Naive Set Theory (ZFC)

Bambordé Baldé | Co-Founder at Zaiku Group | Twitter: @zaikugroup • **zaikugroup.com** • September 18, 2020

Session Agenda

1. Frequent Mathematical Jargons
2. Basic Set Theoretic Concepts
3. Natural Numbers and Integers
4. Maps Between Sets
5. Cardinality of Sets (definition of finite and infinite sets)
6. Countable and Uncountable Sets
7. Measure Theory Hack (without mentioning sigma-algebras)
8. Study Material Comments
9. Session Q&As

Bambordé Baldé | Co-Founder at Zaiku Group | Twitter: @zaikugroup • zaikugroup.com • September 18, 2020

Frequent Mathematical Jargons

1. Axiom
2. Theorem
3. Proposition
4. Lemma
5. Corollary
6. Conjecture
7. Proof
8. Definitions

Naive Set Theory 101

Definition (1.0)

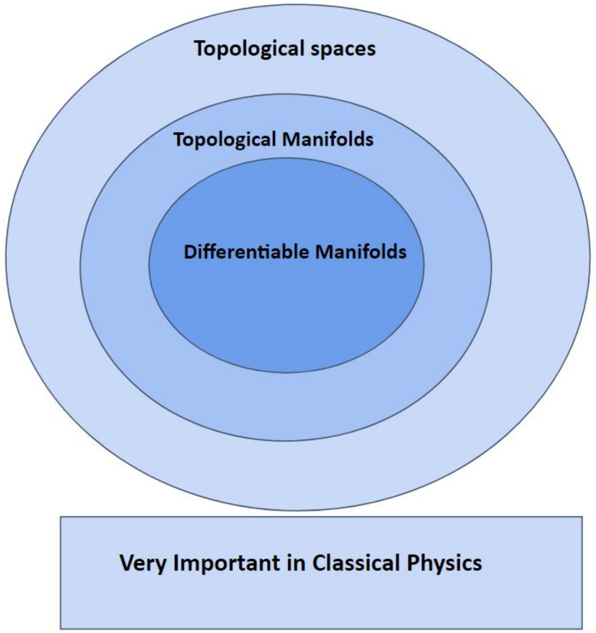
A set is a collection of **distinct** objects called elements of the set.

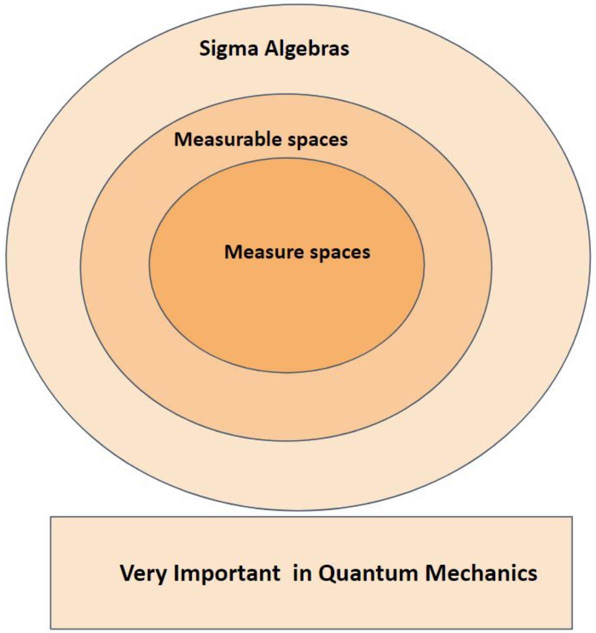
- Let $X = \{a, b, c, d\}$ and $Y = \{a, b, c, d, a\}$. If you give Y to mathematicians they will assume you mean X!
- If X is a set and ψ is an element of X, we write $\psi \in X$ or else we write $\psi \notin X$ to indicate that ψ is not an element of X. So for example, if $X = \{2, 10, 1, 6\}$ then $6 \in X$ but $11 \notin X$.
- **Warning** (Russel Paradox): Let S be the set of all sets which are not elements of themselves or more formally $S = \{A \mid A \notin A\}$. Is S an element of itself i.e. $S \in S$?
  Popular version: Consider the barber who shaves all people who don't shave themselves. Who shaves the barber?

