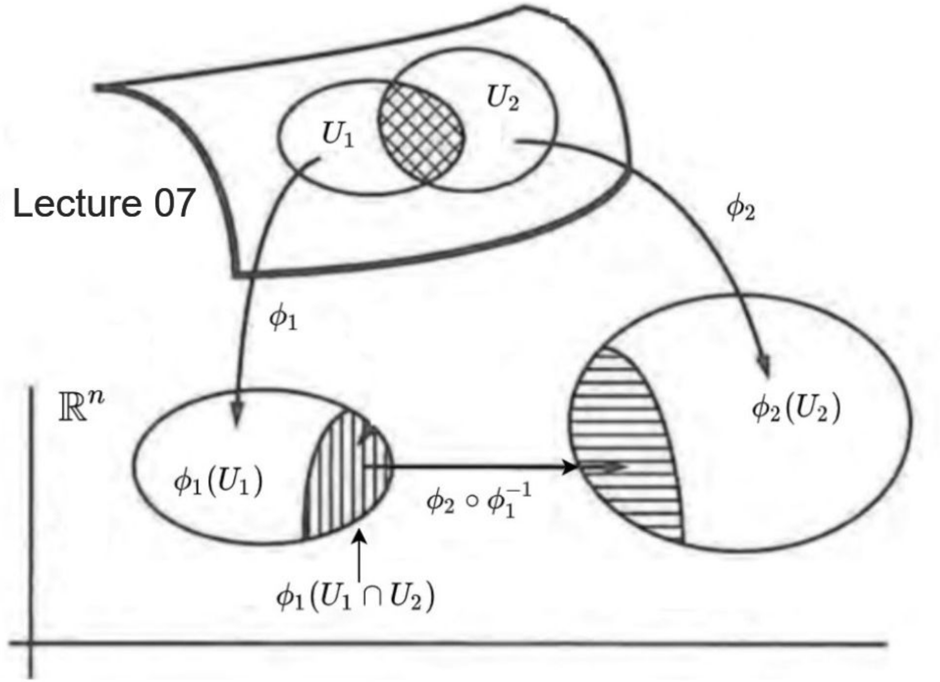
Topology & Differential Geometry Crash Course: Lecture 07



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Session Agenda

Pre-lecture
blah blah blah

PART A

1. Product Manifolds
2. Component Maps (Notation Awareness)
3. Chart Transition Maps
4. Chart Transition Challenge

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PART B

1. Map Differentiability Classes (Brief Recap)
2. Differentiability Classes Challenge
3. Diffeomorphisms (Part I)
4. Chart Compatibility
5. Chart Compatibility Challenge
6. **Study Materials**
7. **Lecture Gap Poll**

Lecture 7 Coverage

General Coverage

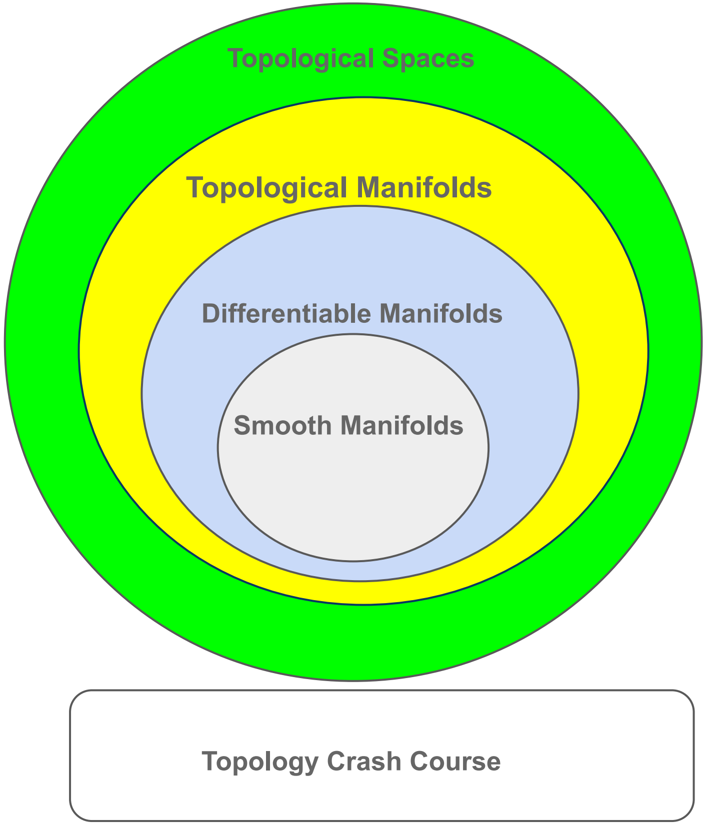
Including a brief intro to Fiber Bundles?

