# QF Group Theory CC2022
# By
# Zaiku Group

Lecture 03

Delivered by Bambordé Baldé

Friday, 25/03/2022

# Session Agenda

1. Learning Journey Timeline
2. Course Approach Overview
3. Upcoming Event

Pre-session Comments

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1. The Zero Element
2. Nilpotent Elements
3. Zero Element Extension
4. Nilpotent Semigroup

Main Session

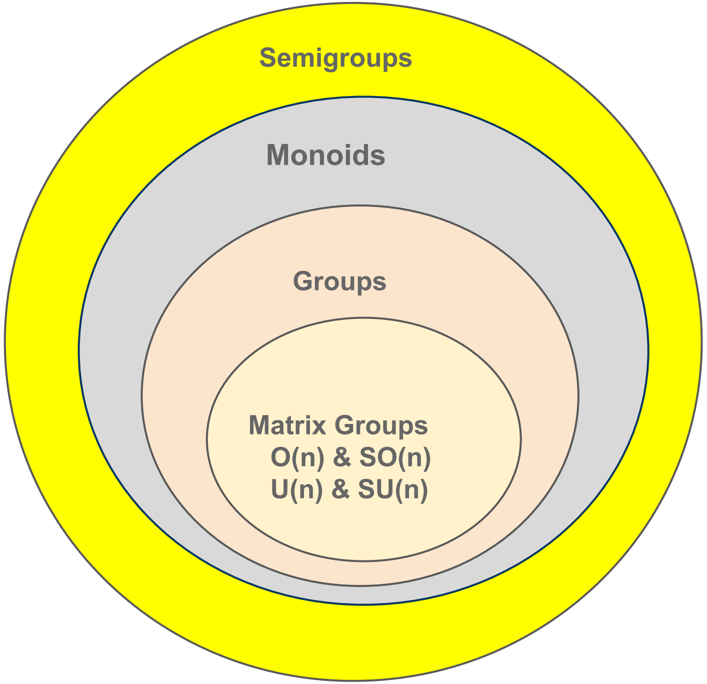
# Learning Journey Timeline

**Topology & Differential Geometry**
**9**

**Group Theory**
**3/14?**

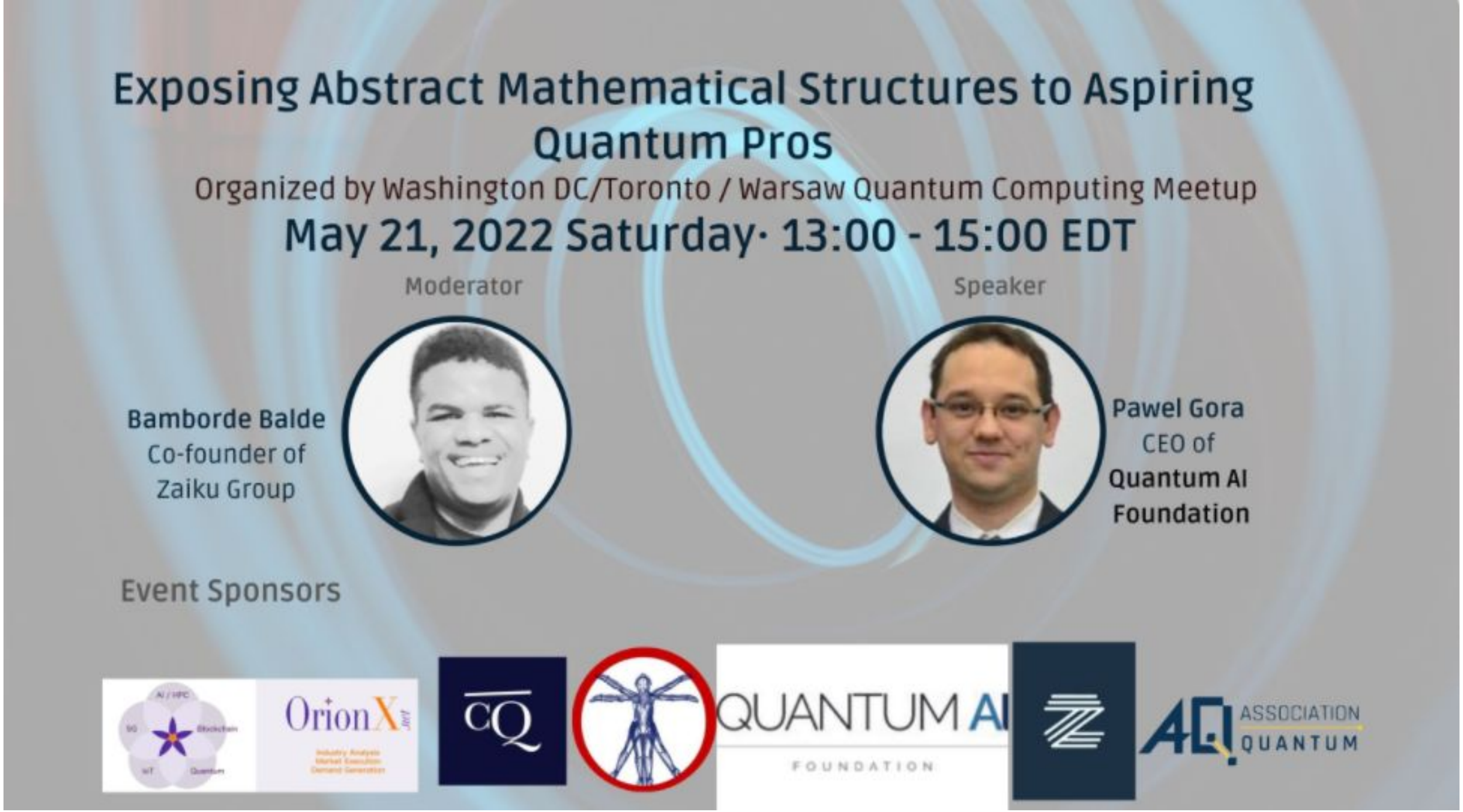
**Lie Groups & Rep.**
**10 - 16?**

Completed | Ongoing | TBC (summer) | n is the number of live lectures |




Course Approach Overview

We’re here!


Exposing Abstract Mathematical Structures to Aspiring Quantum Pros
Organized by Washington DC/Toronto / Warsaw Quantum Computing Meetup
May 21, 2022 Saturday· 13:00 - 15:00 EDT
Moderator
Speaker
Bamborde Balde
Co-founder of
Zaiku Group
Pawel Gora
CEO of
Quantum AI
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# The Zero Element

**Definition 1.0**

Let $(S, *)$ be a semigroup. An element $\boldsymbol{z} \in S$ is called a 'zero element' or 'absorbing element' if $\boldsymbol{z} * a = a * \boldsymbol{z} = \boldsymbol{z}$ for all $a \in S$.

- Obviously it follows that $\boldsymbol{z} * \boldsymbol{z} = \boldsymbol{z}$! Hence, $\boldsymbol{z}$ is idempotent right?!

**Homework Challenge 1**

Let $(S, *)$ be a semigroup with a zero element $\boldsymbol{z} \in S$.

- Is it true that $\boldsymbol{z}$ is unique i.e. if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are two zero elements then $\boldsymbol{z}_1 = \boldsymbol{z}_2$?

