QF Group Theory CC2022 By Zaiku Group

Lecture 19

Delivered by Bambordé Baldé

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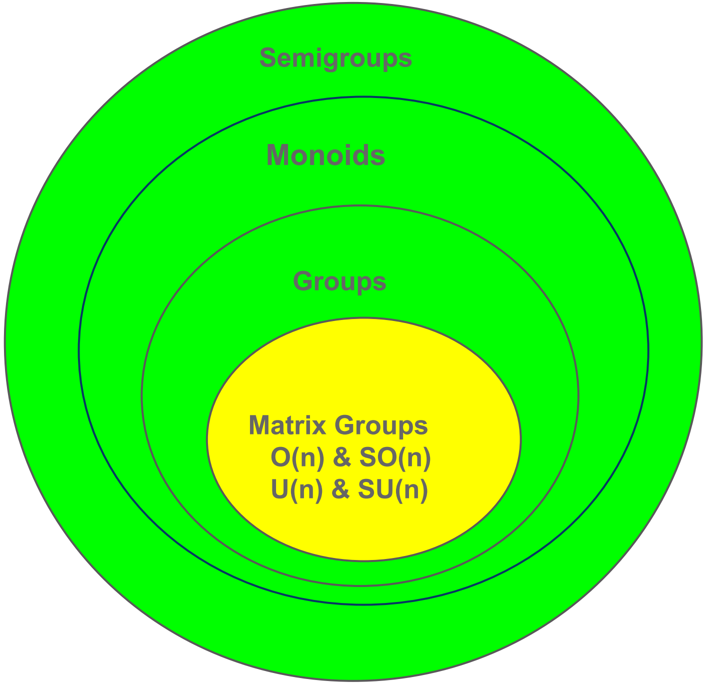
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
19/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We’re here!

A Brief Linear Algebra Recap

Definition 1.0

We'll write $M_n(\mathbb{C})$ to denote the set of all $n \times n$ matrices over ~~the reals~~ $\mathbb{C}$.

- Some authors use the notation $M^{n\times n}(\mathbb{C})$ instead of $M_n(\mathbb{C})$.
- I'll assume everyone knows about the basics of $n \times n$ matrices over the reals $\mathbb{C}$ including; how to compute the transpose, perform addition and multiplication of $n \times n$ matrices.
- When equipped with the ordinary matrix addition or multiplication, which of the following is true?

1. $M_n(\mathbb{C})$ forms an abelian group structure under addition.
2. $M_n(\mathbb{C})$ forms a nonabelian group structure under multiplication.

Important: From linear algebra 101 an element $A \in M_n(\mathbb{C})$ induces a linear map $L_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, with $\mathbb{C}^n$ equipped with the canonical vector space structure over $\mathbb{C}$. Likewise, any linear map $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ induces an element $A_L \in M_n(\mathbb{C})$ i.e. linear maps on $\mathbb{C}^n \equiv n \times n$ matrices over $\mathbb{C}$.

Complex Matrix Groups

Definition 1.1

A subset $G \subset M_n(\mathbb{C})$ is a complex matrix group if it's a group under the ordinary matrix multiplication. This obviously implies the following:

1. If $A, B \in G$ then $AB \in G$ i.e. matrix multiplication is a closed binary operation in G.
2. If $A, B, C \in G$ then $A(BC) = (AB)C$ i.e. matrix multiplication is associative in G. This is trivial to show because it is associative in $M_n(\mathbb{C})$!
3. The identity matrix $I_n \in G$.
4. For any $A \in G$ there exists an inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

- Since G is a group, then all the abstract group-theoretic properties and constructions we've made so far also applies to it! Hence, we can ask about subgroups of G, left $G-$ actions, left cosets, orbits, stabilisers and so on.

The General Linear Group over $\mathbb{C}$

Proposition 1.0

Let us consider the subset of $M_n(\mathbb{C})$ defined as $GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$. Then $GL(n, \mathbb{C})$ is a complex matrix group under the ordinary matrix multiplication.

Proof : Homework challenge!

- As a hint to help you prove the above: Recall from kindergarten linear algebra that if $A \in M_n(\mathbb{C})$ and $det(A) \neq 0$, then A is invertible! In fact A is invertible iff $det(A) \neq 0$!
- $GL(n, \mathbb{C})$ is known in the literature as the general linear group of order n over $\mathbb{C}$. Also, some authors use the notation $GL_n(\mathbb{C})$!

Side note: Observe the following subtle facts about $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ as Lie groups:

1. $GL(n, \mathbb{C})$ is a noncompact connected Lie group of complex dimension n^2 and real dimension $2n^2$.
2. $GL(n, \mathbb{R})$ is a noncompact disconnected Lie group of dimension n^2.

The Complex Special Linear Group

Proposition 1.1

The set $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of $GL(n,\mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- $SL(n,\mathbb{C})$ is known in the literature as the complex special linear group.

Side note: Observe the following subtle facts about $SL(n,\mathbb{C})$ and $SL(n,\mathbb{R})$ as Lie groups:

1. $SL(n,\mathbb{C})$ is a noncompact connected Lie group of complex dimension $n^2 - 1$ and real dimension $2(n^2 - 1)$.
2. $SL(n,\mathbb{R})$ is a noncompact connected Lie group of dimension $n^2 - 1$.

