Context Free Grammars

$G = (V, T, P, S)$

V = (finite) set of **variables** (or **non-terminal** symbols)

T = (finite) set of **terminal** symbols

P = finite subset of $V \times (V \cup T)^*$ called **productions**

$S \in V$ = the **start** symbol

A convention on notation

Lower case **a,b,c** ,… symbols in **T**

Upper case **A,B,C** ,… symbols in **V**

Lower case **u,w,v,z** ,… symbols in **T***

Upper case **X,Y,Z**,… symbols in $V \cup T$

Lower case Greek α, β, γ ,… symbols in $(V \cup T)^*$

Example

$G = (\{S\}, \{0,1\}, P, S)$, *where*

$\{S\}$ = V, $\{0,1\}$ = T

$P : S \rightarrow 0S1 \mid e$ *short hand notation for* $\{ (S,0S1), (S,e) \} \subseteq V \times (V \cup T)^*$

$$S \Rightarrow_G 0S1 \Rightarrow 0(0S1)1 = 0^2S1^2 \Rightarrow 0^2e1^2 = 0^21^2$$

$S \rightarrow 0S1$ $\quad$ $S \rightarrow 0S1$ $\quad$ $S \rightarrow e$

hence $S \Rightarrow^3 0^21^2$ *is a **3-step** derivation*

Derivations

Let $\alpha A \beta \in (V \cup T)^*$, *with* $\alpha, \beta \in (V \cup T)^*$ *and* $A \in V$ *and let*

$A \rightarrow \gamma$ *be a production of a CFG* $\mathbf{G}$ *then :*

$\alpha A \beta \Rightarrow_G \alpha \gamma \beta$ *is called a (one step) derivation in* $\mathbf{G}$ **;**

in a similar manner we have :

$W \Rightarrow_G^n \beta$ *and* $W \Rightarrow_G^* \beta$ *are* ***n-step*** *and finite step derivations in* $\mathbf{G}$

where each step conforms to the rule for the one step derivation

above.

Definition

- The language $L_G(A)$ *generated by a nonterminal variable* A *of a grammar* G *is given by :* $L_G(A) := \{v \in T^* \mid A \Rightarrow^*_G v\}$

- the language L_G *generated by* G *is* $L_G := L_G(S)$ *, where* T *and* S *denote the set of terminal symbols and the start symbol of* G *respectively.*

For the previous example $L_G = \{0^n 1^n, n \geq 0\}$

3 Examples of CFGs

(1) Regular Expressions over Σ where $\Sigma = \{\sigma_1, \sigma_2, \ldots \sigma_n\}$

$$E \rightarrow \sigma_1 \mid \sigma_2 \ldots \mid \sigma_n \mid e \mid \varnothing \mid E+E \mid E.E \mid E^* \mid (E)$$

There are ***n+6*** *productions with* ***n*** = $|\Sigma|$ *where :*

$V = \{ E \}$,

$T = \Sigma \cup \{e, \varnothing, +, \,.\, , *, (,)\}$

P = *the* ***n+6*** *productions above*

S = E

Example of a regular expression derivation with $\Sigma = \{0,1\}$

$$E \Rightarrow^7 0.(1+0)^*$$

$$E \Rightarrow E.E \Rightarrow 0.E \Rightarrow 0.E^* \Rightarrow 0.(E)^* \Rightarrow 0.(E+E)^* \Rightarrow 0.(1+E)^* \Rightarrow 0.(1+0)^*$$

$E \rightarrow E.E$ $\quad E \rightarrow 0$ $\quad E \rightarrow E^*$ $\quad E \rightarrow (E)$ $\quad E \rightarrow E+E$ $\quad E \rightarrow 1$ $\quad E \rightarrow 0$

(2) Simple Arithmetic Expressions *(variables and binary numbers)*

*Two operations : + and * and numbers and variables that are strings in* $x0 \cup x1\{0,1\}^*$

(**i.e.** x_0, x_1, x_2 *... ; variables with binary indices)*

$V = \{ E , I , J \}$

$T = \{ 0 , 1 , x , + , * , (,) \}$

P = 11 *productions given next*

$S = E$

E = *(Arithmetic) Expression*

I = *Identifier* , J = *Identifier trailer*

Productions :

$E \rightarrow I \mid E + E \mid E^*E \mid (E)$

$I \rightarrow x0 \mid x1J \mid 0 \mid 1J \;;\; J \rightarrow 0J \mid 1J \mid e$

Example for arithmetic expressions

$E \Rightarrow^{13} x1*(x0+11)$ *in ordinary notation* : $x_1 \,.\, (x_0 + 3)$

$E \Rightarrow E*E \Rightarrow I*E \Rightarrow x1J*E \Rightarrow x1e*E \Rightarrow^2 x1*(E+E) \Rightarrow x1*(I+E) \Rightarrow x1*(x0+E) \Rightarrow x1*(x0+I)) \Rightarrow x1*(x0+1J) \Rightarrow^2 x1*(x0+11)$

