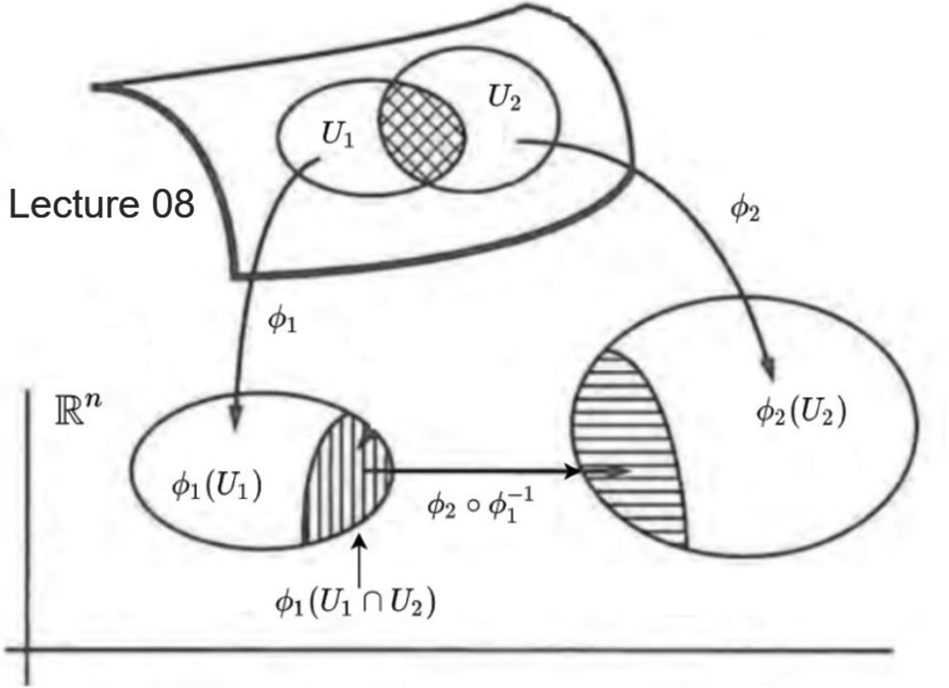
Topology & Differential Geometry Crash Course: Lecture 08



QUANTUM FORMALISM

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Bambordé Baldé | Co-Founder at Zaiku Group | Twitter: @zaikubalde • **zaikugroup.com** • October 29, 2021

Session Agenda

Pre-lecture
blah blah blah

1. Atlases
2. C^k & Smooth Atlases
3. Atlas Challenge
4. Compatibility of Atlases

PART A

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1. Maximal Atlases
2. Maximal Atlas Existence
3. Smooth Manifolds
4. Smooth Manifold Challenge
5. **Study Materials**
6. **Lecture Gap Poll**

PART B

Lecture 8 Coverage

General Coverage

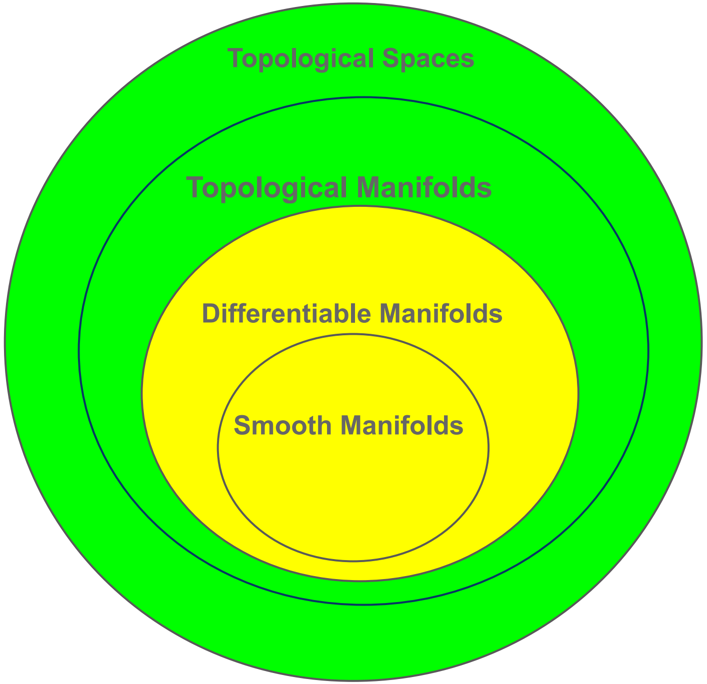
Including a brief intro to Fiber Bundles?