Point-set Topology 101
2/2

Metric Topology 101
3/3

Manifolds 101
2/4

#n is the number of live lectures | Completed | We’re here!



Completed | We’re here!

Crash Course Motivation

Lie Groups, Lie Algebras & Representations

Module II (January 2022 start date?)

Lie group

From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.

Interesting Question from Discord

What is the connection between differential geometry and quantum computing?

My Lazy Answer!

For our purposes, in a nutshell, Lie Groups are the connection between the main parts of Differential Geometry that we're covering in the crash course and quantum computing. This is because the quantum gates form a Lie Group structure i.e. both the unitary group **U**(**n**) and the special unitary **SU**(**n**) are Lie Groups (Compact Lie Groups). This is an important technical detail that introductory quantum computing courses normally don't mention because it is a topic that is more suitable to advanced graduate/research level folks!

Now, Lie Groups don't just carry an algebraic structure (group structure). They do also carry a 'smooth manifold' structure which is part of Differential Geometry. Hence, before you learn what a Lie Group is, it helps if you know what a smooth manifold is. This is exactly the motivation of this crash course!

PART A

Product Manifold

Proposition (1.0)

Let X be an $n-$ dimensional topological manifold and let Y be a $k-$ dimensional topological manifold. Then, the product space $X \times Y$ is a $n+k-$ dimensional topological manifold.

Proof : Homework challenge?

- Hence, the manifold structure is preserved under product topology. Which makes the product topology a great tool for building new topological manifolds out of given manifolds!
- Of course, the above holds for any finite number of products i.e. for any finite product $X_1 \times X_2 \ldots \times X_d$ of topological manifolds for $d \geq 2$.
- If we consider the sphere S^2 and the circle S^1, then their product $S^2 \times S^1$ is $3+1-$ dimensional manifold.

- It also follows that the torus (aka doughnut) $\mathbb{T}^2$ is a $1+1-$ dimensional manifold because it can realised as the product of two circles i.e. $\mathbb{T}^2 = S^1 \times S^1$.

Component Maps

- Consider a coordinate chart on a manifold X $\phi : U \subseteq X \longrightarrow \phi(U) \subseteq \mathbb{R}^n$.

ϕ takes a point $p \in U$ to a point $\phi(p) \in \phi(U) \subseteq \mathbb{R}^n$. Hence, $\phi(p)$ is an $n-$ tupple of reals i.e. $\phi(p) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- Since $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$, we can construct ϕ via several maps $\phi^i : U \subseteq X \longrightarrow \mathbb{R}$ where $i \in \{1, 2, \ldots, n\}$.

$$
\begin{aligned}
&\phi^1 : U \subseteq X \longrightarrow \phi(U) \subseteq \mathbb{R} \\
&\phi^2 : U \subseteq X \longrightarrow \phi(U) \subseteq \mathbb{R} \\
&\vdots \\
&\phi^n : U \subseteq X \longrightarrow \phi(U) \subseteq \mathbb{R}
\end{aligned}
$$

For all $p \in U$, $\phi(p) = (\phi^1(p), \phi^2(p), \ldots, \phi^n(p))$.

- The maps ϕ^i are called components maps of ϕ or local coordinate maps.

- Now, as previously noted, physicists normally use the notation x to denote the map ϕ.

- Hence, for all $p \in U$, physicists would write $x(p) = (x^1(p), x^2(p), \ldots, x^n(p))$!

