QUANTUM FORMALISM

Vector Spaces 101 - Part 2

Bambordé Baldé | Co-Founder at Zaiku Group | Twitter: @zaikubalde • **zaikugroup.com** • October 16, 2020

Lecture Agenda Summary

1. Foundation Module Comments
2. Lecture 04 Recap
3. Linear Independence
4. Bases (Hamel)
5. Dimensions
6. Direct Sum of Subspaces
7. Study Material Comments
8. Tuesday Fireside Chat Comments

Refined Foundation Module

Rings and Fields 101 #1	Matrix Algebra #1	Quantum Matrix Operators #2
Group Theory 101 #1	Linear Operators 101 #2	Complex Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups: GL(n, C) & U(n) #2

Completed | Ongoing | #n is the number of live lectures

Proposition (Lecture 04 homework)

A subset $L \subseteq V$ is a linear subspace of V if only if the following conditions hold:

1. $\psi_1 + \psi_2 \in L$ for all $\psi_1, \psi_2 \in L$ i.e. addition is closed in L.
2. $\alpha\psi \in L$ for all $\psi \in L$ and $\alpha \in \mathbb{C}$ i.e. scalar multiplication in L is also closed.

Proposition (Lecture 04 homework)

If L and W are linear subspaces of V then $L \cap W$ is also a subspace of V.

Proof : By the proposition above, you only needed to prove that:

1. If $\psi_1, \psi_2 \in L \cap W$ then $\psi_1 + \psi_2 \in L \cap W$
2. If $\psi \in L \cap W$ then $\alpha\psi \in L \cap W \; \forall \alpha \in \mathbb{C}$

- We can actually generalise to an arbitrary number of subspaces i.e. if $L_1, L_2, \ldots, L_n$ are linear subspaces, then $L_1 \cap L_2 \cap \ldots \cap L_n$ is also a subspace.

Linear Combinations

Definition (Lecture 04)

Let $\psi_1, \psi_2, ..., \psi_n \in V$. Their linear combination is defined as the sum $\sum_{i=1}^{n} \alpha_i \psi_i = \alpha_1\psi_1 + \alpha_2\psi_2 + ... + \alpha_n\psi_n$ where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$.

- Let $V = \mathbb{C}^2$ and consider the example of the two famous vectors $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If we make $\alpha_1 = \frac{1}{\sqrt{2}}$ and $\alpha_2 = \frac{i}{\sqrt{2}}$, then we can create the following linear combination:
  $\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$
- At some point you'll figure it out that what physicists and quantum computing people call 'superposition' is just a special type of linear combination!☺

Definition (Lecture 04)

Let $\psi_1, \psi_2, \ldots\ldots, \psi_n \in V$. Their linear span is defined as

$$\text{Span}(\psi_1, \psi_2, \ldots\ldots, \psi_n) = \left\{ \sum_{i=1}^{n} \alpha_i \psi_i \mid \alpha_i \in \mathbb{C} \right\}$$ i.e. $\text{Span}(\psi_1, \ldots\ldots, \psi_n)$ is the set of all possible linear combinations of the vectors $\psi_1, \psi_2, \ldots\ldots, \psi_n$.

- So a vector $\psi \in \text{Span}(\psi_1, \psi_2, \ldots\ldots, \psi_n)$ if there exist some $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in \mathbb{C}$ such that $\psi = \sum_{i=1}^{n} \alpha_i \psi_i$.
- Does $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \in \text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$? The answer is yes because $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Proposition (Lecture 04)

$\text{Span}(\psi_1, \psi_2, \ldots\ldots, \psi_n)$ is a linear subspace of V.

You only needed to prove the following two things:

1. $+$ is closed in $\text{Span}(\psi_1, \psi_2, \ldots\ldots, \psi_n)$.
2. The scalar product is closed in $\text{Span}(\psi_1, \psi_2, \ldots\ldots, \psi_n)$.

Dirac Notation (Ket)

- Given a vector space V over $\mathbb{C}$. We'll write $|\psi\rangle$ to denote an element of V instead ψ i.e. we'll put the elements of V inside these strange objects $|\rangle$ called kets!
- As such linear combinations will now look like $|\psi\rangle = \sum_{i=1}^{n} \alpha_i|\psi_i\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle .... + \alpha_n|\psi_n\rangle$.
- In quantum computing there is a special convention of denoting the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ as follows:

  $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. Always be alert to not mistake $|0\rangle$ with the zero vector $\begin{pmatrix}0\\0\end{pmatrix}$ in $\mathbb{C}^2$!
- Later we'll see that for each $|\psi\rangle \in V$ we can construct a strange object $\langle\psi|$ called bra!

Definition (1.0)

We say that vectors $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle \in V$ are linearly independent if $\sum_{i=1}^{n} \alpha_i|\psi_i\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle .... + \alpha_n|\psi_n\rangle = 0$ if only if $\alpha_i = 0$ $\forall i \in \{1, ..., n\}$ i.e. $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are of course linearly independent i.e.

  $\alpha_1|0\rangle + \alpha_2|1\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if only if $\alpha_1 = \alpha_2 = 0$.

