

Lecture Notes for CS 726 - Spring 2022

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Probabilistic Modeling

Probability Theory

We will briefly review probability in a rigorous sense.

We define events considering we have a space of possible outcomes denoted by Ω . \mathcal{S} is a set of measurable events, to which we assign probabilities, and each event $\alpha \in \mathcal{S}$ is a subset of Ω .

The event space necessarily satisfies three properties -

1. It contains the empty event \emptyset and the trivial event Ω .
2. It is closed under union, i.e if $\alpha, \beta \in \mathcal{S}$, so is $\alpha \cup \beta$.
3. It is closed under complementation, i.e if $\alpha \in \mathcal{S}$, so is $\Omega - \alpha$.

Definition 1 (Probability distribution). A probability distribution P over (Ω, \mathcal{S}) is a mapping of events in \mathcal{S} to real values satisfying

- ◇ $P(\alpha) \geq 0$ for all $\alpha \in \mathcal{S}$
- ◇ $P(\Omega) = 1$
- ◇ If $\alpha, \beta \in \mathcal{S}$ and $\alpha \cap \beta = \emptyset$, then $P(\alpha \cup \beta) = P(\alpha) + P(\beta)$

Conditional probability answers the question - after learning that event α is true, how does our belief about β change? Formally, we define

$$P(\beta|\alpha) = \frac{P(\alpha \cap \beta)}{P(\alpha)} \quad (1)$$

It can be checked that this satisfies Definition 1 and is a probability distribution. Noting that $P(\alpha \cap \beta) = P(\alpha)P(\beta|\alpha)$, we define the chain rule of conditional probabilities

$$P(\alpha_1 \cap \dots \cap \alpha_k) = P(\alpha_1)P(\alpha_2|\alpha_1) \dots P(\alpha_k|\alpha_1 \cap \dots \cap \alpha_{k-1}) \quad (2)$$

We further define the Bayes' rule

$$P(\alpha|\beta) = \frac{P(\beta|\alpha)P(\alpha)}{P(\beta)} \quad (3)$$

Here, $P(\alpha|\beta)$ is called the *posterior*, $P(\beta|\alpha)$ is the *likelihood*, $P(\alpha)$ is the *prior* and $P(\beta)$ is the *marginal probability* of the structure in context.

We can generalize Equation 3 as

$$P(\alpha|\beta \cap \gamma) = \frac{P(\beta|\alpha \cap \gamma)P(\alpha|\gamma)}{P(\beta|\gamma)} \quad (4)$$

Now, we formally define the notion of random variables, which intuitively can be considered to be attribute reporters.

In a single coin toss, we have

$$\Omega = \{H, T\}$$

A direct consequence of the properties is:

$$P(\emptyset) = 0$$

$$P(\alpha \cup \beta) = P(\alpha) + P(\beta) - P(\alpha \cap \beta)$$

Definition 2 (Random Variable). A random variable X is a *measurable* function $X : \Omega \rightarrow \mathcal{S}$. The probability that X takes values in a set $s \in \mathcal{S}$ is written as

$$\Pr(X \in s) = \Pr(\{\omega \in \Omega | X(\omega) \in s\}) \quad (5)$$

The marginal distribution over a random variable X is the distribution over events that can be described using X , and is denoted by $P(X)$. More generally, if we want to describe a distribution over a set of random variables $\mathcal{X} = \{x_1, \dots, x_n\}$ called the *joint distribution* denoted as $P(x_1, \dots, x_n)$. The full assignment to the variables is denoted as $\xi \in \text{Val}(\mathcal{X})$. The space corresponding to the joint assignment in \mathcal{X} is called the *canonical outcome space*.

Now, we glance at independencies, a core component of Probabilistic Graphical Models.

Definition 3 (Independence). An event α is independent of an event β denoted by $P \models (\alpha \perp\!\!\!\perp \beta)$, if $P(\alpha|\beta) = P(\alpha)$ or $P(\beta) = 0$.

Proposition 4. A distribution satisfies $(\alpha \perp\!\!\!\perp \beta)$ if and only if

$$P(\alpha \cap \beta) = P(\alpha)P(\beta) \quad (6)$$

Proof. Skipped (hint: Use the definition of conditional probability). \square

Definition 5 (Conditional Independence). An event α is conditionally independent of event β given γ in P , denoted by $P \models (\alpha \perp\!\!\!\perp \beta | \gamma)$ if $P(\alpha|\beta \cap \gamma) = P(\alpha|\gamma)$ or if $P(\beta \cap \gamma) = 0$.

Proposition 6. P satisfies $(\alpha \perp\!\!\!\perp \beta | \gamma)$ if and only if

$$P(\alpha \cap \beta | \gamma) = P(\alpha | \gamma)P(\beta | \gamma) \quad (7)$$

Definition 7. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be sets of random variables. \mathbf{X} is conditionally independent of \mathbf{Y} given \mathbf{Z} in a distribution P if P satisfies $(\mathbf{X} = \mathbf{x} \perp\!\!\!\perp \mathbf{Y} = \mathbf{y} | \mathbf{Z} = \mathbf{z})$ for all values of $\mathbf{x} \in \text{Val}(\mathbf{X})$, $\mathbf{y} \in \text{Val}(\mathbf{Y})$ and $\mathbf{z} \in \text{Val}(\mathbf{Z})$. We say that the variables in \mathbf{Z} are *observed*. If \mathbf{Z} is empty, then we say that \mathbf{X} and \mathbf{Y} are marginally independent.

