QF

QUANTUM FORMALISM

Quantum Axioms & Operators - Part 2

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Eigenvectors & Eigenvalues
3. Spectrum & Eigenspaces
4. Characteristic Polynomial
5. Diagonalization
6. The Spectral Theorem (Hermitian version)

Part A

1. Quantum Axioms Remarks
2. Shut Up & Calculate Challenge
3. Study Materials Reference
4. Certification Comments

Part B

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Axioms & Operators 2/4
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

PART A

''Quantum mechanics is about as abstract as anything can be. Quantum mechanics is about things that your evolutionarily developed neural structure is not prepared to deal with directly.

How do you deal with it? Abstract mathematics! It's also true of relativity, but it's worst in quantum mechanics!''

Professor Leonard Susskind

Pre- Session Key Remarks

- In the previous session we saw that for systems modelled with finite dimensional Hilbert spaces, the observables of the systems such as the energy (aka Hamiltonian) are encoded in 'Self-adjoint' operators i.e. Hermitian matrices.

- For systems modelled with infinite dimensional Hilbert spaces, it is more accurate to say the observables are encoded in 'Self-adjoint' operators.

- We defined the concept of $A \in M_n(\mathbb{C})$ being Hermitian if $A = A^\dagger$. But we can alternatively say A is Hermitian if $\langle A\psi_1, \psi_2 \rangle = \langle \psi_1, A\psi_2 \rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H} = \mathbb{C}^n$ where $\langle \cdot, \cdot \rangle$ is obviously the inner product in $\mathcal{H}$.

- Two legitimate questions that one may ask are:
  1. Why Hermitian operators?
  2. How do physicists decide which Hermitian operator is suitable to encode a particular observable of a system?

Eigenvectors and Eigenvalues

Definition (1.0)

Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. A vector $|\psi\rangle \in \mathcal{H}$ is said to be an eigenvector of A if there exists a $\lambda \in \mathbb{C}$ such that $A|\psi\rangle = \lambda|\psi\rangle$. The scalar λ is called an eigenvalue of A.

- It's easy to prove that if $|\psi\rangle \in \mathcal{H}$ is an eigenvector of A with the corresponding eigenvalue λ. Then $|\psi\rangle$ is also an eigenvector of $A^2 = AA$ with eigenvalue λ^2!
- Does the property above also hold for any $n > 2$ i.e. is $|\psi\rangle$ an eigenvector of A^n with corresponding eigenvalue λ^n ?
- Suppose that $A \in GL(n, \mathbb{C})$. Then $|\psi\rangle$ is an eigenvector of A^{-1} with corresponding eigenvalue $\frac{1}{\lambda}$!
- Is it true that if 0 is an eigenvalue of A with respect to $|\psi\rangle$ then $|\psi\rangle \in \mathrm{Ker}(A)$? Hence, if 0 is an eigenvalue of A then $A \notin GL(n, \mathbb{C})$?

The Spectrum

Definition (1.1)

The set of all eigenvalues of A is called the spectrum of A and often denoted $\sigma(A)$ i.e $\sigma(A) = \{\lambda \in \mathbb{C} \mid A|\psi\rangle = \lambda|\psi\rangle\}$ where $|\psi\rangle \in \mathcal{H}$ is obviously an eigenvector of A.

- The definition above is technically known as 'point spectrum' aka discrete spectrum.
- Since if $A \in GL(n, \mathbb{C})$ and $\lambda \in \sigma(A)$ then $\frac{1}{\lambda} \in \sigma(A^{-1})$. Then this means that if we know $\sigma(A)$ we automatically know $\sigma(A^{-1})$.
- But how big is $\sigma(A)$ i.e. what is the cardinality of $\sigma(A)$?

Theorem (1.0)

If A is Hermitian then $\sigma(A) \subseteq \mathbb{R}$ i.e. the eigenvalues of A are real numbers if A is Hermitian.

Proof : Homework? Try at least proving for the case $A \in M_2(\mathbb{C})$?