**Homework Challenge 2**

Let $(S_1, *_1)$ and $(S_2, *_2)$ be semigroups. Now suppose that a map $f : S_1 \longrightarrow S_2$ is a homomorphism and there is a zero element $\boldsymbol{z} \in S_1$.

- Is it true that $f(\boldsymbol{z})$ is a zero element in $S_2$?

# The Zero Element (Examples)

1. If $(S, *) = (\mathbb{R}, \times)$, then the zero element $\boldsymbol{z}$ is the ordinary 0!
2. What if $(S, *) = (\mathbb{R}, +)$? Is the ordinary 0 still a zero element as per our definition?
3. If we now consider the semigroup $M_2(\mathbb{R})$ of 2 by 2 matrices over the reals under matrix multiplication. Then the zero element is of course the zero matrix i.e. $\boldsymbol{z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
4. Consider $M_2(\mathbb{R})$ under the matrix addition. Is the zero matrix $\boldsymbol{z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ still a zero element?

Obviously the above is true for $M_n(\mathbb{R})$ for any $n \geq 1$.

# The Zero Element (mod 3 Example A)

- Consider $\mathbb{Z}_3 = \{0, 1, 2\}$ with the binary operation $+$ defined by the following table:

| $+$ | $0$ | $1$ | $2$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $2$ |
| $1$ | $1$ | $2$ | $0$ |
| $2$ | $2$ | $0$ | $1$ |

- Clearly there is no zero element right?

# The Zero Element (mod 3 Example B)

- Consider $\mathbb{Z}_3 = \{0, 1, 2\}$ with the binary operation $\times$ defined by the following table:

| $\times$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

- There is now a zero element right?!