Definition (1.1)

The ZFC axiomatic system guarantees the existence of a set called the empty set that has no elements and denoted $\emptyset$.

- It can be proved that $\emptyset$ is unique i.e. there is only one empty set!
- $\emptyset$ is so ubiquitous that modern mathematics built on set theory would not function properly without it!





Definition (1.2)

Let X be a non-empty set. We say a set A is a subset of X and write $A \subseteq X$ if only if $\psi \in A \implies \psi \in X$. We write $A \nsubseteq X$ otherwise.

- It's obvious that $X \subseteq X$. But is $\emptyset \subseteq X$ true?

Proposition (1.0)

Let X, Y, Z be sets. If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.

Proof : Well if $X \subseteq Y$ then $\forall \psi \in X$, $\psi \in Y$. But then since $Y \subseteq Z$ it follows $\psi \in Z$. Hence $X \subseteq Z$.

Definition (1.3)

If X and Y are sets, then we say $X = Y$ if only if $X \subseteq Y$ and $Y \subseteq X$ holds. We write $X \neq Y$ if the two sets are not equal.

- Let $X = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$ and $Y = \{\psi_6, \psi_1, \psi_3, \psi_4, \psi_2, \psi_5\}$. Assuming only the definitions that we have gone through so far, is $X = Y$?

Definition (1.4)

If A is a subset of X then we say A is a proper subset of X if $A \neq X$ i.e. if not all elements of X are in A.

- If $X = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$, then $A = \{\psi_1, \psi_3, \psi_6, \psi_4\}$ is of course a proper subset of X.

Natural Numbers and Integers

Definition (1.5)

The set of natural numbers is very often defined in textbooks as $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$.

- In mathematics it's generally optional whether to consider 0 as a natural number! To use 0 with the natural numbers, mathematicians extend the set $\mathbb{N}$ by creating another set denoted $\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$. $\mathbb{N}$ is a proper subset of $\mathbb{N}_0$ right?

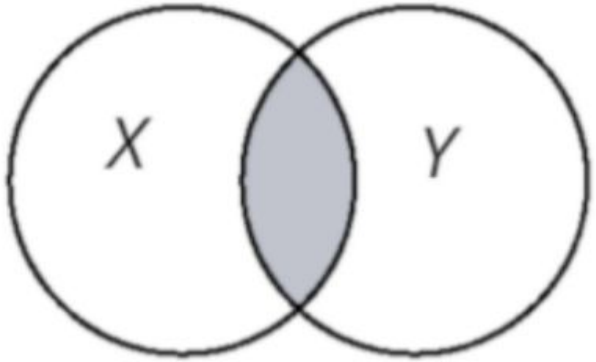
Definition (1.6)

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of all integers.

- $\mathbb{N}_0$ is of course a proper subset of $\mathbb{Z}$.

Definition (1.7)

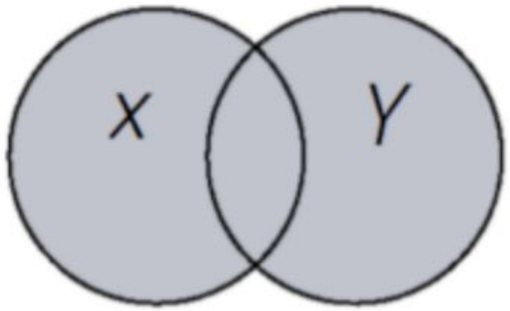
Let X and Y and be sets. The intersection of X with Y is defined as $X \cap Y = \{\psi \mid \psi \in X \text{ and } \psi \in Y\}$.