Point-set Topology 101
2/2

Metric Topology 101
3/3

Manifolds 101
3/4

#n is the number of live lectures | Completed | We're here!



Topology Crash Course

Completed | We're here!

Crash Course Motivation

Lie Groups, Lie Algebras & Representations

Module II (January 2022 start date?)

Lie group

From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.

Community Announcement

- Potential fast track to a pilot incubator program
- Cash prizes for teams that demonstrate the application of concepts covered in Module II to their solutions
- Maximum 3 people per team
- Maximum 15 teams
- After Module II

QF Hackathon 2022

PART A

Atlases on Manifolds

Definition (1.0)

Let X be an $n-$ dimensional topological manifold. An indexed collection of charts on X written $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ is called an atlas if $X = \bigcup\limits_{i \in I} U_i$.

- Note that the index I can be finite or infinite.
- Hence, from the definition of an $n-$ dimensional topological manifold, it follows that an atlas always exists!
- Obviously the chart domains U_i of $\mathcal{A}$ form an open cover for X. What about the other way around i.e. does an open cover $\mathcal{C}$ of X necessarily forms an atlas?
- Let $X = \mathbb{R}^n$ with the standard chart $(\mathbb{R}^n, id_{\mathbb{R}^n})$ where $id_{\mathbb{R}^n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map. Then $\mathcal{A} = \{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$ is an atlas (standard atlas).

C^K and Smooth Atlases

Definition (1.1)

Let X be an $n-$ dimensional topological manifold and $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ be an atlas i.e. $X = \bigcup_{i \in I} U_i$. The we can make the following definitions:

1. $\mathcal{A}$ is called a C^k- atlas if given any two charts (U_1, ϕ_1) and (U_2, ϕ_2) in $\mathcal{A}$, the two charts are C^k-compatible for some $k \geq 1$ i.e. $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ are both C^k maps for some $k \geq 1$.
2. $\mathcal{A}$ is called a smooth- atlas if given any two charts (U_1, ϕ_1) and (U_2, ϕ_2) in $\mathcal{A}$, the two charts are smooth-compatible i.e. $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ are both smooth maps.

- Since the transition maps $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ are inverses of each other, it means they are diffeomorphisms on $\mathbb{R}^n$!

Some Comments

- As you might have guessed, for our purposes, we are interested in smooth atlases. In fact smooth atlases that satisfy some extra conditions that we'll define soon!

- Based on the previous definitions of C^k and smooth atlases, given an $n-$ dimensional manifold X, it is tempting to jump to the conclusion and make the following definitions:
  1. X is a C^k (or differentiable) manifold if it has a C^k- atlas $\mathcal{A}$.
  2. Similarly, X is a smooth manifold if it has a smooth- atlas $\mathcal{A}$.

- From now on, we'll just consider smooth atlases.

- Once we fix a smooth atlas $\mathcal{A}$, it is very important to know which charts on X are we allowed to use with $\mathcal{A}$. Hence, we want only charts that don't destroy the smoothness of $\mathcal{A}$ i.e. charts that are compatible with $\mathcal{A}$!

Atlas Challenge

- Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
    1. Let $U_1 = S^1 \setminus (0, -1)$ and $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = \frac{x}{y+1}$.
    2. Let $U_2 = S^1 \setminus (0, 1)$ with $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_2$ let us define the map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = \frac{x}{1-y}$.

  Challenge question 1: Does $\mathcal{A} = \{U_1, U_2\}$ form atlas on S^1? If yes, is it a smooth or just C^k atlas for some $k \geq 1$.

- Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
    1. Let $U_1 = S^2 \setminus (0, 0, -1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x, y, z) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = (\frac{x}{z+1}, \frac{y}{z+1})$.
    2. Let $U_2 = S^2 \setminus (0, 0, 1)$ and $\hat{U} = \mathbb{R}^2$. For each $p = (x, y, z) \in U_2$ let us define $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = (\frac{x}{1-z}, \frac{y}{1-z})$.

Challenge question 2: Does $\mathcal{A} = \{U_1, U_2\}$ form atlas on S^2? If yes, is it a smooth or just C^k atlas for some $k \geq 1$.