(3) The Grammar of Balanced Parentheses

$V = \{ E \}$

$T = \{ (\, ,\,) \}$

P = *the* **3** *productions below*

$S = E$

P :

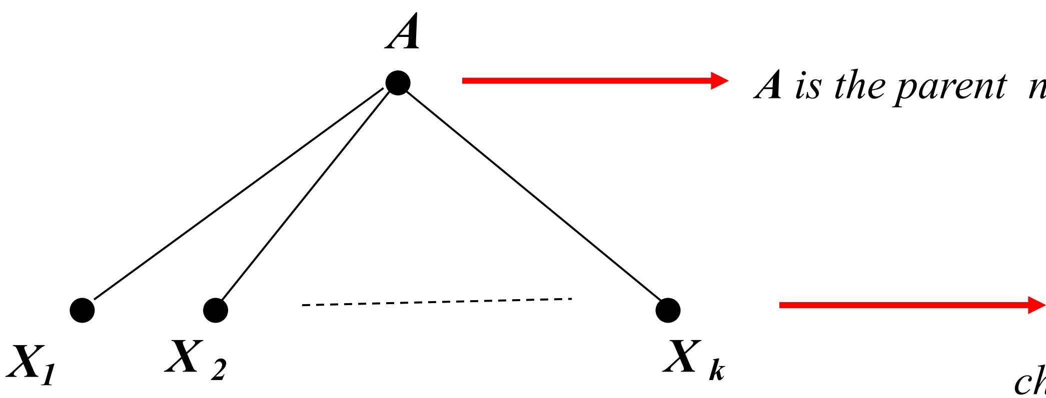
$E \rightarrow EE \mid (E) \mid e$

Example

$E \Rightarrow^* (()())()$

$E \Rightarrow EE \Rightarrow (E)E \Rightarrow (EE)E \Rightarrow ((E)E)E \Rightarrow ((e)E)E \Rightarrow (()(E))E \Rightarrow (()(e))E \Rightarrow (()())(E) \Rightarrow (()())(e) \Rightarrow (()())()$

Every production $A \rightarrow X_1 X_2 \ldots X_k$ *where each* $X_j \in V \cup T$, *corresponds to an* ***ordered tree*** *of height* **1** *as shown below*



A is the parent node of its children

children of A ordered from left to right ; X_1 to X_k

Terminology on ordered trees **:** *root, order, children , siblings, parent , descendants, ancestors, leaves, internal nodes …*

Recursive definition of ordered trees

Basis : *a tree* ***T*** *of depth 1 with a root node* ***r*** *and ordered sequence of children (leaf) nodes* $(n_1, \ldots, n_k)$ *is an* ***ordered tree****.*

Induction : *Let* ***S*** *be an ordered tree with a root node* ***r*** *and ordered sequence of leaf nodes* $(m_1, \ldots, m_p)$ *; let* ***T*** *be an ordered tree of depth* ***1*** *with a root node* ***t*** *and children nodes* $(n_1, \ldots, n_k)$ *then for any* $0 \leq j \leq p$ *,* ***S'*** *is an ordered tree obtained from* ***S*** *by replacing the leaf node* m_j *of* ***S*** *by* ***T*** *; so that the new ordered leaf nodes of* ***S'*** *are* $(m_1, \ldots, m_{j-1}, n_1, \ldots, n_k, m_{j+1}, \ldots m_p)$ *and* ***t*** *is an internal node replacing* m_j *.*

*For a tree **T** of depth one the root node **t** is the **parent** and an **ancestor** of the children nodes **(n_1 , …, n_k)** ; and the children nodes are called **siblings** of each other and **descendants** of the root node **t** .*

*For **S'** defined as above all nodes of **S** retain the **ancestor** and **descendant** relations in **S'** ; every ancestor of the replaced node **m_j** is an ancestor of all the newly added nodes **(n_1 , …, n_k)** as well as the root node **t** ; and if **m_j** was a descendant of a node **n** in **S** then **t** and the nodes **(n_1 , …, n_k)** are descendants of the node **n** ; etc.*