$X \cap Y$

Definition (1.8)

Let X and Y and be sets. The union of X with Y is defined as
$X \cup Y = \{\psi \mid \psi \in X \text{ or } \psi \in Y\}$.



$X \cup Y$

- Please note that $X \cup Y$ may also contain elements that are in both sets!

Proposition (1.1)

Let X, Y, Z be sets. Then the following properties hold:

1. $X \cap X = X$ and $X \cap \emptyset = \emptyset$
2. $X \cap Y = Y \cap X$ i.e. $\cap$ is commutative
3. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$ i.e. $\cap$ associative
4. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ i.e. $\cap$ is distributive
5. $X \cup X = X$ and $X \cup \emptyset = X$
6. $X \cup Y = Y \cup X$ i.e. $\cup$ is commutative
7. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ i.e. $\cup$ is associative
8. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

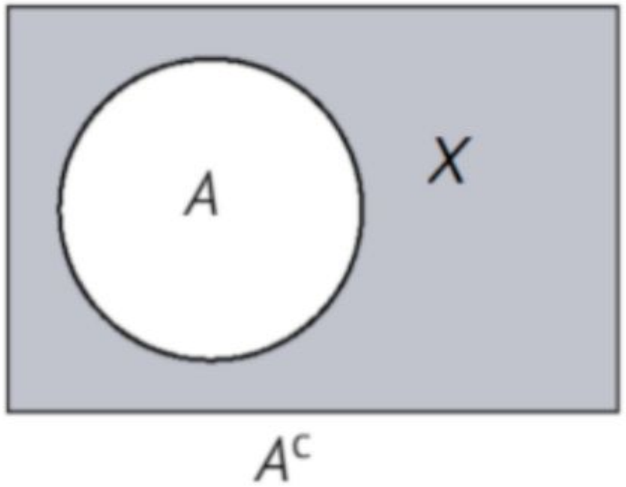
Proof : Homework for you!

Definition (1.9)

Sets X and Y are said to be disjoint if $X \cap Y = \emptyset$.

Definition (2.0)

Let A be a subset of X. The complement of A in X is
$A^c = \{\psi \in X \mid \psi \notin A\}$ i.e. the set of all elements of X that are not in A.



Proposition (1.2)

Let X, Y, Z be sets such that $X \subseteq Z$ and $Y \subseteq Z$. Then the following is true:

1. $(X \cup Y)^c = X^c \cap Y^c$
2. $(X \cap Y)^c = X^c \cup Y^c$

Proof :

1. Let $\psi \in (X \cup Y)^c$ i.e. $\psi \in Z$ such that $\psi \notin X \cup Y$. Hence, $\psi \notin X$ and $\psi \notin Y \implies \psi \in X^c$ and $\psi \in Y^c \implies \psi \in X^c \cap Y^c \implies (X \cup Y)^c \subseteq X^c \cap Y^c$.

Will leave the remaining parts of the proof for you!

