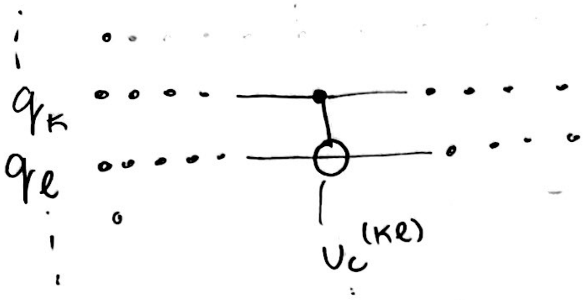
Suppose our quantum circuit is



Assumptions:

1. All CNOTs, regardless of the qubits they act on, have the same error rat ϵ

$\Downarrow$

Probability of losing all information about the control and target qubit i.e mixed state.

After action of that sole CNOT, our initial state ρ becomes

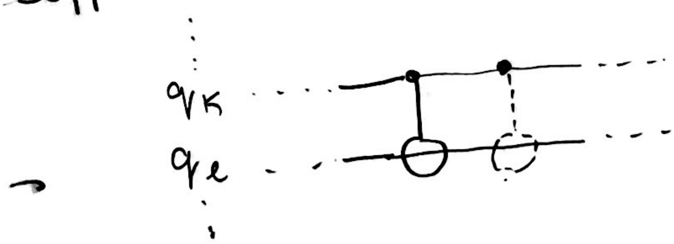
$$(1.)\quad \rho \longrightarrow \underbrace{(1-\epsilon)}_{\substack{\text{Probability that} \\ \text{CNOT acts} \\ \text{ideally}}} \underbrace{U_c^{(K\ell)} \rho\, U_c^{(K\ell)}}_{\substack{\text{CNOT gate} \\ \text{acting on} \\ \text{qubits } K,\ell}} + \underbrace{\epsilon}_{\substack{\text{Probability} \\ \text{that depolarizing} \\ \text{error occurs}}} \left(\underbrace{\frac{I_4}{4} \otimes \rho_{\cancel{K\ell}}}_{} \right)$$

$\Downarrow$

i.e no depolarizing error

qubits K,ℓ in mixed state while all other qubits in initial state

Suppose we add another CNOT, what happens then?



We will substitute ρ_{final} from (1.) into (1).

$$(2)\quad \rho \rightarrow \underbrace{(1-\epsilon)\, U_c^{(K\ell)} \Big[\underbrace{(1-\epsilon) U_c^{K\ell} \rho\, U_c^{K\ell} + \epsilon\left(\frac{I_4}{4} \otimes \rho_{\cancel{K\ell}}\right)}_{\text{From (1) inserted into}} \Big] U_c^{K\ell} + \epsilon\left(\frac{I_4}{4} \otimes \rho_{\cancel{K\ell}}\right)}_{(1)}$$

③ $\Rightarrow \underbrace{(1-\epsilon)^2 \left(U_c^{(k\ell)}\right)^2 \rho \left(U_c^{(k\ell)}\right)^2}_{Ⓐ} + \underbrace{\epsilon(1-\epsilon) U_c^{k\ell}\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right) U_c^{k\ell}}_{Ⓑ}$

$$+ \underbrace{\epsilon\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right)}_{Ⓒ}$$

Ⓐ = Both CNOTs act in ideal fashion

Ⓑ = First CNOT acts noisily i.e. by the time we get to the 2nd CNOT, depolarizing error has already occured. Thus, regardless of how 2nd CNOT acts, the qubits will be depolarized any way.

Thus,

④ $$\epsilon(1-\epsilon) U_c^{(k\ell)}\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right) U^{(k\ell)} = \epsilon(1-\epsilon)\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right)$$

Ⓒ = First CNOT may have acted noisily or ideally ; but second CNOT acts noisily i.e depolarizing error occurs.

