Homework 2

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. We call a collection of sets $\mathcal{R} \subset P(X)$ a ring if it is closed under finite unions and differences, that is, if $E, F \in \mathcal{R}$, then $E \backslash F \in \mathcal{R}$. If it is closed under countable unions, we call it a σ-ring.

   a) Let $\mathcal{R}$ be a ring such that $X \in \mathcal{R}$. Show that $\mathcal{R}$ is also an algebra. (The same holds for σ-rings/σ-algebras).

   b) Construct an example of a ring that is not an algebra.

   c) If $\mathcal{R}$ is a σ-ring, then

   $$\mathcal{A}_1 = \{E \subset X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$$

   and

   $$\mathcal{A}_2 = \{E \subset X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$$

   are both σ-algebras.

2. a) Let X be a non-empty set and $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a collection of σ-algebras on X. Verify that $\cap_{j=1}^{\infty} \mathcal{A}_j$ is a σ-algebra on X. (Recall we state this in lecture but never verified it).

   b) Provide an example to show that the analogous statement about a union of σ-algebras is false.

   c) Suppose we add the condition that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ Is it now the case that $\cup_{j=1}^{\infty} \mathcal{A}_j$ is a σ-algebra? Prove it or provide a counter example.

3. An algebra $\mathcal{A}$ is a σ-algebra if and only if for any collection $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ with $E_1 \subset E_2 \subset \ldots$, $\cup_{j=1}^{\infty} E_j \in \mathcal{A}$.