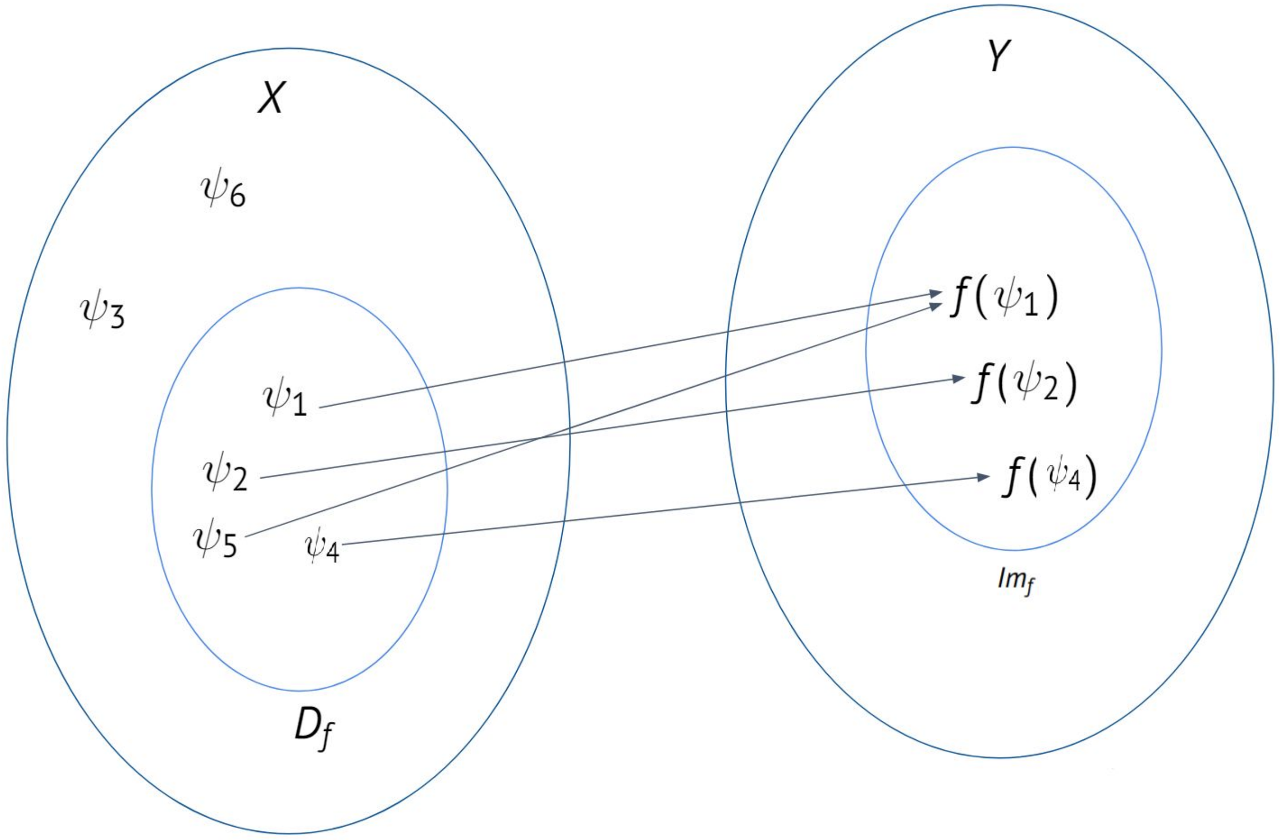
Maps Between Sets

Definition (2.1)

Let X and Y be sets. A map (or function) from X to Y written $f : X \to Y$ is a prescription that associates an element of X with an element of Y.

- The set of all covered elements of X under the map f is called the domain of f and we'll denote it as D_f.
- For each $\psi \in D_f$ we write $f(\psi)$ to denote the corresponding element in Y.
- The image of f is defined as $Im_f = \{f(\psi) \mid \psi \in D_f\}$.

Example of Map



Special Maps

Definition (2.2)

Let X and Y be sets. A map $f : X \to Y$ is called:

1. **Surjective** if $Im_f = Y$ i.e. $\forall \phi \in Y$ there exists a $\psi \in X$ such that $\phi = f(\psi)$.
2. **Injective** if for all $\psi_1, \psi_2 \in D_f$, $f(\psi_1) = f(\psi_2)$ if only if $\psi_1 = \psi_2$.
3. **Bijective** if it's both surjective and injective i.e. f is one-to-one which implies that $D_f = X$ and $Im_f = Y$.

Definition (2.3)

The set X is (set)-isomorphic to Y ($X \simeq Y$) if there is a bijection between the two sets i.e. if there is at least a bijective map $f : X \to Y$.

- Obviously, if $X \simeq Y$ and $Y \simeq Z$ then $X \simeq Z$.

Cardinality of Sets

Definition (2.4)

A non-empty set X is finite if there exists a natural number $k \geq 1$ such that $X \simeq \mathbb{N}_k = \{1, ...k\}$. We call such k the cardinality of X and write $|X| = k$.