Smooth Atlas Compatibility

Definition (1.2)

Let X be an $n-$ dimensional topological manifold and $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ be a smooth atlas. Then a chart $(\tilde{U}, \tilde{\phi})$ on X is said to be compatible with the atlas $\mathcal{A}$ if the transition maps between such a chart $(\tilde{U}, \tilde{\phi})$ and any chart $(U_i, \phi_i) \in \mathcal{A}$ are diffeomorphisms (on $\mathbb{R}^n$).

- Another way to state the above is that $(\tilde{U}, \tilde{\phi})$ is compatible with $\mathcal{A}$ if the union $\mathcal{A} \cup \{(\tilde{U}, \tilde{\phi})\}$ is also a smooth atlas.

Definition (1.3)

Let X be an $n-$ dimensional topological manifold and $\mathcal{A}_1$ and $\mathcal{A}_2$ be two smooth atlases. The atlas $\mathcal{A}_2$ is said to be compatible with $\mathcal{A}_1$ if every chart in $\mathcal{A}_2$ is compatible with $\mathcal{A}_1$ in the sense of definition 1.2 above.

- Another way to state the above is that $\mathcal{A}_2$ is compatible with $\mathcal{A}_1$ iff their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas.

Proposition (1.0)

Let $\mathcal{A}_1 = \{(U_i, \phi_i) \mid i \in I\}$ and $\mathcal{A}_2 = \{(U_j, \phi_j) \mid j \in J\}$ be two compatible smooth atlases on X. If a chart $(\tilde{U}, \tilde{\phi})$ is compatible with $\mathcal{A}_1$, then $(\tilde{U}, \tilde{\phi})$ is also compatible with $\mathcal{A}_2$.

Proof : Homework challenge!

- Note that the proposition above implies that compatibility defines an equivalence relation on smooth atlases.

- Some authors use the equivalence class of smooth compatible atlases on X to define the notion of 'smooth structure' on X. We'll take a more direct and easier to understand approach!

PART B

Maximal Atlases

Definition (1.4)

A smooth atlas $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ on X is said to be maximal if it is not properly contained in any larger smooth atlas i.e. there is no smooth atlas $\tilde{\mathcal{A}}$ on X such that $\mathcal{A} \subset \tilde{\mathcal{A}}$.

- Another equivalent formal way to say the above is that $\mathcal{A}$ is maximal if given any chart $(\tilde{U}, \tilde{\phi})$ on X, $(\tilde{U}, \tilde{\phi})$ is smooth compatible with $\mathcal{A}$ iff $(\tilde{U}, \tilde{\phi}) \in \mathcal{A}$. In plain English, $\mathcal{A}$ is maximal if it contains all the charts that are compatible with itself!

Definition (1.5)

A maximal atlas $\mathcal{A}$ on X is called a smooth structure on X.

- A very natural question that we may ask is: How many maximal atlases are there for X i.e. how many smooth structures are there on X?

Maximal Atlas Existence

Theorem (1.0)

Let X be an $n-$ dimensional topological manifold and $\mathcal{A}$ be a smooth atlas on X. Then there exists a unique maximal atlas $\bar{\mathcal{A}}$ that contains $\mathcal{A}$.

Proof : Textbook or try prove it yourself?!

- $\bar{\mathcal{A}}$ is called the smooth structure determined by $\mathcal{A}$.
- $\bar{\mathcal{A}}$ is normally constructed as the set of all charts in X that are smooth compatible with $\mathcal{A}$.
- Hence, to build a smooth structure on X, you don't need to waste time trying to find or construct a maximal atlas from scratch. All you need is to find a smooth atlas $\mathcal{A}$ and then use $\bar{\mathcal{A}}$!
- Very often people abuse notation and only mention or write a smooth atlas $\mathcal{A}$ when considering a smooth structure. But in reality, $\bar{\mathcal{A}}$ is the actual smooth structure!

- It can be proved that two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ determine the same smooth structure $\bar{\mathcal{A}}$ iff they are smooth compatible i.e. iff their union $\mathcal{A}_1 \cup \mathcal{A}$ is smooth.

Smooth Manifolds

Definition (1.6)

A smooth manifold is a pair $(X, \mathcal{A})$ where X is an $n-$ dimensional topological manifold and $\mathcal{A}$ is a smooth structure on X.

