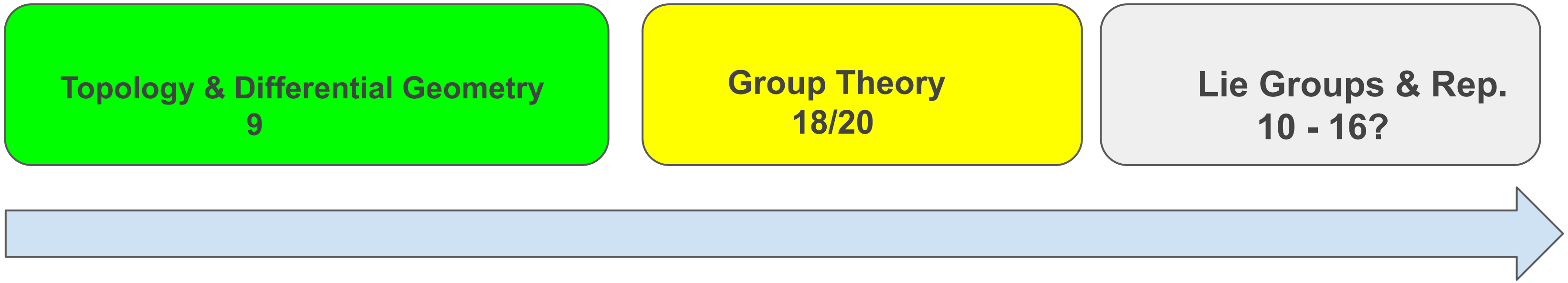
QF Group Theory CC2022
By
Zaiku Group

Lecture 18

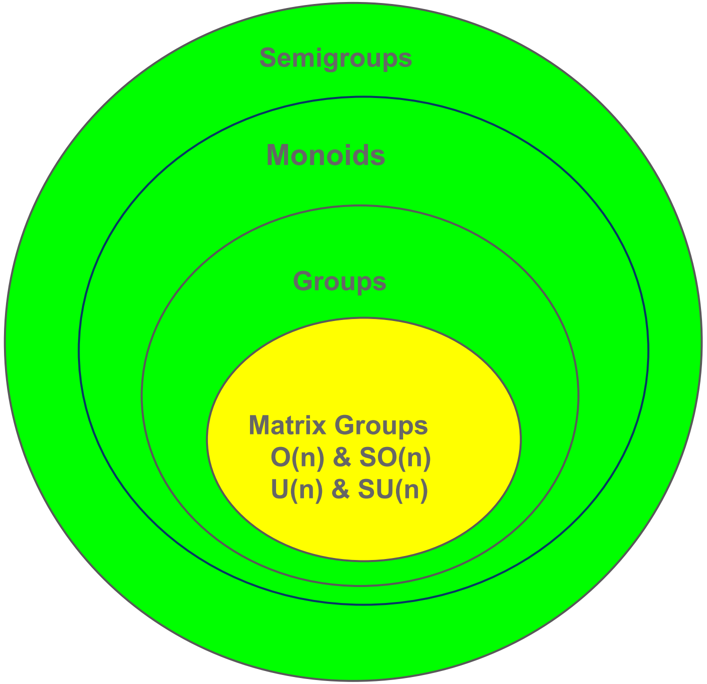
Delivered by Bambordé Baldé

Friday, 11/11/2022

Learning Journey Timeline



Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

Some Warm up

Theorem 1.0

Let $GL(\mathbb{R}^n)$ be the group of all invertible linear maps $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the ordinary vector space structure over $\mathbb{R}$. Then $GL(\mathbb{R}^n) \simeq GL(n, \mathbb{R})$.

Proof : Homework challenge! As a hint, you can use the canonical basis of $\mathbb{R}^n$ to construct the proof!

- The isomorphism above is not canonical i.e. it depends on your choice of basis!
- $GL(n, \mathbb{R})$ is a (noncompact) Lie group of dimension n^2.

Proposition 1.0

The set $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid det(A) > 0\}$ is a subgroup of $GL(n, \mathbb{R})$.

Proof : Homework challenge!

Challenge 1

Is $GL^+(n, \mathbb{R})$ a normal subgroup of $GL(n, \mathbb{R})$?

Challenge 2

Is the set $GL^-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid det(A) < 0\}$ also a subgroup of $GL(n, \mathbb{R})$? If yes, is it a normal subgroup of $GL(n, \mathbb{R})$?

The Real Special Linear Group

Proposition 1.1

The set $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid det(A) = 1\}$ is a subgroup of $GL(n, \mathbb{R})$.

Proof : Homework challenge!

