QUANTUM FORMALISM

Quantum Axioms & Operators - Part 1

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Matrix Conjugate
3. Hermitian Conjugate
4. Hermitian Matrices
5. Skew-Hermitian Matrices
6. Unitary Matrices
7. Matrix Similarity

Part A

1. Dirac Notation Comments
2. Quantum Axioms Remarks
3. Study Material Comments
4. Physics Lecture Reference

Part B

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Axioms & Operators #3
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

Foundation Module 2 ?

Lie Theory (Groups & Algebras)	Measure Theory 101	???
Differential Topology 101	Functional Analysis 101	???
Topology 101	Complex Analysis 101 ?	Quantum Axioms Revisited?

PART A

Some Remarks

- From now on, unless otherwise stated, the Hilbert space will always be $\mathbb{C}^n$ where $n > 0$. So even if we are still using the abstract notation $\mathcal{H}$, just insert $\mathbb{C}^n$ in your mind!
- Recall that the elements of $M_n(\mathbb{C})$ act as linear operators on $\mathbb{C}^n$. In fact, any abstract linear operator acting on $\mathbb{C}^n$ can be represented as an element of $M_n(\mathbb{C})$ by (see study materials for Linear operators 101).
- If you are interested in quantum computing stuff, you'll probably want $n = 2^k$ where k is the number of qubits under consideration.
- To help get concrete picture of things we'll try to use $\mathbb{C}^2$ and $M_2(\mathbb{C})$ in the definitions/examples as much as possible.
- Finally, we'll write $GL(n, \mathbb{C})$ to denote the set of all invertible elements of $M_n(\mathbb{C})$. What algebraic structure $GL(n, \mathbb{C})$ forms?

Matrix Conjugate

Definition (1.0)

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$, we define $A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} \in M_2(\mathbb{C})$.

- Where a_{ij}^* is the conjugate of a_{ij} i.e. if $a_{ij} = x + yi$ then $a_{ij}^* = x - yi$.
- Mathematicians normally use the notation $\bar{a}_{ij}$ to denote the conjugate of a_{ij} and so use $\bar{A}$.
- The definition can be extended to matrices in $M_n(\mathbb{C})$.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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Conjugate Transpose

Definition (1.1)

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$, $A^\dagger = (A^*)^T = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \in M_2(\mathbb{C})$.

- $A^\dagger$ is also known as the Hermitian transpose of A and physicists often call it 'the A dagger'. Mathematicians use A^* instead of $A^\dagger$!
- The definition can be extended to matrices in $M_n(\mathbb{C})$.

Proposition (1.0)

Let $A, B \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$. The following identities hold:

1. $(A^\dagger)^\dagger = A$.
2. $(\lambda A)^\dagger = \lambda^* A^\dagger$.
3. $(A + B)^\dagger = A^\dagger + B^\dagger$.
4. $(AB)^\dagger = B^\dagger A^\dagger$.
5. $\det(A^\dagger) = det(A)^*$.
6. If A is invertible then $A^\dagger$ is also invertible.

Proof : Homework challenge?

- Of course, the properties above also apply to matrices in $M_n(\mathbb{C})$.

Hermitian Matrices

Definition (1.2)

$A \in M_n(\mathbb{C})$ is called Hermitian if $A = A^\dagger$.

- Which of the following $M_2(\mathbb{C})$ matrices are Hermitian?
  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proposition (1.0)

Let $A, B \in M_n(\mathbb{C})$. Then the following properties hold:

1. $A + A^\dagger$ is Hermitian.
2. $AA^\dagger$ and $A^\dagger A$ are Hermitian.
3. If A is Hermitian and invertible, then A^{-1} is Hermitian.
4. If A and B are Hermitian, then $\alpha A + \beta B$ is Hermitian for all $\alpha, \beta \in \mathbb{R}$.
5. If A is Hermitian then the main diagonal entries of A are all real.

Proof : Homework challenge! I really encourage you to try prove the above, even if for just $M_2(\mathbb{C})$!

- Since we are dealing with finite dimensional Hilbert spaces, we'll consider the term 'Self-adjoint' to be equivalent to Hermitian i.e. being Self-adjoint operator is the same as being Hermitian operator. This is not true if the space is infinite dimensional!

Skew-Hermitian Matrices

Definition (1.3)

$A \in M_n(\mathbb{C})$ is called skew-Hermitian if $A = -A^\dagger$.

- Physicists normally use the term 'anti-symmetric' instead of skew-Hermitian!

Proposition (1.1)

Let $A, B \in M_n(\mathbb{C})$. Then the following properties hold:

1. $A - A^\dagger$ is skew-Hermitian.
2. If A and B are skew-Hermitian, then $\alpha A + \beta B$ is skew-Hermitian for all $\alpha, \beta \in \mathbb{R}$.
3. If A is Hermitian, then iA is skew-Hermitian, where i is the imaginary unit in $\mathbb{C}$.
4. If A is skew-Hermitian, then iA is Hermitian!

