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Homework 6

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $f : X \to Y$. Suppose $X = A \cup B$ where $A, B \in \mathcal{M}$. Show that f is measurable on X if and only if f is measurable on A and B.

2. Suppose $f, g : X \to \mathbb{R}$ are measurable, and let $c \in \mathbb{R}$. Prove that fg and cf are both measurable.

   Hint: To show fg is measurable, first prove that f^2 is measurable, then note $2fg = (f+g)^2 - f^2 - g^2$.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

   Hint: Suppose f is increasing. Let $a \in \mathbb{R}$, $z = \sup\{y \mid f(y) \leq a\}$, and consider $f^{-1}((a, \infty))$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Show that there exists a Borel measurable function g such that $f = g$ a.e.

   Hint: Consider a sequence of Lebesgue measurable simple functions that approach the function f, and then modify them to be Borel measurable.
