$$\gamma : [0,1] \longrightarrow X$$

# Topology Crash Course - Lecture 03

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# Session Agenda

Pre-lecture
blah blah blah

1. Metric Spaces
2. Open Balls
3. Open Sets in Metric Spaces
4. Closed Sets in Metric Spaces

**PART A**

**+**

1. Metric Topology
2. Metric Topology’s Hausdorffness
3. Equivalent Metrics
4. Metrizable Topological Spaces
5. **Reference Study Material**
6. **Lecture Gap Poll**

**PART B**

**Lecture 3 Coverage**

# General Coverage

Including a brief intro to Fiber Bundles?

**Point-set Topology 101**
**2/2**

**Metric Topology 101**
**1/3**

**Manifolds 101**
**#3**

**#n is the number of live lectures | Completed | Ongoing**

# Post Crash Course Program?

Lie Groups + Lie Algebras
# ?

Topological Groups 101
#2

Matrix Groups Refresh
#4

Topology Crash Course
#8

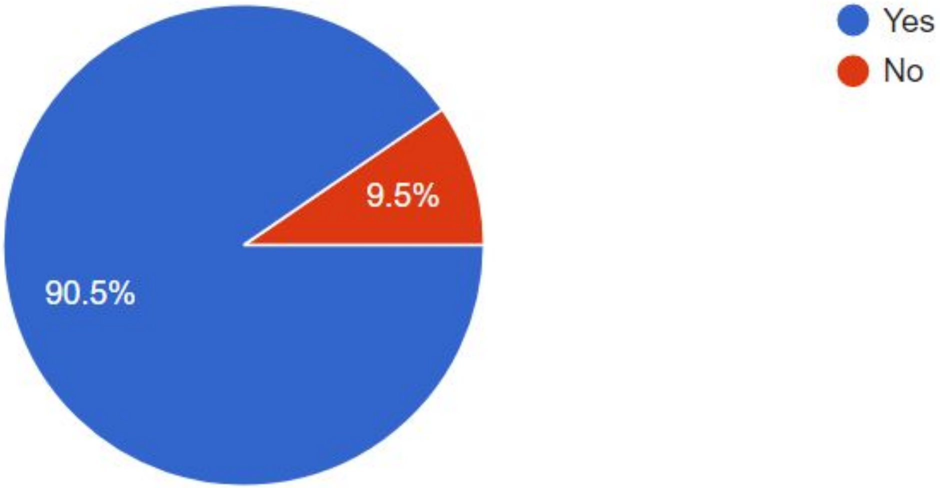
A good mathematical foundation for:

- Quantum Algorithms e.g. Quantum Error Correction
- Theoretical/Mathematical Physics e.g Gauge Theories
- Topological Data Analysis
- Geometric Deep Learning

