QUANTUM FORMALISM

Linear Operators 101 - Part 2

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Lecture Agenda Summary

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Matrix Operators #2
Group Theory 101 #1	Linear Operators 101 #2	Complex Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups: GL(n, C) & U(n) #2

Completed | Ongoing | #n is the number of live lectures

Canonical Example of Linear Operator

- Let $V = W = \mathbb{C}^2$ and let $M_2(\mathbb{C})$ be the ring of all 2×2 matrices over $\mathbb{C}$ i.e. $M_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$.

Then for each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ we can construct a linear operator $T_A : \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

For any $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$, let T_A act on $|\psi\rangle$ as $T_A|\psi\rangle = A|\psi\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \alpha + \begin{pmatrix} b \\ d \end{pmatrix} \beta = \begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix}$.

- Hence, each matrix $A \in M_2(\mathbb{C})$ generates a linear operator for $\mathbb{C}^2$.
- Can we generalise this to $\mathbb{C}^n$ such that any $A \in M_n(\mathbb{C})$ generates a linear operator T_A for $\mathbb{C}^n$?
- Can we also generate a matrix linear operator $A_T \in M_n(\mathbb{C})$ for $\mathbb{C}^n$ given any linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$?
- By convention we just write A to denote the linear operator generated from $A \in M_n(\mathbb{C})$ instead of writing T_A.

Definition (Lecture 06)

If $T : V \to W$ is a linear operator then the range of T is defined as $\mathrm{Ran}(T) = \{T|\psi\rangle \mid |\psi\rangle \in V\}$.

Proposition (Lecture 06)

$\mathrm{Ran}(T)$ is a linear subspace of W.

Proof : Homework 1.3!

- The dimension of $\mathrm{Ran}(T)$ is called the rank of T.

Definition (Lecture 06)

Let $T : V \to W$ be a linear operator. The kernel of T is defined as $\mathrm{Ker}(T) = \{|\psi\rangle \in V \mid T|\psi\rangle = |0_W\rangle\}$.

- Ker(T) is also often called the null-space of T in the literature.

Proposition (Lecture 06)

Ker(T) is a subspace of V.

Proof : Homework 1.4!

Theorem (Lecture 06)

Let $T : V \to W$ be a linear operator. Then $\dim(V) = \dim \mathrm{Ker}(T) + \dim \mathrm{Ran}(T)$.

Proof : Homework challenge?

- The theorem above is often called 'the dimension theorem' or sometimes 'the rank-nullity theorem'.

Vector Space Isomorphisms

Definition (1.0)

A linear operator $T : V \longrightarrow W$ is called:

1. **Surjective** (onto) if Ran $(T) = W$ i.e. for all $|\psi'\rangle \in W$ there exists a vector $|\psi\rangle \in V$ such that $|\psi'\rangle = T|\psi\rangle$.
2. **Injective** (one-to-one) if for all $|\psi_1\rangle, |\psi_2\rangle \in V$, $T|\psi_1\rangle = T|\psi_2\rangle$ if only if $|\psi_1\rangle = |\psi_2\rangle$.
3. **Bijective** if it's both surjective and injective.

Definition (1.1)

We say that V is isomorphic to W ($V \simeq W$) if there is at least a bijective linear operator $T : V \longrightarrow W$.

- Because of the injectivity of T, it follows that T is an isomorphism iff $\mathrm{Ker}(T) = \{0_V\}$.

Isomorphism between $\mathbb{C}^4$ and $M_2(\mathbb{C})$

Lets first recall that $\mathbb{C}^4 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$ and $M_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$.

We can define a linear operator $T : \mathbb{C}^4 \longrightarrow M_2(\mathbb{C})$ naturally as follows:

For each $|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{C}^4$ define $T|\psi\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- The definition above clearly makes T into an isomorphism!

Isomorphism Theorems

Theorem (1.0)

Let V and W be vector spaces over $\mathbb{C}$. Then $V \simeq W$ if and only if $dim(V) = dim(W)$.

Proof : Homework challenge?

- A more elegant way of stating the above theorem is to say that the following statements are equivalent:
  1. $V \simeq W$
  2. $\dim(V) = \dim(W)$

Theorem (1.1)

If V is a vector space over $\mathbb{C}$ and $dim(V) = n$. Then $V \simeq \mathbb{C}^n$.

- So for example, take $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. Because $\dim(\mathbb{C}^2 \otimes \mathbb{C}^2) = 4$ it then follows that $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

Identity Operator

Definition (1.2)

The identity operator $\mathbb{I}_V : V \to V$ is defined as $\mathbb{I}_V|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle \in V$.

- $\mathbb{I}_V$ is of course a linear operator. What can we say about $\mathrm{Ker}(\mathbb{I}_V)$?
- For $V = \mathbb{C}^2$, the identity operator is of course $\mathbb{I}_V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Obviously the example above can be generalised for any $\mathbb{C}^n$, where $\mathbb{I}_V$ is identified as the $n \times n$ identity matrix of the ring $M_n(\mathbb{C})$.

Convention: Whenever V is understood from the context, we shall just write $\mathbb{I}$ instead of $\mathbb{I}_V$!

