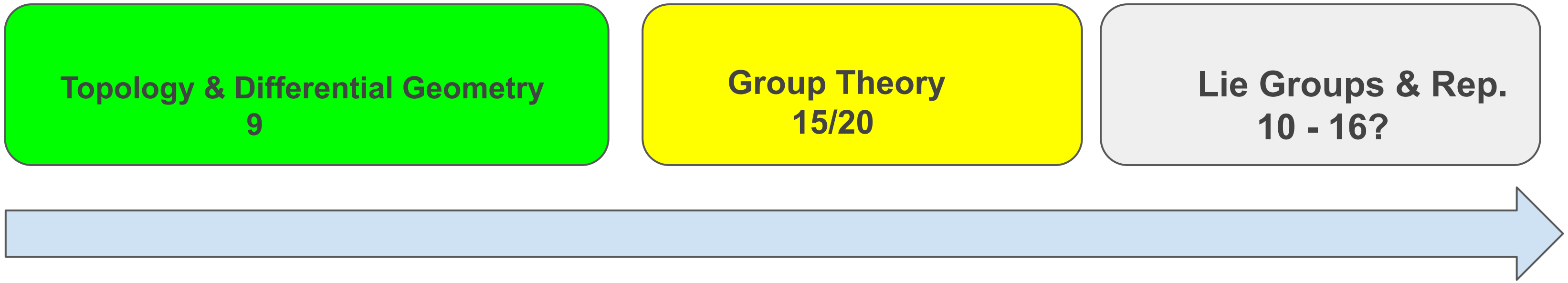
QF Group Theory CC2022
By
Zaiku Group

Lecture 15

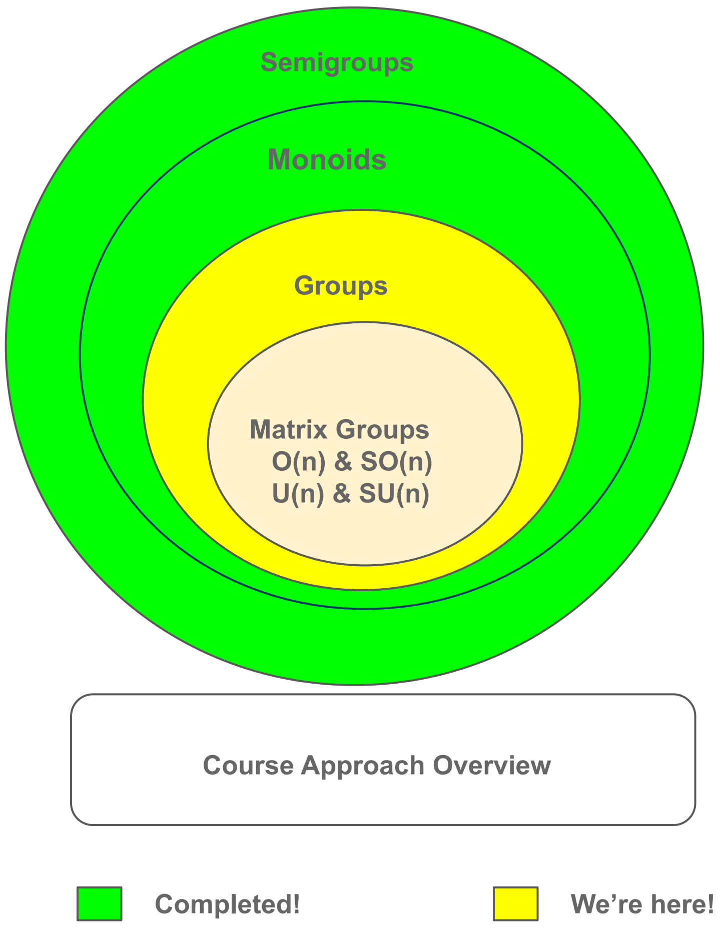
Delivered by Bambordé Baldé

Friday, 23/09/2022

<u>Learning Journey Timeline</u>



Completed | Ongoing | TBC (summer) | n is the number of live lectures |



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Quick Summary of Lecture 14 Concepts

- In lecture 14 we covered the following concepts:
  1. Left and right cosets.
  2. The index of a subgroup.
  3. Normal subgroups.
  4. Quotient groups.
- Today we'll layout some important results related to the concepts above before jumping to group actions.

