Error Mitigation

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1 Introduction

Isolating physical quantum systems is hard. For this reason, most quantum systems are considered *open* in the sense that the system inevitably interacts with its external environment. To illustrate, consider a qubit represented by 2 discrete states of an electron. Being a charged particle, that electron is naturally bound to interact with other charged particles and surrounding electromagnetic fields. This means that the *environment* can affect the state of the electron as well as the qubit it represents. These uncontrollable effects are called *noise* and they can significantly affect our chemistry computations. To understand the nature of such quantum noise, we will first turn our attentention to classical noise.

2 Classical Noise

Imagine a bit inside a computer hard-drive with external magnetic fields. These magnetic fields have potential to change the state of the bit with a probability of let' say p. Let i_0, i_1 be initial probabilities of the bit being in state 0 and 1 respectively. Similarly, let j_1, j_2 be the new probabilities after the bit interacts with the magnetic field. Then, we can represent such *noisy* event as

$$\begin{bmatrix} j_0 \\ j_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} i_0 \\ i_1 \end{bmatrix}.$$

$$\begin{bmatrix} j_0 \\ j_1 \end{bmatrix} = \begin{bmatrix} (1-p)i_0 + pi_1 \\ pi_0 + (1-p)i_1 \end{bmatrix}$$

We can make sense of this equation by looking at the equality $j_0 = (1-p)i_0 + pi_1$. Here, j_0 denotes the final probability of the bit being in state 0. If the probability of initial state being 0 is i_0, then the probability of getting state 0 after interaction is $(1-p)i_0$. In this case, we multiplied the probability of no-bit-flip $1-p$ with i_0. On the other hand, if the probability of initial state being 1 is i_1, then the probability of getting state 0 after interaction is the product of bit-flip probability p and i_1. Thus, the probability of getting the final state of 0 i.e. j_0 is simply the sum of $(1-p)i_0$ and pi_1. Now, we can write the above equations more succintly as

$$\vec{j} = \hat{E}\vec{i}$$

where $\hat{E}$ is called the evolution matrix (or the *noise* matrix), and $\vec{i}, \vec{j}$ are the initial and final probability distributions respectively. Then, the final state of the system $\vec{j}$ is said to be "linearly" related to the initial state of the system $\vec{i}$.

Note that for the *noise* matrix to describe such a linear transformation, it has to abide by 2 rules:

- *Positivity*: All entries of E must be non-negative. If E has negative entries, then the vector $E\vec{q}$ will have negative components i.e. negative probabilities. That would be non-sensical.

- *Completeness*: The entries in each column of E must add up to 1. Suppose $\vec{i} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $E = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ Then

$$\vec{j} = E\vec{i} = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$$

  For $\vec{i}, \vec{j}$ to be valid probability distribution, their components must add up to 1. Then we can assume that $x + y = 1$, and $ax + cy + bx + dy = 1$ or $(a + b)x + (c + d)y = 1$. For the latter equation to be true for any non-trivial inputs x, y such that $x+y = 1$, both coefficients $a+b$ and $c+d$ must be equal to 1. Since these coefficients represent the sum of entries of each column of E, we conclude that the sum of entries of each column of E must be equal to 1.

2.1 Markov Process

Earlier we only looked at one “noise event”. Now suppose we have 2 *noisy* gates A and B. An important assumption that we can make here is whether gate A works correctly is independent of whether gate B works correctly. That is, the *noisiness* of gate A in independent of the *noisiness* of gate B. This assumption can be *physically reasonable* in cases such as the one where gate A and B are placed a significant distance apart from each other. Then the process of gate A and B being applied in any order is known as Markov process.

With each noise event being indpendent and being described by a linear transformation, then the final state after a multiple noise events is still linearly related to the initial state.

3 Quantum Operations

Quantum Operations offer us tools to understand how quantum states react to noise. Just as we used a vector of probabilities to describe initial state and final state, we will use *density matrices* ρ to describe initial and final state.

$$\rho \longrightarrow \mathcal{E}(\rho)$$

Here $\mathcal{E}$ describes a quantum operation. For example, $\mathcal{E}(\rho)$ could be $U\rho U^\dagger$ where U is some unitary operator. Now, before moving onto quantum operations, we will briefly discuss density matrices and outer products. There are 3 ways to understand quantum operations: Operator sum representation, physically motivated axioms and system coupled to environment.