**#n is the number of live lectures**

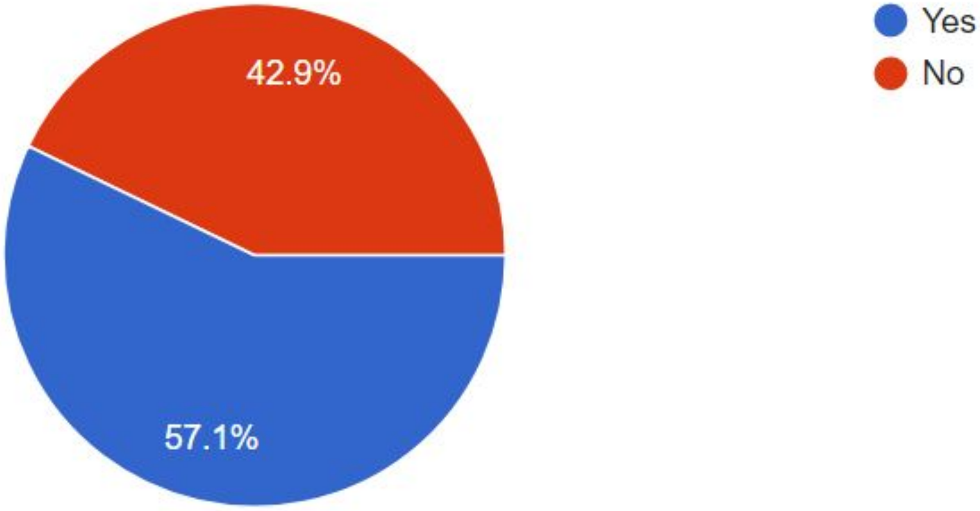
# Audience Background Poll

Have you been exposed to University level mathematics? For example, undergraduate level calculus and real analysis.



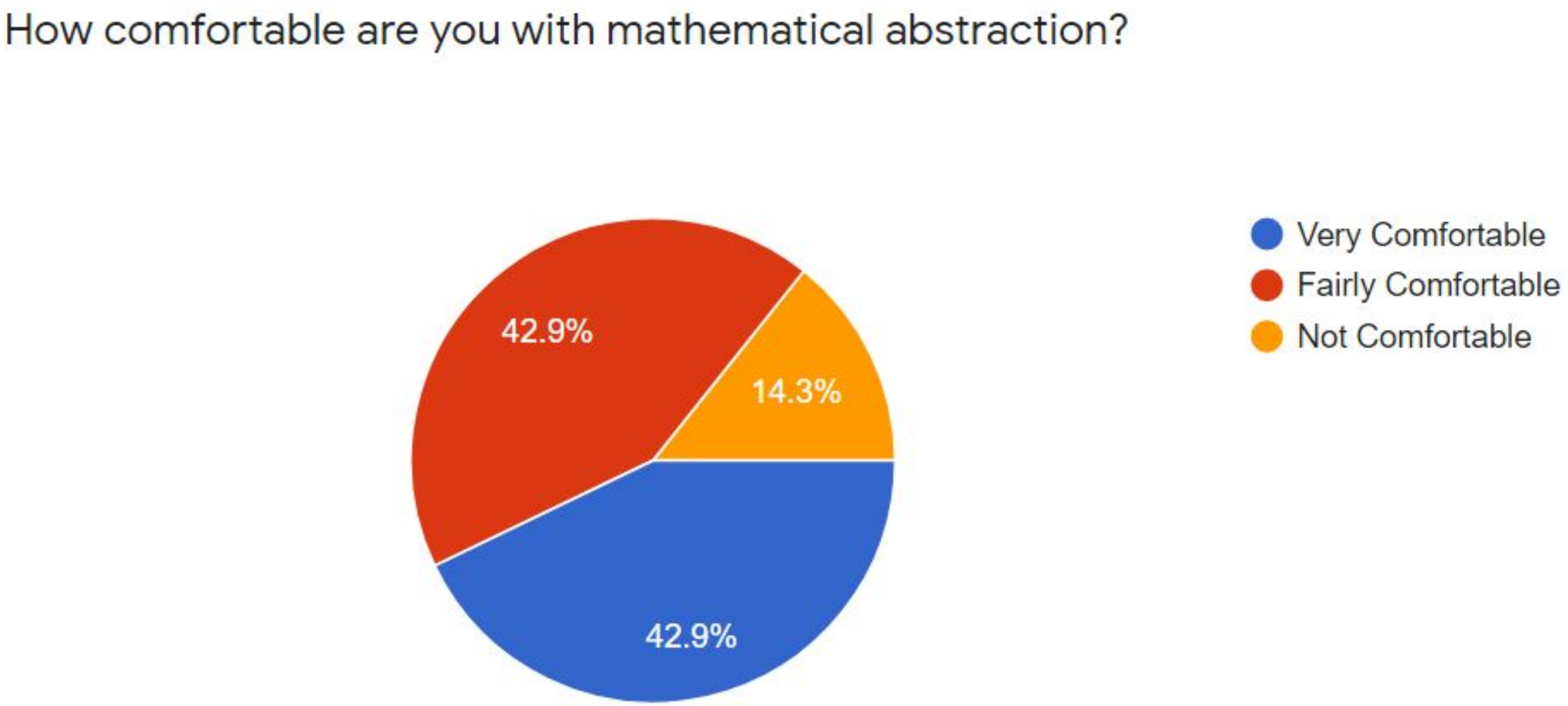
# Audience Background Poll

Have you been exposed to topology before?




