Homework 8

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Suppose f is measurable, non-negative, and $\int f \, d\mu = 0$. Then $f = 0$ a.e.

   Solution: We will suppose that $f \neq 0$ *a.e.* so there is some set A such that $f(x) \geq c > 0$ for all $x \in A$ and $\mu(A) > 0$. But now

   $$0 = \int f \, d\mu \geq \int_A f \, d\mu \geq c\mu(A) > 0,$$

   an obvious contradiction.

2. Suppose f is real-valued, measurable, and for every measurable set A, we have $\int_A f \, d\mu = 0$. Then $f = 0$ a.e.

   Solution: Let $A_n = \{x \mid f(x) > 1/n\}$. Then by the given information,

   $$0 = \int_{A_n} f \geq \frac{1}{n}\mu(A_n) \implies \mu(A_n) = 0.$$

   This is true for all n, and note that $A_n \subset A_{n+1}$, $A_n \to A = \{x \mid f(x) > 0\}$, so by continuity from below, we have that $\mu(A_n) \to \mu(A) \implies \mu(A) = 0$. Likewise, we can show that $B = \{x \mid f(x) < 0\}$ is also a null set. Thus $f = 0$ a.e., as desired.

3. Find an example of a sequence of functions $\{f_n\}$ on $[0, 1]$ such that each f_n is Riemann integrable, $f_n \leq f_{n+1}$, and $f_n \to f$, but f is not Riemann integrable.

   Remark: This further elucidates the strength of the Lebesgue integral compared to the Riemann integral by showing that MCT and DCT do not hold for Riemann integrals. Take some time to reflect upon what the difference is between a Riemann sum is compared to a simple function!

   Solution: Let $\{q_1, q_2, \ldots\}$ be an enumeration of the rationals in the interval $[0, 1]$. Then we define f_n as follows:

   $$f_n(x) = \begin{cases} 1 & x = q_i, 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $f_n \leq f_{n+1}$, each f_n only has finitely many discontinuities, so it is Riemann integrable. However we note that f_n converge pointwise to the Dirchlet Fucntion with $f(x) = 1$ for rational x and 0 for all irrationals, which we know is not Riemann integrable.