Proof : Homework challenge! I really encourage you to try prove the above, even if for just $M_2(\mathbb{C})$!

Hermitian and skew-Hermitian Decomposition

Theorem (1.0)

Let $A \in M_n(\mathbb{C})$. Then $A = \frac{1}{2}(A + A^\dagger) + i((\frac{-i}{2})(A - A^\dagger))$.

An important note:

1. The term $\frac{1}{2}(A + A^\dagger) = H(A)$ is called the Hermitian part of A.
2. The term $i((\frac{-i}{2})(A - A^\dagger)) = S(A)$ is called the skew-Hermitian part of A.

- As homework challenge, consider the following $M_2(\mathbb{C})$ matrices and compute their Hermitian and skew-Hermitian parts.

For $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

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Diagonal Matrices

Definition (1.4)

$A \in M_n(\mathbb{C})$ is called a diagonal matrix if it has the following form:

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \text{ such that } a_{ij} = 0 \text{ if } i \neq j.$$

- The following elements of $M_2(\mathbb{C})$ are examples of diagonal matrices: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

Proposition (1.2)

If $A, B \in M_n(\mathbb{C})$ are diagonal, then AB is also diagonal and $AB = BA$.

Proof : Homework challenge?

Invertible Diagonal Matrices

Proposition (1.3)

Let $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. The following statements are true:

1. D is invertible iff $\lambda_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$.
2. $D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{pmatrix}$.

Proof : Homework challenge?

- What is the kernel of $D : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ if:
  1. $\lambda_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$.
  2. There is one $\lambda_i = 0$.
- D is called real diagonal matrix if λ_i is a real number for all $i \in \{1, 2, \ldots, n\}$.

Unitary Operators

Definition (1.5)

$A \in M_n(\mathbb{C})$ is unitary if $AA^\dagger = A^\dagger A = \mathbb{I}$.

- The following $M_2(\mathbb{C})$ operators are of course unitary: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition (1.6)

Given $\theta_1, \theta_2 \ldots, \theta_n \in \mathbb{R}$, we can construct a diagonal operator of the form $U(\theta_1, \theta_2, \ldots, \theta_n) = \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_n} \end{pmatrix} \in M_n(\mathbb{C})$.

- Is $U(\theta_1, \theta_2, \ldots, \theta_n)$ unitary? What is $U(0, \pi) \in M_2(\mathbb{C})$?

Matrix Similarity

Definition (1.7)

Let $A \in M_n(\mathbb{C})$. $B \in M_n(\mathbb{C})$ is said to be similar to A if there exists a matrix $S \in GL(n, \mathbb{C})$ such that $B = S^{-1}AS$.

- We write $B \sim A$ to denote the similarity relationship between B and A.

Proposition (1.3)

The similarity relationship $\sim$ is an equivalence relation in $M_n(\mathbb{C})$ i.e. the following properties hold:

1. $A \sim A$ for all $A \in M_n(\mathbb{C})$ (Reflexive).
2. If $B \sim A$ then $A \sim B$ for all $A, B \in M_n(\mathbb{C})$ (Symmetric).
3. If $B \sim A$ and $A \sim C$ then $B \sim C$ for all $A, B, C \in M_n(\mathbb{C})$ (Transitive).

Proof : Homework challenge!

PART B

Dirac Notation Comments

- Recall that the bra denoted $\langle\psi|$ is a linear functional i.e. $\langle\psi| \in \mathcal{H}^*$ (or $\mathcal{H}^\dagger$ if you're a physicist). The Riesz representation theory implies that every linear functional can be written as a bra.
- Given two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ we can construct an operator $\kappa_{\psi_1,\psi_2} : \mathcal{H} \longrightarrow \mathcal{H}$ as $\kappa_{\psi_1,\psi_2} |\phi\rangle = \langle\psi_2|\phi\rangle |\psi_1\rangle$. Physicists normally write $|\psi_1\rangle \langle\psi_2|$ to denote the operator κ_{ψ_1,ψ_2}!
- Physicist call the operator $|\psi\rangle\langle\psi|$ projection operator!
- Researchers in QC often write $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ to label the elements of an ON basis for $\mathbb{C}^n$ that they're working with. Where $n = 2^k$, with k being the number of qubits under consideration.
- Be aware physicists use similar notation as above when working with systems of $n-$ particles where $|n\rangle$ denotes a state with n particles e.g. $|1\rangle$ denotes a state with one particle!
- We will stick with the convention of writting $|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle$ to label elements of an abstract ON basis.