Product of subgroups

Definition 1.0

Let G be a group, H and K be subgroups of G. The (internal) product HK is the set defined as $HK = \{hk \mid h \in H, k \in K\}$.

- HK is not necessarily a subgroup of G!

Theorem 1.0

Let G be a group, H and K be subgroups of G. Then the following hold:

1. HK is a subgroup iff $HK = KH$.
2. If either H or K are normal in G then HK is a subgroup of G.

Challenge 1

Let $G = S_3$, $H = \{1, (12)\}$ and $K = \{1, (13)\}$. You're encouraged to:

1. Compute the product HK.
2. Verify whether or not, HK is a subgroup of G. What about the product KH?

Quotient map

Definition 1.1

Let G be a group and $N \lhd G$. The quotient map $\pi : G \longrightarrow G/N$ is defined as $\pi(x) = xN$ for all $x \in G$.

- The quotient map π is a homomorphism right?
- The map π is also known as 'canonical projection'.

Challenge 2

Let G be a group, $N \lhd G$ and $\pi : G \longrightarrow G/N$ be the quotient map. Is it true that $Ker(\pi) = N$? If not, what is $Ker(\pi)$?

- A gentle reminder that $Ker(\pi) = \{x \in G \mid \pi(x) = 1_{G/N}\}$.

Isomorphism theorems (A)

Theorem 1.1 (The first isomomorphism theorem for groups)

If a map $f : G_1 \longrightarrow G_2$ is a group homomorphism, then $G_1/Ker(f) \simeq Im(f)$ i.e there exists an isomorphism $\phi : G_1/Ker(f) \longrightarrow Im(f)$.

Proof : Homework challenge (not very hard to prove)!

- The theorem is equivalent to saying there exists an isomorphism $\phi : G_1/Ker(\pi) \longrightarrow Im(f)$ such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Im(f) \subseteq G_2 \\
& G_1/Ker(\pi) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Challenge 3

Let $G_1 = \mathbb{Z}$ under ordinary addition and $G_2 = \mathbb{Z}_n$ under mod$n$ addition. Construct a homomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ and verify that Theorem 1.1.

Isomorphism theorems (B)

Theorem 1.2 (The second isomorphism theorem for groups)

Let G be a group, $N \triangleleft G$ and H a subgroup of G. Then the following hold:

1. HN is a subgroup of G.
2. $N \cap H \triangleleft G$.
3. $H/(N \cap H) \simeq HN/N$.

Proof : Do you fancy having a go? It's not technically very hard to prove!

Note: The third isomorphism theorem will be included in the final slide after the session!

Group Actions on Sets

Definition 1.2

Let G be a group and X a set. A left action of G on the set X is a 'rule' that takes a pair $(g, x) \in G \times X$ and produces an element $gx \in X$ such that the following conditions hold:

1. $1_G x = x$ for all $x \in X$.
2. $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1, g_2 \in G$ and $x \in X$.

- Observe that for $g \in G$ we can define a map (left translation) $g_L : X \longrightarrow X$ as $g_L(x) = gx$ for all $x \in X$. Also note that:
  1. g_L has an inverse map $(g^{-1})_L$ and so g_L is a bijection in X i.e. $g_L \in Symm(X)$!
  2. We can define a map $\sigma : G \longrightarrow Sym(X)$ as $\sigma(g) = g_L$ for all $g \in G$. The map σ is then a homomorphism right?

Important conclusion: A left $G-$ action on X gives us a homomorphism from G to $Sym(X)$ and conversely, every such homomorphism yields an action. Hence, a $G-$ action on $X \equiv$ group homomorphisms from G to $Sym(X)$!

Group Actions (Examples)

1. Let G be a group, we can define a left action on G that takes $(g, x) \in G \times G$ to $gx \in G$.
2. Let G be a group and H a subgroup of G. We can define a left $H-$ action that takes $(h, g) \in H \times G$ to $hx \in G$.
3. Let G be a group and N a normal subgroup of G. We can define a action on the set of left cosets of N that takes $(g, C) \in G \times G/N$ to $gC \in G/N$.

QUANTUM FORMALISM

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