let $Ax^2 + Bxy + Cy^2 + Dx + Ey = 1$ be the equation of best fit ellipse.

let X be the matrix formed by the data points where first col. contains the coeff. of A, second col = coeff of B, third col = coeff of C, fourth = coeff. of D and fifth = coeff. of E.

We have to find the solution of $(X^T X) z = X^T b$

where b is the column vector containing n 1's, where n is the no. of data points.

$$z = (X^T X)^{-1} X^T b.$$

various intermediate matrices are attached in the pictures ahead along with the best fit ellipse plotted on desmos.

Semi Major Axis of ellipse = 3.914463219

Eccentricity = 0.86666609456

(Z is a 5x1 matrices representing A, B, C, D, E).

A	B	C	D	E
x2	xy	y2	x	y
0.094482464	0.19121495	0.386983526	-0.30738	-0.62208
13.63544246	12.47337495	11.41034353	3.69262	3.37792
22.02068246	1.77343495	0.142823526	4.69262	0.37792
13.63544246	-2.29710505	0.386983526	3.69262	-0.62208
32.40592246	19.22921495	11.41034353	5.69262	3.37792
0.479722464	1.64699495	5.654503526	0.69262	2.37792

Result:

	C_1	C_2	C_3	C_4	C_5
1	1907.1439092260955488	801.7588061479079016	536.51870896414873164	388.81353566621216	156.44573365072256
2	801.7588061479079016	536.518708891637015	370.4887237582116818	156.445733672814	112.985483356224
3	536.51870896414873164	370.4887237582116818	292.68520012914201248	112.98548338980968	90.10491870149888
4	388.81353566621216	156.445733672814	112.98548338980968	82.2716947864	33.0171297024
5	156.44573365072256	112.985483356224	90.10491870149888	33.0171297024	29.3919811584

sult:

