QUANTUM FORMALISM

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Homework 10

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $D = [0,1] \times [0,1]$, and consider the following variant of Thomae's function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ or } y \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ and } y \text{ both rational and } y = p/q \text{ in reduced form} \end{cases}$$

Show the following (note all the following are standard Riemann integrals):

a) $\int_0^1 f(x,y)\, dy = 0$ for any $x \in [0,1]$.

b) $\iint_D f(x,y)\, dA = 0$.

c) $\int_0^1 f(x,y)\, dx = 0$ for any irrational y but does not exist for rational y.

d) Explain why this doesn't contradict Fubini's Theorem.

e) What if instead of using the standard Riemann integral, we replaced dx and dy with $d\lambda$, the Lebsegue integral?

Solution: If x is irrational, then we automatically have the integral is zero, so we will prove it for some fixed rational x_0. In this case we have

$$\int_0^1 f(x_0, y)\, dy = \int_0^1 f_{x_0}(y)\, dy$$

which is just the integral of the standard Thomae's function, which from the first lecture we know is zero. But short of recalling that, let's compute it out using step functions. Since we wish to show the in integral is zero, given $\epsilon > 0$, we need only find some step function h such that $f \le h$ and $\int h < \epsilon$. Note that there is some N such that $0 < 1/N < \epsilon$, and there are only finitely many values of y such that $f_{x_0}(y) > 1/N$, say this happens at $q_1, \ldots q_n$. Cover each q_i in an interval A_i such that the A_i do not intersect, and $\nu(A_i) < \epsilon$. Then define h as follows

$$h = \begin{cases} 1 & x \in A_i, i = 1, 2, \ldots, n \\ \epsilon & x \notin \bigcup A_i \end{cases}$$

Finally note $\int h < \sum \nu(A_i) + \epsilon < (n+1)\epsilon$ which, whatever, good enough. So by Theorem 3.3, this means f_{x_0} is integrable and it must integrate to zero.

Now part (a) follows just by considering the intervals $B_i = A_i \times [0,1]$, and the corresponding step function.

Finally to see part (c), note that if we instead fix $y_0 \in \mathbb{Q}$, then

$$f(x, y_0) = f_{y_0}(x) = \begin{cases} \frac{1}{q} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

which is just a Dirichlet type function, which we know from real analysis is not integrable (note for any step functions $h \leq f \leq k$, $\int k - h > 1$).

The significance of the problem is that we have a situation where we cannot flip the order of integration because

$$\iint f \; dydx = \int 0 \; dx = 0$$

but

$$\iint f \; dxdy$$

just does not exist. The reason is that we see by parts (a) and (c) that while f_x is integrable with respect to y, f_y is not in general integrable with respect to x.

Finally, note that since the set of rationals has Lebesgue measure zero, if we were to use $d\lambda$, then we wouldn't have the issue with flipping the since we've seen in terms of the Lebesgue integral, the Dirichlet function is in fact integrable! Neat!

2. For $i = 1, 2, 3, \ldots$, let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be continuous real valued functions with support in $(1/(i+1), 1/i)$ such that $\int_0^1 \varphi_i = 1$. Define

$$f(x,y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

   a) Prove the following (note the following are both Riemann integrals):

$$\int_0^1 \int_0^1 f(x,y) \; dx \, dy = 0 \qquad \int_0^1 \int_0^1 f(x,y) \; dy \, dx = 1.$$

   b) Explain why the findings above do not contradict Fubini's Theorem.

   c) What if instead of using the standard Riemann integral, we replaced dx and dy with $d\lambda$, the Lebsegue integral?

Solution: First we observe that for any fixed (x, y), at most two terms of the sum are not zero, so it is a finite sum. If $y_0 \in [0, 1]$ is fixed, then only at most one of the $\varphi_i(y_0)$ is not zero, say φ_k

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \int_0^1 \int_0^1 [\varphi_k(x) - \varphi_{k+1}(x)]\varphi_k(y_0)\, dx\, dy = \int_0^1 0\, dy = 0.$$

If instead we start by fixing some $x_0 \in [0, 1]$, then $g_k(x_0)$ is not zero for some k. In particular, if $k \neq 1$, then $f(x_0, y) = \varphi_k(x_0)\varphi_k(y) - \varphi_k(x_0)\varphi_{k-1}(y)$, which will integrate to zero. If $\varphi_1(x_0) \neq 0$, then we will get that the integral of $f(x_0, y) = \varphi_1(x_0)\varphi_1(y)$ is $\varphi_1(x_0)$. So we have that

$$\int_0^1 \int_0^1 f(x, y)\, dy\, dx = \int_0^1 \int_0^1 \varphi_1(x)\varphi_1(y)\, dy\, dx = \int_0^1 \varphi_1(x)\, dx = 1.$$

Since these integrals are not equal this seems like it should be a contradiction to Fubini's Theorem. Especially since we've shown above that both the functions f_x and f_y are in fact integrable! The issue here is that the function $f(x, y)$ itself does not satisfy the definition of integrability. Note that since the φ_i are only supported on $(1/(i + 1), 1/i)$, and they are continuous, this means there must be some point p_i such that $\varphi_i(p_i) > i(1 + i)$, otherwise they could not integrate to a value of 1. In particular this means in any neighborhood U of $(0, 0)$, $f(x, y)$ is unbounded, hence at least one of it's positive or negative sets cannot be covered in finitely many rectangles, thus not integrable.

However in this problem, if we try to use the Lebesgue integral, the outcome will not change since both double integrals exist, i.e., they are Riemann integrable, thus they are Lebesgue integrable and will have the same values.