quantumformalism.com

# Audience Background Poll



# Quantum Industry Fellowship

$$\langle \Omega \rangle = \int_X \Omega \cdot \mu_P$$

**Zaiku Group + Partner Companies**

1. Independent Thinkers
2. Mission Driven
3. Team Players
4. Humble

# PART A

# Metric Spaces

**Definition (1.0)**

Let $M$ be a non-empty set. A metric on $M$ is map $d : M \times M \longrightarrow \mathbb{R}$ satisfying the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in M$.
2. $d(x, y) = 0$ iff $x = y$ for all $x, y \in M$.
3. $d(x, y) = d(y, x)$ for all $x, y \in M$.
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

- The pair $(M, d)$ is called a metric space. Whenever the metric $d$ is understood from the context, we'll just write $M$ instead of $(M, d)$.

**Definition (1.1)**

Let $(M, d)$ be a metric space and $A \subseteq M$ be non-empty. We can construct $d_A : A \times A \longrightarrow \mathbb{R}$ as $d_A(a_1, a_2) = d(a_1, a_2)$ for all $a_1, a_2 \in A$.

- As you might have guessed, $(A, d_A)$ is a metric subspace of $(M, d)$!

## Some Examples

- For any non-empty set $M$, we can define the following metric:
  $$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
  $(M, d)$ with the above metric (aka discrete) is a metric space!

- Let us consider $M = \mathbb{R}^n$. Then for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we can define the famous Euclidean metric $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. Thus $\mathbb{R}^n$ is a metric space under $d$ aka Euclidean metric space!

- Let $(M, d)$ be any metric space and let $\alpha \in \mathbb{R}^+$. Is $d_\alpha(x,y) = \alpha d(x,y)$ a metric on $M$?

- $(M_1, d_1), (M_2, d_2), \ldots, (M_k, d_k)$ be metric spaces. Can you construct a metric space from $M_1 \times M_2 \times \ldots \times M_K$?

- Now suppose that $\mathcal{H}$ is a finite dimensional complex Hilbert space. Can you construct a metric on $\mathcal{H}$?

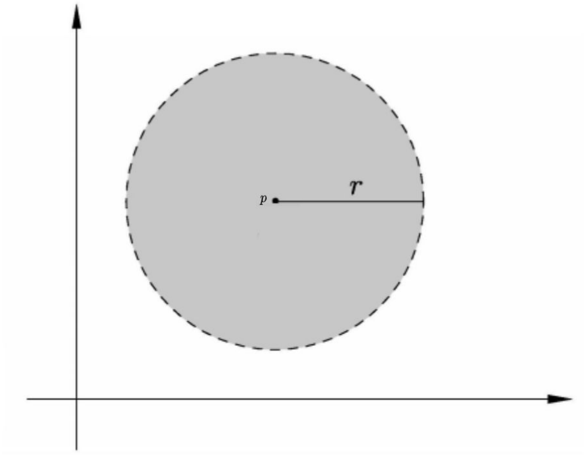
**Hint:** Use the norm induced by the inner product!