Proposition 8. The distribution P satisfies $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z})$ if and only if

$$P(\mathbf{X}, \mathbf{Y} | \mathbf{Z}) = P(\mathbf{X} | \mathbf{Z})P(\mathbf{Y} | \mathbf{Z}) \quad (8)$$

The following properties hold for conditional independencies:

1. *Symmetry:* $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}) \implies (\mathbf{Y} \perp\!\!\!\perp \mathbf{X} | \mathbf{Z})$
2. *Decomposition:* $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y}, \mathbf{W} | \mathbf{Z}) \implies (\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z})$
3. *Weak Union:* $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y}, \mathbf{W} | \mathbf{Z}) \implies (\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}, \mathbf{W})$

Decomposition can also be stated as

$$\mathbf{X} \perp\!\!\!\perp \{\mathbf{Y}, \mathbf{Z}\} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y}, \mathbf{X} \perp\!\!\!\perp \mathbf{Z}$$

Weak Union can also be stated as

$$\mathbf{X} \perp\!\!\!\perp \{\mathbf{Y}, \mathbf{Z}\} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}$$

But note that, if $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$ and $\mathbf{Z} \not\perp\!\!\!\perp \{\mathbf{X}, \mathbf{Y}\}$ then it is not necessary to have $\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}$

4. *Contraction*: $(\mathbf{X} \perp\!\!\!\perp \mathbf{W} | \mathbf{Z}, \mathbf{Y}) \ \& \ (\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}) \implies (\mathbf{X} \perp\!\!\!\perp \mathbf{Y}, \mathbf{W} | \mathbf{Z})$

If our distribution is positive (i.e, for all non-empty $\alpha \in \mathcal{S}, P(\alpha) > 0$), we have another property

Contraction can also be stated as

$$\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}, \mathbf{X} \perp\!\!\!\perp \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \{\mathbf{Y}, \mathbf{Z}\}$$

◇ *Intersection*: $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}, \mathbf{W}) \ \& \ (\mathbf{X} \perp\!\!\!\perp \mathbf{W} | \mathbf{Z}, \mathbf{Y}) \implies (\mathbf{X} \perp\!\!\!\perp \mathbf{Y}, \mathbf{W} | \mathbf{Z})$

Theorem 9. Consider $\mathcal{X} = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. Then $P \models (\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z})$ if and only if we can write

$$P(\mathcal{X}) = \phi_1(\mathbf{X}, \mathbf{Z}) \phi_2(\mathbf{Y}, \mathbf{Z}) \quad (9)$$

Proof. Skipped. □

Probabilistic Graphical Models

Given a set of n random variables $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ where n is large, we want to build a joint probability distribution P over this set. Explicitly representing the joint distribution is computationally expensive, since just having binary values variables requires the joint distribution to specify $2^n - 1$ numbers, and for more practical variables, the count is too large.

We want to efficiently represent, estimate and answer inference queries on the distribution.

An example of a query can be -
Estimate the fraction of people
with a bachelor's degree.

Alternatives to explicit joint distributions

▷ Can we assume all columns are independent? **NO** - this is obviously a very bad assumption.

▷ Can we use data to detect highly correlated column pairs, and estimate their pairwise frequencies? **MAYBE** - but there might be too many correlated pairs, and the method is ad hoc.

To solve the above two not so good ways, we explore conditional independencies. It may be possible that $\text{income} \not\perp\!\!\!\perp \text{age}$ but $\text{income} \perp\!\!\!\perp \text{age} | \text{experience}$.

Note that we write that a set X is conditionally independent of Y given Z , i.e $X \perp\!\!\!\perp Y | Z$ if

$$\Pr(X|Y, Z) = \Pr(X|Z)$$

Probabilistic graphical models use a graph-based representation as the basis for compactly encoding a complex distribution over a high-dimensional space.

It is convenient to represent the independence assumption using a graph. The so called graphical model has nodes as the variables (continuous or discrete), and the edges represent direct interaction. If we consider directed edges, we talk about Bayesian Networks, and if we consider undirected edges, we talk about Markov Random Fields.

Essentially the graphical model is a combination of the graph and potentials.

Definition 10 (Potentials). Potentials $\psi_c(\mathbf{x}_c)$ are scores for assignment of values to subsets c of directly interacting variables. We factorize

the probability as a product of these potentials, i.e

$$\Pr(\mathbf{x} = x_1, \dots, x_n) \propto \prod \psi_s(\mathbf{x}_s) \quad (10)$$

Bayesian Networks

Bayesian Networks, also referred to as *directed graphical models* are a family of probability distributions that has a compact parameterization representable using a directed graph.

It is known that

$$\Pr(x_1, x_2, \dots, x_n) = \Pr(x_1)\Pr(x_2|x_1)\Pr(x_3|x_2, x_1) \cdots \Pr(x_n|x_{n-1}, \dots, x_1) \quad (11)$$

A compact Bayesian Network is a distribution in which each factor in the above equation depends on the *parent* variables represented by $\text{Pa}(x_i)$ for variable x_i . Thus, we have

$$\Pr(x_i|x_{i-1}, x_{i-2}, \dots, x_1) = \Pr(x_i|\text{Pa}(x_i)) \quad (12)$$

and the corresponding potentials at each node in terms of its parents are

$$\psi_i(x_i, \text{Pa}(x_i)) = \Pr(x_i, \text{Pa}(x_i)) \quad (13)$$

Thus,

$$\Pr(x_1, x_2, \dots, x_n) = \prod_{i=1}^n \Pr(x_i|\text{Pa}(x_i)) \quad (14)$$

Consider the situation when each variable can take d values. The naive approach gives us $\mathcal{O}(d^n)$ parameters. If we think of the potentials as probability tables (with the rows corresponding to $\text{Pa}(x_i)$) and columns corresponding to the values of x_i , with entries as $\psi_i(x_i, \text{Pa}(x_i))$, we can notice that if $|\text{Pa}(x_i)| \leq k$, then the number of parameters are $\mathcal{O}(d^{k+1})$, and for n variables, we have $\mathcal{O}(nd^{k+1})$, which provides us the compact representation.

