Category Theory:

a crash course

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§1 Categories, functors, & Naturality

"To understand a structure, it is necessary to understand the morphisms that preserve it" - goguen

	objects	morphisms
Sets	sets	functions
Grp	groups	group homomorphisms
$\mathbf{Vect}_k$	vector spaces	linear maps
Top	topological spaces	continuous f'ns
	posets	order-preserving f'ns
	⋮	⋮
Cat	categories	functors
	⋮	⋮
	?	?

Def: A category is (loosely) a collection of objects & a collection of morphisms between objects

$\mathcal{C}$

(i) a collection of objects $Ob(\mathcal{C})$

(ii) for any X & Y in $Ob(\mathcal{C})$, a set

$f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) :=$ set of morphisms from X to Y

$\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m) = \mathcal{M}_{m\times n}(\mathbb{R})$ space of matrices

(sometimes called "arrows")

$X \xrightarrow{f} Y$

Compatibility

(iii) for any $X, Y, Z \in Ob(\mathcal{C})$,

sets

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$

$\left(X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z\right) \longmapsto X \xrightarrow{g\circ f} Z$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$g\circ f$ = composition of f'ns

Compatibility

(iv) function composition is associative

$$h\circ(g\circ f) = (h\circ g)\circ f$$

$X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$

(v) identity morphisms $\mathbb{1}_X = id_X$

for any $X \in Ob(\mathcal{C})$, $\exists\, id_X \in Hom_{\mathcal{C}}(X,X)$

for any $g \in G$ $\quad e\cdot g = g\cdot e = g$

for any $f: X\to Y$,
$g: Z \to X$,

Equivalent

$$\begin{cases} f\circ id_X = f \\ g = id_X \circ g \end{cases}$$

in terms of "commutative diagrams"

"this diagram commutes"

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & Z \arrow[r, "h"'] & W \end{tikzcd}$$

$$(h\circ g)\circ f = h\circ(g\circ f)$$

- $$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "id_X\circ g"'] & X \arrow[d, "id_X"] \arrow[dr, "f\circ id_X"] & \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

when are two objects "the same"?

Def: two objects X & Y of $\mathcal{C}$ are isomorphic if $\exists$ a pair of morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $\begin{cases} g\circ f = id_X \\ f\circ g = id_Y \end{cases}$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$$

$g = f^{-1}$, etc.

Examples: Set, Grp, Ab, Top, $Vect_k$, $Vect_k^{f.d.}$, Mon, Met, ...

Banach Analytic Manifolds

BanAnaMan

Ex: Let $\mathcal{C}$ be a category, its opposite category, $\mathcal{C}^{op}$, is given by:

(i) $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

(ii) $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$

$$\underset{X}{\bullet} \xrightarrow{f} \underset{Y}{\bullet} \quad \text{in } \mathcal{C}$$

$$X \longleftarrow Y \quad \text{in } \mathcal{C}^{op}$$

Let R be a ring w/ unity,

$\underline{R\text{-mod}}$ = category of left R-modules

the category of right R-modules is $(\underline{R\text{-mod}})^{op}$

in $\underset{\sim}{\mathcal{D}_X}$-modules

sheaf of linear diff operators w/ holomorphic coefficients on a complex manifold

Example

$\Rightarrow$ $\mathcal{C} = \underline{Set}$, $\emptyset$ = emptyset $\in Ob(\underline{Set})$

for any set X, $\emptyset \subseteq X$.

Categorically: for any X, $\exists! \; \emptyset \xrightarrow{\iota} X$

if Y satisfies this property for any X, then $Y \cong \emptyset$ (isomorphic)

"inclusion" $\hookrightarrow$ \hookrightarrow

$Hom_{\underline{set}}(\emptyset, X) = \{\emptyset \hookrightarrow X\}$

initial object

Exercise 1: show that $\emptyset$ is the only set with this property (up to isomorphism)

Example: "the" singleton set $\{*\} \in Ob(set)$.

"a set containing only one element"

are these different?

$$A = \{a\} \cong \{chair\} = B$$

uniquely isomorphic!

$$A \xrightarrow{f} B \qquad B \xrightarrow{g} A$$

$$a \mapsto chair \qquad chair \mapsto a$$

The singleton set is the unique (up to isomorphism) set $\{*\}$ satisfying:

For any set X, $\exists!\ X \xrightarrow{\alpha} \{*\}$

$$Hom_{Set}(X, \{*\}) = \{X \xrightarrow{\alpha} \{*\}\}$$
$$x \mapsto *$$

terminal object

Exercise 2: Formulate this property in terms an arbitrary category $\mathcal{C}$, & prove it uniquely defines this object.