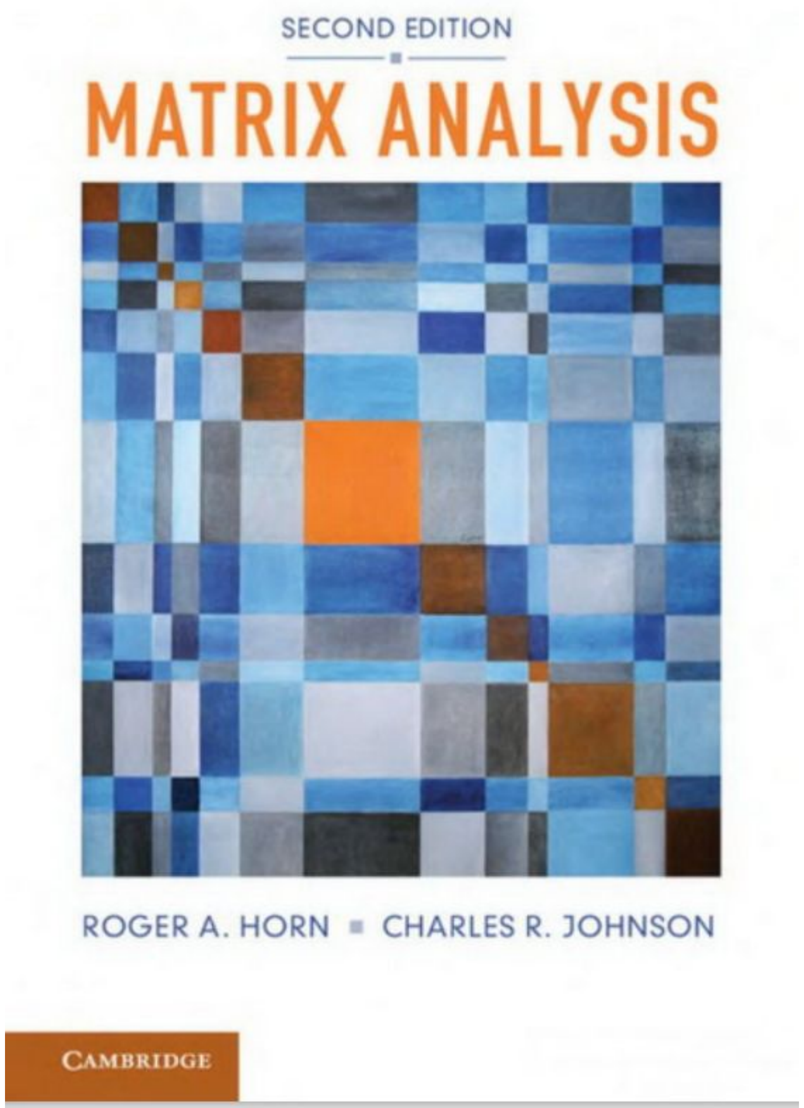
The Axioms of QM (Non Relativistic QM)

Axiom 1: The states of quantum systems are modelled by **normalised vectors** on **separable complex Hilbert spaces**. [✓]

- As previously mentioned, the axiom is actually referring to what physicists call 'pure states'. There are also the so-called mixed states!

Axiom 2: The observables of quantum systems are modelled by **self-adjoint operators** on separable complex Hilbert spaces. [✓]

- Since we are dealing with finite dimensional Hilbert spaces, then we know that self-adjoint operators correspond to Hermitian matrices!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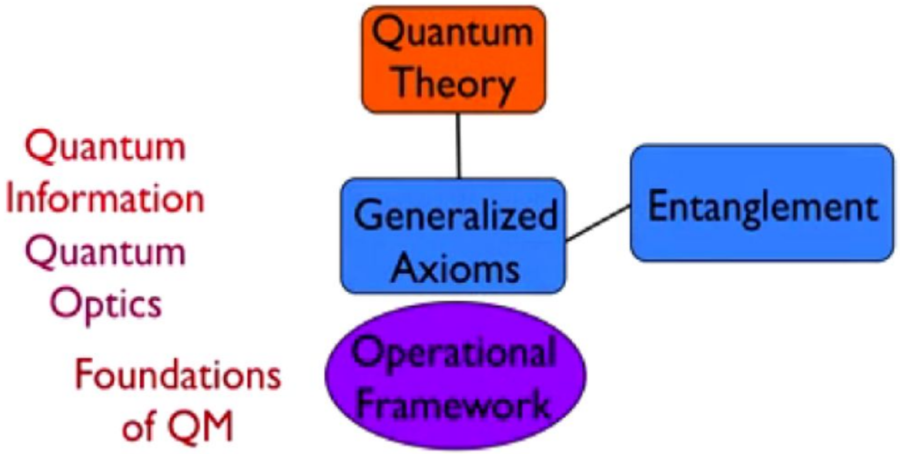

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