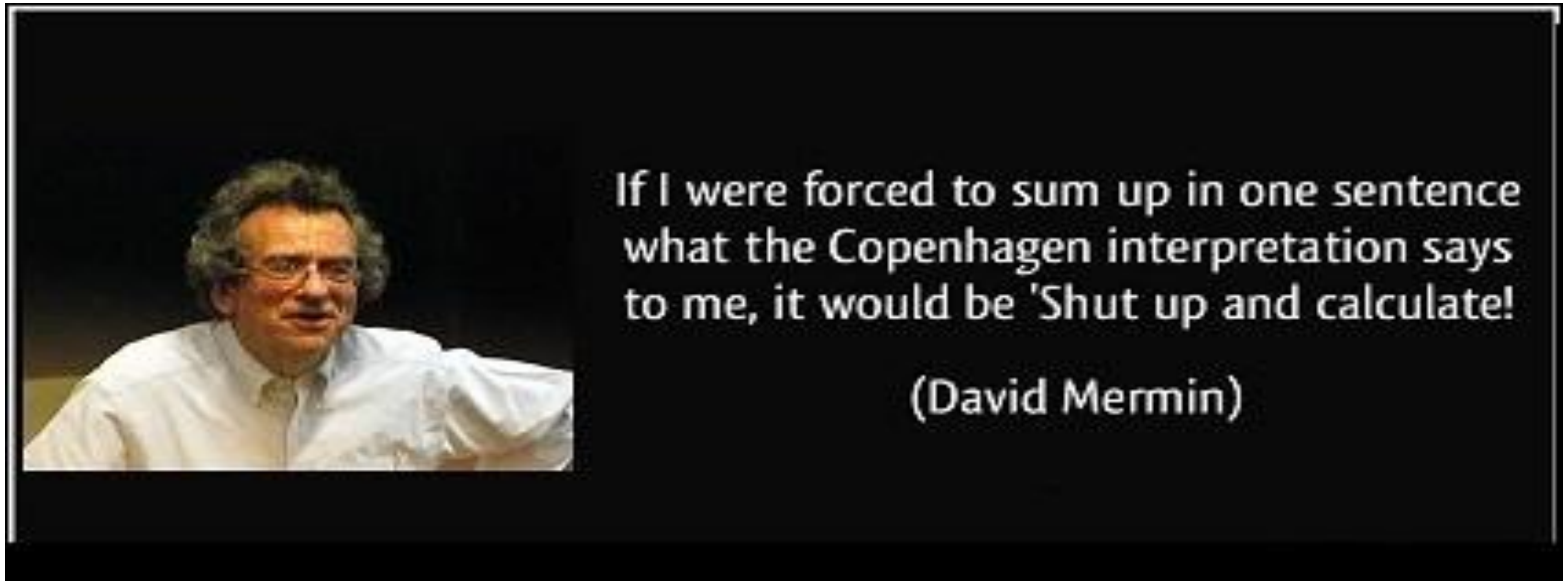
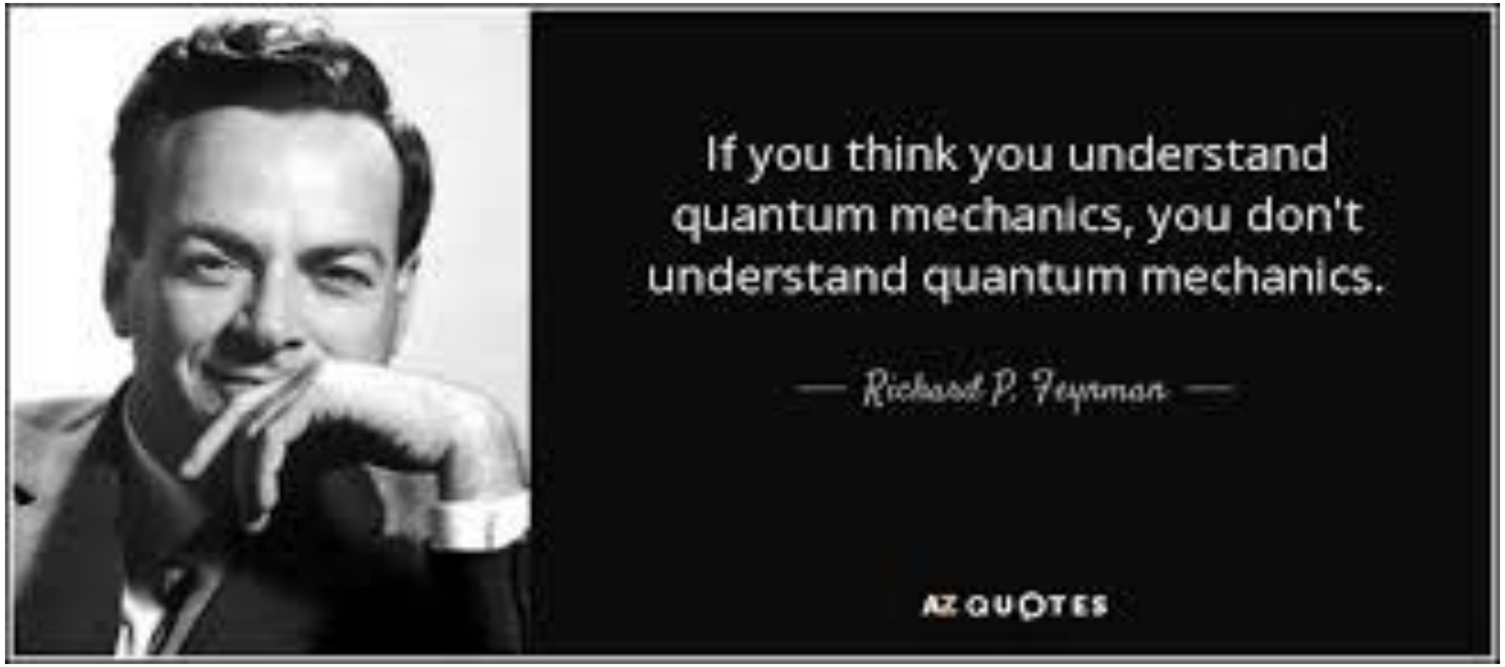


