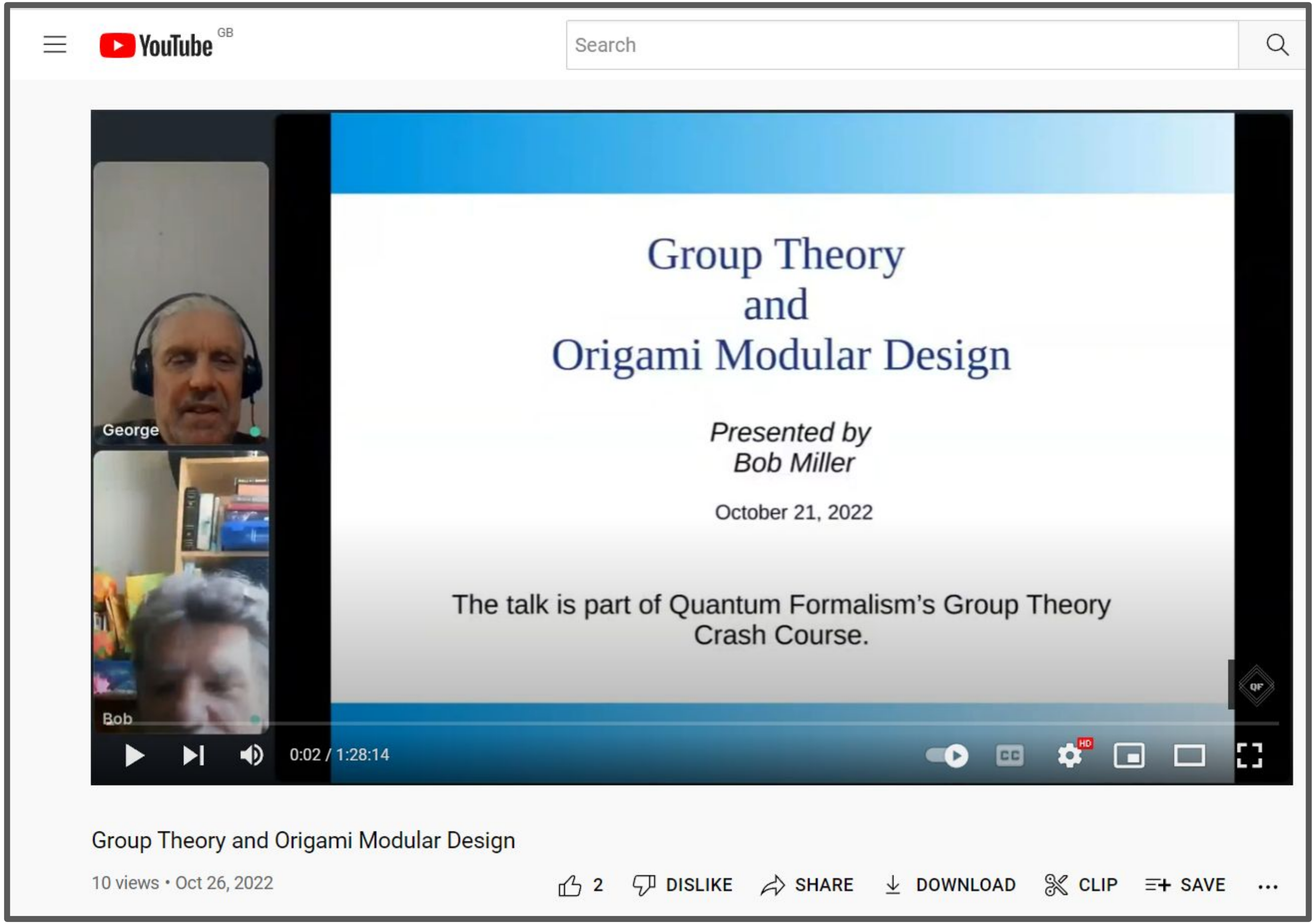
QF Group Theory CC2022
By
Zaiku Group

Lecture 17

Delivered by Bambordé Baldé

Friday, 28/10/2022



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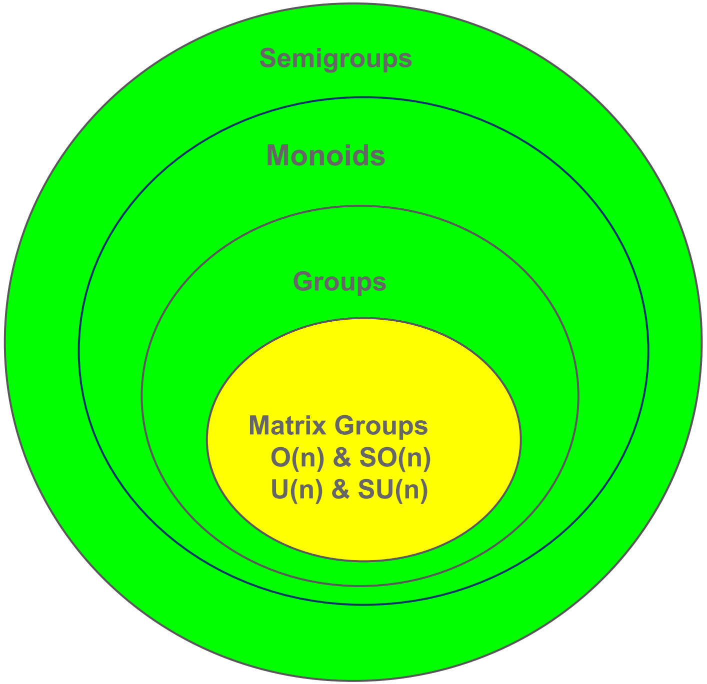
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
17/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

Theorem 1.0 (The Orbit-Stabiliser Theorem)

Let G be a finite group acting on a set X. Then $|G| = |Stab_G(x)||Orb_G(x)|$ for all $x \in X$.

Some Linear Algebra Refresher (A)

Definition 1.0

Let V be an $n-$ dimensional vector space over the reals $\mathbb{R}$. A map $L : V \longrightarrow V$ is linear if it satisfies the following conditions:

1. $L(v_1 + v_2) = L(v_1) + L(v_2)$ for all $v_1, v_2 \in V$.
2. $L(\alpha v) = \alpha L(v)$ for all $v \in V$ and $\alpha \in \mathbb{R}$.

Proposition 1.0

Let $Lin(V)$ be the set of all linear maps $L : V \longrightarrow V$. If $L, L' \in Lin(V)$, then $L' \circ L \in Lin(V)$ where $\circ$ is the ordinary composition of maps.

Proof : Homework challenge!

Side note: We can equip $Lin(V)$ with a real vector structure as well!

The Abstract General Linear Group

Proposition 1.1