- $SL(n, \mathbb{R})$ is known in the literature as the special linear group.
- Geometrically, it is the group of volume and orientation preserving linear maps on $\mathbb{R}^n$.
- On a side note, $SL(n, \mathbb{R})$ is a Lie group of dimension $n^2 - 1$ (noncompact for $n > 1$).

Proposition 1.2

Let $\mathbb{R}^*$ be the multiplicative group of the nonzero real numbers. Then the determinant map $det : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}^*$ taking $A \in GL(n, \mathbb{R})$ to $det(A) \in \mathbb{R}^*$ is a group homomorphism and $Ker(det) = SL(n, \mathbb{R})$.

Proof : Homework challenge!

Proposition 1.3 (Corollary of 1.2)

$SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$.

Proof : Homework challenge (trivial if you recall the isomorphism theorems)!

Challenge 3

Is it true that $GL(n, \mathbb{R})/SL(n, \mathbb{R}) \simeq \mathbb{R}^*$.

The Real Orthogonal Group

Proposition 1.3

The set $O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid A^T A = I_n\}$ is a subgroup of $GL(n,\mathbb{R})$.

Proof : Homework challenge!

- $O(n,\mathbb{R})$ is known in the literature as the orthogonal group i.e. it is the group of all orthogonal $n \times n$ real matrices.
- We'll use the abbreviation $O(n)$ to denote $O(n,\mathbb{R})$. It's what most textbooks do.
- Geometrically, $O(n)$ is the group of 'linear isometries' on $\mathbb{R}^n$ i.e. the linear maps that preserve distances between points in $\mathbb{R}^n$ and so preserve the length of vectors in $\mathbb{R}^n$.
- Recall that not all isometries are linear and $O(n)$ cannot include all the possible isometries!
- Another nice fact geometric about $O(n)$ is that it's the symmetry group for the $n-1$ sphere. So $O(3)$ is then the symmetry group of the ordinary sphere and $O(2)$ the symmetry group of the circle.

- On a side note, $O(n)$ is a (compact) Lie group of dimension $\frac{n(n-1)}{2}$.

The Real Special Orthogonal Group

Proposition 1.4

The set $SO(n) = \{A \in O(n) \mid det(A) = 1\}$ is a subgroup of $O(n)$.

Proof : Homework challenge!

- $SO(n)$ is known in the literature as the real special orthogonal group.
- Geometrically, $SO(n)$ is the group of rotations on $\mathbb{R}^n$. As a side note, it is also a (compact) Lie group of dimension $\frac{n(n-1)}{2}$.

Proposition 1.5

Let $\mathbb{R}^*$ be the multiplicative group of the nonzero real numbers. Then the determinant map $det : O(n) \longrightarrow \mathbb{R}^*$ taking $A \in O(n)$ to $det(A) \in \mathbb{R}^*$ is a group homomorphism and $Ker(det) = SO(n)$!

Proof : Homework challenge!

Proposition 1.6 (Corollary of 1.5)

$SO(n)$ is a normal subgroup of $O(n)$.

$SO(n)$ Challenge (A)

Isomorphism question 1

Which of the following is true (if any):

1. $O(n)/SO(n) \simeq \mathbb{R}^*$.
2. $O(n)/SO(n) \simeq \{-1, 1\}$, where $\{-1, 1\}$ is a group under the ordinary multiplication.
3. $O(n)/SO(n) \simeq \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group under mod 2 addition.

Abelian question

Is it true that $SO(2)$ is abelian? What about $SO(3)$ or even better $SO(n)$ for $n > 2$?

Hint: Recall that the elements of $SO(2)$ are of the following form:

$$R_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \text{ where } \theta \in [0, 2\pi).$$

$SO(n)$ Challenge (B)

Isomorphism question 2

Consider the multiplicative group (circle group) $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Is it true that $\mathbb{S}^1 \simeq SO(2)$?

QUANTUM FORMALISM

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Which of the following dates you prefer us to wrap up the course?

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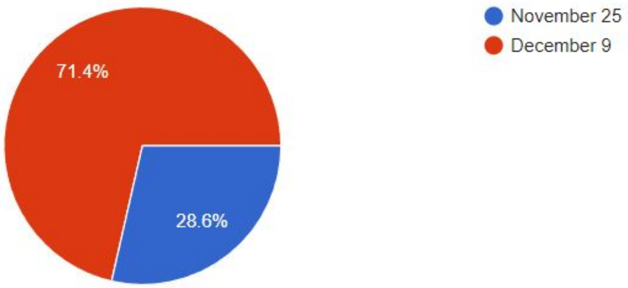
7 responses



Are you currently taking the course?

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