Image credits: Shiro Ishikawa


If you think you understand quantum mechanics, you don't understand quantum mechanics.
— Richard P. Feynman —
AZ QUOTES

Inner Product Space

Definition (1.0)

Let V be a vector space over $\mathbb{C}$. An inner product on V is a map $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ such that the following conditions hold:

1. $\langle\psi,\psi\rangle \geq 0$ for all $\psi \in V$ and $\langle\psi,\psi\rangle = 0$ iff $\psi = 0_V$.
2. $\langle\psi_1,\psi_2\rangle = \langle\psi_2,\psi_1\rangle^*$ for all $\psi_1,\psi_2 \in V$.
3. $\langle\alpha\psi_1 + \beta\psi_2,\psi_3\rangle = \alpha\langle\psi_1,\psi_3\rangle + \beta\langle\psi_2,\psi_3\rangle$ for all $\psi_1,\psi_2,\psi_3 \in V$ and $\alpha,\beta \in \mathbb{C}$.

- As an exercise, what is $\langle\psi_1,\alpha\psi_2 + \beta\psi_3\rangle$? Remember to pay attention to conditions 2 and 3 in the definition!

Definition (1.1)

The pair $(V,\langle\cdot,\cdot\rangle)$ is called an inner product space.

- When the inner product is understood from the context we'll just write V.

Example 1

- If $V = \mathbb{C}^2$ we can build an inner product as follows:

  For any two vectors $\psi_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in $\mathbb{C}^2$, their inner product can be defined as $\langle \psi_1, \psi_2 \rangle = x_1 y_1^* + x_2 y_2^*$ where y_i^* is the complex conjugate of y_i.