Suppose that $\lambda \in \sigma(A)$ with eigenvector $|\psi\rangle$. Then by the definition of eigenvector and the property of the inner product, we have that $\langle A\psi, \psi\rangle = \langle \lambda\psi, \psi\rangle = \lambda^*\langle\psi, \psi\rangle$. Since A is Hermitian, then $\langle A\psi, \psi\rangle = \langle\psi, A\psi\rangle = \langle\psi, \lambda\psi\rangle = \lambda\langle\psi, \psi\rangle$.
Subtracting the first identity above from the latter, we get $(\lambda - \lambda^*)\langle\psi, \psi\rangle = 0$ Now, because $|\psi\rangle \neq |0_{\mathcal{H}}\rangle$ then $\langle\psi, \psi\rangle \neq 0$. Hence, we must have $\lambda = \lambda^*$ for $(\lambda - \lambda^*)\langle\psi, \psi\rangle = 0$ to be true and so $\lambda \in \mathbb{R}$ i.e. λ is a complex number with zero complex part i.e. λ has only real component - complex numbers 101!

The Characteristics Polynomial

Definition (1.2)

The characteristic polynomial of $A \in M_n(\mathbb{C})$ is defined as $p_A(z) = \det(z\mathbb{I} - A)$ where $z \in \mathbb{C}$. $p_A(z)$ is a polynomial of degree n over $\mathbb{C}$.

- An alternative definition is $p_A(z) = \det(A - z\mathbb{I})$. This makes no material difference in terms of $p_A(z) = 0$. In fact, the relationship between the two definitions is given by $\det(A - z\mathbb{I}) = (-1)^n \det(z\mathbb{I} - A)$.

Theorem (1.1)

Let $A \in M_n(\mathbb{C})$. Then $\lambda \in \sigma(A)$ iff $p_A(\lambda) = 0$ i.e. if λ is a zero of the characteristic polynomial.

- For $A \in M_2(\mathbb{C})$, the characteristics polynomial has the form $p_A(z) = z^2 - Tr(A) + det(A)$. Hence, to find the eigenvalues of any $A \in M_2(\mathbb{C})$, we need to solve the second degree polynomial

$$z^2 - Tr(A)z + det(A) = 0.$$

- Interestingly, $p_A(z) = p_B(z)$ for all $A, B \in M_n(\mathbb{C})$ such that $A \sim B$. Hence, if $A \sim B$ then they have the same eigenvalues!

λ-Eigenspace

Definition (1.3)

Fixed an operator A action on $\mathcal{H}$, the set of all its eingenvectors $|\psi\rangle \in \mathcal{H}$ for which λ is an eigenvalue is called λ-eigenspace and often denoted E_λ i.e. $E_\lambda = \{|\psi\rangle \in \mathcal{H} \mid A|\psi\rangle = \lambda \cdot |\psi\rangle\}$.

- E_λ is a subspace of $\mathcal{H}$. The dimension of E_λ is called the geometric multiplicity of λ.
- There is also 'algebraic multiplicity', which is the number of times that λ appears as a root of $p_A(z)$.

Physicist notation awareness:

- Suppose that $A \in M_n(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ with corresponding eigenvectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \in \mathcal{H}$. Then physicists often write $|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_n\rangle$ to denote these eigenvectors!

Matrix Diagonalization

Definition (1.4)

$A \in M_n(\mathbb{C})$ is said to be diagonalizable if $A \sim D$ where $D \in M_n(\mathbb{C})$ is a diagonal matrix.

- Some authors use the term 'diagonable'!

Theorem (1.2)

Let $A \in M_n(\mathbb{C})$. Then the following statements are equivalent:

1. A is diagonalizable.
2. There is a linearly independent set of vectors $B = \{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ in $\mathcal{H}$ such that $|b_i\rangle$ is an eigenvector of A for all $i \in \{1, \ldots, n\}$.
3. A has an n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Proof : Fancy proving it for at least $A \in M_2(\mathbb{C})$?

- Given that $\mathcal{H} = \mathbb{C}^n$, what could such linearly independent set of vectors B give us?

The Spectral Theorem (Hermitian matrices)

Theorem (1.3)

$A \in M_n(\mathbb{C})$ is Hermitian iff there exists a unitary matrix $U \in M_n(\mathbb{C})$ and real diagonal matrix $D \in M_n(\mathbb{C})$ such that $A = UDU^\dagger$.

Proof : Do you fancy proving it for the $M_2(\mathbb{C})$ case?!

- This means that Hermitian matrices are diagonalizable! Hence, the eigenvectors of any Hermitian $A \in M_n(\mathbb{C})$ form a basis in $\mathcal{H} = \mathbb{C}^n$ (theorem 1.2). In fact, it can be proved this basis is orthonormal!
- With some mathematical creativity using results such as Schur decomposition, we can proved that:
  1. The eigenvectors of A coincide with the columns of U!
  2. The eigenvalues of A coincide with the diagonal entries of matrix D!

