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Quantum Axioms & Operators - Part 3

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Quantum Axioms
3. Commutators Refresh
4. A Brief Comment on CCR
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6. Commuting Family of Operators

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1. Operator Exponential
2. Diagonal Operator Exponential
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4. Shut Up & Calculate Challenge
5. Study Materials Reference

Part B

Thank you!

Amir Claudia

Diego CJ Volz

Giri Nicolas

Soham Vesselin

Foundation Module Review

Rings and Fields 101 #1	Matrix Algebra #2	Quantum Axioms & Operators 3/4
Group Theory 101 #1	Linear Operators 101 #2	Finite dim. Hilbert Spaces #2
Naive Set Theory Overview #1	Complex Vector spaces 101 #2	Matrix Groups 101: U(2) + SU(2) #2

Completed | Ongoing | #n is the number of live lectures

PART A

‘’A good deal of my research work in physics has consisted in not setting out to solve some particular problems, but simply examining mathematical quantities of a kind that physicists use and trying to get them together in an interesting way regardless of any application that the work may have. It is simply a search for pretty mathematics. It may turn out later that the work does have an application. Then one has had good luck. ‘’

Paul M. Dirac

Pre- Session Key Remarks

- In the previous session we learned that Hermitian operators are diagonalizable i.e for any Hermitian A there exists a unitary operator $U \in M_n(\mathbb{C})$ and real diagonal operator $D \in M_n(\mathbb{C})$ such that $A = UDU^\dagger$. In particular, we commented that:
  1. The eigenvectors of A coincide with the columns of U!
  2. The eigenvalues of A coincide with the diagonal entries of operator D!
- We also commented that for any Hermitian $A \in M_n(\mathbb{C})$ encoding a physical observable, it can be proved that the only possible values of measurement are the eigenvalues of A.
- In particular, the eigenvectors of a Hermitian operator A encode physical states of the system in which the observable encoded in A can be measured without uncertainty?
- We defined $U \in M_n(\mathbb{C})$ being unitary if $UU^\dagger = U^\dagger U = \mathbb{I}$. But we can alternatively say U is unitary if $\langle U\psi_1, U\psi_2\rangle = \langle \psi_1, \psi_2\rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H} = \mathbb{C}^n$ where $\langle \cdot, \cdot\rangle$ is obviously the inner product in $\mathcal{H}$. Hence, U preserves the inner product and so the induced norm too!

The Axioms of QM (Non Relativistic QM)

Axiom 1: The states of quantum systems are modelled by **normalised vectors** on **separable complex Hilbert spaces**. [✓]

- Recall that the axiom above is actually referring to what physicists call 'pure states'. There are also the so-called mixed states!

Axiom 2: The observables of quantum systems are modelled by **self-adjoint operators** on separable complex Hilbert spaces. [✓]

- Recall that self-adjoint operators $\Leftrightarrow$ Hermitian matrices iff $\mathcal{H} = \mathbb{C}^n$!

Axiom 3: Given a state vector $|\psi\rangle \in \mathcal{H}$ that encodes a particular state of a quantum system, and a self-adjoint operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ that encodes a particular observable of the same system. The measurement expectation value of the observable encoded in A in the state encoded with $|\psi\rangle$ is computed as $\langle A\rangle_\psi = \langle \psi, A\psi\rangle$. [✓]

Axiom 4: Provided a quantum system is undisturbed (e.g. no measurement made) then the system evolves smoothly over time $t \in \mathbb{R}$ according to the equation $\frac{d|\psi(t)\rangle}{dt} = \frac{-i}{\hbar} H |\psi(t)\rangle$, where $H \in M_n(\mathbb{C})$ is the Hermitian operator encoding the energy of the system (aka the Hamiltonian).