- Also, it also becomes a convention to identify/label $x^i(p)$ with the notation x^i i.e. $x(p) = (x^1(p), x^2(p), \ldots, x^n(p)) = (x^1, x^2 \ldots, x^n)$!

Chart Transition Maps

Proposition (1.1)

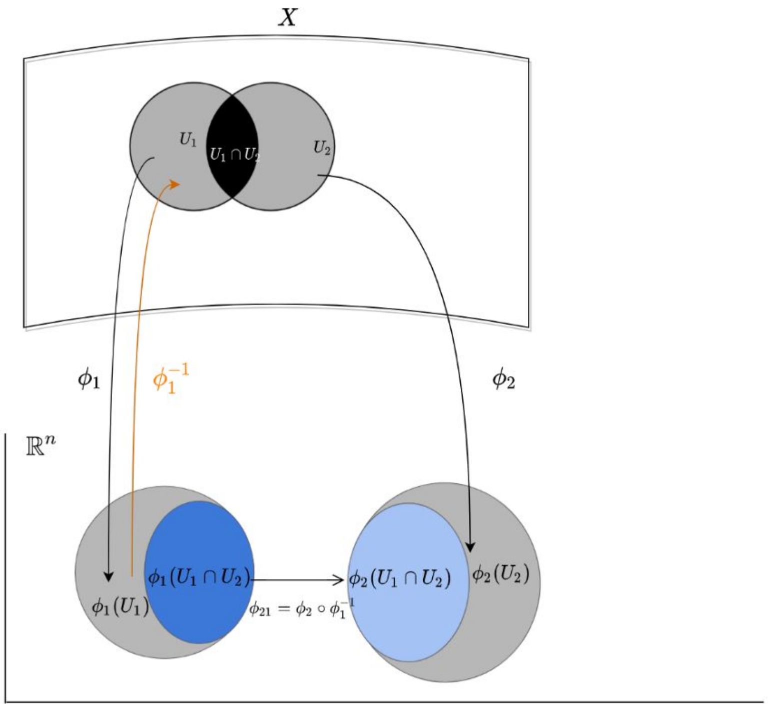
Let (U_1, ϕ_1) and (U_2, ϕ_2) be two non-disjoint charts (i.e. $U_1 \cap U_2 \neq \emptyset$) on an $n-$ dimensional manifold X. Then the composition map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^n \longrightarrow \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^n$ is a homeomorphism.

Proof : Trivial to prove!

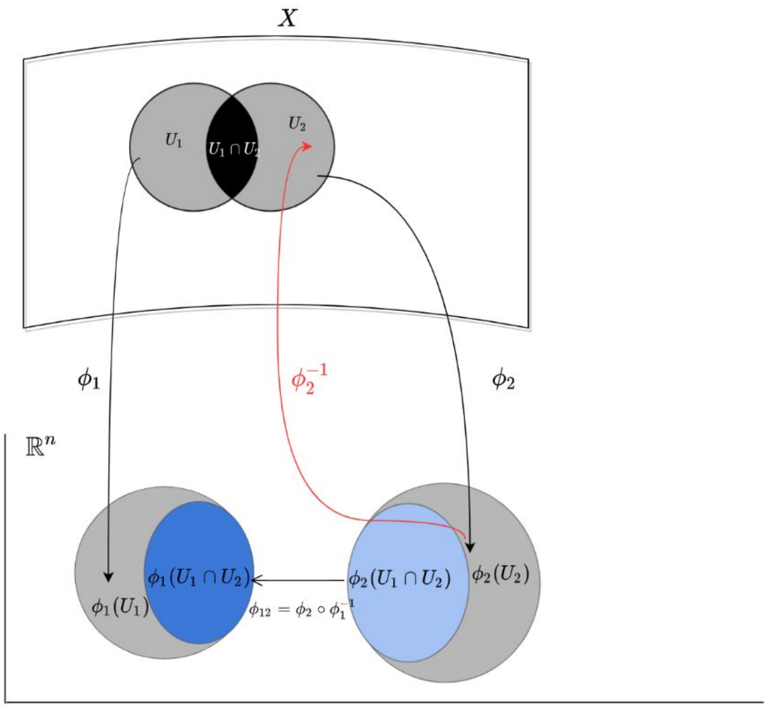
- The composition map $\phi_2 \circ \phi_1^{-1}$ is called a 'chart transition map' from chart the (U_1, ϕ_1) to the chart (U_2, ϕ_2) . Let us by convention abbreviate it as ϕ_{21} i.e. $\phi_{21} = \phi_2 \circ \phi_1^{-1}$.
- Obviously, we can also decide to make a transition from the chart (U_2, ϕ_2) to (U_1, ϕ_1) via $\phi_1 \circ \phi_2^{-1}$ which we'll abbreviate as ϕ_{12}!
- Is there an interesting relationship between the transition maps ϕ_{21} and ϕ_{12}?

