

Category Theory:

a crash course



§1 Categories, functors, & Naturality

"to understand a structure, it is necessary to understand the morphisms that preserve it"

Objects

morphisms

joguen

Sets

Sets

functions

Grp

groups

group homomorphisms

Vect

vector spaces

linear maps

Top

topological spaces

continuous f's

posets

order-preserving f's

⋮

⋮

⋮

⋮

Cat

categories

functors

⋮

⋮

⋮

⋮

⋮

⋮

⋮

⋮

Def:

A category is (loosely) a collection of objects & a collection of morphisms between objects

(i) a collection of objects $Ob(\mathcal{C})$

(ii) for any $X, Y \in Ob(\mathcal{C})$, a set

$Hom_{\mathcal{C}}(X, Y) :=$ set of morphisms from X to Y

(sometimes called "arrows")

Compatibility

$X \xrightarrow{f} Y$

(iii) for any $X, Y, Z \in Ob(\mathcal{C})$,

$Hom_{\mathcal{C}}(X, Y) \times Hom_{\mathcal{C}}(Y, Z) \rightarrow Hom_{\mathcal{C}}(X, Z)$

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$(X \xrightarrow{f} Y, Y \xrightarrow{g} Z) \mapsto X \xrightarrow{g \circ f} Z$

$g \circ f =$ composition of f and g

space of matrices

$Hom(\mathbb{R}^n, \mathbb{R}^m) = M_{m \times n}(\mathbb{R})$

sets

↓

↓

Compatibility

(iv) function composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$X \xrightarrow{f} Y, Y \xrightarrow{g} Z, Z \xrightarrow{h} W$$

(v) identity morphisms id_X

for any $X \in \text{Ob}(\mathcal{C})$, $\exists id_X \in \text{Hom}_{\mathcal{C}}(X, X)$

for any $g \in G$ $g \circ id_X = g$

equivalent

for any $f: X \rightarrow Y$,
 $g: Z \rightarrow X$,

$$\begin{cases} f \circ id_X = f \\ g = id_X \circ g \end{cases}$$

when are two objects "the same"?

Def: two objects X & Y of \mathcal{C} are isomorphic if \exists a pair of morphisms

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \xleftarrow{g} & \\ & g & \end{array}$$

such that $\begin{cases} g \circ f = id_X \\ f \circ g = id_Y \end{cases}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \xleftarrow{g} & \\ & g & \end{array}$$

$g = f^{-1}$, etc.

in terms of "commutative diagrams"

• $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$ "this diagram commutes"

• $\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{h \circ g} & W \\ & \searrow g \circ f & \downarrow g & \searrow h & \\ & & Z & \xrightarrow{h} & W \end{array}$
 $(h \circ g) \circ f = h \circ (g \circ f)$

• $\begin{array}{ccccc} Z & \xrightarrow{g} & X & \xrightarrow{f \circ id_X} & Y \\ & \searrow id_X \circ g & \downarrow id_X & \searrow f & \\ & & X & \xrightarrow{f} & Y \end{array}$

Examples: Set, Grp, Ab, Top, Vect _{\mathbb{K}} , Vect _{\mathbb{K}} ^{f.d.}, Mon, Met, ...

Banach Analytic Manifolds

Ban Ana Man

Ex: Let \mathcal{C} be a category, its opposite category, \mathcal{C}^{op} , is given by:

(i) $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

(ii) $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$

$$\begin{array}{ccc}
 & f & \\
 \bullet & \longrightarrow & \bullet \\
 X & & Y
 \end{array} \quad m \in \mathcal{C}$$

$$X \longleftarrow Y \quad m \in \mathcal{C}^{op}$$

Let R be a ring w/ unity,

$R\text{-mod}$ = category of left R -modules

the category of right R -modules

is $(R\text{-mod})^{op}$

$\left\{ \begin{array}{l} \text{in } \underline{D_X}\text{-modules} \\ \text{sheaf of linear diff operators} \\ \text{w/ holomorphic coefficients} \\ \text{on a complex manifold} \end{array} \right\}$

Example

$\Rightarrow \mathcal{C} = \text{Set}, \emptyset = \text{empty set} \in Ob(\text{Set})$

for any set X , $\emptyset \in X$.

$\left\{ \begin{array}{l} \text{Categorically: for any } X, \\ \exists! \emptyset \hookrightarrow X \end{array} \right\} \begin{array}{l} \text{if } Y \text{ satisfies} \\ \text{this property for any } X, \\ \text{then } Y \cong \emptyset \\ \text{isomorphic} \end{array}$

"inclusion"



"hook right arrow"

$$Hom_{\text{Set}}(\emptyset, X) = \{ \emptyset \hookrightarrow X \}$$

initial object

Exercise 1: Show that \emptyset is
 the only set with this property
 (up to isomorphism)

Example: "the" singleton set $\{*\} \in \text{Ob}(\text{set})$.

a set containing only one element.

are these different?

$$A = \{a\} \simeq \{\text{chair}\} = B$$

uniquely isomorphic!

