QUANTUM FORMALISM

Matrix Algebra - Part 1

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Lecture 07 Recap
3. General Matrices
4. Square Matrices as Operators
5. Basic Matrix Operations
6. Invertible Matrices
7. Diagonal Matrices
8. Study Materials Comments

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Matrix Operators #2
Group Theory 101 #1	Linear Operators 101 #2	Complex Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups: GL(n, C) & U(n) #2

Completed | Ongoing | #n is the number of live lectures

Lecture 07 Recap

Isomorphism Theorems

Theorem (1.0)

Let V and W be vector spaces over $\mathbb{C}$. Then $V \simeq W$ if and only if $dim(V) = dim(W)$.

Proof : Homework challenge?

- A more elegant way of stating the above theorem is to say that the following statements are equivalent:
  1. $V \simeq W$
  2. $\dim(V) = \dim(W)$

Theorem (1.1)

If V is a vector space over $\mathbb{C}$ and $dim(V) = n$. Then $V \simeq \mathbb{C}^n$.

- So for example, take $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. Because $\dim(\mathbb{C}^2 \otimes \mathbb{C}^2) = 4$ it then follows that $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

Theorem (1.2)

A linear operator $T : V \longrightarrow V$ is invertible if and only if $Ker(T) = \{0\}$.

Proof : Homework challenge?

Hence, the theorem above implies T is invertible iff T is an isomorphism from V to V?

Theorem (1.3)

Let $T_1 : V \longrightarrow V$ and $T_2 : V \longrightarrow V$ be invertible linear operators. Then T_2T_1 is invertible.

Proof : Homework challenge?

- If $V = \mathbb{C}^2$ then the following linear operators are invertible:
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$
- Do the set of all invertible linear operators acting on V form a group?

Abstract Eigenvectors and Eigenvalues

Definition (1.6)

Let $T : V \longrightarrow V$. A vector $|\psi\rangle \in V$ is said to be an eigenvector of T if there exists a $\lambda \in \mathbb{C}$ such that $T|\psi\rangle = \lambda|\psi\rangle$. The scalar λ is called an eigenvalue of T.

- The set of all eigenvalues of T is called the spectrum of T and often denoted Spec(T) i.e Spec $(T) = \{\lambda \in \mathbb{C} \mid T|\psi\rangle = \lambda|\psi\rangle\}$ where $|\psi\rangle \in V$ is obviously an eigenvector of T.

Theorem (1.4)

Let $T : V \longrightarrow V$ be a linear operator with distinct set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ with corresponding eigenvectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \in V$. Then the vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ are linearly independent.

- Can the eigenvectors of the distinct eigenvalues above form a basis for V if $\dim(V) = n$?
- In quantum mechanics, the eigenvalues and eigenvectors of linear operators encoding physical observables such as the energy (aka the Hamiltonian) are very important!
- Be aware that physicists often write $|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_n\rangle$ to denote the eigenvectors of $\lambda_1, \lambda_2, \ldots, \lambda_n$!

Lecture 08

Complex Matrices

Definition (1.0)

Let m and n be two natural numbers. We define a $m \times n$ matrix over $\mathbb{C}$ as $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ where $a_{ij} \in \mathbb{C}$.

- The index i indicates the row whereas j indicates the column e.g. a_{11} is the entry in the first row and first column.
- The entries $a_{11}, a_{22}, ...a_{mn}$ are called the diagonal elements of matrix A. Often $\text{diag}(A)$ or $\text{diag}(a_{11}, a_{22}, ...a_{mn})$ is to denote the diagonal of a matrix A.
- A is called a square matrix if $m = n$. We'll also continue using $M_n(\mathbb{C})$ to denote the set of all $n \times n$ matrices.

$M_n(\mathbb{C})$ Matrices as Operators for $\mathbb{C}^n$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C})$ and $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{C}^n$.

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The matrix A is of course a linear operator acting on $|\psi\rangle$ as follows:

$$A|\psi\rangle = \alpha_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \alpha_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \ldots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

- Since A is a linear operator, then we can talk about the eigenvectors and eigenvalues of A. Hence, anything we said about linear operators previously also applies to A!

Addition in $M_n(\mathbb{C})$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{pmatrix}$$

- It's easy to verify the following properties:

1. $A + B = B + A$ for all $A, B \in M_n(\mathbb{C})$.
2. $A + (B + C) = (A + B) + C$ for all $A, B, C \in M_n(\mathbb{C})$.