Definition

Now we formally define these -

Definition 12 (Bayesian Network). A Bayesian Network is a directed graph $G = (V, E)$ together with

- ◇ a random variable x_i for each node $i \in V$
- ◇ a potential $\psi_i(x_i, \text{Pa}(x_i))$ for each node $i \in V$

For a variable x_i in our Bayesian Network \mathcal{G} , denote $\text{ND}(x_i)$ as the non-descendants of x_i . The following local conditional independencies hold in \mathcal{G} -

$$x_i \perp\!\!\!\perp \text{ND}(x_i) | \text{Pa}(x_i) \quad (15)$$

Example 11 shows the independencies in a simple Bayesian Network.

Example 11.

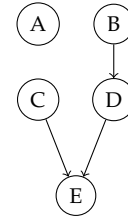


Figure 1: Sample BN

Consider the BN above. We will consider each variable at a time.

- ◇ A has no parent, and has no descendent. Thus,

$$A \perp\!\!\!\perp B, C, D, E$$

- ◇ B has no parent, but has D as a descendent. Thus,

$$B \perp\!\!\!\perp A, C$$

- ◇ C has no parent, but has E as a descendent. Thus,

$$C \perp\!\!\!\perp A, B, D$$

- ◇ D has B as a parent, and has E as the descendent. Thus,

$$D \perp\!\!\!\perp A, C | B$$

- ◇ E has C and D as parents, but has no descendent. Thus,

$$I \perp\!\!\!\perp A, B | C, D$$

Definition 13 (Factorization). Let \mathcal{G} be a Bayesian Network graph over the variables $\{X_i\}_{i=1}^n$. We say that a distribution P over the same space factorizes according to \mathcal{G} if P can be expressed as a product described in Equation 14. Such factorization is also known as the chain rule for Bayesian Networks, and is denoted as $\text{Factorize}(P, \mathcal{G})$.

Definition 14. Let P be a distribution over \mathcal{X} . We define $\mathcal{I}(P)$ to be the set of independent assertions of the form $\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}$ that hold in P .

We can now write " P satisfies the local independencies associated with \mathcal{G} " as $\mathcal{I}_\ell(\mathcal{G}) \subseteq \mathcal{I}(P)$.

Definition 15 (Independency-Map). Let \mathcal{K} be any graph object associated with a set of independencies $\mathcal{I}(\mathcal{K})$. We call \mathcal{K} an I-map for a set of independencies \mathcal{I} if $\mathcal{I}(\mathcal{K}) \subseteq \mathcal{I}$.

Thus for \mathcal{G} to be an I-map for P , any independence that asserts in \mathcal{G} must also assert in P , but P can have additional independencies not reflected in \mathcal{G} .

Remark 16 (Notation Alert). Note that we will use the following interchangeably - P satisfies the local conditional independencies satisfied by \mathcal{G} and \mathcal{G} is an I-map for P , i.e

$$\text{Local-CI}(P, \mathcal{G}) \equiv \mathcal{I}_\ell(\mathcal{G}) \subseteq \mathcal{I}(P) \quad (16)$$

Definition 17 (I-equivalence). Two Bayesian Networks \mathcal{G}_1 and \mathcal{G}_2 are \mathcal{I} -equivalent, if they encode the same dependencies, i.e

$$\mathcal{I}(\mathcal{G}_1) = \mathcal{I}(\mathcal{G}_2) \quad (17)$$

Theorem 18. If \mathcal{G}_1 and \mathcal{G}_2 have the same skeleton, and the same v -structures (see *D-separation*), then they are \mathcal{I} -equivalent.

Proof. Skipped. □

Theorem 19. Given a distribution $P(x_1, x_2, \dots, x_n)$ and a directed acyclic graph (DAG) \mathcal{G} ,

$$\text{Local-CI}(P, \mathcal{G}) \iff \text{Factorize}(P, \mathcal{G}) \quad (18)$$

Proof. (\implies) We essentially need to show that if \mathcal{G} is an I-map for P , then P factorizes according to \mathcal{G} . Consider a topologically sorted order x_1, x_2, \dots, x_n in \mathcal{G} . Local-CI(P, \mathcal{G}) tells us that

$$\Pr(x_i | x_1, \dots, x_{i-1}) = \Pr(x_i | \text{Pa}(x_i))$$

We can write

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1, \dots, x_{i-1})$$

Each term in the product can be simplified due to the notion of Local-CI stated above, and we reach Equation 14, proving factorization.

(\impliedby) Proof has been skipped. □

Minimal Construction

Our goal is to construct a minimal and correct BN \mathcal{G} to represent P . A DAG \mathcal{G} is correct if all Local-CIs that are implied in \mathcal{G} hold in P , and a DAG \mathcal{G} is minimal if we cannot remove any edge(s) from \mathcal{G} and still get a correct BN for P .