- The equation above is known as the time-dependent Schrodinger equation.
- For the curious, you are encouraged to ask a physicist or do research about the difference between time-dependent and time independent Schrodinger equations! It's something that we'll cover in the advanced module!
- Physicists often use the convention that $\hbar = 1$ and so the Schrodinger equation becomes $\frac{d|\psi(t)\rangle}{dt} = -iH |\psi(t)\rangle$
- Now, if we are given an initial state $|\psi(0)\rangle$, how do we find out the state at any later time $|\psi(t)\rangle$? Part B of this session will be about helping find out!

Commutators Refresh

Definition (1.0)

Let $A, B \in M_n(\mathbb{C})$. The commutator of A and B is defined as $[A, B] = AB - BA$.

The following properties hold for commutators:

1. For all $A, B \in M_n(\mathbb{C})$, $[A, B] = 0$ iff A and B commute.
2. $[A, B] = -[B, A]$ for all $A, B \in M_n(\mathbb{C})$.
3. $[A, B + C] = [A, B] + [A, C]$ for all $A, B, C \in M_n(\mathbb{C})$.
4. $[AB, C] = A[B, C] + [A, C]B$ for all $A, B, C \in M_n(\mathbb{C})$.
5. $[A, BC] = B[A, C] + [A, B]C$ for all $A, B, C \in M_n(\mathbb{C})$.
6. $[\alpha A, B] = [A, \alpha B] = \alpha[A, B]$ for all $A, B, C \in M_n(\mathbb{C})$ and $\forall \alpha \in \mathbb{C}$.

- Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Is $[A, B]$ Hermitian? What about $i[A, B]$?

A Brief Comment On CCR

Let X be a self-adjoint operator encoding position and P a self-adjoint operator encoding momentum.

- In physics textbooks the Heisenberg canonical commutation relation is written as $[X, P] = i\hbar$ or in a cleaner unambiguous mathematical fashion $[X, P] = i\hbar\mathbb{I}$ where $\mathbb{I}$ is the identity operator acting on the Hilbert space $\mathcal{H}$ under consideration.
- You should be aware that the above commutation cannot be satisfied for operators acting on a finite dimensional Hilbert space $\mathcal{H}$ i.e. if $\mathcal{H} = \mathbb{C}^n$!
- Hence, the canonical commutation relation only holds for operators acting on infinite dimensional Hilbert spaces!

Simultaneous Diagonalization

Definition (1.1)

Let $A, B \in M_n(\mathbb{C})$ be diagonalizable. We say A and B are simultaneously diagonalizable if there exists $S \in M_n(\mathbb{C})$ such that $S^{-1}AS$ and $S^{-1}BS$ are diagonal matrices.

- The above essentially means that there is a basis of $\mathcal{H}$ whose elements are eigenvectors of both A and B!
- Is it true that if A is diagonizable then A and $\lambda\mathbb{I}$ are simultaneously diagonalizable for all $\lambda \neq 0$?

Theorem (1.0)

Let $A, B \in M_n(\mathbb{C})$ be two diagonalizable matrices. A and B are simultaneously diagonalizable iff $[A, B] = 0$.

Proof : Homework challenge for at least $M_2(\mathbb{C})$!

- Hence, if A and B are Hermitian and $[A, B] = 0$. Then A and B are simultaneously diagonalizable.

Commuting Family of Operators

Definition (1.2)

Let $\mathrm{C} = \{C_1, C_2, \dots, C_n\} \subseteq M_n(\mathbb{C})$ such that $[C_i, C_j] = 0$ for all $i \neq j$. C is called a commuting family of operators.

- Physicists will be mostly be interested when the family above is made of Hermitian operators i.e. if C_i is Hermitian $\forall\, i$.

Theorem (1.1)

Let $\mathrm{C} = \{C_1, C_2, \dots, C_n\} \subseteq M_n(\mathbb{C})$ be a commuting family of operators. Then there exists a $|\psi\rangle \in \mathcal{H}$ such that $|\psi\rangle$ is an eigenvector of C_i $\forall i$.

Proof : See study material!