3.1 Mathematical Interlude

3.1.1 Outer Products

Inner Products such as $\langle\phi|\psi\rangle$ are **complex numbers** and describe the *probability amplitude of the system to* ***be*** *in state* ϕ *if the system* ***is*** *in state* ψ . In contrast, outer products like $|\phi\rangle\langle\psi|$ are **linear operators** which act on states in a special way. To see such actions, we are going to consider a special case of outer products: projection operators.

Projection Operators are of the of the form $|\psi\rangle\langle\psi|$. Assume $|\psi\rangle$ is normalized. When acting on some state $|A\rangle$, we obtain

$$|\psi\rangle\langle\psi|A\rangle = \langle\psi|A\rangle|\psi\rangle .$$

Here the action of projection operator gave us the component of $|A\rangle$ in the direction of state $|\psi\rangle$ i.e. the projection of $|A\rangle$ onto $|\psi\rangle$.

Here are some properties of projection products:

1. Projection operataors are hermitian.

2. Eigenvector of the projection operator $|\psi\rangle\langle\psi|$ is $|\psi\rangle$ itself with an eigenvalue of 1.

3. Any vector orthogonal $|\phi\rangle$to $|\psi\rangle$ is an eigenvector with an eigenvalue of 0. This is because the term $\langle\phi|\psi\rangle = 0$.

4. The square of a projection operator P is P itself. So if $P = |0\rangle\langle 0|$, then

$$P^2 = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0| = P$$

5. Trace of a projection operator is equal to 1 or

$$\mathrm{Tr}(|\psi\rangle\langle\psi|) = \sum_{i=0}^{n} \langle i|\psi\rangle\langle\psi|i\rangle$$

$$= \sum_{i=0}^{n} |\langle i|\psi\rangle|^2 = 1$$

This is true because we are summing over the *shadows of* $|\psi\rangle$ *in each direction of the Hilbert space defined by the basis* $\{i\}$. Since $|\psi\rangle$ is of length 1 i.e. normalized, the whole sum must be equal to 1.

6. The sum of all projection operators of a basis is equal to identity.

$$\sum_{i=0}^{n} |i\rangle\langle i| = I$$

This is true because all vectors are eigenvectors of $\sum_{i=0}^{n} |i\rangle\langle i|$ with an eigenvalue of 1. Since this is only true for the identity matrix, the 2 operators are equal to each other.

7. The expectation value of any observable A in state $|\psi\rangle$is given by

$$\langle\psi| A |\psi\rangle = Tr(|\psi\rangle\langle\psi| A)$$

To see why this is true, note that

$$Tr\,|\psi\rangle\,\langle\psi|\,A = \sum_{i=0}^{n}\langle i|\psi\rangle\,\langle\psi|\,A\,|i\rangle$$

$$= \sum_{i=0}^{n}\langle\psi|\,A\,|i\rangle\,\langle i|\psi\rangle$$

$$= \langle\psi|\,A(\sum_{i=0}^{n}|i\rangle\,\langle i|)\,|\psi\rangle$$

$$= \langle\psi|\,AI\,|\psi\rangle$$

$$= \langle\psi|\,A\,|\psi\rangle$$

which is the expectation value of A.

3.1.2 Density Matrices

In most cases, you will not know the true definite state of a quantum system. Instead you would be told that the quantum system has λ_a probability of being in $|A\rangle$ and λ_b probability of being in $|B\rangle$. So how would you package all this information? Consider the expectation values of some operator L with respect to states $|A\rangle$ and $|B\rangle$:

$$\langle A|\,L\,|A\rangle = Tr|A\rangle\,\langle A|\,L$$

$$\langle B|\,L\,|B\rangle = Tr|B\rangle\,\langle B|\,L$$

Now accounting for the probabilities, the new expectation value for observable L becomes

$$\langle L\rangle = \lambda_A Tr|A\rangle\,\langle A|\,L + \lambda_B Tr|B\rangle\,\langle B|\,L$$

This "mixture of states" becomes more compact if we define

$$\rho = \lambda_A\,|A\rangle\,\langle A| + \lambda_B\,|B\rangle\,\langle B|$$

So,

$$\langle L\rangle = Tr\rho L$$

This quantity ρ is known as a density matrix or a density operator. It encodes the probability of the system being in a mix of states. In the example above, we had more than one state, each given with some associated probability. This is known as a "mixed state". On the other hand, if we know that our quantum system is definitely in state $|A\rangle$, our resulting density operator $\rho = |A\rangle\,\langle A|$ would represent a "pure state".