- We can prove that for all $k_1, k_2 \in \mathbb{N}$, $\mathbb{N}_{k1} \simeq \mathbb{N}_{k2}$ if only if $k_1 = k_2$.

Definition (2.5)

A set X is infinite if it contains a proper subset Λ such that $\Lambda \simeq X$.

- Let $\mathbb{N}_0 = \{0, 1, 2, 3, 4...\}$. It's obvious that $\mathbb{N} = \{1, 2, 3, 4...\}$ is a proper subset of $\mathbb{N}_0$. Is it true that $\mathbb{N} \simeq \mathbb{N}_0$?!
- $\mathbb{N}$ is a proper subset of the integers set $\mathbb{Z}$. But is $\mathbb{N} \simeq \mathbb{Z}$?!

Bijection Between $\mathbb{N}_0$ and $\mathbb{N}$

$$
\begin{array}{cccccccc}
\mathbb{N}_0 & 0 & 1 & 2 & 3 & \cdots\cdots\cdots\cdots & n \\
f\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\mathbb{N} & 1 & 2 & 3 & 4 & \cdots\cdots\cdots\cdots & n+1
\end{array}
$$

- The above map $f : \mathbb{N}_0 \to \mathbb{N}$ defined as $f(n) = n + 1$ is clearly a bijection and so $\mathbb{N}_0 \simeq \mathbb{N}$!

Bijection Between $\mathbb{N}_0$ and $\mathbb{Z}$

$$\begin{array}{ccccccc} \mathbb{N}_0 & 0 & 1 & 2 & 3 & 4 & \cdots\cdots \\ f\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & \cdots\cdots \end{array}$$

- Can you define a map f with the pattern above?

Countable and Uncountable Sets

Definition (2.6)

A set X is countably infinite if $X \simeq \mathbb{N}$. Else if X is infinite and not isomorphic to $\mathbb{N}$, we say X is uncountably infinite or just uncountable.

- By definition it's obvious that both $\mathbb{N}$ and $\mathbb{Z}$ are countably infinite.
- What about the sets $\mathbb{Q}$ and $\mathbb{R}$?

Definition (2.7)

If X is countably infinite then its cardinality is defined as $|X| = \aleph_0$ (read as aleph-null).

- The cardinality of $\mathbb{R}$ is called continuum and denoted $\mathfrak{c}$.
- Continuum Hypothesis (open problem): Is there a set $\mathbb{S}$ with cardinality between $\aleph_0$ and $\mathfrak{c}$?

Power Sets

Definition (2.8)

Let X be a non-empty set. The power set of X denoted $\mathcal{P}(X)$ is defined as the set of all subsets of X i.e. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

- It's obvious that both X and $\emptyset$ are in $\mathcal{P}(X)$ right?
- Let $X = \{\psi_1, \psi_2, \psi_3\}$. Then we get

  $$\mathcal{P}(X) = \{\{\psi_1\}, \{\psi_2\}, \{\psi_3\}, \{\psi_1, \psi_2\}, \{\psi_1, \psi_3\}, \{\psi_2, \psi_3\}, \{\psi_1, \psi_2, \psi_3\}, \emptyset\}.$$

- Let now $X = \{h, t\}$ where we call the element h 'heads' and t tails! So then $\mathcal{P}(X) = \{\{h\}, \{t\}, \{h, t\}, \emptyset\}$.
- If X is finite i.e. $X \simeq \mathbb{N}_k$ for some k then $|\mathcal{P}(X)| = 2^k$.
- Interestingly, it can be proved that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$!