- As we have seen in the topological spaces section, the choice of a topology matters in determining the key properties of a topological space e.g, choosing a particular underlying topology may create a whole different topological space than another choice of topology.
- It turns out that in some dimensions (many of them!), the choice of the smooth structure matters too! In other words, the same topological manifold X may generate two completely different smooth manifolds $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$! We'll get back to this in the next session when we cover diffeomorphisms between manifolds.

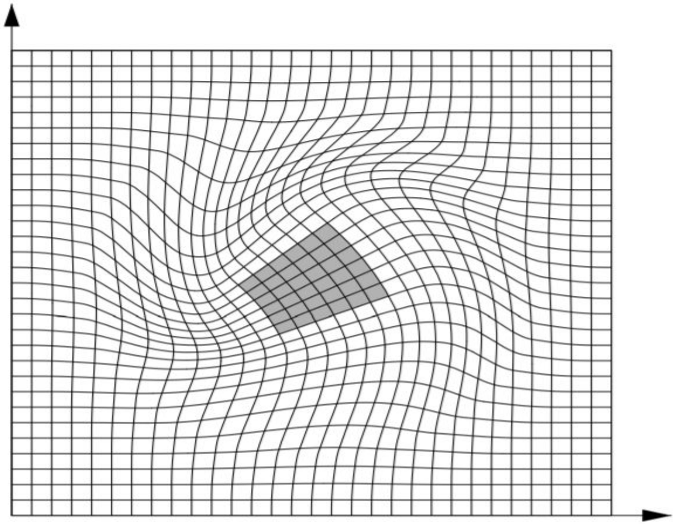
- Worth nothing that not all topological manifolds can be made into smooth manifolds i.e. there are topological manifolds that have no smooth structures!

Side note: For dimension $n \leq 3$ it was proved that any topological manifold can be equipped with a smooth structure. However, for dimension $n \geq 4$ there are manifolds that do not have smooth structures!

Smooth Manifold Challenge

- Prove that the following topological manifolds are smooth manifolds by constructing concrete smooth structures on them:
  1. $\mathbb{R}^n$ for $n \geq 1$.
  2. The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
  3. The torus $\mathbb{T}^2 = S^1 \times S^1$.
  4. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

Multivariable Calculus



4.6 Higher Order Derivatives

Partial differentiation can be carried out more than once on nice enough functions. For example if

$$f(x,y) = e^{x \sin y}$$

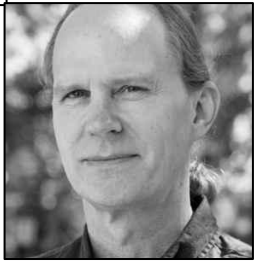
then

$$D_1 f(x,y) = \sin y e^{x \sin y}, \quad D_2 f(x,y) = x \cos y e^{x \sin y}.$$

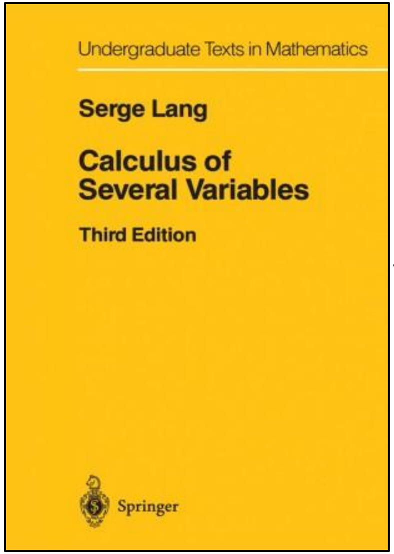
Taking partial derivatives again yields

$$\begin{aligned}
D_1 D_1 f(x,y) &= \sin^2 y e^{x \sin y},\\
D_1 D_2 f(x,y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y},\\
D_2 D_1 f(x,y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y} = D_1 D_2 f(x,y),\\
D_2 D_2 f(x,y) &= -x \sin y e^{x \sin y} + x^2 \cos^2 y e^{x \sin y},
\end{aligned}$$

and some partial derivatives of these in turn are,



Prof. Jerry Shurman



CHAPTER VI

Higher Derivatives

In this chapter, we discuss two things which are of independent interest. First, we define partial differential operators (with constant coefficients). It is very useful to have facility in working with these formally.