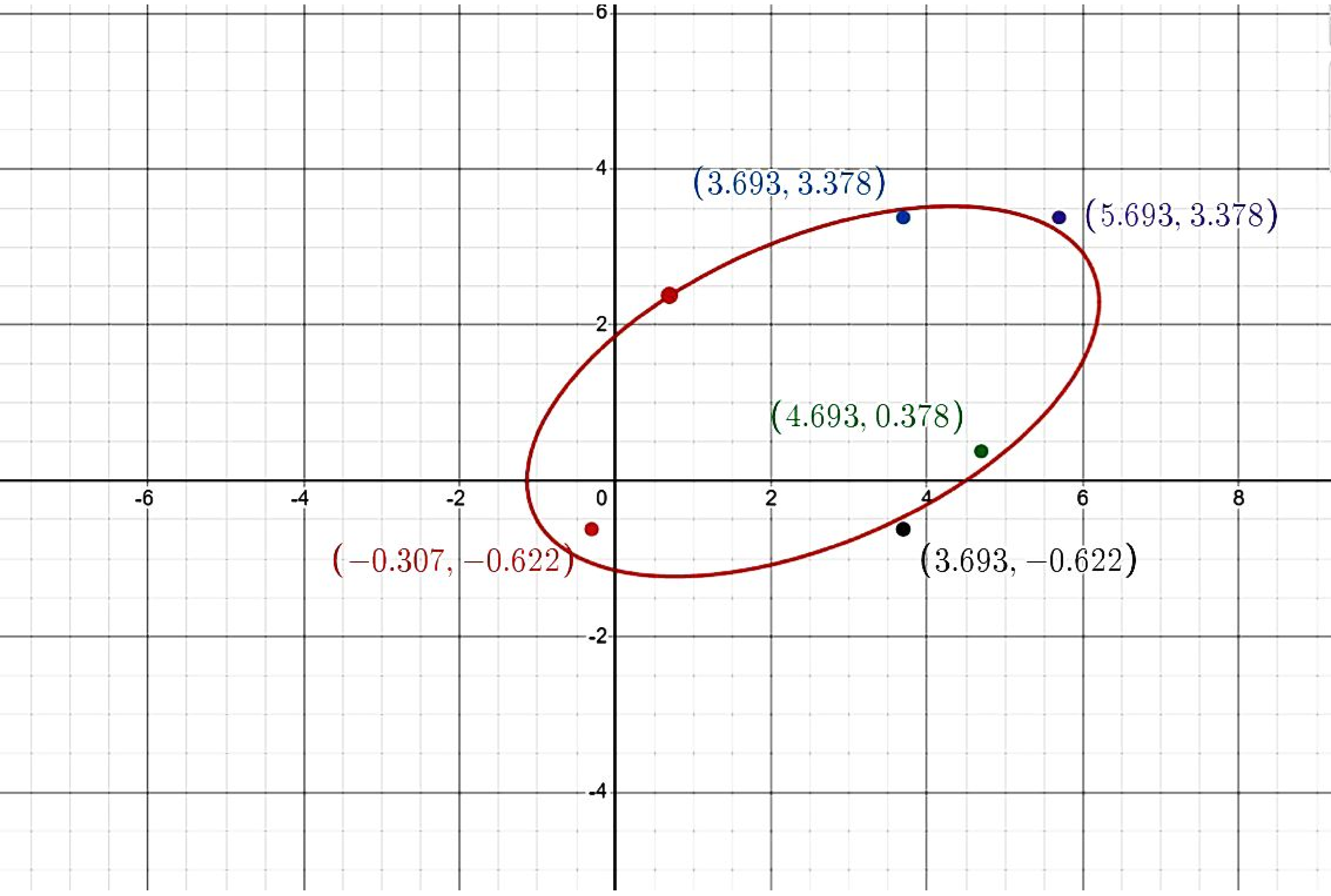
	B_1	B_2	B_3	B_4	B_5
1	0.075458426332025164382	-0.071418899761633170238	0.079594449130395413926	-0.32990923066445813624	-0.00051189017648089104065
2	-0.071418899761633170237	0.083532472527524378822	-0.094882518377727795709	0.30692536334649971476	0.0051310749864569135673
3	0.079594449130395413853	-0.094882518377727795626	0.18260688516667496632	-0.36090985825765210686	-0.21330429191855199503
4	-0.32990923066445813623	0.30692536334649971476	-0.36090985825765210716	1.471360025562161241	0.029751302802891109897
5	-0.00051189017648089084137	0.0051310749864569133336	-0.21330429191855199476	0.029751302802891109052	0.63751433441276016344

Result:

	C_1	C_2	C_3	C_4	C_5	C_6
1	0.12600081323957095005	-0.17368422461304648566	-0.0019748754003126915645	0.0058564995596712402354	0.10039353658915719213	0.13892066299416959711
2	-0.1250278048985162615	0.13615251841752085833	0.0041182715221991813388	-0.072262338182815617261	-0.026231765286869619162	-0.20841329388574662839
3	0.30367198610869802725	-0.06782020053462444334	-0.16368810220352025764	0.17391587278609847606	0.06337498550618727936	0.15726956482761868354
4	-0.58292242362511565864	0.74550231920966314239	0.14371643429946367759	0.071501463762603099792	-0.43079359895779613891	-0.6036868127922371428
5	-0.48734234600280679188	-0.1135205978031195873	0.34790356141880671196	-0.38803637570361594937	-0.0289616948806633332	0.3386398520437506315

Computation time: 0.221 sec

	c_1
1	0.19551241236920985
2	-0.29166441231422809
3	0.46672410649045774
4	-0.65668261810341903
5	-0.33131760092764828



6
4
2
-2
-4
-6
-4
-2
0
2
4
6
8
(3.693, 3.378)
(5.693, 3.378)
(4.693, 0.378)
(−0.307, −0.622)
(3.693, −0.622)

(ii) for second part, we consider earth to be circular orbit.

draw a circle of unit AU radius around focus.

we then parametrise the ellipse and circle & then find the min. distance.