Transition Maps in Pictures

- The following diagrams give a clear intuitive picture of what the transition maps ϕ_{21} and ϕ_{12} are doing:



Transition from (U_1, ϕ_1) to (U_2, ϕ_2)



Transition from (U_2, ϕ_2) to (U_1, ϕ_1)

Some Comments

- The ϕ_{21} and ϕ_{12} are what our friends in physics call 'change of coordinates'!
- Hence, chart transition maps are really just abstraction of changing coordinates i.e. they enable us to switch from one coordinate system to another e.g. from Cartesian to polar coordinates vice-versa.
- Now, we could ask if the maps $\phi_{21} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^n \longrightarrow \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^n$ and $\phi_{12} : \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^n \longrightarrow \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^n$ are differentiable?
- In fact, we could ask questions that we would ask for any interesting map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. For example, how many times can we continuously differentiate ϕ_{21} and ϕ_{12}?
- To summarise, transition maps enable us to outsource concrete computations from an abstract $n-$ dimensional manifold X to $\mathbb{R}^n$ i.e. they enable us to do Calculus on X by proxy!

Chart Transition Challenge I

- Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
  1. Let $U_1 = S^1 \setminus (0, -1)$ and $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = \frac{x}{y+1}$.
  2. Let $U_2 = S^1 \setminus (0, 1)$ with $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_2$ let us define the map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = \frac{x}{1-y}$.
- Obviously $U_1 \cap U_2 = S^1 \setminus \{(0, -1), (0, 1)\}$. The challenge is to:
  1. Find the chart transition from (U_1, ϕ_1) to (U_2, ϕ_2) i.e. find $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ which implies finding ϕ_1^{-1}.
  2. Find the chart transition from (U_2, ϕ_2) to (U_1, ϕ_1) i.e. find $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ which implies finding ϕ_2^{-1}.

Chart Transition Challenge II

- Consider the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.
  Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
  1. Let $U_1 = S^2 \setminus (0,0,-1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x,y,z) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = (\frac{x}{z+1}, \frac{y}{z+1})$.
  2. Let $U_2 = S^2 \setminus (0,0,1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x,y,z) \in U_2$ let us define the chart map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = (\frac{x}{1-z}, \frac{y}{1-z})$.
- Obviously $U_1 \cap U_2 = S^2 \setminus \{(0,0,-1),(0,0,1)\}$. The challenge is to:
  1. Find the chart transition from (U_1, ϕ_1) to (U_2, ϕ_2) i.e. find $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ which implies finding ϕ_1^{-1}.
  2. Find the chart transition from (U_2, ϕ_2) to (U_1, ϕ_1) i.e. find $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ which implies finding ϕ_2^{-1}.

PART B

Map Differentiability Classes

Definition (1.0)

Recall from undergrad that given a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we can put f in the following categories:

1. C^0 class if f is continuous.
2. C^1 class if f is continuous and when differentiated once, the result of the differentiation is still continuous.
3. C^k class if f is continuous and when differentiated $k-$ times, the result of the differentiation is still continuous for some $k > 1$.
4. C^∞ class if f is continuous and when differentiated arbitrarily many times, the result of the differentiation is still continuous. Another name for a C^∞ class map f is 'smooth map'.