## Open Balls in Metric Spaces

**Definition (1.2)**

Let $(M, d)$ be a metric space. For each point $p \in M$ and $r \in \mathbb{R}^+$ we can define $B_r(p) = \{x \in M \mid d(p, x) < r\}$

- $B_r(p)$ is called the open ball centred around the point $p$ with radius $r$.
- Obviously $B_r(p_1) = B_r(p_2)$ iff $p_1 = p_2$ right?
- If $M = \mathbb{R}$ and $d$ is the Euclidean metric, then $B_r(p)$ is just the open interval $(p - r, p + r)$. For example, $B_1(1) = (0, 2)$.
- If $M = \mathbb{R}^2$ and $d$ is the Euclidean metric, then $B_r(p)$ is just the open disc of radius $r$ centred around $p$ i.e. the set of all points inside the circle of radius of $r$ centred at point $p$.




- The concept of open ball is very important as it will enable us to realise the concept of open sets that we covered abstractly in Point-set Topology!

## Proposition (1.0)

Let $B_r(p)$ be an open ball in a metric $(M, d)$. Then for any $p' \in B_r(p)$ there exists a $r' > 0$ such that $B_{r'}(p') \subseteq B_r(p)$.

*Proof* : Homework challenge!

**Hint** : Use $r$ to construct such $r'$!

# Open Sets in Metric Spaces

**Definition (1.3)**

Let $(M, d)$ be a metric space. A subset $U \subseteq M$ is open in respect to the metric $d$ if for all $p \in U$ there exists an $r \in \mathbb{R}^+$ such that $B_r(p) \subseteq U$.

- Hence, for any point $p \in M$, $B_r(p)$ is open?
- Let $M = \mathbb{R}$ and $d$ the Euclidean metric. Then the open intervals (in the **Real Analysis** sense) $(a, b)$, $(-\infty, a)$, $(b, \infty)$ are also open in $M$. Is $\mathbb{R}$ itself open?
- Is it true that the intervals $[a, b]$ and $[a, b)$ are not open if $M = \mathbb{R}$? However, the same intervals are open if $M = [a, b]$ equiped with Euclidean metric $d$?!
- Let now $M = \mathbb{R}^2$ also equiped with the Euclidean metric. Is $\mathbb{R}$ viewed as a subset $\mathbb{R}^2$ open?

**Note:** Whether or not a subset $U$ is open really depends on the metric space that we are considering i.e. being 'open' depends on the metric space structure being considered!

## Properties of Open Sets

**Proposition (1.1)**

For any given metric space $(M, d)$, the following is true:

1. $\emptyset$ and $M$ are open in respect to $d$.
2. If $U_1, U_2, \ldots, U_k$ are open in respect to $d$, then $U_1 \cap U_2 \cap \ldots \cap U_k$ is open in respect to $d$.
3. If $\{U_i\}_{i \in I}$ is a collection of open sets in respect to $d$, then their union $\bigcup_{i \in I} U_i$ is open in respect to $d$.

*Proof* : Homework challenge!

- Did you get any Déjà vu feeling regarding the proposition above?

**Definition (1.4)**

Let $(M, d)$ be a metric space. A subset $N \subseteq M$ is a neighbourhood of a point $p \in M$ if there exists a $\delta \in \mathbb{R}^+$ such that $B_\delta(p) \subseteq N$.

- As home challenge, you can verify that the above definition of neighbourhood is equivalent to the one we defined abstractly in point-set topology i.e. the above definition satisfies all the properties that the abstract one satisfies.

# Closed Sets in Metric Spaces

**Definition (1.5)**

Let $(M, d)$ be a metric space. We say that a subset $A \subseteq M$ is closed in respect to $d$ if $M \setminus A$ is open i.e. for any $p \in M \setminus A$ there exists some $r \in \mathbb{R}^+$ such that $B_r(p) \subseteq M \setminus A$.

- Let $M = \mathbb{R}$ and $d$ the Euclidean metric. Is it true that the intervals $[a, b]$ and $(-\infty, 0)$ are closed?