PART B

The Axioms of QM (Non Relativistic QM)

Axiom 1: The states of quantum systems are modelled by **normalised vectors** on **separable complex Hilbert spaces**. [✓]

- As previously mentioned, the axiom is actually referring to what physicists call 'pure states'. There are also the so-called mixed states!

Axiom 2: The observables of quantum systems are modelled by **self-adjoint operators** on separable complex Hilbert spaces. [✓]

- Since we are dealing with finite dimensional Hilbert spaces, then we know that self-adjoint operators correspond to Hermitian matrices!

Axiom 3: Given a state vector $|\psi\rangle \in \mathcal{H}$ that encodes a particular state of a quantum system, and a self-adjoint operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ that encodes a particular observable of the same system. The measurement expectation value of the observable encoded in A in the state encoded with $|\psi\rangle$ is computed as $\langle A \rangle_\psi = \langle \psi, A\psi \rangle$.

- In Dirac notation the expectation value is $\langle A \rangle_\psi = \langle \psi | A | \psi \rangle$.
- In quantum mechanics, the expectation value is the probabilistic expected value of the result (measurement) of an experiment. [Wikipedia]
- With a little bit of mathematical creativity, axiom 3 can be used to deduct that if $\sigma(A)$ is discrete. Then the only possible measurement values for the observable encoded in A are its eigenvalues!
- The famous 'Born rule' can also be deducted from axiom 3!

Shut Up & Calculate Challenge

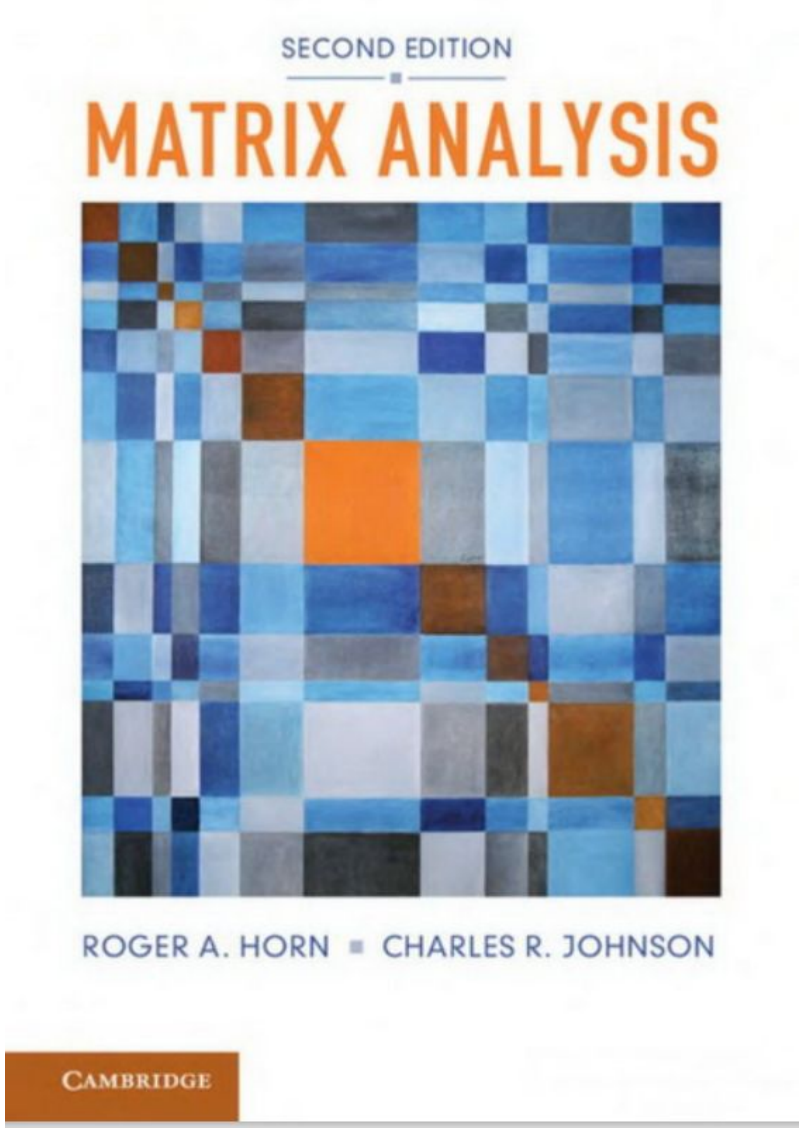
Consider a system S modelled on the Hilbert space $\mathbb{C}^2$ with three observables encoded with the following Hermitian operators in $\mathbb{C}^2$:

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Calculate $\langle \sigma_X \rangle, \langle \sigma_Y \rangle$ and $\langle \sigma_Z \rangle$ in the following state vectors:

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \ |\psi_4\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ and } |\psi_5\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}.$$

- Find the eigenvalues of σ_X, σ_Y and σ_Z. You can use software but encouraged to manually calculate!