2.(a) $-mg = \frac{dm}{dt}v_e + \frac{mdv}{dt}$

$-mg\,dt = v_e\,dm + m\,dv$

Now, $\frac{dm}{dt} = -\dot{m}$

$\therefore \frac{mg}{\dot{m}}dm = v_e\,dm + m\,dv$

$\Rightarrow \left(\frac{mg}{\dot{m}} - v_e\right)dm = m\,dv$

$dv = \left(\frac{g}{\dot{m}} - \frac{v_e}{m}\right)dm$

Let initial mass of rocket is M_0.

Integrating within proper limits,

$\int_0^v dv = \int_{M_0}^{m} \frac{g}{\dot{m}} - \frac{v_e}{m}\,dm$

$\Rightarrow v = \frac{g}{\dot{m}}(m - M_0) - v_e \ln\left(\frac{m}{M_0}\right)$

Now, $M_0 - m = \dot{m}t \Rightarrow m - M_0 = -\dot{m}t$

$\therefore v = -gt - v_e \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$. Ans.

$$v = -gt - v_e \ln\left(\frac{m}{m_0}\right)$$

$$\Rightarrow h = \int_0^t -gt\,dt - \int_0^t v_e \ln\left(\frac{m}{m_0}\right) dt$$

$$= \frac{-gt^2}{2} + \int \frac{v_e}{\dot{m}} \ln\left(\frac{m}{m_0}\right) dm$$

$$= \frac{-gt^2}{2} + \frac{v_e m_0}{\dot{m}} \left[\frac{m}{m_0} \ln\left(\frac{m}{m_0}\right) - \frac{m}{m_0} + 1 \right]$$

$$= \frac{-gt^2}{2} + \frac{v_e m_0}{\dot{m}} \left[\frac{(m_0 - \dot{m}t)}{m_0} \ln\left(\frac{m_0 - \dot{m}t}{m_0}\right) + \frac{\dot{m}t}{m_0} \right]$$

$$h = \frac{-gt^2}{2} + v_e t + \frac{v_e}{\dot{m}} (m_0 - \dot{m}t) \ln\left(\frac{m_0 - \dot{m}t}{m_0}\right)$$

At $t = t_b$, $h_b = \frac{-gt_b^2}{2} + v_e t_b + \frac{v_e}{\dot{m}} (m_0 - \dot{m}t_b) \ln\left(\frac{m_0 - \dot{m}t_b}{m_0}\right)$

Now, v at t_b $= V_1 = -gt_b - v_e \ln\left(\frac{m_0 - \dot{m}t_b}{m_0}\right)$

Max altitude $= h_b + \frac{V_1^2}{2g_0}$ (as rocket is only under gravity once fuel is burnt).

(3)

<u>Speed of spaceship quadrant at time t:</u>

$$V = v + u_e \ln\left(\frac{m_f}{m/5}\right)$$

$$= v + u_e \ln\left(\frac{\frac{m}{5} - \dot{m}t}{m/5}\right) = v + u_e \ln\left(\frac{m - 5\dot{m}t}{m}\right)$$

at $t = t_0$, $\quad v' = v + u_e \ln\left(\frac{m - 5\dot{m}t}{m}\right)$

for escaping sphere of destruction

$$v' > \frac{v}{2}$$

$$v + u_e \ln\left(\frac{m - 5\dot{m}t}{m}\right) > \frac{v}{2}$$

$$\ln\left[\frac{m - 5\dot{m}t}{m}\right] > -\frac{v}{2u_e}$$

$$1 - \frac{5t_0\dot{m}}{m} > e^{-v/2u_e}$$

$$\dot{m} < \frac{m}{5t_0}\left[1 - e^{-\frac{v}{2u_e}}\right].$$