Exercise 3: For $Vect_k$, is there an initial object or a terminal object?

$\emptyset \notin Vect_k$ ... and?

partially ordered set

Example Let $(X, \le)$ be a poset.

Consider as a category via:

$$Ob(X,\le) = X$$

for any two $x, y \in X$,

$$Hom_{(X,\le)}(x,y) = \begin{cases} \{x \to y\}, & \text{if } x \le y \\ \emptyset, & \text{else} \end{cases}$$

$$\begin{tikzcd} y & & z \\ & x \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

$Hom(x,y) = \{i\}$

$Hom(y,z) = \emptyset$

$Hom(x,z) = \{j\}$

Let X be any set.

$\mathcal{P}(X) = 2^X =$ set of subsets of X,

power set of X

partially ordered by inclusion.

$\rightsquigarrow$ Complete Boolean algebra

Let X be a topological space,

$Op(X) =$ set of open subsets of X,

partially ordered by inclusion.

Comes up all the time in algebraic geometry, for Topos theory, Grothendieck topology ...

Ex: Monoids.

like groups, but w/o inverses

$(\mathbb{Z},+)$ vs. $(\mathbb{N},+)$

group   monoid

$$M \times M \longrightarrow M$$
$$(x,y) \longmapsto x * y$$
$$e * x = x * e = x$$

turn into a category: $\underline{M}$

only one object, $Ob(\underline{M}) = \{\bullet\}$

$Hom_{\underline{M}}(\bullet,\bullet) = M$

monoid multiplication by x

$$\begin{tikzcd} M \arrow[loop above, "x"] \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[dr, "y \circ x := y * x"'] & M \arrow[d, "y"] \\ & M \end{tikzcd}$$

Exercise 4: prove the statement

"a group is a category with one element, where every morphism is an isomorphism"

what is a "structure-preserving" morphism between categories?

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

?   "Functor"

F should define a function

(i) $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$

$X \mapsto F(X)$

(ii) $(X \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(f)} F(Y))$

in $\mathcal{C}$ in $\mathcal{D}$

$$Hom_{\mathcal{C}}(X,Y) \xrightarrow[F_*]{f'n} Hom_{\mathcal{D}}(F(X), F(Y))$$

respect composition

(iii) $F(g \circ f) = F(g) \circ F(f)$ covariant ☆

$F(g \circ f) = F(f) \circ F(g)$ contravariant

(iv) $F(id_X) = id_{F(X)}$

"Every sufficiently good analogy yearns to become a functor"

–Baez

category theory := mathematics of analogies

Ex: for any category $\mathcal{C}$, there's an identity functor: $\quad id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$$X \mapsto X$$
$$f \mapsto f$$

"forgetful functor"

Ex: $\mathcal{U} : \underline{Grp} \to \underline{Set}$

$$(G, *) \mapsto G$$

forget the binary operation!

$\mathcal{F} : \underline{Set} \to \underline{Grp}$

$$S \mapsto \text{Free}(S) = \{\text{free group on the set } S\}$$

look this up, (if hard, talk about it later.)

$$\begin{tikzcd} \underline{Grp} \arrow[r, bend left, "\mathcal{U}"] & \underline{Set} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

generalized inverse functors to each other

they form an <u>adjoint pair</u> of functors.

Ex: $\mathcal{C} = \underline{Vect}_k$ dual vector space functor.

$$D : \underline{Vect}_k^{op} \to \underline{Vect}_k$$

$Hom_{\underline{Vect}_k}(V, k)$

$$V \mapsto V^* = Hom_k(V, k)$$

$$Hom_k(W, k) \to Hom_k(V, k)$$

$$(W \xrightarrow{\xi} k) \mapsto (V \to \xi)$$

$$(\xi \circ f) = f^*(\xi)$$

first problem session next Sunday

Exercise: describe $D^2 : \underline{Vect}_k \to \underline{Vect}_k$ functor of "double dual"

$$D^2(V) = D(D(V)) = Hom_k(Hom_k(V, k), k)$$

& its relationship w/ the identity functor

$$id_{\underline{Vect}_k} : \underline{Vect}_k \to \underline{Vect}_k .$$