SECOND EDITION
MATRIX ANALYSIS
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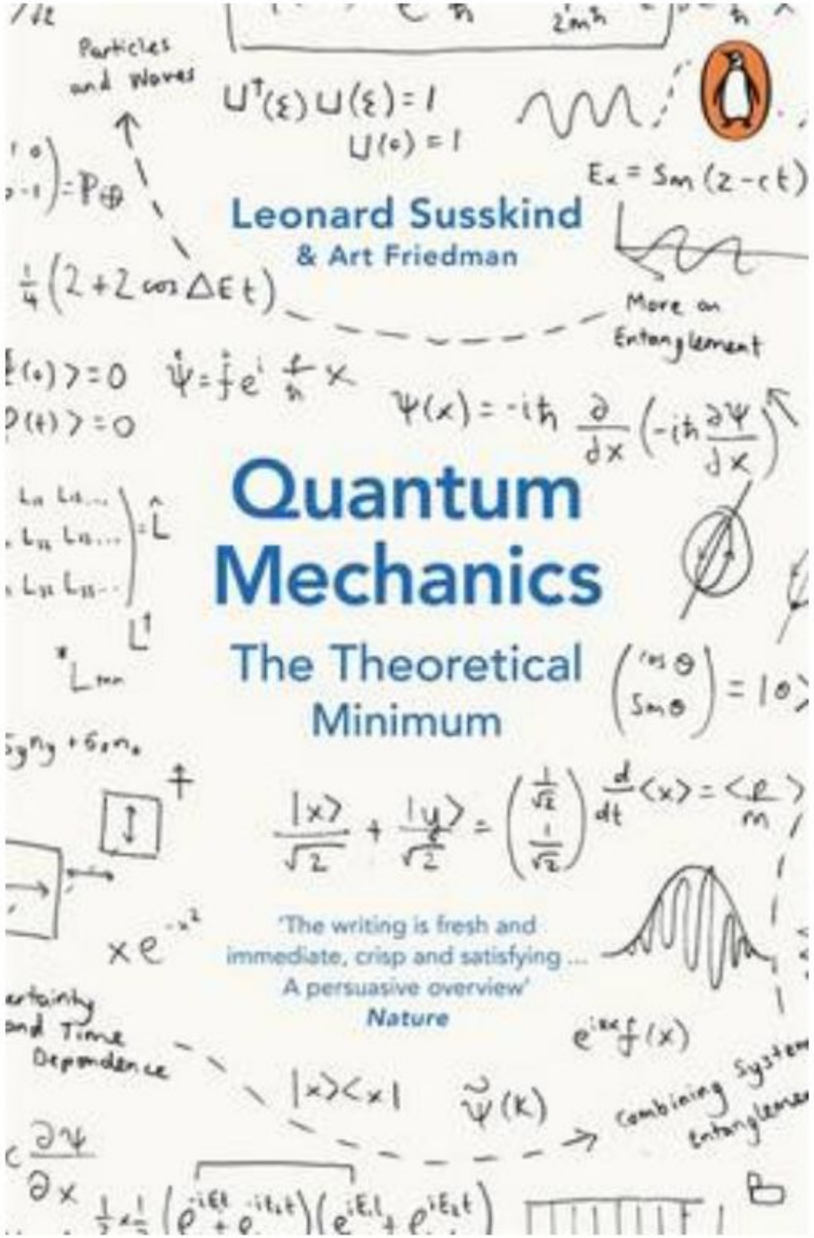
Leonard Susskind
& Art Friedman
Quantum Mechanics
The Theoretical Minimum
'The writing is fresh and immediate, crisp and satisfying ... A persuasive overview'
Nature

Module Certification

Career Recommendation to Zaiku Group and Partners

Individual submission

OR

Group submission

*Theoretical or practical work e.g. contribution to an existing open source project or create a new one!

The work has to contain a mathematical element to it.

- **GitHub (Curated study materials):** *github.com/quantumformalism*
- **YouTube:** youtube.com/zaikugroup
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