Now, *density matrices* are really *operators* that don't become matrices until you choose an orthonormal basis. If we choose an orthonomal basis $|a\rangle$. Then the matrix representation of ρ with respect to this basis is

$$\rho_{a,a'} = \langle a|\,\rho\,|a'\rangle$$

i.e. $\rho_{a,a'}$ is the entry of the a'th row and a''th column of the matrix.

Let's choose a special orthonormal basis for the density operator such that the resulting density matrix is a diagonal matrix i.e. all the elements off the main diagonal are 0. What we would find is that the sum of the diagonal entries is 1. Why would that be case? Consider the eigenvectors, $e_1, e_2, \ldots, e_n$. Each diagonal element is then an eigenvalue $_i$ of some eigenvector $\vec{e_i}$ A special property of density matrices is that the sum of the diagonal entries is 1. In other words,

$$\sum_a \langle a| \rho |a\rangle = \sum_a \rho_{a,a} = 1$$

ρ (*can be verified easily*). So if all the eigenvalues i.e. probabilities are equal, then the density matrix is just proportional to the unit matrix. What does this mean? That we have absolutely no knowledge about the system as system is equally likely to be in any of those eigenstates.

Let's say we wanna convert the expectation value of observable $\mathbf{L}$ in this representation. Then, we would proceed as follows:

$$\langle \mathbf{L} \rangle = Tr\mathbf{L}\rho = \sum_{a'} \langle a'| \mathbf{L}\rho |a'\rangle$$

Suppose we have an identity matrix I sandwhiched between $\mathbf{L}$ and ρ. Using the trick $I = \sum_a |a\rangle \langle a|$ from outer products,

$$\langle \mathbf{L} \rangle = \sum_{a'} \langle a'| \mathbf{L} \sum_a |a\rangle \langle a| \rho |a'\rangle$$

$$= \sum_{a'} \sum_a \langle a'| \mathbf{L} |a\rangle \langle a| \rho |a'\rangle$$

$$= \sum_{a,a'} L_{a',a} \rho_{a,a'}$$

3.1.3 Density Matrices and Entanglement

Alice and Bob each have their own systems. Alice's system can be in any of the states $|a\rangle$ and Bob's system can be in any of the states $|b\rangle$. Let's say Alice knows that the state of the combined system is described the wavefunction $\psi(a,b)$ where a and b are discrete variables.

$$\text{state of the system} = |\Psi\rangle = \sum_{a,b} \psi(a,b) |ab\rangle$$

Suppose Alice wants to do a measure some observable $\mathbf{L}$ on her own system, completely ignoring Bob's system. In other words, we apply an observable $\tilde{\mathbf{L}} = \mathbf{L}_a \otimes \mathbf{I}_b$ on the combined system. The expectation value $\tilde{\mathbf{L}}$ is

$$\left\langle \tilde{\mathbf{L}} \right\rangle = \langle \Psi| \tilde{\mathbf{L}} |\Psi\rangle$$

$$= (\sum_{a',b'} \psi(a',b') \left|a'b'\right\rangle)\tilde{\mathbf{L}}(\sum_{a,b} \psi(a,b) \left|ab\right\rangle)$$

$$= \sum_{a,a',b,b'} \psi(a',b') \left|a'b'\right\rangle \tilde{\mathbf{L}} \left|ab\right\rangle \psi(a,b)$$

Since $\tilde{\mathbf{L}}$ is just the identity on $|b\rangle$ and $\langle b'|b\rangle = 0$ if $b \neq b'$, all the terms in the grand summation with $b \neq b'$ are 0. So getting rid of those 0 terms, we have

$$\sum_{a,a',b} \psi(a',b) \left|a'b\right\rangle \tilde{L} \left|ab\right\rangle \psi(a,b)$$

$$\sum_{a,a',b} \psi(a',b) \left|a'\right\rangle \mathbf{L} \left|a\right\rangle \psi(a,b)$$

$$\sum_{a,a',b} \psi(a',b) L_{a,a'} \psi(a,b)$$

$$\sum_{a,a'} L_{a,a'} \sum_{b} \psi(a',b)\psi(a,b)$$

Let

$$\rho_{a,a'} = \sum_{b} \psi(a,b)\psi(a',b)$$

Here $\rho_{a,a'}$ is known as the reduced density matrix of subsystem A.