# Nilpotent Elements

**Definition 1.1**

Let $(S, *)$ be a semigroup with a zero element $\boldsymbol{z} \in S$. An element $a \in S$ is called nilpotent if there exists a natural number $n \in \mathbb{N}$ such that $a^n = \boldsymbol{z}$.

- Obviously the zero element $\boldsymbol{z}$ itself is nilpotent right?
- We'll denote by $Nilp(S)$ the set of all nilpotent elements in $S$ i.e $Nilp(S) = \{a \in S \text{ such that there exists some } n \in \mathbb{N} \mid a^n = \boldsymbol{z}\}$.
- Obviously, we may have $Nilp(S) = \{\boldsymbol{z}\}$ or even $Nilp(S) = \emptyset$!
- Is it true that if $Nilp(S) \neq \emptyset$ then $Nilp(S)$ is a subsemigroup?

**Homework Challenge 3**

Let $(S_1, *_1)$ and $(S_2, *_2)$ be semigroups with zero elements. Now suppose that $f : S_1 \longrightarrow S_2$ is a homomorphism.

- Is it true that if $a \in S_1$ is nilpotent then $f(a)$ is nilpontent in $S_2$?

# Nilpotent Elements (Boring Examples)

- Let $(S, *) = (\mathbb{Z}, \times)$ where $\times$ is the ordinary multiplication in $\mathbb{Z}$. Then 0 is the only nilpotent element i.e. $Nilp(\mathbb{Z}) = \{0\}$?
- Now if $(S, *) = (\mathbb{Z}, +)$ where $+$ is the ordinary addition in $\mathbb{Z}$ then $Nilp(\mathbb{Z}) = \emptyset$?

- Let now $(S, *) = (\mathbb{R}, \times)$ where $\times$ is the ordinary multiplication in $\mathbb{R}$. Then $Nilp(\mathbb{R}) = \{0\}$?

- If $(S, *) = (\mathbb{R}, +)$. Then again $Nilp(\mathbb{R}) = \emptyset$?
- Similarly, if now $(S, *) = (\mathbb{C}, \times)$ where $\times$ is the ordinary multiplication in $\mathbb{C}$. Then $Nilp(\mathbb{C}) = \{0\}$. Obviously if $(S, *) = (\mathbb{C}, +)$, then $Nilp(\mathbb{C}) = \emptyset$.

- Can you notice anything interesting when the binary operation $*$ is the notion of 'addition'?

**Question:** Are there examples of semigroup structure where $Nilp(S)$ is not trivial/boring like the examples above?

# Nilpotent Elements (Matrix Examples)

- Consider the set $M_2(\mathbb{R})$ of two by two matrices over the reals, then:

1. Trivially, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is nilpotent in respect to matrix multiplication!
2. Nontrivial examples are $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix}$.

**Question:** Do you notice anything about the trace and determinant?

# Nilpotent Elements (mod 4 Example A)

- Consider $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with the binary operation $+$ defined by the following table:

| $+$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

$Nilp(\mathbb{Z}_4) = \emptyset$ right?

## Nilpotent Elements (mod 4 Example B)

- Consider $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ again but now with $\times$ defined by the following table:

| $\times$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

- We finally have a nontrivial example because $Nilp(\mathbb{Z}_4) = \{0, 2\}$ right?!

**Question 1:** Are there more nontrivial examples for $n > 4$?

**Spoiler alert:** Unfortunately $\mathbb{Z}_5$ is boring too i.e. $Nilp(\mathbb{Z}_5) = \{0\}$!

**Question 2:** What about $\mathbb{Z}_6$? Is it boring too i.e. $Nilp(\mathbb{Z}_6) = \{0\}$?

## Nilpotent Elements (mod 6 Example)

- Consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with $\times$ defined by the following table:

| $\times$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

- Unfortunately $Nilp(\mathbb{Z}_6) = \{0\}$ right?!

**Spoiler alert:** Unfortunately $\mathbb{Z}_7$ is boring too i.e. $Nilp(\mathbb{Z}_7) = \{0\}$!

**Question:** Are there really more nontrivial examples for $n > 4$?

# Nilpotent Elements (mod 8 Example)

- Consider $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with $\times$ defined by the following table:

| $\times$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |
| 3 | 0 | 3 | 6 | 1 | 4 | 7 | 2 | 5 |
| 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| 5 | 0 | 5 | 2 | 7 | 4 | 1 | 6 | 3 |
| 6 | 0 | 6 | 4 | 2 | 0 | 6 | 4 | 2 |
| 7 | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

- We have another nontrivial example because $Nilp(\mathbb{Z}_8) = \{0, 4\}$ right?!