In the setting, we define our oracle \mathcal{O} to whom we can ask any query of the type "Is $X \perp\!\!\!\perp Y|Z$?" pertaining to P and get a boolean answer. We will query the oracle several times to build up our BN. The following algorithm constructs such a BN -

```

1 Variables:  $x_1, x_2, \dots, x_n \leftarrow$  ordered variables in  $\mathcal{X}$ 
2 Independencies:  $\mathcal{I} \leftarrow$  set of independencies
3  $\mathcal{G} \leftarrow$  Empty graph over  $\mathcal{X}$ 
4 for  $i = 1$  to  $n$  do
5    $\mathbf{U} \leftarrow \{x_1, \dots, x_{i-1}\}$  // Set of candidate parents of  $x_i$ 
6   for  $U' \subseteq \{x_1, \dots, x_{i-1}\}$  do
7     if  $U' \subset \mathbf{U}$  and  $(x_i \perp\!\!\!\perp \{x_1, \dots, x_{i-1}\} - U' | U') \in \mathcal{I}$  then
8        $\mathbf{U} \leftarrow U'$ 
9     end
10  end
11  // Now we have the minimal set  $\mathbf{U}$  satisfying
     $(x_i \perp\!\!\!\perp \{x_1, \dots, x_{i-1}\} - \mathbf{U} | \mathbf{U})$ 
12  // Now we set  $\mathbf{U}$  to be the parents of  $x_i$ 
13  for  $x_j \in \mathbf{U}$  do
14    Add  $x_j \rightarrow x_i$  in  $\mathcal{G}$ 
15  end
16 end
17 return  $\mathcal{G}$ 

```

Algorithm 1: Minimal Bayesian Network Construction (I-Map)

We know sketch rough proofs for the claims of the algorithm.

Theorem 20. *The BN \mathcal{G} constructed by algorithm 1 is minimal, i.e we cannot remove any edge from the BN while maintaining the correctness of the BN for P .*

Proof. By construction. A subset of $\text{ND}(x_i)$ were available when we chose parents of \mathbf{U} minimally. \square

Theorem 21. *\mathcal{G} constructed by the above algorithm is correct, i.e, the local-CIs induced by \mathcal{G} hold in P .*

Proof. The construction is such that $\text{Factorize}(P, \mathcal{G})$ holds everytime. Since $\text{Factorize}(P, \mathcal{G}) \implies \text{Local-CI}(P, \mathcal{G})$, the constructed BN satisfies the local-CIs of P . \square

Question 22 (Construction of BN). *Draw a Bayesian network over five variables x_1, \dots, x_5 assuming the variable order x_1, x_2, x_3, x_4, x_5 . For this*

ordering, assume that the following set of local CIs hold in the distribution:

$$x_1 \perp\!\!\!\perp x_2 \quad x_3 \perp\!\!\!\perp x_2 | x_1 \quad x_4 \perp\!\!\!\perp x_1, x_3 | x_2 \quad x_5 \perp\!\!\!\perp x_1, x_2 | x_3, x_4$$

Answer 23. Due to the ordering, we begin by inserting x_1 into the BN. Then we follow Algorithm 1 as follows:

1. x_2 : Predecessor - x_1

- Query 1 - Is $x_2 \perp\!\!\!\perp x_1 | \emptyset$? : Result 1 - True

Thus, x_2 has no parents.

2. x_3 : Predecessors - x_1, x_2

- Query 1 - Is $x_3 \perp\!\!\!\perp \{x_1, x_2\} | \emptyset$? : Result 1 - False
- Query 2 - Is $x_3 \perp\!\!\!\perp x_2 | x_1$? : Result 2 - True

Thus, x_3 has x_1 as a parent, and x_2 as a non-descendent.

3. x_4 : Predecessors - x_1, x_2, x_3

- Query 1 - Is $x_4 \perp\!\!\!\perp \{x_1, x_2, x_3\} | \emptyset$? : Result 1 - False
- Query 2 - Is $x_4 \perp\!\!\!\perp \{x_2, x_3\} | x_1$? : Result 2 - False
- Query 3 - Is $x_4 \perp\!\!\!\perp \{x_1, x_3\} | x_2$? : Result 3 - True

Thus, x_4 has x_2 as a parent, and x_1, x_3 as non-descendents.

4. x_5 : Predecessors - x_1, x_2, x_3, x_4

Check that it has x_3, x_4 as parents and x_1, x_2 as non-descendents.

Thus, finally we get the BN as in Figure 2

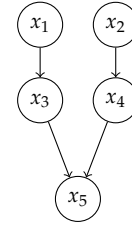


Figure 2: BN Example

Remark 24 (Importance of ordering). It is possible that a different ordering in \mathcal{X} gives rise to a different BN, which although may be minimal, but may not be *optimal*. A minimal BN is defined for a given ordering, while an optimal BN is defined over all orderings. Example 25 shows such a case.

Example 25. Consider the BN shown in Figure 3, with the ordering x_1, x_2, x_3 . We have $x_1 \perp\!\!\!\perp x_2$ as the only CI holding. Now consider our order changes to x_3, x_2, x_1 . We follow the procedure as before, first inserting x_3 into the BN. Now

1. x_2 : Predecessor - x_3

- Query 1 - Is $x_2 \perp\!\!\!\perp x_3 | \emptyset$? : Result 1 - False

Thus, we have x_3 as a parent of x_2 .

2. x_1 : Predecessor - x_3, x_2

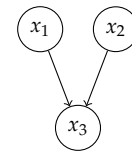


Figure 3: BN Ordering (optimal)

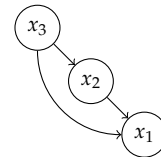


Figure 4: BN Ordering (non-optimal)

- *Query 1* - Is $x_1 \perp\!\!\!\perp \{x_2, x_3\} \mid \emptyset$? : *Result 1* - False
- *Query 2* - Is $x_1 \perp\!\!\!\perp x_2 \mid x_3$? : *Result 2* - False
- *Query 3* - Is $x_1 \perp\!\!\!\perp x_3 \mid x_2$? : *Result 3* - False

Thus, we have both x_2 and x_3 as parents of x_1 .