3.2 Quantum Mechanics Interlude: Measurement

Measurement in quantum mechanics is described by measurement operator $\{M_m\}$ where m denotes the various measurement outcomes. In this section, I will list some properties of the quantum measurement. To make these properties concrete, I will assume these operators $\{M_m\}$ to be projectors. *Why such assumption?* Projective measurements are simple, ideal and intuitive–and make explanations a lot easier!

Now suppose we have some initial state $|\psi\rangle$.

- *The probability of observing a measurement outcome m is $\langle\psi| M_m^\dagger M_m |\psi\rangle$.*

  Since M_m is a projection operator, M_m is hermitian ($M_m = {M_m}^\dagger$) and $M_m^2 = M_m$. Then $M_m {M_m}^\dagger$ is just M_m. So

  $$\langle\psi| M_m^\dagger M_m |\psi\rangle = \langle\psi| M_m |\psi\rangle$$

  Let's say $|m\rangle$ is the associated eigenvector associated with measurement outcome m. Then $M_m = |m\rangle \langle m|$. So,

  $$\langle\psi|m\rangle \langle m|\psi\rangle$$

  which is equal to

  $$|\langle m|\psi\rangle|^2$$

Here, $\langle m|\psi\rangle$ is the probability amplitude of taking $|\psi\rangle$ to the state $|m\rangle$ or the shadow (i.e. projection) of $|\psi\rangle$ onto the state $|m\rangle$. From elementary quantum mechanics, we know that the square of the norm of such complex number $|\langle m|\psi\rangle|^2$ must be the *probability of measuring the system to be in state* $|m\rangle$.

Hence, $\langle\psi| M_m^\dagger M_m |\psi\rangle$ is indeed The probability of observing a measurement outcome m.

- *Completeness Relation:* $\sum_m {M_m}^\dagger M_m = I$.

  Recall from above that $M_m {M_m}^\dagger = M_m$. Then

  $$\sum_m {M_m}^\dagger M_m = \sum_m M_m$$

  And

  $$\sum_m M_m = \sum_m |m\rangle\langle m|$$

  Since we are summing over all the measurement outcomes, we are also summing over all the projectors of some basis. So, as noted in the "Outer-products" section,

  $$\sum_m |m\rangle\langle m| = I$$

  Another way to look at the completeness relation is that it implies that $\sum_m p(m) = 1$ i.e. the sum of probabilities of measuring all outcomes is 1.

- *The state of the system immediately followed by measurement of outcome m is*

  $$|\psi\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}.$$

  Let's take it apart. The numerator is

  $$M_m |\psi\rangle = |m\rangle\langle m|\psi\rangle = \alpha |m\rangle$$

  Here α is the *probability amplitude of being in state* $|\psi\rangle$ *if in state* $|m\rangle$. On the othe hand, the denomiator is

  $$\begin{aligned}\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle} &= \sqrt{\langle\psi| M_m |\psi\rangle} \\ &= \sqrt{\langle\psi|m\rangle\langle m|\psi\rangle} \\ &= \sqrt{|\alpha|^2} = |\alpha|\end{aligned}$$

  So the fraction becomes

  $$\frac{\alpha |m\rangle}{|\alpha|}$$

  Here $\frac{\alpha}{|\alpha|} = e^{i\phi}$ for some $\phi \in [0, 2\pi)$. Since global phases aren't of much importance, we can equivalently say that the final state is just $|m\rangle$. This makes sense because after "observing " the outcome m, the whole state $|\psi\rangle$ "collapses" to the state $|m\rangle$.

3.3 Basic Operations

3.3.1 Unitary Evolution

Under unitrary transformations, density matrices ρ evolve as $U\rho U^\dagger$

If normal ket states evolve under unitary transformations as

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

Then, its hermitian conjugate evolves as

$$\langle\psi| \longrightarrow \langle\psi| U^\dagger$$

So ince ρ is combination of both states i.e. $\rho = |\psi\rangle\langle\psi|$, then ρ evolves as

$$|\psi\rangle\langle\psi| \longrightarrow U|\psi\rangle\langle\psi| U^\dagger$$

So,

$$\rho \longrightarrow U\rho U^\dagger \equiv \mathcal{E}(\rho)$$

3.3.2 Measurement

If the measurement operator M_m describes the unitary transformation of density matrix ρ, we know from the previous section that the resulting state would be $M_m\rho M_m^\dagger$. To see an example of this in action, suppose that we are making simple projective measurements on a system which is in a pure state. So $M_m = |m\rangle\langle m|$ and $\rho = |\psi\rangle\langle\psi|$. Also recall that M_m is hermitian i.e. $M_m = M_m^\dagger$. Then,