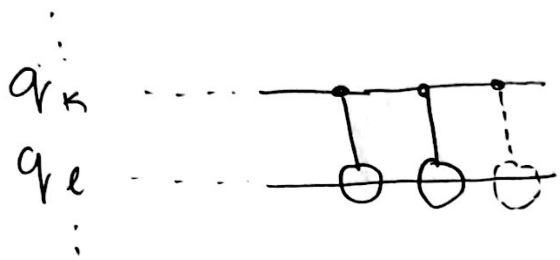
Making a substitution in ③ using ④, and also recognizing $U_c^{(k\ell)\,2} = I_4$ → 2 CNOTs form identity

⑤ $\Rightarrow (1-\epsilon)^2 \rho + [\epsilon + \epsilon(1-\epsilon)]\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right)$

⑥ $\Rightarrow (1-\epsilon)^2 \rho + (1-(1-\epsilon)^2)\left(\frac{I_4}{4} \otimes P_{\cancel{k}\ell}\right)$

⑥ makes sense since if $\epsilon = 0$, then $\rho \Rightarrow \rho$ i.e. acting 2 ideal CNOTs on ρ is doing nothing at all.

Now what if we added another CNOT?



We would have insert (6) into (1). So

(7)
$$\rho \rightarrow \underbrace{(1-\epsilon)U_c^{(kl)}\left[\underbrace{(1-\epsilon)^2\rho + (1-(1-\epsilon)^2)\left(\frac{I_4}{4}\otimes P_{\cancel{kl}}\right)}_{(6)}\right]U_c^{(kl)} + \epsilon\left(\frac{I_4}{4}\otimes P_{\cancel{kl}}\right)}_{(1)}$$

(8)
$$\rightarrow (1-\epsilon)^3 U_c^{(kl)}\rho U_c^{(kl)} + (1-\epsilon)(1-(1-\epsilon)^2)\underbrace{U_c^{(kl)}\left[\frac{I_4}{4}\otimes P_{\cancel{kl}}\right]U_c^{(kl)}}_{(D)} + \epsilon\left(\frac{I_4}{4}\otimes P_{\cancel{kl}}\right)$$

Using (4) for expression (D), we have,

(9)
$$\rightarrow (1-\epsilon)^3 U_c^{(kl)}\rho U_c^{(kl)} + (1-(1-\epsilon)^3)\left[\frac{I_4}{4}\otimes P_{\cancel{kl}}\right]$$

In general, if we replace that sole CNOT with r CNOTs, we would ~~to~~ have (where r is odd)

(10)
$$\Rightarrow (1-\epsilon)^r U_c^{(kl)}\rho U_c^{(kl)} + (1-(1-\epsilon)^r)\left[\frac{I_4}{4}\otimes P_{\cancel{kl}}\right]$$

Using taylor expansion, we have

(11)
$$\rightarrow (1-r\epsilon) U_c^{(kl)}\rho U_c^{(kl)} + r\epsilon\left[\frac{I_4}{4}\otimes P_{\cancel{kl}}\right] + O(r^2\epsilon^2)$$

Consider Equation 10

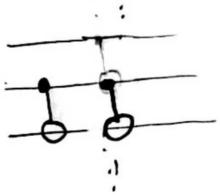
$$\rho \to \underbrace{(1-\epsilon)^r \, U_c^{(ke)} \, \rho \, U_c^{(ke)}}_{\text{All the } r \text{ gates act ideally}} + \underbrace{(1-(1-\epsilon)^r)\left[\frac{I_4}{4} \otimes \rho_{se}\right]}_{\text{At least one of the } r \text{ CNOT gates acted noisily}}$$

This is true when our circuit contains a single CNOT. But what if our circuit contains multiple CNOT gates, each of which has to be amplified by factor r?

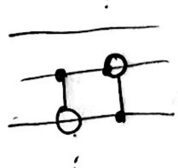
⇓

Let's say our circuit has 2 CNOTs. Then there are 5 cases of how they are arranged.