- Hence, the operators in C have an eigenvalue in common!
- What could be the implication if C comprises of Hermitian operators that encode observables of a particular system?

Simultaneous Diagonalizable Family

Definition (1.3)

$\mathrm{F} = \{F_1, F_2, \ldots, F_n\} \subseteq M_n(\mathbb{C})$ is simultaneously diagonalizable if there exists $S \in M_n(\mathbb{C})$ such that $S^{-1}F_iS$ is diagonal for all $F_i \in \mathrm{F}$

- Hence, the notion of simultaneously diagonalization can be extended to an arbitrary number of operators.

Theorem (1.2)

Let $\mathrm{D} = \{D_1, D_2, \ldots, D_n\} \subseteq M_n(\mathbb{C})$ *be a family of diagonalizable operators. Then D is a commuting family iff it's also simultaneously diagonalizable family.*

Proof : See study material!

- Hence, there is a strong relationship between commuting families and diagonalizable families!

- You're challenged to go think about the possible implications of the above in the modelling aspects of quantum observables.

PART B

Operator Exponential

Definition (1.4)

Given $A \in M_n(\mathbb{C})$ its exponential is defined as the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

- The series expansion $\sum_{k=0}^{\infty} \frac{1}{k!} A^k = \mathbb{I} + A + \frac{A^2}{2!} + \frac{A^3}{3!} \ldots$.
- It can be proved that this series converges for all $A \in M_n(\mathbb{C})$.
- The matrix exponentiation has the following properties:
  1. $e^0 = \mathbb{I}$ where in this context $0 \in M_n(\mathbb{C})$.
  2. $e^{-A} = (e^A)^{-1}$ for all $A \in M_n(\mathbb{C})$.
  3. $e^{\alpha A} e^{\beta A} = e^{(\alpha+\beta)A}$ for all $A \in M_n(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$.
  4. $e^{A^\dagger} = (e^A)^\dagger$ for all $A \in M_n(\mathbb{C})$.
  5. $e^{BAB^{-1}} = Be^A B^{-1}$ for all $A, B \in M_n(\mathbb{C})$ with B being invertible of course i.e. $B \in GL(n, \mathbb{C})$.
  6. $e^A e^B = e^B e^A = e^{(A+B)}$ iff $[A, B] = 0$ for all $A, B \in M_n(\mathbb{C})$.
  7. If $A \in M_n(\mathbb{C})$ is diagonalizable then $\det\left(e^A\right) = e^{Tr(A)}$.

Diagonal Operator Exponential

If $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \in M_n(\mathbb{C})$. Then it turns out that $e^D = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}$.

- Computing the exponential of diagonal operators is very easy!
- If $A \in M_n(\mathbb{C})$ is diagonalizable i.e. there exists $G \in GL(n, \mathbb{C})$ and a diagonal $D \in M_n(\mathbb{C})$ such that $A = GDG^{-1}$. Then it turns out that $e^A = Ge^D G^{-1}$!
- Hence, if $A \in M_n(\mathbb{C})$ is Hermitian then $e^A = Ue^D U^\dagger$?!

Hamiltonian Exponential

Theorem (1.3)

Let $H \in M_n(\mathbb{C})$ be the Hermitian encoding the Hamiltonian and $U(t) = e^{\frac{-i}{\hbar}Ht}$ where $t \in \mathbb{R}$. Then $U(t)$ is unitary for all $t \in \mathbb{R}$.

Proof : You're welcome to try proving this for at least $H \in M_2(\mathbb{C})$!

- With a little bit of thinking we can conclude that $U(t) = e^{\frac{-i}{\hbar}Ht}$ is a solution to the Schrodinger's differential equation $\frac{d|\psi(t)\rangle}{dt} = \frac{-i}{\hbar}H|\psi(t)\rangle$?
- This is why it's called 'unitary evolution'!
- Hence, given an initial state $|\psi(0)\rangle$, the state at later time t is $|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{\frac{-i}{\hbar}Ht}|\psi(0)\rangle$.
- Although we expressed the theorem above in terms of the Hamiltonian. There is actually a general version where it can be proved that e^{iA} is unitary for any Hermitian operator A!