Definition (1.1)

A subset $B = \{|e_1\rangle, |e_2\rangle ......, |e_n\rangle\}$ of V forms a basis (Hamel) in V if:

1. $|e_1\rangle, |e_2\rangle ......, |e_n\rangle$ are linearly independent i.e. $\sum_{i=1}^{n} \alpha_i|e_i\rangle = 0$ if only if $\alpha_i = 0$ $\forall i \in \{1, ..., n\}$.
2. Span$(e_1, ..., e_n) = V$ i.e. any $|\psi\rangle \in V$ can be written (uniquely) as linear combination of $|e_1\rangle, |e_2\rangle ......, |e_n\rangle$.

- Let $V = \mathbb{C}^2$ and $B = \left\{ |0\rangle, |1\rangle \right\}$. It's clear that B forms a basis in $\mathbb{C}^2$ right?

- Are there more bases in $\mathbb{C}^2$ other than B ?

Bases Home Challenge

- From the list of the following vectors in $\mathbb{C}^2$, are there any pairs that form a basis in $\mathbb{C}^2$?
- $e_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, $e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $e_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$, $e_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$, $e_5 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$, $e_6 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{pmatrix}$.

- Consider the vector space $\mathbb{C}^4$. Can you think of a trick to construct bases for $\mathbb{C}^4$ based on bases of $\mathbb{C}^2$?
- A natural question that one may ask is, how do quantum physicists determine which basis to work with when dealing with a particular system e.g. spin 1/2 system?

Definition (1.2)

If $B = \{|e_1\rangle, |e_2\rangle......, |e_n\rangle\}$ forms a basis in V then its cardinality is called the dimension of V and denoted $\dim(V)$ or just $\dim V$.

- It can be proved that all bases of V have the same cardinality (homework?). Hence, the dimension of V does not depend on the choice of basis.
- It's obvious that $\dim(\mathbb{C}^2) = 2$ because the cardinality of $B = \left\{ |0\rangle, |1\rangle \right\}$ is 2.
- As you might have noticed, in general, $\dim(\mathbb{C}^n) = n$.
- Later we'll see that if V_1 and V_2 are two vector spaces over $\mathbb{C}$. Then, $dim(V_1) = dim(V_2)$ iff $V_1 \simeq V_2$ vise versa. This will be important when we talk about tensor products e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

Direct Sum of Subspaces

Definition (1.3)

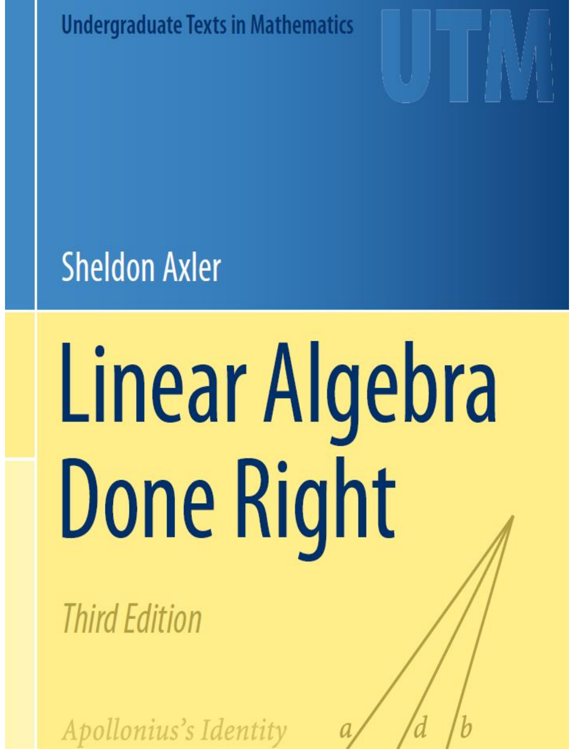
Let $L_1, L_2, ..., L_n$ be linear subspaces of V. Their sum is defined as $L_1 + L_2 + ... + L_n = \{|\psi_1\rangle + |\psi_2\rangle + ... + |\psi_n\rangle \mid |\psi_i\rangle \in L_i\}$.

- $L_1 + L_2 + ... + L_n$ is of course a linear subspace of V.

Definition (1.4)

If each $|\psi\rangle \in L_1 + L_2 + ... + L_n$ can be uniquely written as $|\psi\rangle = \sum_{i=1}^{n} |\psi_i\rangle$ where $|\psi_i\rangle \in L_i$, we write $L_1 \oplus L_2 \oplus ...... \oplus L_n$ and call it an internal direct sum or just direct sum when it's understood from the context.

- It can be proved that $\dim(L_1 \oplus L_2 \oplus ..... \oplus L_n) = \sum_{i=1}^{n} \dim(L_i)$.
- Suppose that $B_1, B_2, \ldots, B_n$ are bases of the respective subspaces. What could be a basis for $L_1 \oplus L_2 \oplus ..... \oplus L_n$?



Prof. Sheldon Axler

Where should you focus?

Vector spaces (*Pages 20 - 50)*

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Tuesday Fireside Chat