QUANTUM
FORMALISM

Linear Operators 101 - Part 1

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Lecture Agenda Summary

1. Pre-Lecture Comments
2. Lecture 05 Recap
3. Linear Operators
4. Operator Range
5. Operator Kernel
6. Study Materials Comments

Foundation Module Review

Rings and Fields 101
#1

Matrix Algebra
#1

Quantum Matrix Operators
#2

Group Theory 101
#1

Linear Operators 101
#2

Complex Hilbert Spaces
#2

Naive Set Theory Overview
#1

Complex Vector spaces 101
#2

Matrix Groups: GL(n, C) & U(n)
#2

Completed | Ongoing | #n is the number of live lectures

Linear Algebra Section

- More Abstract and Algebraic Approach
- More tailored to QM/QC

- More Aligned to undergrad LA textbooks
- Not tailored to QM/QC

Live Sessions ****Versus*** **Study Materials**

**When we get to Hilbert spaces, the study materials will then be aligned with the live sessions!*

“I think it is really important to have this mindset of not to be intimidated by the things you don’t know, but rather be excited about the things you don’t know. “ **Amira Abbas**

Definition (Lecture 05)

We say that vectors $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle \in V$ are linearly independent if $\sum_{i=1}^{n} \alpha_i|\psi_i\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle .... + \alpha_n|\psi_n\rangle = 0$ if only if $\alpha_i = 0$ $\forall i \in \{1, ..., n\}$ i.e. $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are of course linearly independent i.e.

  $\alpha_1|0\rangle + \alpha_2|1\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if only if $\alpha_1 = \alpha_2 = 0$.

Definition (Lecture 05)

A subset $B = \{|e_1\rangle, |e_2\rangle ......, |e_n\rangle\}$ of V forms a basis (Hamel) in V if:

1. $|e_1\rangle, |e_2\rangle ......, |e_n\rangle$ are linearly independent i.e. $\sum_{i=1}^{n} \alpha_i|e_i\rangle = 0$ if only if $\alpha_i = 0$ $\forall i \in \{1, ..., n\}$.
2. $\text{Span}(|e_1\rangle, |e_2\rangle, ..., |e_n\rangle) = V$ i.e. any $|\psi\rangle \in V$ can be written (uniquely) as linear combination of $|e_1\rangle, |e_2\rangle, ..., |e_n\rangle$.

- Let $V = \mathbb{C}^2$ and $B = \Big\{ |0\rangle, |1\rangle \Big\}$. It's clear that B forms a basis in $\mathbb{C}^2$ right?

- Are there more bases in $\mathbb{C}^2$ other than B ?

Definition (Lecture 05)

If $B = \{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ forms a basis in V then its cardinality is called the dimension of V and denoted $\dim(V)$ or just $\dim V$.

- It's obvious that $\dim(\mathbb{C}^2) = 2$ because the cardinality of $B = \left\{|0\rangle, |1\rangle\right\}$ is 2.
- As you might have noticed, in general, $\dim(\mathbb{C}^n) = n$.
- Later we'll see that if V_1 and V_2 are two vector spaces over $\mathbb{C}$. Then, $dim(V_1) = dim(V_2)$ iff $V_1 \simeq V_2$ vise versa. This will be important when we talk about tensor products e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

Proposition (1.0)

If $B = \{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ and $B' = \{|e_1'\rangle, |e_2'\rangle, ..., |e_m'\rangle\}$ are bases in V then $|B| = |B'|$ i.e. $n = m$.

Proof: Homework 1.0!

- Hence, the dimension of V does not depend on the choice of basis.

Linear Operators 101

Definition (1.0)

Let V and W be vector spaces over $\mathbb{C}$. A map $T : V \to W$ is called linear operator if it satisfies the following axioms:

1. $T(|\psi_1\rangle + |\psi_2\rangle) = T(|\psi_1\rangle) + T(|\psi_2\rangle)$ for all $|\psi_1\rangle, |\psi_2\rangle \in V$
2. $T(\alpha|\psi\rangle) = \alpha T(|\psi\rangle)$ for all $|\psi\rangle \in V$ and $\alpha \in \mathbb{C}$

Convention: Most of the times I'll just write $T|\psi\rangle$ instead of $T(|\psi\rangle)$.

Proposition (1.1)

If $T : V \to W$ is a one-to-one linear operator and $B = \{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ is a basis of V. Then $B' = \{|e'_1\rangle = T|e_1\rangle, |e'_2\rangle = T|e_2\rangle, ..., |e'_n\rangle = T|e_n\rangle\}$ is a basis of W.