**Proposition (1.2)**

For any given metric space $(M, d)$, the following is true:

1. $\emptyset$ and $M$ are closed in respect to $d$.
2. If $U_1, U_2, \ldots, U_k$ are closed in respect to $d$, then $U_1 \cup U_2 \cup \ldots \cup U_k$ is closed in respect to $d$.
3. If $\{U_i\}_{i \in I}$ is a collection of closed sets in respect to $d$, then their intersection $\bigcap_{i \in I} U_i$ is closed in respect to $d$.

*Proof* : Homework challenge!

# PART B

# The Metric Topology

**Proposition (1.3)**

For any given metric space $(M, d)$, the collection of subsets $\mathcal{T}_d = \{U \subseteq M \mid U \text{ is open in respect to } d\}$ is a topology on $M$.

*Proof* : Trivial to prove because it is a consequence (corollary) of **Proposition 1.1**!

- $\mathcal{T}_d$ is called a metric topology on $M$ i.e. the topology induced by the metric $d$. Hence, we'll write $(M, \mathcal{T}_d)$ to denote the topological space induced by $d$.
- When $M = \mathbb{R}^n$ and $d$ is the Euclidean metric, then $\mathcal{T}_d$ is called the 'standard topology' on $\mathbb{R}^n$!
- Most topological spaces of interest in applied subjects such as physics will be subspaces of $(\mathbb{R}^n, \mathcal{T}_d)$. So from now on, always assume $\mathcal{T}_d$ to be the standard topology whenever $M \subseteq \mathbb{R}^n$.
- Can you see why the metric induced topological space $(M, \mathcal{T}_d)$ is a Hausdorff space?

# VIP Spaces

- Consider the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. The subspace topology induced by the standard topology on $\mathbb{R}^2$ makes $S^1$ into a topological space.
- More generally, let $S^n = \{(x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$.

Then the subspace topology induced by the standard topology on $\mathbb{R}^{n+1}$ makes $S^n$ into a topological space.

- In Topology lingo, $S^n$ is known as the '$n-$ sphere'. Hence, $2-$ sphere corresponds to the ordinary sphere whereas $1-$ sphere corresponds to the circle.
- The interval $[0, 1]$ equipped with the subspace topology from the standard topology on $\mathbb{R}$ is a topological space. This is the space that will help construct 'Path-Connected' topological spaces. Not to be mistaken with 'Connected' topological spaces!
- Can you think of a reason why the VIP spaces above are Hausdorff?

# Homeomorphism Challenge

- Consider the topological spaces $X = (-1, 1)$ and $Y = (0, 5)$ constructed from the standard topology on $\mathbb{R}$. Is the map $f : X \longrightarrow Y$ defined as $f(x) = \frac{5}{2}(x + 1)$ a homeomorphism?
- Let now $X = (-1, 1)$ and $Y = \mathbb{R}$. Is the map $f : X \longrightarrow Y$ defined as $f(x) = tan(\frac{\pi x}{2})$ a homeomorphism?
- Let $X = [0, 1]$ and $Y = \mathbb{R}$. Can you construct a homeomorphism from $X$ to a subset of $\mathbb{R}$ that is not $X$ itself i.e. is $X = [0, 1]$ homeomorphic to any other subset of $\mathbb{R}$?
- Is it true that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $X = [0, 1]$ i.e. there is a homeomorphism $f : S^1 \longrightarrow X$?

# Equivalent Metrics

**Definition (1.6)**

Let $d_1$ and $d_2$ be two metrics on $M$ with corresponding topologies $\mathcal{T}_{d1}$ and $\mathcal{T}_{d2}$. The metrics $d_1$ and $d_2$ are said to be equivalent if $\mathcal{T}_{d1} = \mathcal{T}_{d2}$ i.e. if the two metrics generate the same open sets.