  Concrete example: If $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\langle \psi_1, \psi_2 \rangle = \langle \psi_2, \psi_1 \rangle = 0$.

  Is it true that $\langle \psi_1, \psi_1 \rangle = \langle \psi_2, \psi_2 \rangle = 1$?

- We can generalise the construction above for $\mathbb{C}^n$ as follows:

  For any $\psi_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in $\mathbb{C}^n$, $\langle \psi_1, \psi_2 \rangle = \sum_{i=1}^n x_i y_i^*$.

  Hence, $\mathbb{C}^n$ is an inner product space for any $n \geq 1$.

Example 2: Home challenge

- If we consider $M_2(\mathbb{C})$ as a vector space over $\mathbb{C}$. Which of the following constructions is an inner product? Do both work?

  1. $\langle A, B\rangle = Tr(A^\dagger B)$ for all $A, B \in M_2(\mathbb{C})$.
  2. $\langle A, B\rangle = Tr(AB^\dagger)$ for all $A, B \in M_2(\mathbb{C})$.

  Now consider the following operators:

- For $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Compute the inner products between them including:

1. $\langle Y, X\rangle$ and $\langle X, Y\rangle$.
2. $\langle Y, Z\rangle$ and $\langle Z, X\rangle$.
3. $\langle Y, H\rangle$ and $\langle H, Y\rangle$.

- We can of course generalise the construction above to $M_n(\mathbb{C})$.

Induced Normed Spaces

Definition (1.2)

Given an inner product $\langle \cdot, \cdot \rangle$ on V, its induced norm is defined for any $\psi \in V$ as $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$.

- Important to note that the concept of norm is in general independent i.e. we can define the norm on its own right.

Proposition (1.0)

Let $\|\cdot\|$ be a norm on V induced by an inner product $\langle \cdot, \cdot \rangle$. Then the following properties hold:

1. $\|\psi\| \geq 0$ for all $\psi \in V$ and $\|\psi\| = 0$ iff $\psi = 0_V$.
2. $\|\alpha\psi\| = |\alpha|\|\psi\|$ for all $\psi \in V$ and $\alpha \in \mathbb{C}$ where $|\alpha|$ is the absolute value in $\mathbb{C}$.
3. $\|\psi_1 + \psi_2\| \leq \|\psi_1\| + \|\psi_2\|$ for all $\psi_1, \psi_2 \in V$.

Proof : Homework challenge!

- V together with a norm $\|\cdot\|$ is called a normed vector space. In our case induced normed space because our norm was induced.
- $\mathbb{C}^n$ and $M_n(\mathbb{C})$ are natural examples of normed vector spaces!
- We can use a norm $\|\cdot\|$ to define the notion of metric (distance) between two vectors $\psi_1, \psi_2 \in V$ as $d(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|$.
- Is there a way to check if a norm $\|\cdot\|$ was induced by an inner product?

Some Properties of Norms

Theorem (1.0)

Let $\|\cdot\|$ be a norm on V induced by an inner product $\langle\cdot,\cdot\rangle$. Then the following is true (Cauchy–Schwarz inequality):

1. $|\langle\psi_1,\psi_2\rangle| \leq \|\psi_1\|\|\psi_2\|$.
2. $|\langle\psi_1,\psi_2\rangle| = \|\psi_1\|\|\psi_2\|$ iff ψ_1 and ψ_2 are linearly dependent.

Proof : Homework challenge? Or check the study materials!

Theorem (1.1)

A norm $\|\cdot\|$ on V is induced by an inner product $\langle\cdot,\cdot\rangle$ if only if for all $\psi_1,\psi_2\in V$, $\|\psi_1+\psi_2\|^2+\|\psi_1-\psi_2\|^2 = 2(\|\psi_1\|^2+\|\psi_2\|^2)$.

Proof : Homework challenge? Or check the study materials!

- The property above is known as the parallelogram identity.