Proposition 1.2

Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. Then the determinant map $det : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}^*$ taking $A \in GL(n,\mathbb{C})$ to $det(A) \in \mathbb{C}^*$ is a group homomorphism and $Ker(det) = SL(n,\mathbb{C})$.

A Special Complex Matrix Group in Disguise

Complex Numbers 101

Given a complex number $a = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$, the complex conjugate of a is defined as $\bar{a} = x - iy$.

Attention: Physicists often use the notation a^* instead of $\bar{a}$!

Proposition 1.3

The set $G = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$ is a subgroup of $GL(2, \mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- The group G above is a very special type of group in disguise! Can anyone unmask it? Can the quantum folks unmask it?

Side note: You'll learn in the next course that as a smooth manifold, G is diffeomorphic to the $3-\textit{sphere}\ S^3$!

Matrix Conjugate Refresh

Definition 1.2

Given $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C})$, we define the conjugate as:

$\bar{A} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{pmatrix}$ where $\bar{a}_{ij} = x - iy$ for all $a_{ij} = x + iy$.

- Physicists often use the notation A^* instead of $\bar{A}$!

Conjugate Transpose Refresh

Definition 1.2 (using the mathematician's notation)

Given $A \in M_n(\mathbb{C})$, we define the conjugate transpose of A as $A^* = (\bar{A})^T$.

- Physicists use the notation $A^\dagger$ instead of A^*!
- We'll adopt the physicist notation for the conjugate transpose of a matrix and adopt the mathematician's notation for the conjugate of complex numbers!

Proposition 1.4

Let $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then the following identities hold:

1. $(A^\dagger)^\dagger = A$.
2. $(\lambda A)^\dagger = \bar{\lambda} A^\dagger$.
3. $(A+B)^\dagger = A^\dagger + B^\dagger$.
4. $(AB)^\dagger = B^\dagger A^\dagger$.
5. $det(A^\dagger) = \overline{det(A)}$.
6. If A is invertible then $A^\dagger$ is also invertible.

Proof : Homework challenge!

Side note: A matrix $A \in M_n(\mathbb{C})$ is said to be Hermitian if $A = A^\dagger$!

The Unitary Matrix Group

Proposition 1.5

The set $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^\dagger A = AA^\dagger = I_n\}$ is a subgroup of $GL(n, \mathbb{C})$ i.e. it is a complex matrix group.

Proof : Homework challenge!

- The group $U(n)$ is known in the literature as the unitary group.
- The group elements of $U(n)$ are indeed linear isometries in $\mathbb{C}^n$ i.e. they preserve the inner product in $\mathbb{C}^n$ and so the norm.
- So $U(n)$ is the complex version of the real orthogonal group $O(n)$!
- $U(n)$ is a very important group with applications in many topics such as theoretical physics and quantum information science.

Side note: Observe the following subtle facts about $U(n)$ and $O(n)$ as Lie groups:

1. $U(n)$ is compact and connected Lie group with 'real' dimension n^2.
2. $O(n)$ is compact and disconnected Lie group with dimension $\frac{n(n-1)}{2}$.

The Special Unitary Group

Proposition 1.4

The set $SU(n) = \{A \in U(n) \mid det(A) = 1\}$ is a subgroup of $U(n)$.

Proof : Homework challenge!

- $SU(n)$ is known in the literature as the special unitary group.
- So $SU(n)$ is the complex version of the real orthogonal group $SO(n)$!
- It's clear than $SU(n) = U(n) \cap SL(n, \mathbb{C})$ right?
- The complex matrix group in disguise we were playing with is indeed $SU(2)$!

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Side note: Observe the following subtle facts about $SU(n)$ and $SO(n)$ as Lie groups:

1. $SU(n)$ is compact and connected Lie group with 'real' dimension $n^2 - 1$.
2. $SO(n)$ is compact and connected Lie group with dimension $\frac{n(n-1)}{2}$.

SU(n) homework challenge

Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. Then the determinant map $det : U(n) \longrightarrow \mathbb{C}^*$ taking $A \in U(n)$ to $det(A) \in \mathbb{C}^*$ is a group homomorphism. What is $Ker(det)$?

- Also, Is it true $SU(n)$ is a normal subgroup of $U(n)$?

Side note tables

G	$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{O}(n,\mathbb{R})$	$\mathrm{SO}(n,\mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(2n,\mathbb{R})$
$\mathfrak{g}$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathrm{tr}\, x = 0$	$x + x^t = 0$	$x + x^t = 0$	$x + x^* = 0$	$x + x^* = 0,\ \mathrm{tr}\, x = 0$	$x + Jx^tJ^{-1} = 0$
$\dim G$	n^2	$n^2 - 1$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}$

G	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$

Credits for the tables: Prof Alexander Kirillov, Math. Department of State Univ. of New York at Stony Brook.

QUANTUM FORMALISM

GitHub: github.com/quantumformalism

YouTube: youtube.com/ZaikuGroup

Discord: discord.gg/SPcmcsXMD2

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