Thus we get the BN as in Figure 4, and see that the BN is minimal, but not optimal.

D-Separation

Our goal is to know when we can guarantee $\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}$ holds given a BN \mathcal{G} . The further discussion provides some cases where we can guarantee $\mathbf{X} \not\perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}$.

1. **Direct Connection:** If there is an edge $X \rightarrow Y$, then regardless of any Z , we can find examples where they influence each other.
2. **Indirect Connection:** This means that there is a trail between the nodes in the graph. We consider the simple case when we have a 3-node graph and Z is between X and Y . Consider the 4 diagrams to the left for reference.
 - (a) *Indirect causal effect:* X cannot influence Y via Z if Z is observed.
 - (b) *Indirect evidential effect:* This is similar to the previous case as dependence is a symmetric notion. Thus, X can influence Y via Z , only if Z is not observed.
 - (c) *Common cause:* The conclusion is similar to (a) and (b).
 - (d) *Common effect:* (v-structure) This case is a bit tricky to understand, but the crux is that X can influence Y when either Z or one of Z 's descendants is observed.

If we have flow of influence from X to Y via Z , we say that the trail $X \rightleftharpoons Y \rightleftharpoons Z$ is active.

$$\begin{aligned}
 & \left. \begin{array}{l} \text{Causal trail: } X \rightarrow Z \rightarrow Y \\ \text{Evidential trail: } Y \rightarrow Z \rightarrow X \\ \text{Common cause: } X \leftarrow Z \rightarrow Y \end{array} \right\} \text{Active if and only if } Z \text{ is observed} \\
 & \star \text{ Common effect: } X \rightarrow Z \leftarrow Y \} \\
 & \hookrightarrow \text{Active if and only if } Z \text{ or one of } Z\text{'s descendant is observed}
 \end{aligned} \tag{19}$$

Now, we can create a general notion of trails -

Definition 26. Let \mathcal{G} be a BN, and $x_1 \rightleftharpoons \dots \rightleftharpoons x_n$ be a trail in \mathcal{G} . Let $\mathbf{Z} \subset \{\text{observed variables}\}$. The trail is active given \mathbf{Z} if

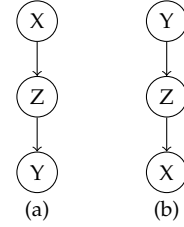


Figure 5: Causal and evidential effect

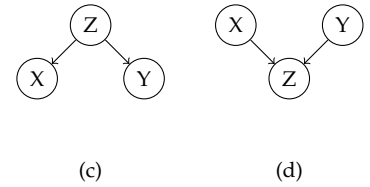


Figure 6: Common cause and common effect

- ◇ Whenever we have a v-structure $x_{i-1} \rightarrow x_i \leftarrow x_{i+1}$, then x_i or one of its descendants are in \mathbf{Z}
- ◇ No other node along the trail is in \mathbf{Z} .

We can see that if $x_1 \in \mathbf{Z}$ or $x_n \in \mathbf{Z}$, then the trail is inactive.

Definition 27 (d-separation). Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be three sets of nodes in \mathcal{G} . We say that \mathbf{X} and \mathbf{Y} are d-separated given \mathbf{Z} , i.e $\text{d-sep}_{\mathcal{G}}(\mathbf{X}; \mathbf{Y} | \mathbf{Z})$ if there is no active trail between any node $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ given \mathbf{Z} .

Definition 28 (Global Markov independencies). The set

$$\mathcal{I}(\mathcal{G}) \stackrel{\text{def}}{=} \{(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z}) : \text{d-sep}_{\mathcal{G}}(\mathbf{X}; \mathbf{Y} | \mathbf{Z})\} \quad (20)$$

denoting the set of independencies corresponding to d-separation is the set of global Markov independencies.

Theorem 29. *The d-separation test identifies the complete set of conditional independencies that hold in all distributions that conform to a given Bayesian Network.*

Proof. Skipped. □

Now, we look at another way to check d-separation over BNs, but first we define some terms.

Definition 30 (Ancestral Graph). Given a graph $G = (V, E)$ and a set of nodes to focus on, say $V^* \subseteq V$, the ancestral graph G^A is a subgraph induced by $V^A = V^* \cup \mathcal{A}(V^*)$ where $\mathcal{A}(V^*)$ denotes the ancestors of V^* . Thus,

$$G^A = G \langle V^A \rangle = (V^A, \{(u, v) | (u, v) \in E \text{ and } u, v \in V^A\}) \quad (21)$$

Definition 31 (Markov Blanket). Given a random variable Y in a random variable set $\mathcal{X} = X_1, X_2, \dots, X_n$, its Markov Blanket is any subset \mathcal{S} of \mathcal{X} , conditioned on which other variables are independent with Y , i.e

$$Y \perp\!\!\!\perp \mathcal{X} \setminus \mathcal{S} | \mathcal{S} \quad (22)$$

Thus, we can infer Y from \mathcal{S} itself, and the rest of the elements are redundant in observation.

Definition 33 (Moral graph). A moral graph of a directed acyclic graph G is an undirected graph in which each node of the original G is now connected to its *Markov Blanket*.

Essentially in a DAG, \mathbf{Z} d-separates \mathbf{X} from \mathbf{Y} if all paths \mathcal{P} from any \mathbf{X} to \mathbf{Y} is blocked by \mathbf{Z} .