Theorem (1.2)

If a norm $\|\cdot\|$ on V satisfies the parallelogram identity then:
$\langle\psi_1,\psi_2\rangle = \frac{1}{4}(\|\psi_1+\psi_2\|^2-\|\psi_1-\psi_2\|^2+i\|\psi_1-i\psi_2\|^2-i\|\psi_1+i\psi_2\|^2)$.

Proof : Homework challenge? Or check the study materials!

- The property above is known as the polarization identity.

Complete Normed Spaces

Definition (1.3)

A norm $\|\cdot\|$ on V is said to be complete if there exists a map (sequence) $\phi : \mathbb{N} \longrightarrow V$ such that $\forall \epsilon > 0$ there exists $\kappa \in \mathbb{N}$ such that $\|\phi(n_1) - \phi(n_2)\| < \kappa$ for all $n_1, n_2 \geq \kappa$.

- A sequence satisfying the above is called a Cauchy sequence.
- Analysts normally write ψ_n instead of $\phi(n)$. So an $n \in \mathbb{N}$ yields a vector $\psi_n = \phi(n) \in V$.

Definition (1.4)

V is called a Banach space if its norm $\|\cdot\|$ is complete.

- Hence, V is a Banach space in respect to the norm induced by its inner product if such norm is complete.

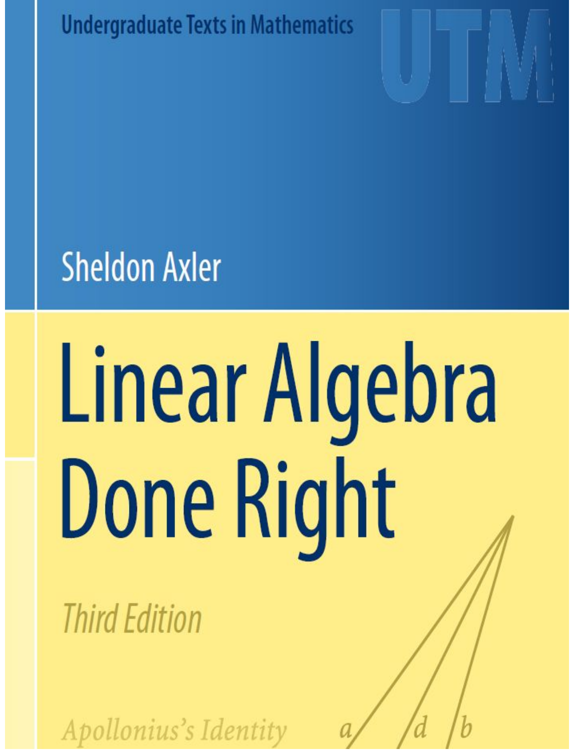
- $\mathbb{C}^n$ and $M_n(\mathbb{C})$ are of course Banach spaces!

Complex Hilbert space

Definition (1.5)

A complex Hilbert space is vector space over $\mathbb{C}$ denoted $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ that induces a complete norm $\| \cdot \|$.

- Hence, by definition a Hilbert space is a Banach space in respect to the induced norm.
- $\mathbb{C}^n$ and $M_n(\mathbb{C})$ are obvious examples of complex Hilbert spaces!
- Hilbert spaces are often presented as the rock stars of quantum formalism. In reality, Banach spaces also play a very important role in the formalism. This becomes more apparent when dealing with infinite dimensional Hilbert spaces where the Banach space of bounded self-adjoint operators play a very important role in the formalism.
- By the way, is a Banach space necessarily a Hilbert space?