$$\begin{aligned} M_m\rho M_m^\dagger &= |m\rangle\langle m|\psi\rangle\langle\psi|m\rangle\langle m| \\ &= |m\rangle\alpha\alpha^\star\langle m| \\ &= |m\rangle|\alpha|^2\langle m| \end{aligned}$$

where $\alpha = \langle m|\psi\rangle$ is the probability amplitude of the system **being** in $|m\rangle$ if the system **is** in state $|\psi\rangle$. Then $|\alpha|^2$ is just real number denoting the probability of measuring the system to be in state $|m\rangle$. So finally

$$M_m\rho M_m^\dagger = |\alpha|^2|m\rangle\langle m|$$

To get rid of the coefficient $|\alpha|^2$, we introduce another term $Tr(M_m\rho M_m^\dagger)$. We will now verify that $Tr(M_m\rho M_m^\dagger)$ is indeed equal to $|\alpha|^2$.

$$Tr(M_m\rho M_m^\dagger) = \sum_i \langle i| M_m|\psi\rangle\langle\psi| M_m^\dagger|i\rangle$$

$$= \sum_i \langle\psi| M_m^\dagger|i\rangle\langle i| M_m|\psi\rangle$$

$$= \langle\psi| M_m^\dagger(\sum_i |i\rangle\langle i|)M_m|\psi\rangle$$

$$= \langle\psi| M_m^\dagger M_m |\psi\rangle$$

As noted in the quantum mechanics interlude, $M_m^\dagger M_m = M_m$. So,

$$= \langle\psi| M_m |\psi\rangle$$

$$= \langle\psi|m\rangle \langle m|\psi\rangle$$

$$= \alpha^\star \alpha$$

So,

$$Tr(M_m \rho M_m^\dagger) = |\alpha|^2$$

Thus, using this term to get rid of the coefficient from the state $M_m \rho M_m^\dagger = |\alpha|^2 |m\rangle \langle m|$, the final state after the measurement becomes

$$\frac{M_m \rho M_m^\dagger}{Tr(M_m \rho M_m^\dagger)}$$

If we make $\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$, then the above expression simplifies to

$$\frac{\mathcal{E}_m(\rho)}{Tr\mathcal{E}_m(p)}$$

where $Tr\mathcal{E}_m(p)$ is the probability of measuring the system to be in state $|m\rangle$.

3.4 Systems Coupled to Environment

Closed Systems can be easily described by unitary transformations as shown in figure 1. However, to model open system, we consider the closed system of the

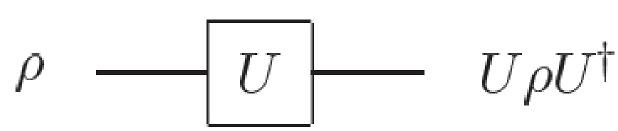

Figure 1: The input state ρ is trasnformed into the output state $U\rho U^\dagger$.

system and environment Suppose the system-environment system starts out in some product state $\rho \otimes \rho_{env}$. (why such assumption? Are the experimentalists

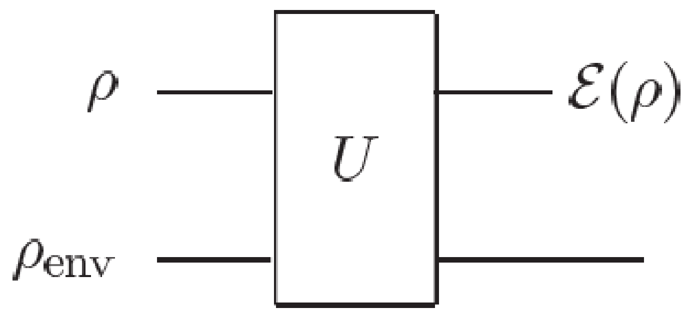

Figure 2: Closed system consisting of the system of interest and the environment

controlling the qubit so it doesn't further interact with environment?) Then after the unitary transformation, the combined system is in state $U(\rho\otimes\rho_{env})U^\dagger$. To get rid of the environment state ρ_{env} in order to obtain only the state of the system of interest after the unitary transformation, we perform a partial trace over the environment state. So, the final state of our system of interest is

$$\mathcal{E}(\rho) = Tr_{env}(U(\rho \otimes \rho_{env})U^\dagger)$$

Why partial trace? Suppose you have some product state $\rho_A\otimes\rho_B$. To isolate the state ρ_A from the product state, at least mathematically, we need to get rid of ρ_B. Recall from the property of density matrices that $Tr\rho = 1$. So tracing over B, we have

$$Tr_B(\rho_A \otimes \rho_B) = \rho_A \otimes Tr_B\rho_B = \rho_A$$

Intuitively, we are averaging over the states of the environment to isolate the state of system A.