Measure Hack (without mentioning sigma-algebras)

Definition (2.9)

Let Ω be a non-empty set. We'll define a measure on Ω as a map $\mu : \mathcal{P}(\Omega) \to \mathbb{R}$ satisfying the following axioms:

1. $\mu(\mathbb{E}) \geq 0$ for all $\mathbb{E} \in D_\mu$, where D_μ is the domain of μ
2. For all $\mathbb{E} \in D_\mu$, $\mathbb{E}^c \in D_\mu$
3. For all $\mathbb{E}_1, \mathbb{E}_2 \in D_\mu$ such that $\mathbb{E}_1 \cap \mathbb{E}_2 = \emptyset$, $\mu(\mathbb{E}_1 \cup \mathbb{E}_2) = \mu(\mathbb{E}_1) + \mu(\mathbb{E}_2)$
4. $\Omega \in D_\mu$ and $\mu(\Omega) = 1$

- We can actually generalise axiom 3 to include an arbitrary countable number of disjoint subsets $\mathbb{E}_1, \mathbb{E}_2, \ldots \mathbb{E}_n$ i.e. $\mathbb{E}_1 \cap \mathbb{E}_2 \cap \ldots \cap \mathbb{E}_n = \emptyset$ so that $\mu(\mathbb{E}_1 \cup \mathbb{E}_2 \cup \ldots \cup \mathbb{E}_n) = \sum_{i=1}^{n} \mu(\mathbb{E}_i) = \mu(\mathbb{E}_1) + \mu(\mathbb{E}_2) + \ldots + \mu(\mathbb{E}_n)$
- Can you recognise what this abstract map μ might be?

Proposition (1.3)

If μ satisfies the axioms above, then the following is true:

1. $\emptyset \in D_\mu$ and $\mu(\emptyset) = 0$
2. $\mu(\mathbb{E}) \leq 1$ for all $\mathbb{E} \in D_\mu$

Proof :

1. Axioms 2 and 4 imply $\Omega^c = \emptyset \in D_\mu$. To prove $\mu(\emptyset) = 0$, just notice that $1 = \mu(\Omega \cup \Omega^c) = \mu(\Omega) + \mu(\Omega^c) = \mu(\Omega) + \mu(\emptyset) = 1 + \mu(\emptyset)$ and so $\mu(\emptyset) = 0$.
2. Since $\mathbb{E}$ is a subset of Ω then $\mathbb{E} \cup \mathbb{E}^c = \Omega$. Now, because $\mathbb{E} \cap \mathbb{E}^c = \emptyset$, we have that $1 = \mu(\Omega) = \mu(\mathbb{E} \cup \mathbb{E}^c) = \mu(\mathbb{E}) + \mu(\mathbb{E}^c)$ which implies $\mu(\mathbb{E}) = 1 - \mu(\mathbb{E}^c) \implies \mu(\mathbb{E}) \leq 1$

- Proving 2 implies that $\mu(\mathbb{E})$ can only take values between 0 and 1! Can you now see what μ might be??!
- The pair (Ω, D_μ) is an example of a measurable space and the triple (Ω, D_μ, μ) is an example of measure space!

