Homework 7

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Suppose $\{f_n\}$ is a sequence of non-negative decreasing integrable functions such that they converge to some f for every x. Prove that

$$\lim_{n\to\infty} \int f_n = \int f.$$

2. Suppose that f_n, g_n, f, and g are all integrable functions with $f_n \to f$, $g_n \to g$, $|f_n| \leq g_n$ for all n, and $\int g_n \to \int g$. Prove that $\int f_n \to \int f$.

   Remark: This is often refered to as the "Generalized Dominated Convergence Theorem".

3. For the following integrals, prove the limit exists, then evaluate.

   a)

$$\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dx.$$

   b) Given that $g(x)$ is a non-negative integrable function, and $f(x)$ is measurable, bounded, and continuous at 1, evaluate

$$\lim_{n\to\infty} \int_{-n}^{n} f\left(1 + \frac{x}{n^2}\right) g(x)\, dx.$$

4. Give an example of a sequence of non-negative functions f_n such that $f_n \to 0$ pointwise, $\int f_n \to 0$, but there is no integrable $g(x)$ such that $f_n \leq g$ for all n.