Proof: Homework 1.1!

- If $V = W = \mathbb{C}^2$, can you think of anything that can be a linear operator?

Canonical Example of Linear Operator

- Let $V = W = \mathbb{C}^2$ and let $M_2(\mathbb{C})$ be the ring of all 2×2 matrices over $\mathbb{C}$ i.e. $M_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$.

Then for each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ we can construct a linear operator $T_A : \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

For any $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$, let T_A act on $|\psi\rangle$ as $T_A|\psi\rangle = A|\psi\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \alpha + \begin{pmatrix} b \\ d \end{pmatrix} \beta = \begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix}$.

- Hence, each matrix $A \in M_2(\mathbb{C})$ generates a linear operator for $\mathbb{C}^2$.
- Can we generalise this to $\mathbb{C}^n$ such that any $A \in M_n(\mathbb{C})$ generates a linear operator T_A for $\mathbb{C}^n$?
- Can we also generate a matrix linear operator $A_T \in M_n(\mathbb{C})$ for $\mathbb{C}^n$ given any linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$?
- By convention we just write A to denote the linear operator generated from $A \in M_n(\mathbb{C})$ instead of writing T_A.

Proposition (1.2)

If $T : V \to W$ is a linear operator, then the following statements are true:

1. $T|0_V\rangle = |0_W\rangle$
2. $T(-|\psi\rangle) = -T|\psi\rangle$

Proof : Homework 1.2!

Definition (1.1)

If $T : V \to W$ is a linear operator then the range of T is defined as $\mathrm{Ran}(T) = \{T|\psi\rangle \mid |\psi\rangle \in V\}$.

Proposition (1.3)

$\mathrm{Ran}(T)$ is a linear subspace of W.

Proof : Homework 1.3!

- The dimension of $\mathrm{Ran}(T)$ is called the rank of T.

Definition (1.2)

Let $T : V \to W$ be a linear operator. The kernel of T is defined as $\text{Ker}(T) = \{|\psi\rangle \in V \mid T|\psi\rangle = |0_W\rangle\}$.

- Ker(T) is also often called the null-space of T in the literature.

Proposition (1.4)

Ker(T) is a subspace of V.

Proof : Homework 1.4!

Theorem (1.0)

Let $T : V \to W$ be a linear operator. Then $dim(V) = dim\ Ker(T) + dim\ Ran(T)$.

Proof : Homework challenge?

- The theorem above is often called 'the dimension theorem' or sometimes 'the rank-nullity theorem'.

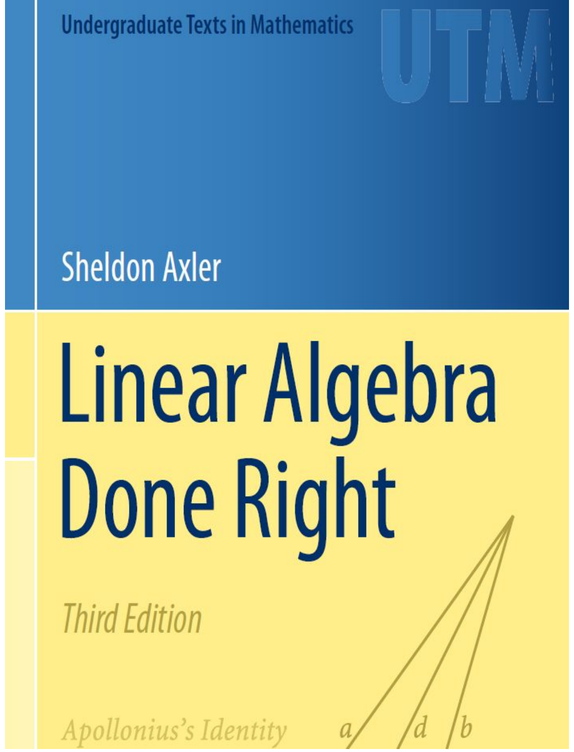
Home Challenge

Let $V = W = \mathbb{C}^2$ and consider the following linear operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

- Find out the kernel of each of the operators above. What is the dimension of each kernel of the operators?
- What is the range of each of the operators? What about their dimensions?

Software aid: Feel free to use any suitable software to help you find the answers!



Prof. Sheldon Axler

Where should you focus?

Linear Maps (*Pages 51- 57)*

QF QUANTUM FORMALISM

- **GitHub (Curated study materials):** *github.com/quantumformalism*
- **YouTube:** youtube.com/zaikugroup
- **Twitter:** *@ZaikuGroup*
- ***Gitter:*** *gitter.im/quantumformalism/community*