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a & \mapsto & \text{chair} \end{array} \quad \begin{array}{ccc} B & \xrightarrow{g} & A \\ \text{chair} & \mapsto & a \end{array}$$

The singleton set is the unique (up to isomorphism)

set $\{*\}$ satisfying:

for any set X , $\exists! X \xrightarrow{\alpha} \{*\}$

$$\text{Hom}_{\text{Set}}(X, \{*\}) = \{X \xrightarrow{\alpha} \{*\} \mid \alpha \mapsto *\}$$

terminal object

Exercise 2: Formulate this property in terms an arbitrary category \mathcal{C} , & prove it uniquely defines this object.

Exercise 3: For Vect_k , is there an initial object or a terminal object?

$\phi \in \text{Vect}_k \dots$ and?

partially ordered set

Example

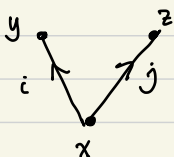
Let (X, \leq) be a poset.

Consider as a category via:

$$\text{Ob}(X, \leq) = X$$

for any two $x, y \in X$,

$$\text{Hom}_{(X, \leq)}(x, y) = \begin{cases} \{x \rightarrow y\}, & \text{if } x \leq y \\ \emptyset, & \text{else} \end{cases}$$



$$\text{Hom}(x, y) = \{i\}$$

$$\text{Hom}(y, z) = \emptyset$$

$$\text{Hom}(x, z) = \{j\}$$

Let X be any set.

$$\mathcal{P}(X) = 2^X = \text{set of subsets of } X,$$

power set of X

partially ordered by inclusion.

Complete Boolean algebra

Let X be a topological space,

$$\mathcal{O}_p(X) = \text{set of open subsets of } X,$$

partially ordered by inclusion.

comes up all the time in algebraic

geometry, for Topos theory,

Grothendieck topology. --

Ex:

Monoids.

like groups, but w/o inverses

$(\mathbb{Z}, +)$ is a group

$(\mathbb{N}, +)$ is a monoid

monoid multiplication by x



$$M \xrightarrow{x} M$$

$$y \mapsto y \cdot x \quad \searrow \quad \downarrow y$$

$$M \times M \rightarrow M$$

$$\{(x, y) \mapsto x \cdot y\}$$

$$e \cdot x = x \cdot e = x$$

turn into a category: \mathcal{M}

only one object, $\text{Ob}(\mathcal{M}) = \{*\}$

$$\text{Hom}_{\mathcal{M}}(*, *) = M$$

Exercise 4: prove the statement

"a group is a category with one element, where every morphism is an isomorphism"

What is a "structure-preserving morphism" between categories?

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

F should define a function

"Functor"

$$(i) \quad F: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$$

$$X \mapsto F(X)$$

$$(ii) \quad (X \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(f)} F(Y))$$

in \mathcal{C} in \mathcal{D}

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow[F_*]{f_*} \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

respect composition

$$(iii) \quad F(g \circ f) = F(g) \circ F(f)$$

$$F(g \circ f) = F(f) \circ F(g) \quad \begin{matrix} \text{Covariant} \\ \text{Contravariant} \end{matrix} \quad \star$$

$$(iv) \quad F(\text{id}_X) = \text{id}_{F(X)}$$

"Every sufficiently good analogy

yearns to become a functor"

-Baez

Category theory := mathematics of analogies

Ex: For any category \mathcal{C} , there's an identity functor: $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$

$$x \mapsto x$$

"forgetful functor" $\mathcal{F} \mapsto \mathcal{F}$

Ex: $U: \text{Grp} \rightarrow \text{Set}$

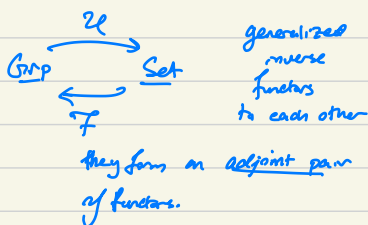
$$(G, *) \mapsto G$$

forget the binary operation!

$F: \text{Set} \rightarrow \text{Grp}$

$$S \mapsto \text{Free}(S) = \{\text{free group on the set } S\}$$

look this up,
if hard, talk about it later!



Ex: $\mathcal{C} = \text{Vect}_k$ dual vector space functor.

$$D: \text{Vect}_k^{\text{op}} \rightarrow \text{Vect}_k$$

$$V \mapsto V^* = \text{Hom}_k(V, k)$$

$$\text{Hom}_k(W, k) \rightarrow \text{Hom}_k(V, k)$$

$$(w \mapsto k) \mapsto (v \mapsto f)$$

$$(f \circ f) = f^*(f)$$

first problem session
next Sunday

Exercise: describe $D^2: \text{Vect}_k \rightarrow \text{Vect}_k$
functor
of "double dual"

$$D^2(V) = D(D(V)) = \text{Hom}_k(\text{Hom}_k(V, k), k)$$

Δ its relationship w/ the identity function

$$\text{nat}_{\text{Vect}_k}: \text{Vect}_k \rightarrow \text{Vect}_k$$