We define the set $GL(V)$ to be the set of all invertible linear maps on V i.e. $GL(V) = \{L \in Lin(V) \mid \text{L is invertible}\}$. Then $GL(V)$ is a group under ordinary composition of maps $\circ$.

- $GL(V)$ is known in the literature as the general linear group of V.
- For our purposes V is finite dimensional, however even if V is infinite dimensional, $GL(V)$ exists and very interesting to study!

Challenge 1

Let V_1 and V_2 be two real vector spaces. Is it true that if $V_1 \simeq V_2$ then $GL(V_1) \simeq GL(V_2)$ i.e. isomorphism of two vector spaces implies isomorphism of their corresponding general linear groups?

Some Linear Algebra Refresher (B)

Definition 1.1

We'll write $M_n(\mathbb{R})$ to denote the set of all $n \times n$ matrices over the reals $\mathbb{R}$.

- Some authors use the notation $M^{n\times n}(\mathbb{R})$ instead of $M_n(\mathbb{R})$.
- I'll assume everyone knows about the basics of $n \times n$ matrices over the reals $\mathbb{R}$ including; how to compute the transpose, perform addition and multiplication of $n \times n$ matrices.
- When equipped with the ordinary matrix addition and multiplication, which of the following algebraic structures $M_n(\mathbb{R})$ forms?