A path \mathfrak{P} is *blocked* if it is inactive, i.e there is no flow of influence.

We use the same notation as $\mathcal{I}(P)$ as we can show that the independencies in $\mathcal{I}(\mathcal{G})$ are those guaranteed to hold for every distribution over \mathcal{G} (Theorem 29).

Remark 32. Essentially we are finding an equivalent undirected graph for a DAG. We find all pairs of non-adjacent nodes having a common child, and add an undirected edge between them. Then we transform all directed edges in the resulting graph to undirected edges.

```

1 Given: Bayesian Network  $\mathcal{G}$ , Condition to check  $\mathcal{C}$ :  $X \perp\!\!\!\perp Y | Z$ 
2  $\mathcal{C} \leftarrow \text{False}$ 
3  $G = (V, E) \leftarrow$  Underlying DAG in  $\mathcal{G}$ .
4  $G^A = (V^A, E^A) \leftarrow$  Ancestral graph of  $G$ 
5  $G_M^A = (V_M^A, E_M^A) \leftarrow$  Moral graph of  $G^A$  using Note 32
6 // Delete the nodes in  $Z$  and all its connections
7 for  $z \in Z$  do
8    $\Xi \leftarrow \{\}$ 
9   for  $u \in V$  such that  $\xi = (u, z) \in E_M^A$  do
10     $\Xi \leftarrow \Xi \cup \xi$ 
11  end
12   $E_M^A \leftarrow E_M^A \setminus \Xi$ 
13   $V_M^A \leftarrow V_M^A \setminus \{z\}$ 
14 end
15 if  $X$  and  $Y$  are disconnected in the resulting graph then
16    $\mathcal{C} \leftarrow \text{True}$ 
17 end

```

Algorithm 2: Checking for independence in a BN

Definition 34 (Perfect map). A graph \mathcal{G} is a perfect map (P-map) for a set of independencies \mathcal{I} if $\mathcal{I}(\mathcal{G}) = \mathcal{I}$. Also, \mathcal{G} is P-map for a distribution P if $\mathcal{I}(\mathcal{G}) = \mathcal{I}(P)$.

Limitations

Consider the following set of CIs

$$x \perp\!\!\!\perp y \quad y \perp\!\!\!\perp z \quad z \perp\!\!\!\perp x \quad x \not\perp\!\!\!\perp \{y, z\} \quad (23)$$

If you try to draw the BN for any ordering of the variables $\{x, y, z\}$, you can check that you get extraneous edges, and we fail to capture at least one of the conditions above. Thus, a symmetric dependency of this sort is not possible to be represented by Bayesian Networks. This, gives rise to a different field of graphical modeling using *Markov Networks*, which can represent some of these situations.

Markov Random Fields

Intuition

We saw previously that we cannot draw a *perfect* I-map such that $\mathcal{I}(\mathcal{G}) = \mathcal{I}(P)$ for any distribution P using directed graphical models. Such too is the case with undirected graphical models, but they help us to represent some of these independencies which directed graphs couldn't.

To be added.

Cliques

Definition 35 (Complete Graph). A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

Definition 36 (Clique). A clique C in an undirected graph $G = (V, E)$ is a subset of vertices, $C \subseteq V$ such that every two distinct vertices are adjacent. Thus, the subgraph induced by C , i.e $G \langle C \rangle$, is a complete graph.

Definition 37 (Maximal Clique). A clique that cannot be extended by including one more adjacent vertex (i.e it does not exist exclusively within the vertex set of a larger clique) is a maximal clique.

Definition 38 (Maximum Clique). A maximum clique of a graph G , is a clique such that there is no other clique with more vertices.

With each clique C , we associate a potential function ψ , which is a provisional function of its arguments that assigns a pre-probabilistic score of their joint distribution. It is to note that ψ must be non-negative, but it shouldn't be interpreted as probability.

Gibbs Fields

A Gibbs Field is a representation of a set of random variables and their relationships. An example is in Figure 7. In this, the edges are undirected and imply some correlation between the connected nodes.

Consider clique potentials as $\psi_i(c_i)$. Then the joint probability for any set of random variables $\mathcal{X} = \{x_1, \dots, x_n\}$ represented by a Gibbs Field can be written as a product of clique potentials

$$P(\mathcal{X}) = \frac{1}{Z} \prod_{c_i \in C} \psi_i(c_i) \quad (24)$$

Z is a normalizing constant required to create a valid probability distribution, i.e

$$Z = \sum_{\mathcal{X}} \prod_{c_i \in C} \psi_i(c_i) \quad (25)$$

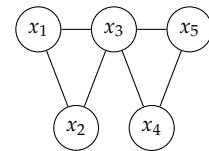


Figure 7: A Gibbs Field

Remark 39. For any Gibbs Field, there is a subset \hat{C} of C consisting of only maximal cliques, which are not proper subsets of any other cliques. We write the potentials for these maximal cliques as products of all potentials of their sub-cliques, and thus state the joint probability as

$$P(\mathcal{X}) = \frac{1}{Z} \prod_{c_i \in \hat{C}} \hat{\psi}_i(c_i) \quad (26)$$

Formal Definition

Definition 40 (Markov Random Field). A Markov Random Field (MRF) is a probability distribution P over variables x_1, \dots, x_n defined by an undirected graph G in which nodes correspond to variables x_i and has the form

$$P(x_1, x_2, \dots, x_n) = \frac{1}{Z} \prod_{c \in C} \psi_c(x_c) \quad (27)$$

where

$$Z = \sum_{x_1, \dots, x_n} \prod_{c \in C} \psi_c(x_c) \quad (28)$$

is the *partition function* which is the normalizing constant ensuring the distribution sums to 1.

