QF Group Theory CC2022
By
Zaiku Group

Lecture 20 (PART B)

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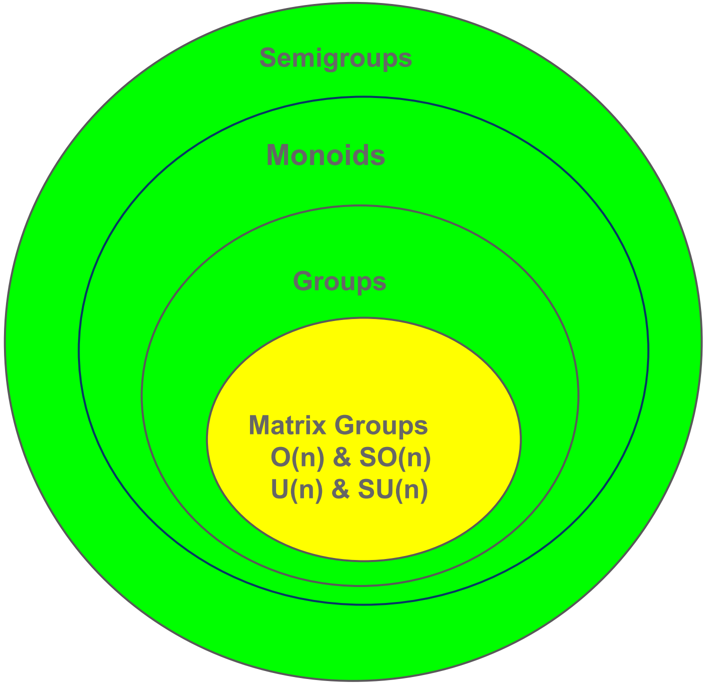
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
20B

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

Linear Algebra 101 (External Direct Sum)

Definition 1.0

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$ (for our purposes $\mathbb{F} = \mathbb{C}$). We can equip the Cartersian product set $V_1 \times V_2 = \{(x, y) \mid x \in V_1, y \in V_2\}$ with a vector space structure over $\mathbb{F}$ as follows;

1. For all $v_1 = (x_1, y_1) \in V_1 \times V_2$ and $v_2 = (x_2, y_2) \in V_2$, we define the addition $v_1 + v_2 = (x_1 + x_2, y_1 + y_2)$ i.e. we do the addition entry-wise.
2. For $v = (x, y) \in V_1 \times V_2$ and $\alpha \in \mathbb{F}$, we define the scalar product $\alpha v = (\alpha x, \alpha y)$ i.e. we do the scalar multiplication entry-wise.

- With the two operations above, we can clearly see that $V_1 \times V_2$ forms a vector space, which is denote $V_1 \oplus V_2$.
- If $Dim(V_1) = d_1$ and $Dim(V_2) = d_2$, then $Dim(V_1 \oplus V_2) = d_1 + d_2$.
- Note that $V_1 \oplus V_2$ contains a copy of both V_1 and V_2 with the identification of V_1 as the subspace of the elements of the form $(x, 0_{V_2})$ and V_2 as the subspace of the elements of the form $(0_{V_1}, y)$.

Linear Algebra 101 (Internal Direct Sum)

Definition 1.1

Let V be a vector space over a field $\mathbb{F}$. If W and U are subspaces of V, we can define their (internal) direct sum as
$W + U = \{w + u \mid w \in W, u \in U\}$.

- It's not hard to see that $W + U$ is a subspace of V. This subspace is known as the internal direct sum of W and U in V, and it's also denoted as $W \oplus U$!
- Indeed, if we use the external direct sum on W and U, the resulting space is isomorphic to the one we get above provided $W \cap U = \{0_V\}$!

Definition 1.2

If W and U are subspaces of V, we say V is a direct sum of the two subspaces and write $V = W \oplus U$ if the following holds:

1. $W \cap U = \{0_V\}$.
2. $W + U = \{w + u \mid w \in W, u \in U\} = V$.

- Hence, every vector $v \in V$ can be expressed (uniquely) as $w + u$ for some $w \in W$ and $u \in U$.

Direct Sum of Representations

Definition 1.3

Let (V_1, ρ_1) be a k_1- dimensional representation of a group G and (V_2, ρ_2) be k_2- dimensional representation of G. We can construct a $(k_1 + k_2)-$dimensional representation written $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ where $\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$ as follows:

For all $v = (x, y) \in V_1 \oplus V_2$ and $g \in G$, we define $\rho_1 \oplus \rho_2 v = (\rho_1(g)x, \rho_2(g)y)$.

- It's not hard to see that $\rho_1 \oplus \rho_2$ defined as above is indeed a group homomorphism!
- As you can guess, $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is called a direct sum of the representations (V_1, ρ_1) and (V_2, ρ_2)!
- Obviously, we can equivalently view the homomorphism $\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$ as $\rho_1 \oplus \rho_2 : G \longrightarrow GL(k_1 + k_2, \mathbb{C})$.