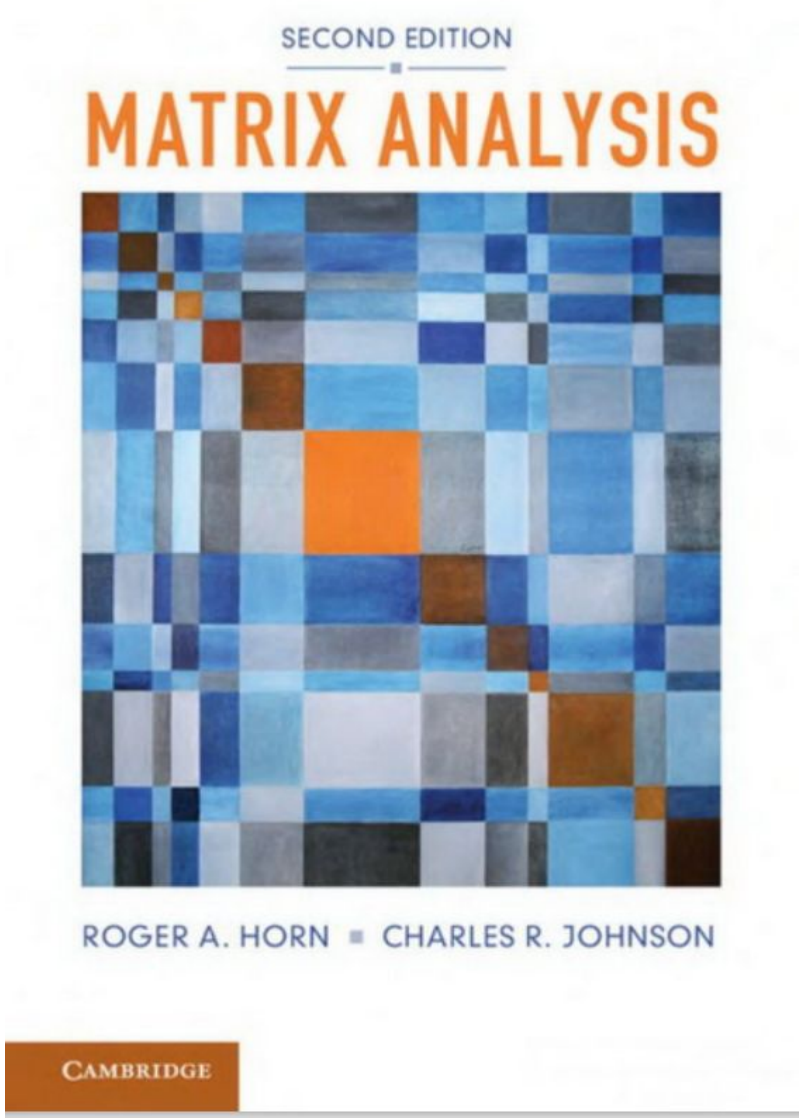
Shut Up and Calculate Challenge

Consider a system S modelled on the Hilbert space $\mathbb{C}^2$ with three observables encoded with the following Hermitian operators in $\mathbb{C}^2$:

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Compute e^{σ_X}, e^{σ_Y} and e^{σ_Z}.
2. Apply e^{σ_X}, e^{σ_Y} and e^{σ_Z} to the following kets:
$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix},$$
$$|\psi_4\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ and } |\psi_5\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}.$$
3. Suppose that the Hamiltonian of the system is encoded as
$$H = \mathbb{I} + \tfrac{1}{2}\sigma_Y \text{ where } \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If the initial state is $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is the state $|\psi(t)\rangle$ at a later time t?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Robert van de Geijn, Margaret Myers

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Advanced Linear Algebra:

Foundations to Frontiers

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