As we saw earlier, if we have symmetric interactions, then UGMs become useful (such as labeling pixels in an image - see Figure 8). Define $y_i = 1$ if the pixel is a part of the foreground, and 0 else. Taking cliques of size 1, we have the potential functions $\psi_1(0)$ to $\psi_9(0)$ and $\psi_1(1)$ to $\psi_9(1)$. Now considering cliques of size 2, we have $\psi(0,0)$, $\psi(0,1)$, $\psi(1,0)$ and $\psi(1,1)$. Thus we write

$$\Pr(y_1, \dots, y_9) \propto \prod_{k=1}^9 \psi_k(y_k) \prod_{(i,j) \in E(G)} \psi(y_i, y_j) \quad (29)$$

Conditional Independencies

From now on, we will work on the UGM in Figure 8. Let

$$V = \{y_1, \dots, y_9\}$$

. We define three types of CIs in UGMs as follows

1. **Local CI:** $y_i \perp\!\!\!\perp V - \mathcal{N}(y_i) - \{y_i\} | \mathcal{N}(y_i)$

$$y_1 \perp\!\!\!\perp y_3, y_5, y_6, y_7, y_8, y_9 | y_2, y_4$$

2. **Pairwise CI:** $y_i \perp\!\!\!\perp y_j | V - \{y_i, y_j\}$ if $(y_i, y_j) \notin E(G)$

$$y_1 \perp\!\!\!\perp y_3 | y_2, y_4, y_5, y_6, y_7, y_8, y_9$$

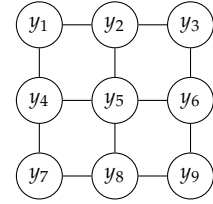


Figure 8: Relations in image pixels

In a graph G with vertices $V = \{x_1, \dots, x_n\}$, $\mathcal{N}(x_i)$ denotes the neighbors of x_i in the graph

3. Global CI: $X \perp\!\!\!\perp Y | Z$ if Z separates X and Y in the graph

$$y_1, y_2, y_3 \perp\!\!\!\perp y_7, y_8, y_9 | y_4, y_5, y_6$$

Checking for CI in MRFs is much more easier than BNs. The way to check is through graph separability. Consider the example given in **Global CI**. If we remove y_4, y_5 and y_6 from the graph along with their edges, we see that the components y_1, y_2, y_3 is disconnected from y_7, y_8, y_9 , and hence the CI holds.

Theorem 41. Let G be an undirected graph of $V = \{x_1, \dots, x_n\}$ nodes, and let $P(x_1, \dots, x_n)$ be a distribution. If P is represented by G , that is, if it can be factorized as per the cliques of G , then P will also satisfy the global-CIs of G . Thus

$$\text{Factorize}(P, G) \implies \text{Global-CI}(P, G) \quad (30)$$

Note that for any arbitrary distribution, the converse doesn't hold, i.e in general for a distribution P

$$\text{Factorize}(P, G) \not\Rightarrow \text{Global-CI}(P, G) \quad (31)$$

We see this through a counter example. Consider the UGM in Figure 9, for the probability distribution $P(x_1, x_2, x_3, x_4)$ such that $P(x_1, x_2, x_3, x_4) = \frac{1}{8}$ when x_1, x_2, x_3, x_4 can take values from $\{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\}$ else 0. It can be manually checked that all 4 Global-CIs hold in the graph, for example $x_1 \perp\!\!\!\perp x_3 | x_2, x_4$. Now consider the factors in the edges as $\psi(x_i, x_j)$. These will be positive, but that cannot represent the probability for $x_1, x_2, x_3, x_4 = 0101$.

Also, it is trivial to see that

$$\text{Global-CI} \implies \text{Local-CI} \quad (32)$$

But again through a counter example, we will show that the converse doesn't hold, i.e

$$\text{Local-CI} \not\Rightarrow \text{Global-CI} \quad (33)$$

Consider a distribution over 5 binary variables $P(x_1, \dots, x_5)$ where $x_1 = x_2, x_4 = x_5$ and $x_3 = x_2 \wedge x_4$. Consider G as in Figure 10. Notice that all 5 Local-CIs hold in the graph, for example $x_1 \perp\!\!\!\perp \{x_3, x_4, x_5\} | x_2$. But notice that the graph also tells us that $x_2 \perp\!\!\!\perp x_4 | x_3$, but this is not present in the distribution P .

We also notice that

$$\text{Local-CI} \implies \text{Pairwise-CI} \quad (34)$$

But again through a counter example, we show that the converse doesn't hold, i.e

$$\text{Pairwise-CI} \not\Rightarrow \text{Local-CI} \quad (35)$$

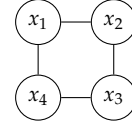


Figure 9: Sample UGM

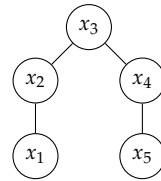


Figure 10: Sample UGM

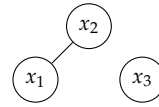


Figure 11: Sample UGM

Consider $P(x_1, x_2, x_3)$ defined over 3 binary variables such that $P(x_1, x_2, x_3) = \frac{1}{2}$ if $x_1 = x_2 = x_3$ and 0 else. Let G be as in Figure 11. See that both the Pairwise-CIs, i.e $x_1 \perp\!\!\!\perp x_3 | x_2$ and $x_2 \perp\!\!\!\perp x_3 | x_1$ hold in the graph, but the local CI $x_1 \perp\!\!\!\perp x_3$ doesn't hold.