- C^∞ maps are very important to us because they will lead us to 'smooth manifolds' and their structure preserving maps i.e. 'diffeomorphisms'.

Differentiability Classes Challenge

- For each of the following maps, find out if they are C^k for some $k \geq 1$ or if they are smooth (C^∞):
  1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x, y) = x^2 + 2xy - y^2$.
  2. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x, y) = x \cos y$.
  3. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x, y) = x \sin y$.
  4. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x, y) = e^{xy}$.
  5. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined as $f(x, y) = (x \cos y, x \sin y, e^{xy})$.

Diffeomorphisms (Part I)

Definition (1.1)

Let $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ be two open subsets. A map $f : U_1 \longrightarrow U_2$ is called a diffeomorphism if the following hold:

1. f is smooth.
2. f is invertible i.e. it has an inverse map f^{-1}.
3. The inverse f^{-1} is also smooth.

- As $n-$ dimensional manifolds, we say the two open subsets $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ are diffeomorphic if there exists a diffeomorphism between them.
- The set of all diffeomorphisms $f : U_1 \longrightarrow U_2$ is denoted $C^\infty(U_1, U_2)$. When $U_1 = U_2$ then we just write $C^\infty(U_1)$.
- Let $U_1 = \mathbb{R}$ and $U_2 = \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive reals. Now let define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as the exponential map $f(x) = e^x$. Is f a diffeomorphism?

Compatibility of Charts

Definition (1.2)

Let (U_1, ϕ_1) and (U_2, ϕ_2) be two non-disjoint charts on an $n-$ dimensional manifold X. We say the two charts are C^k- compatible if $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ are both C^k maps for some $k \geq 1$.

- Note that the above definition is for when $U_1 \cap U_2 \neq \emptyset$. When $U_1 \cap U_2 = \emptyset$ then we always assume the two charts are compatible!

Definition (1.3)

Let X be an $n-$ dimensional manifold. Given two non-disjoint charts (U_1, ϕ_1) and (U_2, ϕ_2) with respective chart transition maps $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{12} = \phi_1 \circ \phi_2^{-1}$. Then we say the two charts are smooth-compatible if ϕ_{21} and ϕ_{12} are both smooth maps.

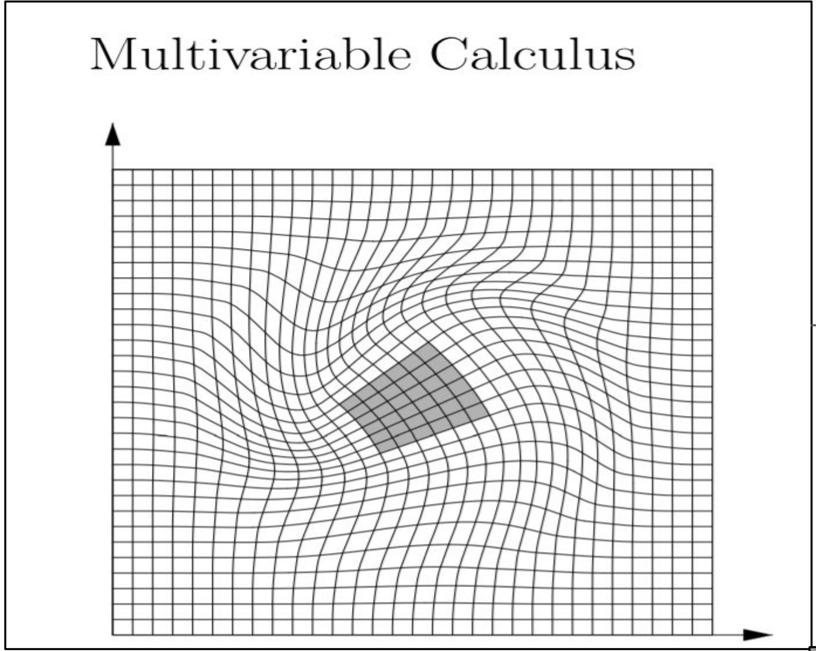
- For our purposes, we'll be interested in smooth-compatible charts when building our atlasses (smooth atlasses).