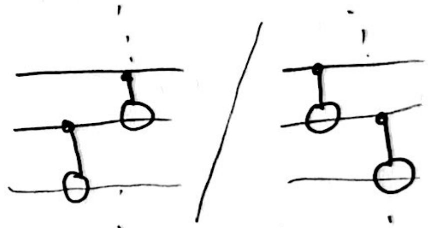
(i)



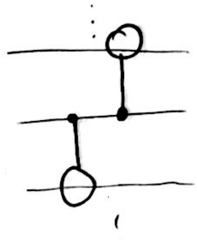
(ii)



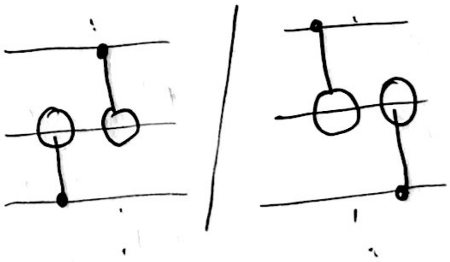
(iii)



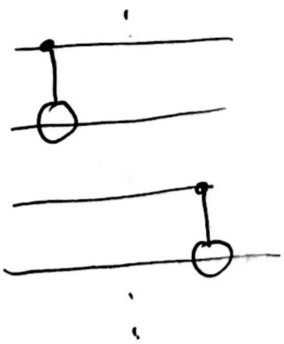
(iv)



(v)



(vi)



Amplifying noise by adding CNOTs to each of the 2 original CNOTs and writing an equation similar to eq. 10 is a daunting task, especially when there are > 5 cases you have to account for.

Guiding ourselves by intuition then, here is what the final density matrix would look like if we amplified noise in both of those CNOT gates by a fact of r.

$$\rho \to \underbrace{(1-\epsilon)^r (1-\epsilon)^r}_{^{*}(1-2r\epsilon)+O(2r^2\epsilon^2)} \begin{bmatrix} \text{Both CNOTs}^{**} \\ \text{act ideally} \end{bmatrix} + \epsilon \begin{bmatrix} \text{First CNOT}^{***} \text{ fails, second} \\ \text{acts ideally} \end{bmatrix}$$

(12)

$$+ \epsilon \begin{bmatrix} \text{second CNOT fails,} \\ \text{first acts ideally} \end{bmatrix} + O(2r^2\epsilon^2) \begin{bmatrix} \text{Both} \\ \text{CNOTs} \\ \text{fail} \end{bmatrix}$$

More generally, suppose we have N CNOT gates in the circuit. Then our final ρ_{final} after amplifying noise in all the CNOTs is

(13)

$$\rho \to (1 - rN_{CNOT}\epsilon) \begin{bmatrix} \text{All CNOTs} \\ \text{act ideally} \end{bmatrix} + \sum_{i=1}^{N_{CNOT}} \epsilon \begin{bmatrix} i^{th} \text{ CNOT} \\ \text{fails, o} \\ \text{-s are go} \end{bmatrix}$$

$$+ O((rN_{CNOT}\epsilon)^2)$$

Then our observable's expectation value $\langle M \rangle = Tr(M\rho$ will have

(14)

$$\langle M \rangle = \underbrace{A(rN_{CNOT}\epsilon)}_{\text{Some term w/ linear dependence}} + O((rN_{CNOT}\epsilon)^2)$$

FootNote

** By CNOT, I mean all the r CNOTs that were substituted for that orginal CNOT in the circuit

*** Then by 1 CNOT failing, I mean 1 of the r CNOT gates failing in that cluster.

After linear extrapolation, we are left with

(15) $$\langle M \rangle \approx O\left((r\, N_{CNOT}\, \epsilon)^2 \right)$$

Thus for linear extrapolation to be accurate, we want

$$\boxed{r \cdot N_{CNOT}\, \epsilon \ll 1}$$