3.5 Operator Sum Representation

Assume the basis e_k for the ρ_{env} and let ρ_{env} be a pure state equal to the projector $|e_0\rangle\langle e_0|$. Then we can expand the expression

$$Tr_{env}(U(\rho_A \otimes \rho_{env})U^\dagger)$$

as

$$\sum_k (\mathbb{I}_A \otimes \langle e_k|)U(\rho_A \otimes |e_0\rangle\langle e_0|)U^\dagger(\mathbb{I}_A \otimes |e_k\rangle)$$

Now the expression

$$\rho_A \otimes |e_0\rangle\langle e_0| = (\rho_A \otimes \mathbb{I}_e)(\mathbb{I}_A \otimes |e_0\rangle)(\mathbb{I}_A \otimes \langle e_0|)$$

where the subexpression

$$(\rho_A \otimes \mathbb{I}_e)(\mathbb{I}_A \otimes |e_0\rangle) = (\rho_A \mathbb{I}_A) \otimes (\mathbb{I}_e |e_0\rangle)$$

Since the product $\rho_A \mathbb{I}_A$ is commutative and $\mathbb{I}_e |e_0\rangle = |e_0\rangle$, we have

$$= (\mathbb{I}_A \rho_A) \otimes (|e_0\rangle \cdot 1)$$

$$= (\mathbb{I}_A \otimes |e_0\rangle)(\rho_A \otimes 1)$$

So,

$$(\rho_A \otimes \mathbb{I}_e)(\mathbb{I}_A \otimes |e_0\rangle) = (\mathbb{I}_A \otimes |e_0\rangle)\rho_A$$

Then plugging this subexpression back into the former expression, we have

$$\rho_A \otimes |e_0\rangle \langle e_0| = (\mathbb{I}_A \otimes |e_0\rangle)\rho_A(\mathbb{I}_A \otimes \langle e_0|)$$

So, the equation for

$$\mathcal{E}(p)$$

becomes

$$\mathcal{E}(p) = \sum_k (\mathbb{I}_A \otimes \langle e_k|)U(\mathbb{I}_A \otimes |e_0\rangle)\rho_A(\mathbb{I}_A \otimes \langle e_0|)U^\dagger(\mathbb{I}_A \otimes |e_k\rangle)$$

$$\mathcal{E}(p) = \sum_k E_k \rho_A E_k^\dagger$$

where

$$E_k = (\mathbb{I}_A \otimes \langle e_k|)U(\mathbb{I}_A \otimes |e_0\rangle)$$

$$E_k^\dagger = (\mathbb{I}_A \otimes \langle e_0|)U^\dagger(\mathbb{I}_A \otimes |e_k\rangle)$$

Environment starts in pure state. Operators act on principal system's Hilbert space alone. Properties of E_k:

1. Completeness Since $\mathcal{E}(p)$ is a density matrix representing resultant state of system A, then

   $$Tr\mathcal{E}(p) = 1$$

   So,

   $$Tr(\sum_k E_k E_k^\dagger \rho) = 1$$

   What's Happening? As of now, we have extracted state of principal system $\mathcal{E}(\rho)$ from the evolved product state of principal system and environment $U(\rho_A \otimes \rho_{env})U^\dagger$. Here we verify that if $\mathcal{E}(\rho)$ is made up of different states, then the probabilities corresponding to all those states add up to one.

2. Trace Preserving

   Looking carefully at the completeness relation, we see that

   $$Tr\mathcal{E}(p) = Tr(\sum_k E_k E_k^\dagger) = 1$$

   is true for arbitrary state ρ.

Importance of Operator Sum Representation

not explicitly consider properties of environment

Measurement

In this subsection, we will solve exercise 8.4 of Nielsen and Chuang.