Chart Compatibility Challenge

- Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
  1. Let $U_1 = S^1 \setminus (0, -1)$ and $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = \frac{x}{y+1}$.
  2. Let $U_2 = S^1 \setminus (0, 1)$ with $\hat{U} = \mathbb{R}$. For each point $p = (x, y) \in U_2$ let us define the map $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = \frac{x}{1-y}$.

- Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Now lets construct two charts (U_1, ϕ_1) and (U_2, ϕ_2) as follows:
  1. Let $U_1 = S^2 \setminus (0, 0, -1)$ and $\hat{U} = \mathbb{R}^2$. For each point $p = (x, y, z) \in U_1$ let us define the chart map $\phi_1 : U_1 \longrightarrow \hat{U}$ as $\phi_1(p) = (\frac{x}{z+1}, \frac{y}{z+1})$.
  2. Let $U_2 = S^2 \setminus (0, 0, 1)$ and $\hat{U} = \mathbb{R}^2$. For each $p = (x, y, z) \in U_2$ let us define $\phi_2 : U_2 \longrightarrow \hat{U}$ as $\phi_2(p) = (\frac{x}{1-z}, \frac{y}{1-z})$.

Challenge question: Are the charts above C^k- compatible? Or even smooth-compatible?



4.6 Higher Order Derivatives

Partial differentiation can be carried out more than once on nice enough functions. For example if

$$f(x, y) = e^{x \sin y}$$

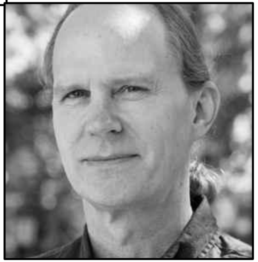
then

$$D_1 f(x, y) = \sin y e^{x \sin y}, \quad D_2 f(x, y) = x \cos y e^{x \sin y}.$$

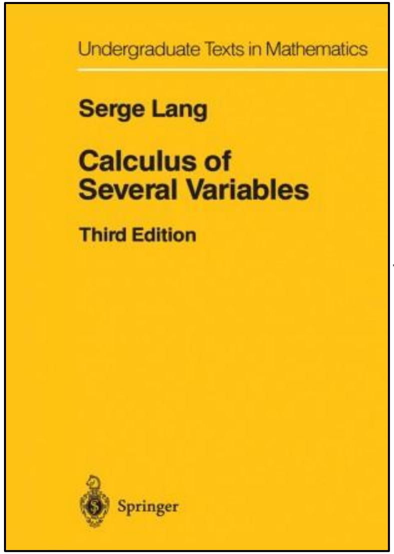
Taking partial derivatives again yields

$$\begin{aligned}
D_1 D_1 f(x, y) &= \sin^2 y e^{x \sin y}, \\
D_1 D_2 f(x, y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y}, \\
D_2 D_1 f(x, y) &= \cos y e^{x \sin y} + x \sin y \cos y e^{x \sin y} = D_1 D_2 f(x, y), \\
D_2 D_2 f(x, y) &= -x \sin y e^{x \sin y} + x^2 \cos^2 y e^{x \sin y},
\end{aligned}$$

and some partial derivatives of these in turn are,



Prof. Jerry Shurman



CHAPTER VI

Higher Derivatives

In this chapter, we discuss two things which are of independent interest. First, we define partial differential operators (with constant coefficients). It is very useful to have facility in working with these formally.

Secondly, we apply them to the derivation of Taylor's formula for functions of several variables, which will be very similar to the formula for one variable. The formula, as before, tells us how to approximate a function by means of polynomials. In the present theory, these polynomials involve several variables, of course. We shall see that they are hardly more difficult to handle than polynomials in one variable in the matters under consideration.

The proof that the partial derivatives commute is tricky. It can be omitted without harm in a class allergic to theory, because the technique involved never reappears in the rest of this book.