- We write $d_1 \simeq d_2$ to indicate the equivalence.
- For any non-empty set $M$, lets define the following:
  $d_1(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if x = y} \end{cases}$ and $d_2(x,y) = \begin{cases} 2021 & \text{if } x \neq y \\ 0 & \text{if x = y} \end{cases}$
  We already know that $d_1$ defines a metric (discrete) on $M$. Is $d_2$ also a metric on $M$? If yes, is $d_1 \simeq d_2$?
- Let $M = \mathbb{R}^n$ and $d$ the Euclidean metric with corresponding standard topology $\mathcal{T}_d$. Can you find other metrics defined in $\mathbb{R}^n$ that generate the same open sets as $d$ i.e. can you find other topologies in $\mathbb{R}^n$ that are equivalent to $\mathcal{T}_d$ ?

# Metric Topology's Hausdorrfness

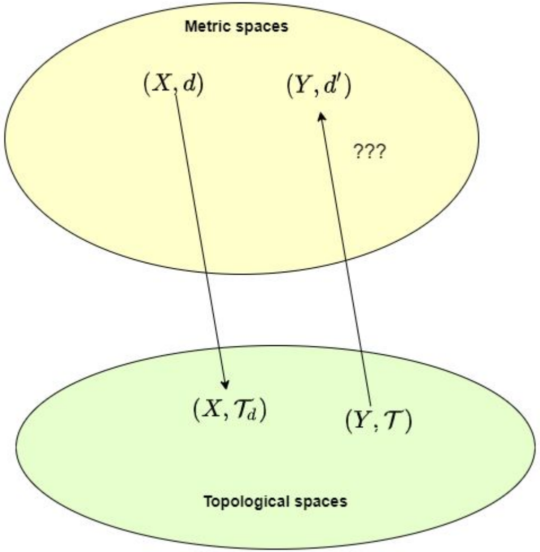
**Theorem (1.0)**

*For any metric space $(M, d)$, the induced topological space $(M, \mathcal{T}_d)$ is a Hausdorff space.*

*Proof* : Let $p_1, p_2 \in M$ with $p_1 \neq p_2$. Recall that for $(M, \mathcal{T}_d)$ to be Hausdorff, there must exist some neighbourhoods $N_{p_1}$ and $N_{P_2}$ of the respective points such that $N_{p_1} \cap N_{p_2} = \emptyset$.
Let $r = d(p_1, p_2), N_{p_1} = B_{\frac{r}{2}}(p_1)$ and $N_{p_2} = B_{\frac{r}{2}}(p_2)$. Clearly $N_{p_1}$ and $N_{p_2}$ are open neighbourhoods. Now it must be true that $N_{p_1} \cap N_{p_2} = \emptyset$ or else there exists some point $p' \in N_{p_1} \cap N_{p_2}$. However, the existence of such $p'$ leads to contradiction because this would mean $d(p_1, p') < \frac{r}{2}$ and $d(p_2, p') < \frac{r}{2}$. But then this implies
$r = d(p_1, p_2) \leq d(p_1, p') + d(p_2, p') < \frac{r}{2} + \frac{r}{2}$ i.e. $r < r$ (contradiction!).
Hence, we must have $N_{p_1} \cap N_{p_2} = \emptyset$ and so $(M, \mathcal{T}_d)$ is a Hausdorff space.

# A Comment on Metrizability

- It's very cool that we can construct topological spaces from metric spaces! But can we construct metric spaces from topological spaces?!