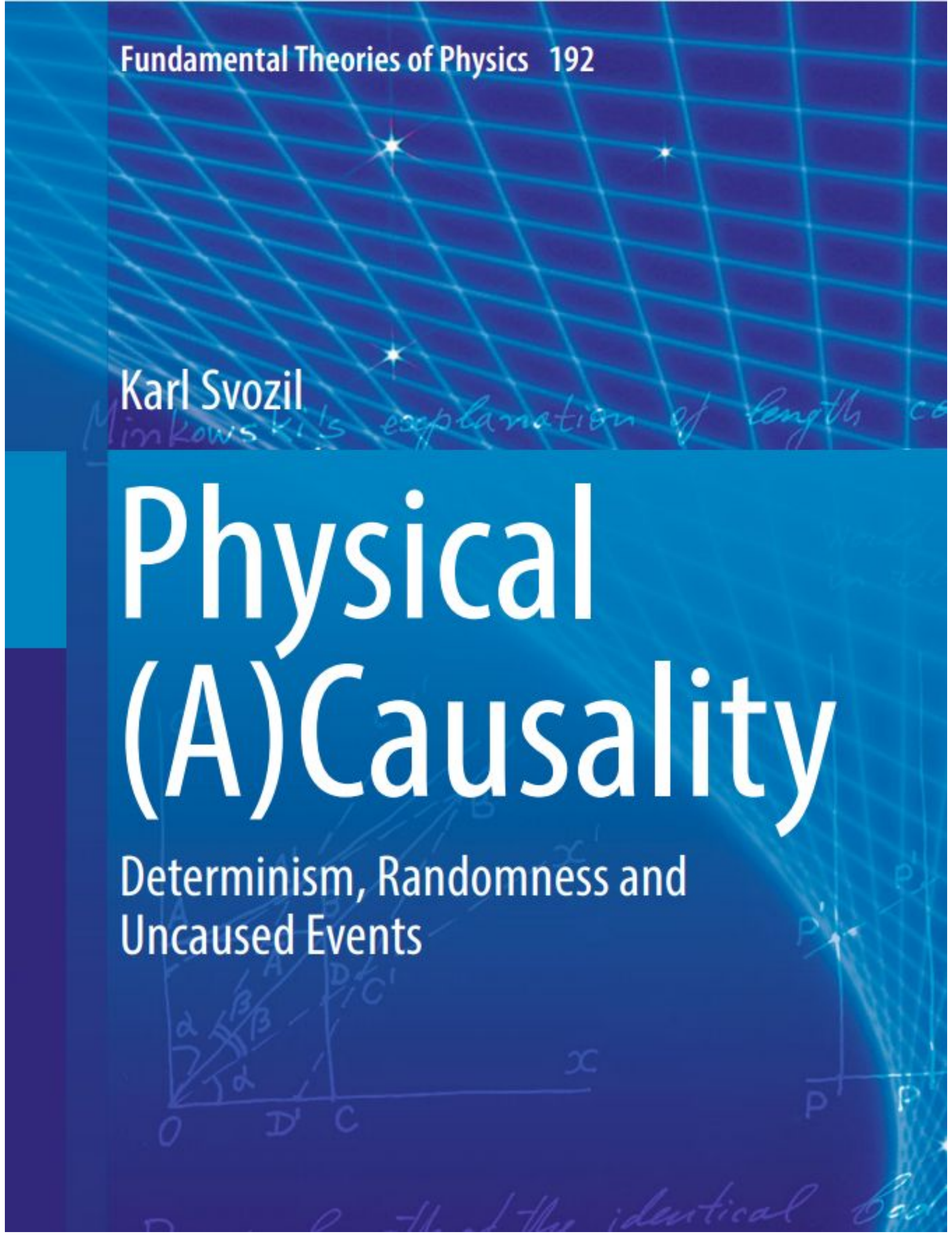


Prof. Sheldon Axler

Where should you focus?

3F: Duality (*101 - 113)*

6.A: Inner Product Spaces (163 - 175)


Fundamental Theories of Physics 192
Karl Svozil
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Guest Bio

Laurentiu Nita is the founder of Quarks Interactive: a company with the mission of making quantum computing fun (through gamification) and accessible (by removing the need-to-know mathematics for understanding computation). He managed to secure funding to create Quantum Odyssey, a video game where the

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