1. An abelian group under addition.
2. A nonabelian group under multiplication.
3. A real vector space of dimension n^2.

Important: From linear algebra 101 an element $A \in M_n(\mathbb{R})$ induces a linear map $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with $\mathbb{R}^n$ equipped with the canonical vector space structure over $\mathbb{R}$. Likewise, any linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induces an element $A_L \in M_n(\mathbb{R})$ i.e. linear maps on $\mathbb{R}^n \equiv n \times n$ matrices over $\mathbb{R}$.

Real Matrix Groups

Definition 1.2

A subset $G \subset M_n(\mathbb{R})$ is a matrix group if it's a group under the ordinary matrix multiplication. This obviously implies the following:

1. If $A, B \in G$ then $AB \in G$ i.e. matrix multiplication is a closed binary operation in G.
2. If $A, B, C \in G$ then $A(BC) = (AB)C$ i.e. matrix multiplication is associative in G. This is trivial to show because it is associative in $M_n(\mathbb{R})$!
3. The identity matrix $I_n \in G$.
4. For any $A \in G$ there exists an inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

- Since G is a group, then all the abstract group-theoretic properties and constructions we've made so far also applies to it! Hence, we can ask about subgroups of G, left $G-$ actions, left cosets, orbits, stabilisers and so on.

Concrete Matrix Group Example

- Let $n = 2$ and G be the set of all 2×2 real matrices defined as $R_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ where $\theta \in [0, 2\pi]$. It's not hard to verify that G is a matrix group under ordinary matrix multiplication. If you're not convinced G is group, then you're encouraged to try prove or disprove it!

- Let us consider a left action of G above on the plane $\mathbb{R}^2$ defined as: For each $R_\theta \in G$ and $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we produce an element $v' \in \mathbb{R}^2$ via $v' = R_\theta v = \begin{pmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{pmatrix}$.

Side note: The group G above is indeed an interesting group acting on the plane $\mathbb{R}^2$! Does anyone know what it is?

Challenge 2

Let G be the rotation group in the previous example together with the constructed action on $\mathbb{R}^2$. You're encouraged to identify the orbit and stabiliser of each of the following elements:

1. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
2. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
3. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The General Linear Group over $\mathbb{R}$

Proposition 1.2

Let us consider the subset of $M_n(\mathbb{R})$ defined as $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) \neq 0\}$. Then $GL(n, \mathbb{R})$ is a group under ordinary matrix multiplication.

Proof : Homework challenge!

- As a hint to help you prove the above: Recall from kindergarten linear algebra that if $A \in M_n(\mathbb{R})$ and $det(A) \neq 0$, then A is invertible! In fact A is invertible iff $det(A) \neq 0$!
- $GL(n, \mathbb{R})$ is known in the literature as the general linear group of order n over $\mathbb{R}$. Also, some authors use the notation $GL_n(\mathbb{R})$!
- In general, $GL(n, \mathbb{R})$ is obviously nonabelian right?
- Again, since $GL(n, \mathbb{R})$ is a group, then all the general properties and constructions we made abstractly also applies to it! Hence, we can ask about subgroups of $GL(n, \mathbb{R})$, left cosets, orbits, stabilisers and so on.

Side note: $GL(n, \mathbb{R})$ is a real Lie group of dimension n^2 that will be covered in the next course!

Theorem 1.1

Let $GL(\mathbb{R}^n)$ be the group of all invertible linear maps $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the ordinary vector space structure over $\mathbb{R}$. Then $GL(\mathbb{R}^n) \simeq GL(n, \mathbb{R})$.

Proof : Homework challenge! As a hint, you can use the canonical basis of $\mathbb{R}^n$ to construct your proof!

Challenge 3

Is it true that every matrix group $G \subset M_n(\mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ i.e. $GL(n, \mathbb{R})$ is indeed the largest possible matrix group?

QF

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