$\int_S f\,\mathrm{d}\mu.$
QUANTUM FORMALISM
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Homework 3

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_{j=1}^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

   Solution: This one is just a definition check. We will let $\nu(E) = \sum_{j=1}^n a_j \mu_j(E)$. First, since each μ_j is a measure, $\mu_j(\emptyset) = 0$, so it follows that $\nu(\emptyset) = \sum_{j=1}^n a_j \mu_j(\emptyset) = 0$. Now let $\{E_k\}_{k=1}^\infty$ be a sequence of disjoint sets in $\mathcal{M}$. It follows then that

$$\nu(\cup_{k=1}^\infty E_k) = \sum_{j=1}^n a_j \mu_j(\cup_{k=1}^\infty E_k) = \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu_j(E_k) = \sum_{k=1}^\infty \left(\sum_{j=1}^n a_j \mu_j(E_k) \right) = \sum_{k=1}^\infty \nu(E_k).$$

   Since all the terms are non-negative, we can safely permute the summations, and we see that ν satisfies the countable additivity condition. Thus we have that ν is a measure.

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and that $E, F \in \mathcal{M}$. Show that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

   Solution: We simply observe that by the definition of a measure, since $E \backslash (E \cap F)$, $F \backslash (E \cap F)$ and $E \cap F$ are all disjoint, and union to $E \cup F$, we have that

$$\mu(E \backslash (E \cap F)) + \mu(F \backslash (E \cap F)) + \mu(E \cap F) = \mu(E \cup F)$$

   Adding to each side another $\mu(E \cap F)$, we get

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E \backslash (E \cap F)) + \mu(F \backslash (E \cap F)) + 2\mu(E \cap F) = \mu(E) + \mu(F).$$

3. Let X be an uncountable set and let $\mathcal{A}$ be the collection of subsets A of X such that either A or A^c is countable. Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ is A is uncountable. Prove that μ is a measure.

   Solution: Since the empty set contains no elements, it is countable, so we have that $\mu(\emptyset) = 0$. We let $A_1, A_2, \ldots$ be a collection of disjoint sets in $\mathcal{A}$. We must consider a few cases. If all of the A_k are countable, then their union remains countable, so we have

$$\mu(\cup A_j) = 0 = \sum \mu(A_j).$$

Suppose now $\mu(\cup A_j) = 1$, which means at least one of the A_k is uncountable (such that A_k^c is countable). Then since the sets are disjoint, we cannot have two disjoint uncountable sets with countable compliments. Thus it must be the case $\mu(A_j) = 0$ for all but one of the A_k, thus $\sum \mu(A_j) = 1$. Thus we've shown μ is indeed a measure.

4. Let X be a set with σ-algebra $\mathcal{M}$. We say μ is a **finitely additive measure** if $\mu\mathcal{M} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and given any finite collection of disjoint sets $E_1, \ldots, E_n \in \mathcal{M}$, $\mu(\cup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$. Note that every measure is finitely additive.

   a) Suppose that μ is a finitely additive measure on $(X, \mathcal{M})$ and μ is continuous from below. Prove that μ is a measure.

   b) Suppose that μ is a finitely additive measure on $(X, \mathcal{M})$, $\mu(X) < \infty$ and μ is continuous from above. Prove that μ is a measure.

Solution:

(a) We suppose that μ is finitely additive and continuous from below. Since we already know $\mu(\emptyset) = 0$, all we need to do is show that μ is countably additive. So we let $\{E_j\}_{j=1}^\infty$ be a collection of disjoint sets in $\mathcal{M}$. We will also define $F_k = E_1 \cup \ldots \cup E_k$. Notice then that $F_1 \subset F_2 \subset \ldots$, so we have a nested sequence of sets, and since σ-algebras are closed under countable unions, each $F_k \in \mathcal{M}$. Since μ is continuous from below, it follows form part (c) of the theorem from class that

$$\mu\left(\cup_{k=1}^\infty F_k\right) = \lim_{k\to\infty} \mu(F_k).$$

We now note that since $\cup_{k=1}^\infty F_k = \cup_{j=1}^\infty E_j$ that

$$\mu\left(\cup_{j=1}^\infty E_j\right) = \mu\left(\cup_{k=1}^\infty F_k\right) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \mu(\cup_{j=1}^k E_j) = \lim_{k\to\infty} \sum_{j=1}^k \mu(E_j) = \sum_{j=1}^\infty \mu(E_j),$$

where the penultimate equality comes from the finite additivity of μ. But now we've shown that μ is countably additive, thus μ is in fact a measure.

(b) As before, let $\{E_j\}_{j=1}^\infty$ be a collection of disjoint sets in $\mathcal{M}$. We now define the sets $F_k = X \backslash \left(\cup_{j=1}^k E_j\right)$ Note that this means $F_1 \supset F_2 \supset \ldots$. Moreover, since $\mu(X) < \infty$, we have that $\mu(X\backslash E_1) + \mu(E_1) = \mu(X)$, so $\mu(F_1) < \infty$. Thus we can apply part (d) of the theorem from class to get that

$$\mu\left(\cap_{k=1}^\infty F_k\right) = \lim_{k\to\infty} \mu(F_k).$$

We observe that $\cap_{k=1}^\infty F_k = X\backslash(\cup_{j=1}^\infty E_j)$, so we have that

$$\mu(X) - \mu(\cup_{j=1}^{\infty} E_j) = \mu\left(\cap_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \left[\mu(X) - \mu(\cup_{j=1}^{k} E_j)\right].$$

Since $\mu(X)$ is finite, we can subtract it from both sides and we're left with

$$\mu(\cup_{j=1}^{\infty} E_j) = \lim_{k\to\infty} \mu(\cup_{j=1}^{k} E_j),$$

and from here we use the finite additivity of μ to finish up as we did in part (a).