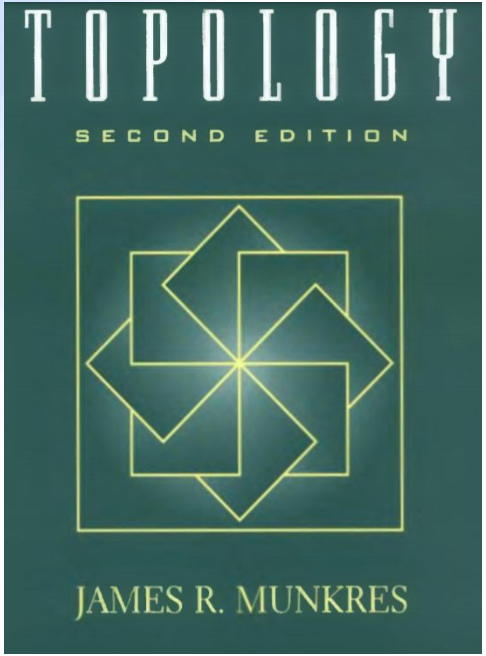
# Metrizable Topological Spaces

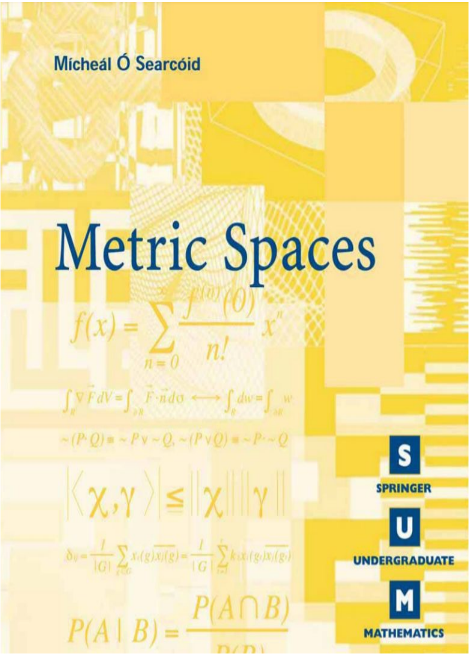
**Definition (1.7)**

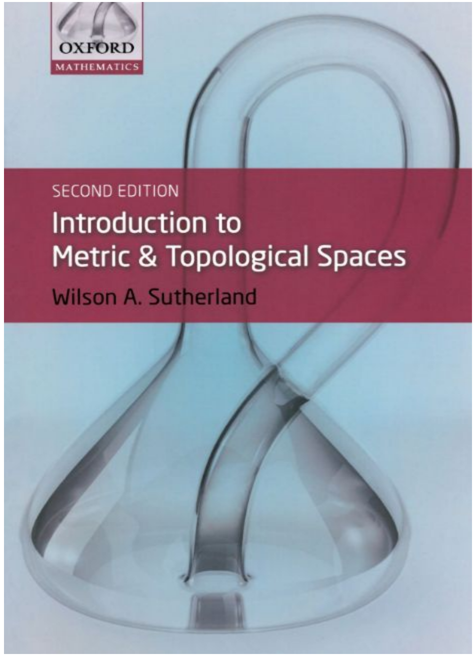
A topological space $(X, \mathcal{T})$ is metrizable if there exists a metric $d$ on $X$ such that $\mathcal{T} = \mathcal{T}_d$.

- Let $X$ be any set and $\mathcal{T} = \mathcal{P}(X)$. Is $(X, \mathcal{T})$ metrizable?
- What if we now have $\mathcal{T} = \{\emptyset, X\}$?
- Now consider $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, X, \{1\}\}$ (Sierpinski topology). Is $(X, \mathcal{T})$ metrizable?
- Suppose now that $X = \mathbb{R}$, can you find a non trivial example of non-metrizable topology on $\mathbb{R}$?
- Is it true that any metrizable space is Hausdorff?

**Side note:** Let $X$ be any non-empty set and $\mathcal{T} = \{\emptyset, Z \subseteq X \mid X \setminus Z \text{ is finite }\}$. $\mathcal{T}$ is a topology on $X$ known as the Zariski topology on $X$. Interestingly, the space $(X, \mathcal{T})$ is not metrizable if the set $X$ is infinite!




Reference Materials

QF

QUANTUM FORMALISM

**GitHub:** github.com/quantumformalism

**YouTube:** youtube.com/ZaikuGroup

**Discord:** discord.gg/SPcmcsXMD2

**Twitter:** twitter.com/ZaikuGroup

**LinkedIn:** linkedin.com/company/zaikugroup