FOUNDATIONS
OF THE
THEORY OF PROBABILITY

BY

A.N. KOLMOGOROV

Second English Edition

TRANSLATION EDITED BY
NATHAN MORRISON

WITH AN ADDED BIBLIOGRPAHY BY
A.T. BHARUCHA-REID

UNIVERSITY OF OREGON

CHELSEA PUBLISHING COMPANY
NEW YOURK

Probability Measure Challenge

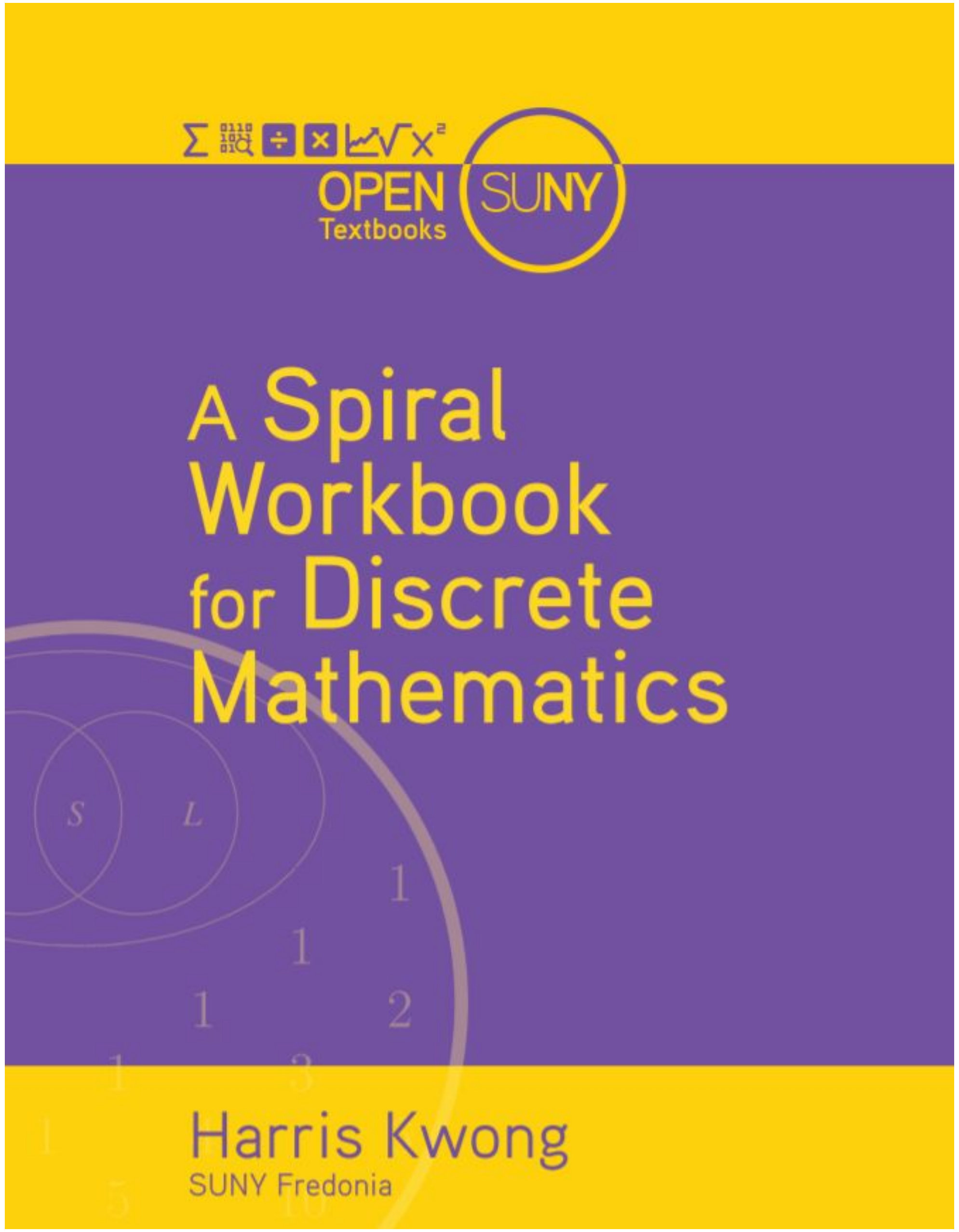
- Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ for some natural number *n* be a discrete sample space of your choice. Can you build a probability measure μ on Ω?

Important Missing Concepts

Some important set-theoretic stuff that were deliberately left out but that are important include:

1. Cartesian product
2. Composition of maps
3. Equivalence classes
4. Indexing sets

However, we'll have the opportunity to introduce them as we go along at the right time!



Where should you focus?

*4 **Sets** (pages 81 - 109)*
*6 **Functions** (pages 157 - 189)*

What else could be helpful?

*2 **Logic** (pages 9 - 36)*

''All mathematics courses are difficult. It takes hard work and patience to learn mathematics.
Rote memorization does not work.'' Harris Kwong