We have made a lot of statements about converses not holding in arbitrary distributions, but the natural question to arise is, can we find distributions where all the relations hold? The answer is yes, and is shown by the following theorem, also called the *fundamental theorem of random fields* -

Theorem 42 (Hammersley Clifford Theorem). *If a positive distribution $P(x_1, \dots, x_n)$ confirms to the Pairwise-CIs of a UDGM G , then it can be factorized as per the cliques C of G as*

$$P(x_1, \dots, x_n) \propto \prod_{C \in G} \psi_C(\mathbf{y}_C) \quad (36)$$

A distribution $P(\mathbf{x})$ is positive, if $P(\mathbf{x}) > 0 \forall \mathbf{x}$.

Proof. Skipped. □

Thus, in summary, for any arbitrary distribution P and UGM H ,

$$\begin{aligned} \text{Factorize}(P, H) &\implies \text{Global-CI}(P, H) \implies \\ \text{Local-CI}(P, H) &\implies \text{Pairwise-CI}(P, H) \end{aligned} \quad (37)$$

and if P is positive, then

$$\text{Pairwise-CI}(P, H) \implies \text{Factorize}(P, H) \quad (38)$$

Hence, for a positive distribution, all three types of CIs are *equivalent*.

Minimal Construction

The question to answer is, given a positive distribution $P(x_1, \dots, x_n)$ as an oracle \mathcal{O} to which we can ask the query - is $X \perp\!\!\!\perp Y | Z$ and get a boolean answer, we need to draw a minimal and correct UGM G to represent P .

Denote $V = \{x_1, x_2, \dots, x_n\}$ as the set of all variables. We see that there are two methods to draw the UGM -

1. *Using Pairwise-CIs:* For each pair of vertices (x_i, x_j) , if $x_i \perp\!\!\!\perp x_j | V - \{x_i, x_j\}$ in P , add an edge between x_i and x_j in G .
2. *Using Local-CIs:* For each vector x_i , find the smallest subset U such that $x_i \perp\!\!\!\perp V - U - \{x_i\} | U$ in P . Then, add U to $\mathcal{N}(x_i)$ in P .

Example 43. To be added.

We had seen Markov Blankets before, but we re-define them in terms of a UGM.

Definition 44 (Markov Blanket). The Markov Blanket (MB) of a variable x_i is the smallest subset of variables V that makes x_i conditionally independent of others given the MB, i.e

$$x_i \perp\!\!\!\perp V - MB(x_i) - \{x_i\} | MB(x_i) \quad (39)$$

Theorem 45. The MB of a variable is always unique for a positive distribution.

Proof. We will prove the following by contradiction. Let $x_i \in V$ and M_1, M_2 be two MBs. Let $\alpha = M_1 - M_2$ and $\beta = M_2 - M_1$, $M = M_1 \cap M_2$, $W = V - (M_1 \cup M_2)$. Note that, by definition, $x_i \perp\!\!\!\perp V - M_2 | M_2$ and $x_i \perp\!\!\!\perp V - M_1 | M_1$. Using this, we can write

$$x_i \perp\!\!\!\perp W, \alpha | M, \beta \quad x_i \perp\!\!\!\perp W, \beta | M, \alpha$$

For positive distributions, using intersection property, we can write

$$x_i \perp\!\!\!\perp W, \alpha, \beta | M$$

This implies that M is also a MB, but that is a contradiction since M_1 and M_2 were supposed to be minimal. Hence, the MB is unique. \square

Definition 46 (Immortality). In a directed acyclic graph, the structure of the form $x \rightarrow y \leftarrow z$ is an immortality provided there is no edge between x and z .

With this, we can restate the equivalence of BNs -

Two BNs \mathcal{G}_1 and \mathcal{G}_2 are equivalent *iff* they have the same skeleton structure and the same set of immortalities.

Conversion to and from Bayesian Networks

Theorem 47. In a Bayesian Network \mathcal{G} , the Markov Blanket of a variable x_i is given as

$$MB(x_i) = Pa(x_i) \cup Ch(x_i) \cup Sp(x_i) \quad (40)$$

where $Pa(x_i)$, $Ch(x_i)$ and $Sp(x_i)$ denote the parents, children and spouses (unmarried shared parent) of the children of x_i (if exists).

Proof. Only a flavor of the proof is provided. We have seen moralization of the Bayesian Network \mathcal{G} , and when we get \mathcal{G}^M (i.e the moralized graph), notice that removing the parents, children and spouses disconnects the node from the graph. \square

For example, in Figure 12, the MB of x_2 is given as

$$MB(x_2) = \{x_4\} \cup \{x_1\} \cup \{x_3\}$$

Interestingly, UGMs were initially used to model interactions of atoms in gases and solids in 1800. A few other places where they are used are in

1. Markov Random Fields - Image Segmentation
2. Conditional Random Fields - Information Extraction
3. Social Networks
4. Bio-informatics - Annotating active sites in proteins

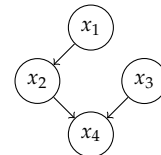


Figure 12: Sample BN

Theorem 48. *A Bayesian Network will have a perfect MRF if it has no immoralities.*

Proof. Skipped. □

We can ask the reverse question too. What condition should be posed on the MRF to have a perfect BN?

Definition 49 (Chordal Graph). A chordal graph is a simple graph in which every graph cycle of length four or greater has a cycle chord.

With this, we can state that

Theorem 50. *An MRF can be perfectly converted to a BN if and only if it is chordal.*

Proof. Skipped. □