Problem 1 *Suppose you have a single qubit principal system in some state ρ and environment in pure state $|0\rangle\langle 0|$. Given the unitary transformation*

$$U = P_0 \otimes I + P_1 \otimes X$$

where $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ act on the the principal system and the pauli matrix X acts on the environment, suppose the system "interacts with the environment through the transform" (???? wierd unclear phrasing alert) U. Give the quantum operation for this process in the operator-sum representation.

Solution Recall that a quantum operation is described by

$$\mathcal{E}(\rho) = Tr_{env}(U(\rho \otimes \rho_{env})U^\dagger)$$

Now the basis for environment will be the usual computation basis as we are assuming all systems to be single qubit. So,

$$\mathcal{E}(\rho) = \sum_{k=0}^{1} (I \otimes \langle k|)(U(\rho \otimes |0\rangle\langle 0|)U^\dagger)(I \otimes |k\rangle)$$

In the previous section, we saw that

$$\rho \otimes |0\rangle\langle 0| = (I \otimes |0\rangle)\rho(I \otimes \langle 0|)$$

So,

$$\mathcal{E}(\rho) = \sum_{k=0}^{1} (I \otimes \langle k|)U(I \otimes |0\rangle)\rho(I \otimes \langle 0|)U^\dagger(I \otimes |k\rangle)$$

Now let's simplify smaller subexpressions,

$$(P_0 \otimes I + P_1 \otimes X)(I \otimes |0\rangle)$$

$$\begin{aligned} U(I \otimes |0\rangle) &= (P_0 \otimes I + P_1 \otimes X)(I \otimes |0\rangle) \\ &= (P_0 \otimes I)(I \otimes |0\rangle) + (P_1 \otimes X)(I \otimes |0\rangle) \\ &= P_0 \otimes |0\rangle + P_1 \otimes |1\rangle \end{aligned} \tag{1}$$

Then the following subexpression can be simplified as

$$\begin{aligned} (I \otimes \langle 0|)U(I \otimes |0\rangle) &= (I \otimes \langle 0|)(P_0 \otimes |0\rangle + P_1 \otimes |1\rangle) \\ &= P_0 \otimes \langle 0|0\rangle + P_1 \otimes \langle 0|1\rangle \\ &= P_0 \otimes 1 + P_1 \otimes 0 \\ &= P_0 \end{aligned} \tag{2}$$

Similarly we can show that

$$(I \otimes \langle 1|)U(I \otimes |0\rangle) = P_1$$

$$(I \otimes \langle 0|)U(I \otimes |1\rangle) = P_1$$

So the quantum operation $\mathcal{E}(\rho)$ can be simplified as

$$\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1$$

$$\mathcal{E}(\rho) = \rho$$

need to verify that last step

Spin Flips

Composition of quantum operations

Physical Interpretation of Operator Sum Representation

Discuss quantum channels and how it relates to classical information theory.

Measurement and Operator Sum Representation

Given a description of open quantum system, how do we determine its operator sum representation to decribe its dynamics?

System environment models for any operator sum representation

Given a set of operators $\{E_k\}$ is there some reasonalble modell environmental system and dynamics?

Mocking up a quantum operation

3.6 Physically Motivated Axioms

some axioms a map has to follow in order to have operator-sum representation

1. $0 \leq tr[\mathcal{E}(\rho)] \leq 1$ for any ρ.
2. $\mathcal{E}(\sum_i p_i\rho_i) = \sum_i p_i\mathcal{E}(\rho_i)$.
3. $\mathcal{E}$ is a completely positive map.

4 Examples of Quantum Noise and Quantum Operations

In the following sections, we will explore how quantum noise deforms the state space of a single qubit. That is, after some noise operation $\mathcal{E}$, we will find certain states changed and other untouched. However, even if $\mathcal{E}$ *can* change the state of qubit, it does so with some probability p. So if the initial state of our single qubit is given by ρ, then we expect $\mathcal{E}$ to act as follows

$$\rho \longrightarrow^{\mathcal{E}} (1-p)\rho + p\mathcal{N}(\rho)$$

where $\mathcal{N}(\rho)$ is some unitary operation on ρ, deforming the state. We first begin with a mathematical interlude on bloch spheres and density matrices. Then we explore "deforming" operations $\mathcal{N}$.