**Spoiler alert:** $\mathbb{Z}_9$ is nontrivial too because $Nilp(\mathbb{Z}_9) = \{0, 3, 6\}$!

**Question:** Can you figure out why $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_9$ are special?

# Nilpotent Elements (mod $5, 7, 9$ Tables)

| × | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 0 | 2 | 4 | 6 | 8 | 1 | 3 | 5 | 7 |
| 3 | 0 | 3 | 6 | 0 | 3 | 6 | 0 | 3 | 6 |
| 4 | 0 | 4 | 8 | 3 | 7 | 2 | 6 | 1 | 5 |
| 5 | 0 | 5 | 1 | 6 | 2 | 7 | 3 | 8 | 4 |
| 6 | 0 | 6 | 3 | 0 | 6 | 3 | 0 | 6 | 3 |
| 7 | 0 | 7 | 5 | 3 | 1 | 8 | 6 | 4 | 2 |
| 8 | 0 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

# Nilpotent Elements (mod $n > 9$ Multiplication)

- For $\mathbb{Z}_{10} = \{0,1,2,3,4,5,6,7,8,9\}$, $Nilp(\mathbb{Z}_{10}) = \{0\}$.
- For $\mathbb{Z}_{11} = \{0,1,2,3,4,5,6,7,8,9,10\}$, $Nilp(\mathbb{Z}_{11}) = \{0\}$.
- For $\mathbb{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$, $Nilp(\mathbb{Z}_{12}) = \{0,6\}$.
- For $\mathbb{Z}_{13} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$, $Nilp(\mathbb{Z}_{13}) = \{0\}$.
- For $\mathbb{Z}_{14} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$, $Nilp(\mathbb{Z}_{14}) = \{0\}$.
- For $\mathbb{Z}_{15} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$, $Nilp(\mathbb{Z}_{15}) = \{0\}$.
- For $\mathbb{Z}_{16} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$, $Nilp(\mathbb{Z}_{16}) = \{0,4,8,12\}$.
- For $\mathbb{Z}_{17} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$, $Nilp(\mathbb{Z}_{17}) = \{0\}$.
- For $\mathbb{Z}_{18} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17\}$, $Nilp(\mathbb{Z}_{18}) = \{0,6\}$.

By the way, $\mathbb{Z}_{20}$, $\mathbb{Z}_{24}$, $\mathbb{Z}_{25}$, $\mathbb{Z}_{27}$, $\mathbb{Z}_{28}$, $\mathbb{Z}_{32}$, and $\mathbb{Z}_{36}$ are also part of the nontrivial gang!

**Question:** Can you now figure out what's going on?

- What if a semigroup $(S, *)$ does not have a 'zero element'? Can we do anything about it?

# Adding Zero Element to a Semigroup

## Definition 1.2

Let $(S, *)$ be a semigroup without a zero element. We define the set $S^0 = S \cup \{\mathbf{0}\}$. We can construct a binary operation $\hat{*}$ on $S^0$ as follows:

1. $a \hat{*} b = a * b$ for all $a, b \in S$.
2. $x \hat{*} \mathbf{0} = \mathbf{0} \hat{*} x = \mathbf{0}$ for all $x \in S^0$.

- With $\hat{*}$ define above, $(S^0, \hat{*})$ forms a semigroup structure right?
- In particular, $\mathbf{0}$ is nilpotent right?

## Homework Challenge 4

Consider the semigroups $(\mathbb{Z}_3, +)$, $(\mathbb{Z}_4, +)$, $(\mathbb{Z}_5, +)$. Try construct the tables for $(\mathbb{Z}_3^0, \hat{+})$, $(\mathbb{Z}_4^0, \hat{+})$ and $(\mathbb{Z}_5^0, \hat{+})$!

# Nilpotent Semigroup

**Definition 1.3**

Let $(S, *)$ be a semigroup with a zero element $\boldsymbol{z}$. We say $(S, *)$ is a nilpotent semigroup if all the elements of $S$ are nilpotent i.e. for all $a \in S$ there exists some $n \in \mathbb{N}$ such that $a^n = \boldsymbol{z}$.

- Can you find a nontrivial example of nilpotent semigroup?

**Hint:** Consider starting your hunt with matrices!

QF QUANTUM FORMALISM

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