Homework 5

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let E be a Lebesgue measurable set.

   a) Let N be the non-measurable set given in lecture 2, (one of the so-called Vitali sets). If $E \subset N$, then $m(E) = 0$, where m is the standard Lebesgue measure.

   b) If $m(E) > 0$, then E contains a nonmeasurable set. You may assume $E \subset [0, 1]$. **Hint:** Using N_r as in the Vitali set example, you may write $E = \cup_{r \in R} N_r$.

   Solution:

   (a) Suppose $m(E) > 0$. We know that $m(E) = m(E + r)$ for any $r \in R = \mathbb{Q} \cap [0, 1)$ (as in the Vitali set example). Making the same adjustment that we did in the Vitali set example (by shifting $E_r \cap [1, 2)$ to $[0, r]$), we can consider each $E_r \subset N_r$. However now note that since the N_r are all disjoint, the E_r are disjoint, and $\cup E_r \subset [0, 1]$, so by the countable additivity of a measure, we have that

$$\sum_{r \in R} m(E_r) = m(\cup_{r \in R} E_r) \leq m([0, 1)) = 1.$$

   However since $m(E_r) = m(E) > 0$ for all $r \in R$, we have that $\infty = 1$, which is a contradiction, so it must be the case that $m(E) = 0$.

   (b) We suppose that $m(E) > 0$, and suppose that it did not contain any nonmeasurable subsets. Since the N_r are disjoint, we have that $E \cap N_r$ are all disjoint, and since we're assuming they are all measurable, we'd have that

$$m(E) = m(\cup_{r \in R} E \cap N_r) = \sum_{r \in R} m(E \cap N_r).$$

   It must be the case that for at least one r, $m(E \cap N_r) > 0$. However by part (a) since we know N_r is not measurable, and $E \cap N_r \subset N_r$, it must be the case that $m(E \cap N_r) = 0$. Thus $E \cap N_r$ cannot be a measurable set.

2. Let $\epsilon \in (0, 1)$ and suppose that A is a Borel measurable subset of $\mathbb{R}$ with $m(A) > 0$. Prove that if

$$m(A \cap I) \leq (1 - \epsilon) m(I)$$

for every interval I, then $m(A) = 0$.

Solution: We will let m^* be the corresponding outer measure and we note that $m(A) = m^*(A)$ since A is in $\mathcal{B}_r$. Let I_k be a collection of intervals that covers A, that is $A \subset \cup I_k$. So we have that

$$m(A) = m^*(A) = m^*(A \cap (\cup I_k)) = m^*(\cup(A \cap I_k)) \leq \sum m^*(A \cap I_k) \leq (1-\epsilon) \sum m^*(I_k).$$

However, taking infimums we now have that

$$m(A) \leq (1-\epsilon) \inf\{\sum m(I_k)\} = (1-\epsilon) m^*(A) < m^*(A) = m(A),$$

a contradiction (note the use of $m(a) \neq 0$ comes in at the final inequality). Thus it must be the case that $m(A) = 0$.

Remark: This problem is oft erroneously over simplified by first time learners. We cannot assume A is itself an interval, nor can we assume a single interval covers A, nor that the intervals that cover A are disjoint (e.g., what if $A = \mathbb{Q}$). We also cannot assume A contains an interval even though it has positive measure (e.g, $A = \mathbb{R} \backslash \mathbb{Q}$ has positive measure but no interval is a subset of A).