4.1 Mathematical Interlude: Bloch Spheres and Density Matrices

4.1.1 Bloch Sphere Representation of a qubit

Recall that the state of a single qubit can be specified in terms of complex number α, β as

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle .$$

where $|\alpha|^2 + |\beta|^2 = 1$. Since each complex number requires 2 real parameters $\alpha = \underline{a} + \underline{b}i$, it seems that the state $|\psi\rangle$ will require 4 real parameters. However, if we express α, β as

$$\alpha = r_\alpha e^{i\phi_\alpha} \text{ and } \beta = r_\beta e^{i\phi_\beta}$$

where $r_\alpha, r_\beta, \phi_\alpha, \phi_\beta \in \mathbb{R}$, then we can extract a global phase factor from $|\psi\rangle$ as follows:

$$|\psi\rangle = r_\alpha e^{i\phi_\alpha} |0\rangle + r_\beta e^{i\phi_\beta} |1\rangle$$

$$|\psi\rangle = e^{i\phi_\alpha}(r_\alpha |0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)} |1\rangle)$$

$$|\psi\rangle = e^{i\phi_\alpha}(r_\alpha |0\rangle + r_\beta e^{i\phi} |1\rangle)$$

Here $\phi = \phi_\beta - \phi_\alpha$ and $e^{i\phi_\alpha}$ is the global phase factor. Since the global phase factor does not really affect the observables, we will ignore that term. That leaves us with 3 real parameters: r_α, r_β and ϕ. Recognizing that $|\alpha|^2 + |\beta|^2 = 1$ implies $|r_\alpha|^2 + |r_\beta|^2 = 1$, it is possible to get rid of another parameter by making the following substitutions

$$r_\alpha = \cos\left(\frac{\theta}{2}\right) \text{ and } r_\beta = \sin\left(\frac{\theta}{2}\right)$$

Thus we have,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle$$

where $0 \leq \theta \leq pi$ and $0 \leq \phi < 2pi$. So the state of a single qubit only requires 2 parameters. This has a nice geometrical interpretation in terms of a unit sphere. Now each pure state has an associated bloch vector $\vec{r} = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ Note that $|r|^2 = 1$ since this is a bloch vector for a pure state. This is so because pure states lie on the surface of the bloch sphere. It is then natural to ask geometrical interpreation of a mixed state. To

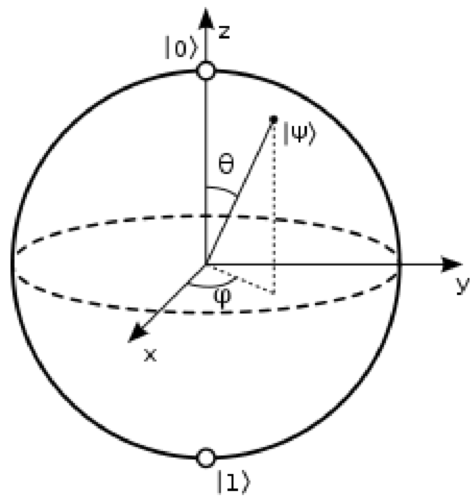

Figure 3: Wikipedia's Bloch Sphere

answer that question, we will first show how a density matrix of some pure state $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$ relates to its bloch vector $\vec{r}$. Recall that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So,

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Then the density matrix $\rho = |\psi\rangle\langle\psi|$ can be expressed as

$$\rho = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)^2 & e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)^2 \end{pmatrix}$$

Using some nice double angle trig identities and expanding $e^{i\phi}$, we simplify the above matrix as follows

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\cos(\theta) & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & 1-\cos\theta \end{pmatrix}$$

Now we will convert spherical coordinates into rectangular coordinates as follows

$$x = \sin\theta\cos\phi$$

$$y = \sin\theta\sin\phi$$

$$z = \cos\theta$$

Thus, ρ now becomes

$$\rho = \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

which can be expressed in terms of the pauli matrices as

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4.2 Bit Flip and Phase Flip Channels

4.3 Depolarizing Channel

4.4 Amplitude Damping

4.5 Phase Damping

5 Error Mitigation

5.1 Extrapolation

Intentionally increasing dominant error rate by some factor and using that to determine error free result.

5.1.1 Richardson's Approach

5.2 Probabilistic Error Cancellation

Cancelling errors by resampling randomized circuits according to quasi probability distributions ?!?!?? Requires full knowledge of the noise model associated with each gate. This could be obtained from wither process tomography or combo of gate set and process tomography.

We are dealing with markovian noise here which means maps must be trace preserving and completely positive.

5.3 Quantum Subspace Expansion

5.4 Other Methods

6 References