Assignment : 02

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Course: Linear Algebra
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Question: 01 What is a determinant? Explain its properties with examples.

Determinant of a square matrix A is a real number. It is defined via its behavior with respect to row operations.

Determinant for 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{Det}A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a)(d) - (b)(c).$$

Determinant for 3x3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \text{Det}(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a[(e)(i) - (f)(h)] - b[(d)(i) - (f)(g)] + c[(d)(h) - (e)(g)]$$

Properties of determinants:

1. Reflection Property:

The determinant remains unaltered if its rows and columns are interchanged.

Example:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -5 & 4 \\ 2 & 0 & 6 \end{bmatrix} = 1[(-5)(6)-(4)(0)] - 2[(3)(6)-(4)(2)] + 1[(3)(0)-(-5)(2)]$$

$$= 1(-30-0) - 2(18-8) + 1(0+10)$$

$$= -30 - 20 + 10$$

$$= -40.$$

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -5 & 0 \\ 1 & 4 & 6 \end{bmatrix} = 1[(-5)(6)-(0)(4)] - 3[(2)(6)-(0)(1)] + 2[(2)(4)-(-5)(1)].$$

$$= 1(-30-0) - 3(12-0) + 2(8+5)$$

$$= -30 - 36 + 26$$

$$= -40.$$

$\rightarrow$ $Det(A) = Det(A^T)$

2. All zero property:

If all elements of a row (or column) are zero, then determinant is zero.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 3 & -2 & 1 \end{bmatrix} = 1(0-0) - 2(0-0) + 1(0-0).$$

$$= 0 - 0 + 0$$

$$= 0. \rightarrow \text{Proved.}$$

3. Proportionality property:

If any two rows (or columns) are identical then determinant is zero.

$$A = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & -4 \\ 1 & 1 & 5 \end{vmatrix} = 1(10+4) - 1(10+4) + 3(2-2)$$

$$= 14 - 14 + 0$$

$$= 0. \rightarrow \text{Proved.}$$

4. Switching property:

The interchange of any two rows (or columns) of determinant changes its sign.

Example:

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 6 \\ 3 & 3 & -6 \end{vmatrix} = 1(0-18) - 2(-12-18) + 1(6-0)$$

$$= -18 + 60 + 6$$

$$= 48$$

$$C_1 \leftrightarrow C_3$$

$$B = \begin{vmatrix} 1 & 2 & 1 \\ 6 & 0 & 2 \\ -6 & 3 & 3 \end{vmatrix} = 1(0-6) - 2(18+12) + 1(18+0)$$

$$= -6 - 60 + 18$$

$$= -48.$$

$\rightarrow$ $Det(B) = -Det(A)$.

5. Scalar Multiple Property:

If a row (or column) of a matrix is multiplied by a non-zero constant, then its determinant is a multiple of same constant.

Example:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 2 \end{bmatrix} = 1(6-0) - 2(4-0) + 1(8-0)$$
$$= 6 - 8 + 8$$
$$= 6.$$

$2R_1 \rightarrow R_1$

$$B = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 2 \end{bmatrix} = 2(6-0) - 4(4-0) + 2(8-0)$$
$$= 12 - 16 + 16$$
$$= 12.$$

$\rightarrow$ $Det(B) = 2\,Det(A)$.

6. Sum property:

If a row (or column) is expressed as sum of two (or more) terms, then determinant can be expressed as sum of those terms.

Example:

$$A = \begin{bmatrix} (1+1) & 2 & 1 \\ 3+(-4) & 0 & 1 \\ 6+(-2) & 8 & 3 \end{bmatrix} = 2(0-8) - 2(-3-4) + 1(-8-0)$$
$$= -16 + 14 - 8$$
$$= -10.$$

$$B+C = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 6 & 8 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ -4 & 0 & 1 \\ -2 & 8 & 3 \end{vmatrix}$$

$= 1(0-8)-2(9-6)+1(24-0) + 1(0-8)-2(-12+2)+1(-32+0)$

$= -8-6+24-8+20-32$

$= 10-20$

$= -10.$

$\rightarrow$ Det(A) = Det(B) + Det(C).

7. Property of Invariance:

If each element of a row and column of a matrix is added with equimultiples of another row and column, then the determinant remains unchanged.

Example:

$$A = \begin{vmatrix} 1 & 1 & 2 \\ -2 & 0 & 5 \\ 2 & 3 & 7 \end{vmatrix} = 1(0-15)-1(-14-10)+2(-6-0)$$

$= -15+24-12$

$= -3.$

$$B = \begin{vmatrix} (1+1+2) & 1 & 2 \\ (-2+0+5) & 0 & 5 \\ (2+3+7) & 3 & 7 \end{vmatrix} = 4(0-15)-1(21-60)+2(9-0)$$

$= -60+39+18$

$= -3.$

$\rightarrow$ Det(A) = Det(B)

8. Factor Property:

If determinant becomes zero when we put $x=\alpha$, then $(x-\alpha)$ is one factor.

Important: Product of diagonal elements gives the degree of determinant which gives same number of linear factors.

Example:

$$A = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

Putting $x=y$.

$$|A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 1(y^2z^3 - y^3z^2) - y^2(z^3-y^3) + y^3(z^2-y^2)$$

$$= y^2z^3 - y^3z^3 - y^2z^3 + y^5 + y^3z^2 - y^5$$

$$= 0.$$

Therefore, $(x-y)$ is a factor.

9. Triangle Property:

In a triangular matrix, the product of diagonal elements equals the determinant.

Example:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 9 & -1 \\ 0 & 0 & 4 \end{vmatrix} = 1(36+0) - 2(0-0) + 3(0-0)$$

$$= 36.$$

$\rightarrow$ $Det(A) = 1 \times 9 \times 4$

$36 = 36.$

16. Determinant of a cofactor matrix

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 2 & 0 & 2 \end{vmatrix} \begin{aligned} &= 0 - 2(6-8) + 0 \\ &= -2(-2) \\ &= 4. \end{aligned}$$

$$\text{Cofactor matrix of } A = \begin{vmatrix} 0 & 2 & 0 \\ -4 & 0 & 4 \\ 8 & -1 & -6 \end{vmatrix} \begin{aligned} &= 0 - 2(24-32) + 0 \\ &= 0 - 2(-8) + 0 \\ &= 16. \end{aligned}$$

→ Det(cofactor matrix of A) = 4 Det(A).