QF $\int_S f \, \mathrm{d}\mu.$ QUANTUM FORMALISM

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Homework 12

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $A \subset X$ for some topological space X. Prove $\overline{A}$ is closed. Reference the previous homework for the definition of $\overline{A}$ if you forgot it!

   Hint: Show that $\overline{A}^c$ is an open set by showing an arbitrary point of $\overline{A}^c$ is an interior point.

2. Suppose X and Y are topological spaces and $f : X \to Y$ is a function. Show that the following are equivalent:

   i) The function f is continuous.
   ii) For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
   iii) For all closed sets $B \subseteq Y$, $f^{-1}(B)$ is closed.

   Hint: It's easiest to show (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i), and this chain of implications gives us that they are all equivalent.

   Hint for (i) $\implies$ (ii): Suppose f is continuous. Start by letting $y \in f(\overline{A})$. So there is some $x \in \overline{A}$ with $f(x) = y$. We wish to show $y \in \overline{f(A)}$. If $x \in A$, this is trivial (why?). So we suppose $x \in A' \backslash A$. Let V be an open set containing y. Using the continuity of f, show that $V \cap f(A) \backslash y \neq \emptyset$.

   Hint for (ii) $\implies$ (iii): If you can show $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$, then this means $\overline{f^{-1}(B)} = f^{-1}(B)$, so $f^{-1}(B)$ is closed by the first problem.

3. Prove that any compact subset of a Hausdorff space is closed

   Hint: Let C be our compact subset, $x \notin C$, and since our space is Hausdorff, for all $y \in C$, we can find some U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. The collection V_y covers V. Use the definition of compactness to reduce this to a finite subcover, and consider what this means for the corresponding open sets containing x.

   Remark: All of these are actually common theorems in a standard topology course, but the proofs can be quite sneaky, hence the copious hints!