§1. *Repeated partial derivatives*

Let f be a function of two variables, defined on an open set U in 2-space. Assume that its first partial derivative exists. Then $D_1 f$ (which we also write $\partial f/\partial x$ if x is the first variable) is a function defined on U. We may then ask for its first or second partial derivatives, i.e. we may form $D_2 D_1 f$ or $D_1 D_1 f$ if these exist. Similarly, if $D_2 f$ exists, and if the first partial derivative of $D_2 f$ exists, we may form $D_1 D_2 f$.

Suppose that we write f in terms of the two variables (x, y). Then we can write

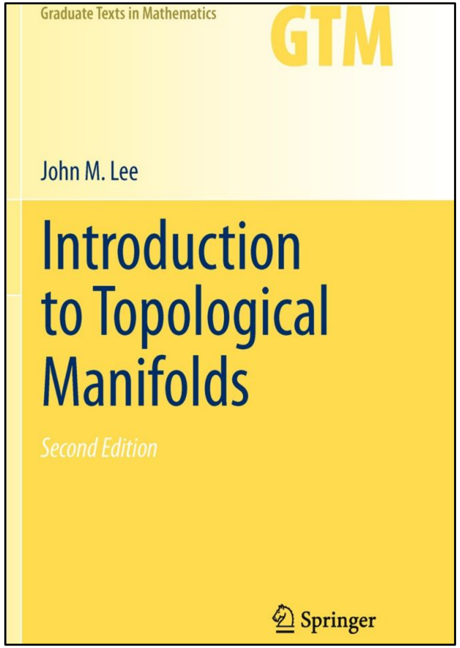
$$D_1 D_2 f(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (D_1(D_2 f))(x, y),$$

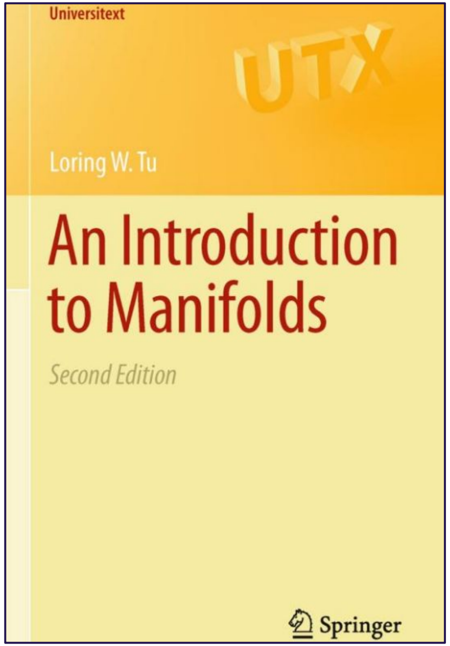
and

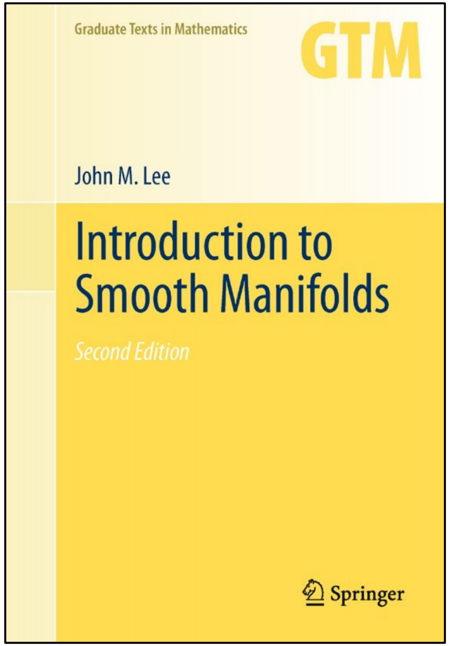
$$D_2 D_1 f(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (D_2(D_1 f))(x, y).$$

For example, let $f(x, y) = \sin(xy)$. Then

$$\frac{\partial f}{\partial x} = y\cos(xy) \quad \text{and} \quad \frac{\partial f}{\partial y} = x\cos(xy).$$







Topological & Diff. Manifolds

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