Example of Direct Sum of Representations

- Consider the additive group $\mathbb{Z}_n$ (often also written as $\mathbb{Z}/n\mathbb{Z}$) and the following representations:

1. $(\mathbb{Z}_n, \rho_1)$ with $\rho_1 : \mathbb{Z}_n \longrightarrow GL(1, \mathbb{C})$ defined as $\rho_1(x) = e^{\frac{2\pi ix}{n}}$ for all $x \in \mathbb{Z}_n$.
2. $(\mathbb{Z}_n, \rho_2)$ with $\rho_2 : \mathbb{Z}_n \longrightarrow GL(1, \mathbb{C})$ defined as $\rho_1(x) = e^{\frac{-2\pi ix}{n}}$ for all $x \in \mathbb{Z}_n$.

The direct product is $(\mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2, \rho_1 \oplus \rho_2)$ with $\rho_1 \oplus \rho_2(x) = \begin{pmatrix} e^{\frac{2\pi xi}{n}}) & 0 \\ 0 & e^{\frac{-2\pi xi}{n}} \end{pmatrix}$ for all $x \in \mathbb{Z}_n$.

Indecomposable Representations

Proposition 1.0

Let $(W, \rho_{|W})$ and $(U, \rho_{|U})$ be subrepresentations of (V, ρ). Then $(W \oplus U, \rho_{|W} \oplus \rho_{|U}) \simeq (V, \rho)$ iff $W \oplus U \simeq V$.

Proof : Homework challenge!

Definition 1.4

A representation (V, ρ) is indecomposable if it is not a direct sum of proper subrepresentations.

- If the opposite of the above happens i.e. (V, ρ) can be written as direct sum of proper subrepresentations, then (V, ρ) is said to be decomposable.

Proposition 1.1

If (V, ρ) is an irreducible representation, then (V, ρ) is indecomposable.

Proof : Homework challenge!

Curiosity question: Is an indecomposable representation necessarily irreducible?

Completely Reducible Representations

Definition 1.5

A representation (V, ρ) of a group G is said to be completely reducible if the following conditions hold:

1. $V = V_1 \oplus \ldots \oplus V_n$ where n is the dimension of the representation and for all $i \in \{1, \ldots, n\}$, V_i is a $G-$ invariant subspace.
2. For all $i \in \{1, \ldots, n\}$, the subrepresentations $(V_i, \rho_{|V_i})$ is irreducible.

- Another way of stating the above is to say the representation (V, ρ) is equivalent to the direct sum of the subrepresentations $(V_1 \oplus \ldots \oplus V_n, \rho_1 \oplus \ldots \oplus \rho_n)$.

Some Facts

1. If a representation (V, ρ) is equivalent to a decomposable representation (V', ρ'), then (V, ρ) is also decomposable.
2. If a representation (V, ρ) is equivalent to an irreducible representation (V', ρ'), then (V, ρ) is also irreducible.
3. If a representation (V, ρ) is equivalent to a completely reducible representation (V', ρ') , then (V, ρ) is also completely reducible.
4. Every n-dimensional representation of a finite group over a field of characteristic 0 is completely reducible.

Challenge 1

Let $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z} \right\}$ under matrix multiplication and consider a $2-$ dimensional representation $(\mathbb{C}^2, \rho)$ where $\rho : G \longrightarrow GL(2, \mathbb{C})$ defined as $\rho(g)v = gv$ (normal multiplication between a matrix $g \in G$ and vector $v \in \mathbb{C}^2$) for all $g \in G$ and $v \in \mathbb{C}^2$. Is this representation completely reducible?

Challenge 2

Let $G = S_3$. You're encouraged to try construct the following representations:

1. A completely reducible $2-$ dimensional representation $(\mathbb{C}^2, \rho)$.
2. A completely reducible $4-$ dimensional representation $(\mathbb{C}^4, \rho)$.
3. A completely reducible $8-$ dimensional representation $(\mathbb{C}^8, \rho)$.

Unitary Representations

Definition 1.6

An $n-$ dimensional representation (V, ρ) of a group G is called unitary if for all $g \in G$, $\rho(g) \in U(n)$ where $U(n) \subset GL(n, \mathbb{C})$ is the unitary group.

- Note that this means the representation preserves the inner product on V i.e $\langle \rho(g)\psi_1, \rho(g)\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle$ for all $g \in G$ and $\psi_1, \psi_2 \in V$.
- An interesting fact is that every representation of a finite group G is equivalent to a unitary representation.

Cool Examples:

1. Let $G = \mathbb{R}$ with the group structure under addition. Then we can build a $1-$ dimensional unitary representation of $\mathbb{R}$ $(\mathbb{C}, \rho)$ with $\rho : \mathbb{R} \longrightarrow U(1)$ defined as $\rho(t) = e^{2\pi i t}$ for all $t \in \mathbb{R}$.
2. Let again $G = \mathbb{R}$ with the group structure under addition. Then we can build a $n-$ dimensional unitary representation of $\mathbb{R}$ $(\mathbb{C}, \rho)$ with $\rho : \mathbb{R} \longrightarrow U(n)$ defined as $\rho(t) = e^{-\frac{i}{\hbar}Ht}$ for all $t \in \mathbb{R}$, where H is a Hermitian matrix and $\hbar$ is the Planck's constant.

Challenge 3

Is any of the two cool examples in the previous slide a completely reducible representation?

QF

QUANTUM FORMALISM

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