Secondly, we apply them to the derivation of Taylor's formula for functions of several variables, which will be very similar to the formula for one variable. The formula, as before, tells us how to approximate a function by means of polynomials. In the present theory, these polynomials involve several variables, of course. We shall see that they are hardly more difficult to handle than polynomials in one variable in the matters under consideration.

The proof that the partial derivatives commute is tricky. It can be omitted without harm in a class allergic to theory, because the technique involved never reappears in the rest of this book.

§1. *Repeated partial derivatives*

Let f be a function of two variables, defined on an open set U in 2-space. Assume that its first partial derivative exists. Then $D_1 f$ (which we also write $\partial f/\partial x$ if x is the first variable) is a function defined on U. We may then ask for its first or second partial derivatives, i.e. we may form $D_2 D_1 f$ or $D_1 D_1 f$ if these exist. Similarly, if $D_2 f$ exists, and if the first partial derivative of $D_2 f$ exists, we may form $D_1 D_2 f$.

Suppose that we write f in terms of the two variables (x, y). Then we can write

$$D_1 D_2 f(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (D_1(D_2 f))(x, y),$$

and

$$D_2 D_1 f(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (D_2(D_1 f))(x, y).$$

For example, let $f(x, y) = \sin(xy)$. Then

$$\frac{\partial f}{\partial x} = y\cos(xy) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x\cos(xy).$$

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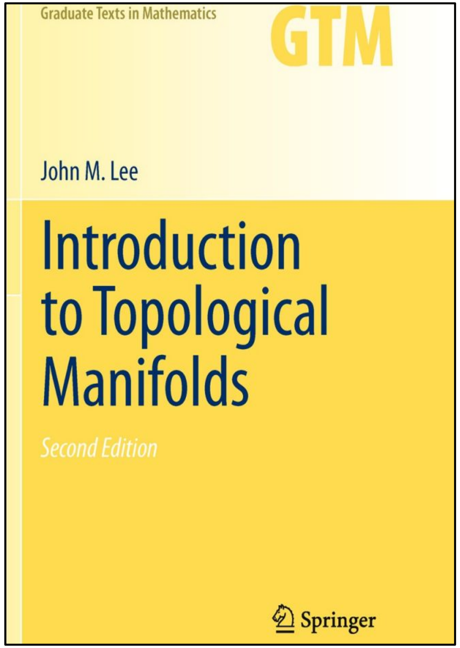

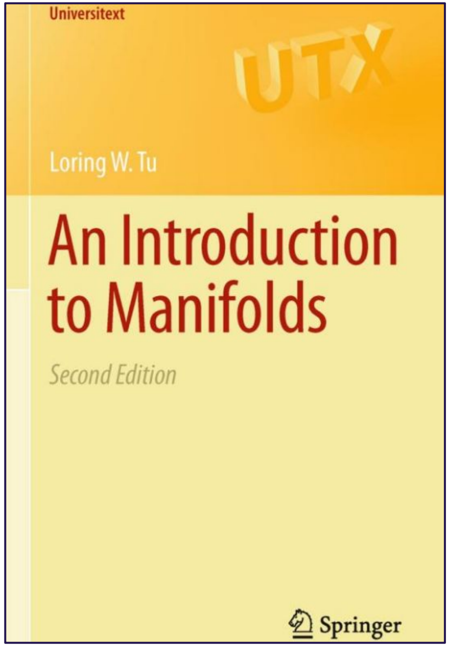

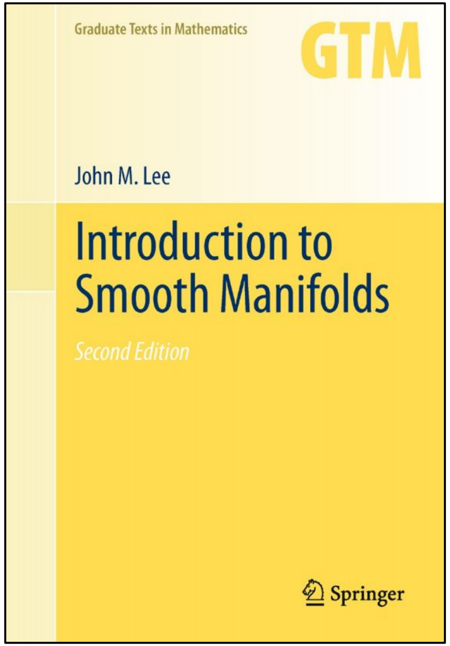

Topological & Diff. Manifolds

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