Scalar Multiplication in $M_n(\mathbb{C})$

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{pmatrix} \text{ for all } A \in M_n(\mathbb{C}) \text{ and } \alpha \in \mathbb{C}.$$

- It's easy to verify that the scalar multiplication above satisfies all the scalar product properties of a vector space.Hence, $M_n(\mathbb{C})$ is of course a vector space $\mathbb{C}$ with dimension $n \times n$.
- $M_n(\mathbb{C})$ becomes far more interesting if you give it the extra structure of a ring e.g. $M_n(\mathbb{C})$ as a $\mathbb{C}^*$ - algebra!
- Behind the scenes, the so-called density matrix formalism of quantum mechanics uses $M_n(\mathbb{C})$ as a $\mathbb{C}^*$ - algebra!

Multiplication in $M_n(\mathbb{C})$

For $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$.

- An easy way to help you see the product AB is as follow:

$AB = \begin{pmatrix} AB_1 & AB_2 & AB_3 & \ldots & AB_n \end{pmatrix}$ where B_j is the j - colunm of B.

$$AB = \left(\begin{matrix} A\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} & A\begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & A\begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix} \end{matrix} \right).$$

- It's easy to verify the following properties:
  1. In general $AB \neq BA$.
  2. $A(BC) = (AB)C$ for all $A, B, C \in M_n(\mathbb{C})$.
  3. $A(B + C) = AB + AC$ for all $A, B, C \in M_n(\mathbb{C})$.
  4. Hence, $M_n(\mathbb{C})$ is a non abelian ring with identity $\mathbb{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.

Invertible Matrices

Definition (1.1)

A matrix $A \in M_n(\mathbb{C})$ is invertible if there exists $A^{-1} \in M_n(\mathbb{C})$ such that $AA^{-1} = A^{-1}A = \mathbb{I}$.

Proposition (1.0)

If $A, B \in M_n(\mathbb{C})$ are invertible, then the following statements are true:

1. AB is also invertible.
2. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : Homework challenge?

- Hence, the set of all invertible matrices in $M_n(\mathbb{C})$ denoted $GL(n, \mathbb{C})$ is a group!
- Please note that some authors use the notation $GL_n(\mathbb{C})$.

Diagonal Matrices

Definition (1.2)

$A \in M_n(\mathbb{C})$ is called a diagonal matrix if it has the following form:

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \text{ such that } a_{ij} = 0 \text{ if } i \neq j.$$

- The following elements of $M_2(\mathbb{C})$ are examples of diagonal matrices: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

Proposition (1.1)

If $A, B \in M_n(\mathbb{C})$ are diagonal, then AB is also diagonal and $AB = BA$.

Proof : Homework challenge?

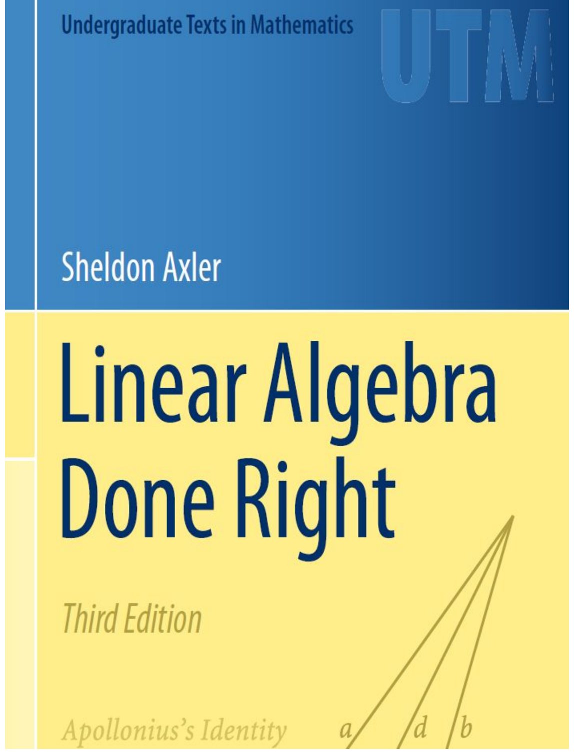
Invertible Diagonal Matrices

Proposition (1.2)

Let $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. The following statements are true:

1. D is invertible iff $\lambda_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$ i.e if all the diagonal elements are nonzero.
2. $$D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{pmatrix}.$$

Proof : Homework challenge?



Prof. Sheldon Axler

Where should you focus?

3.C Matrices (*Pages 70 - 78)*

- **GitHub (Curated study materials):** *github.com/quantumformalism*
- **YouTube:** youtube.com/zaikugroup
- **Twitter:** *@ZaikuGroup*
- ***Gitter:*** *gitter.im/quantumformalism/community*