Product of Operators

Definition (1.3)

Let $T_1 : V \longrightarrow V$ and $T_2 : V \longrightarrow V$ be linear operators. Their composition $T_2 \circ T_1 : V \longrightarrow V$ is defined as $T_2 \circ T_1|\psi\rangle = T_2T_1|\psi\rangle$ for all $|\psi\rangle \in V$.

- We can prove that $T_2 \circ T_1$ is indeed a linear operator.

Convention: We'll just write T_2T_1 instead of $T_2 \circ T_1$ and rename it as 'operator product'!

Definition (1.4)

A linear operator $T : V \longrightarrow V$ is invertible if there exists a linear operator denoted $T^{-1} : V \longrightarrow V$ such that $TT^{-1} = T^{-1}T = \mathbb{I}_V$.

- So not every T is invertible. But, is there a necessary and sufficient condition for T to be invertible?

Theorem (1.2)

A linear operator $T : V \longrightarrow V$ is invertible if and only if $Ker(T) = \{0\}$.

Proof : Homework challenge?

Hence, the theorem above implies T is invertible iff T is an isomorphism from V to V?

Theorem (1.3)

Let $T_1 : V \longrightarrow V$ and $T_2 : V \longrightarrow V$ be invertible linear operators. Then $T_2 T_1$ is invertible.

Proof : Homework challenge?

- If $V = \mathbb{C}^2$ then the following linear operators are invertible: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.
- Do the set of all invertible linear operators acting on V form a group?

Abstract General Linear Group

Definition (1.5)

The set $GL(V) = \{T : V \longrightarrow V \mid T \text{ is invertible } \}$ is called the general linear group of V i.e. $GL(V)$ is the set of all invertible linear operators acting on V.

- As home challenge, verify that $GL(V)$ is a group under the operator product defined previously. Is it abelian or non abelian?
- In the matrix group section, we'll see that if $\dim(V) = n$ then $GL(V) \simeq GL(n, \mathbb{C})$!
- As you'll see, $GL(n, \mathbb{C})$ contains some interesting subgroups such as the unitary group $U(n)$ and the special unitary group $SU(n)$.
- As the name suggests, in quantum mechanics $U(n)$ is mathematically behind the so-called 'unitary evolution'!

Abstract Eigenvectors and Eigenvalues

Definition (1.6)

Let $T : V \longrightarrow V$. A vector $|\psi\rangle \in V$ is said to be an eigenvector of T if there exists a $\lambda \in \mathbb{C}$ such that $T|\psi\rangle = \lambda|\psi\rangle$. The scalar λ is called an eigenvalue of T.

- The set of all eigenvalues of T is called the spectrum of T and often denoted Spec(T) i.e Spec $(T) = \{\lambda \in \mathbb{C} \mid T|\psi\rangle = \lambda|\psi\rangle\}$ where $|\psi\rangle \in V$ is obviously an eigenvector of T.

Theorem (1.4)

Let $T : V \longrightarrow V$ be a linear operator with distinct set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ with corresponding eigenvectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \in V$. Then the vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ are linearly independent.

- Can the eigenvectors of the distinct eigenvalues above form a basis for V if $\dim(V) = n$?
- In quantum mechanics, the eigenvalues and eigenvectors of linear operators encoding physical observables such as the energy (aka the Hamiltonian) are very important!
- Be aware that physicists often write $|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_n\rangle$ to denote the eigenvectors of $\lambda_1, \lambda_2, \ldots, \lambda_n$!

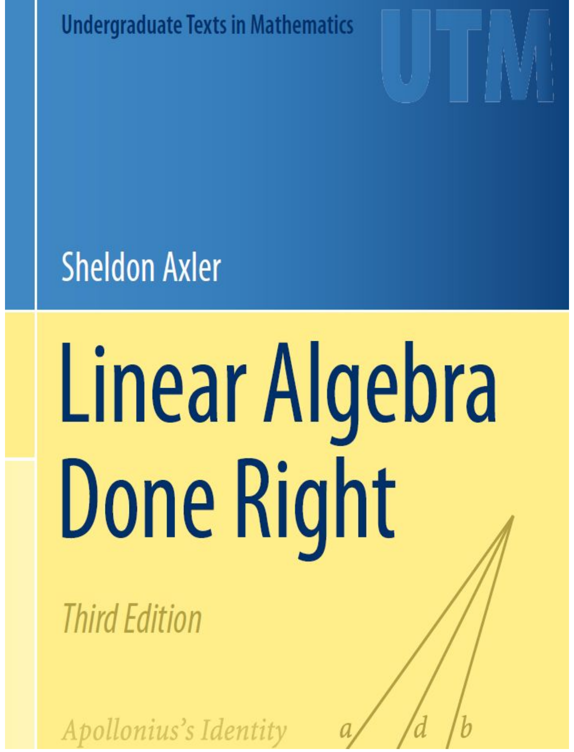
Home Challenge

Let $V = \mathbb{C}^2$ and consider the following linear operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

- Try find the eigenvalues and eigenvectors of each operator above!
- Do their eigenvectors form a basis for $\mathbb{C}^2$?

Software aid: Feel free to use any suitable software to help you find the answers!



Prof. Sheldon Axler

Where should you focus?

3.D Invertible Linear Maps (*Pages 80 - 88)*

5.A Eigenvalues & Eigenvectors (131 - 133)

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