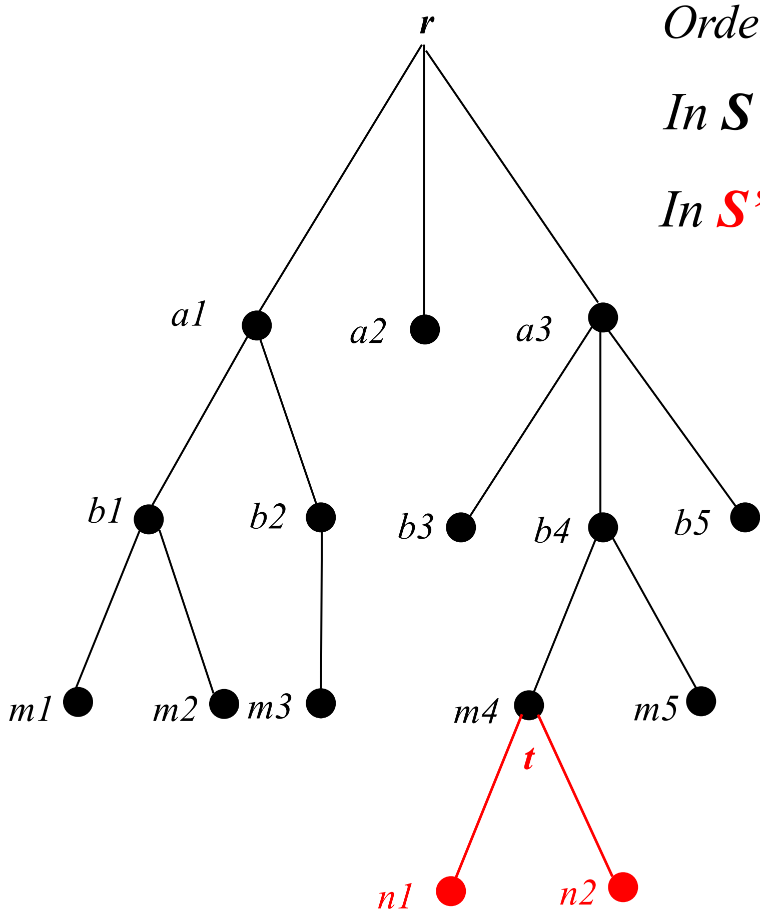
S to S'

Ordered leaves of S → *m1 m2 m3 a2 b3 m4 m5 b5*

Ordered leaves of S' → *m1 m2 m3 a2 b3 n1 n2 m5 b5*

*In **S : a3 ancestor** of **m4** and **m4 descendant** of **a3***

*In **S' : a3 ancestor** of **t,n1,n2** ; **t,n1,n2 descendants** of **a3***



Derivations and Parse Trees

Consider the derivation $\boldsymbol{S \Rightarrow^*_G \omega \in T^*}$ **;** *then for each step of the derivation a production of a non-terminal is used until all symbols are terminals as in* $\boldsymbol{\omega}$.

The parse tree is obtained by replacing each non-terminal corresponding to the production used– starting from ***S*** *– by the production tree of that non-terminal.*

Order on the leaves of the parse tree is the induced order of the children in the productions (every pair of nodes have a unique common youngest ancestor whose corresponding children set the order !)

Leftmost** (lm) **and Rightmost** (rm) **derivations

Definition *A derivation is called a **leftmost** (**rightmost**) **derivation** if at each step of the derivation a production is applied to **the** nonterminal at the **leftmost** (**rightmost**) position of the string*

Theorem *For every derivation* $A \Rightarrow^* \omega$ *of a variable **A** there is a **leftmost** (**rightmost**) derivation shown* $A \Rightarrow^*_{lm} \omega$ ($A \Rightarrow^*_{rm} \omega$) *with the same parse tree as the original derivation.*

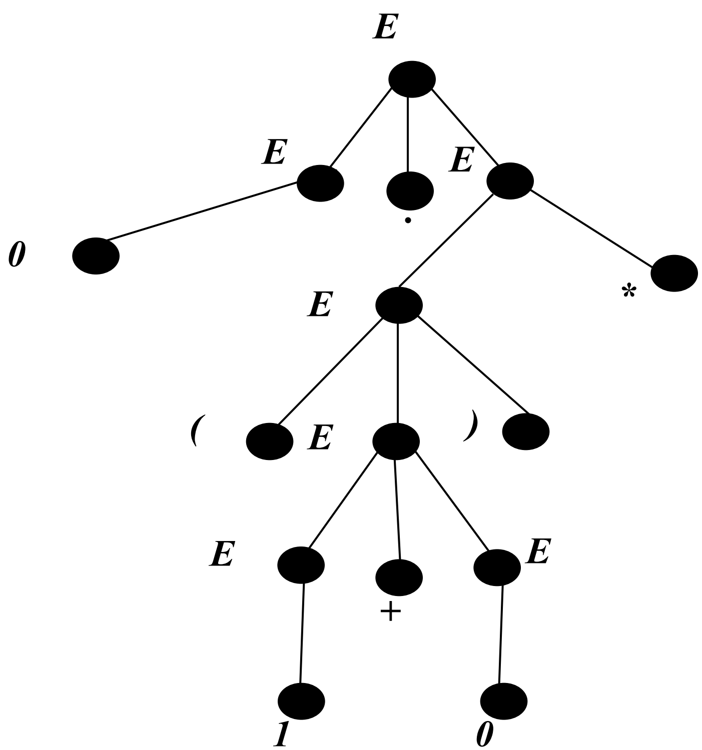
Example

$E \Rightarrow^7_{lm} 0.(1+0)^*$ $E \Rightarrow E.E \Rightarrow 0.E \Rightarrow 0.E^* \Rightarrow 0.(E)^* \Rightarrow 0.(E+E)^* \Rightarrow 0.(1+E)^* \Rightarrow 0.(1+0)^*$

$E \rightarrow E.E$ $E \rightarrow 0$ $E \rightarrow E^*$ $E \rightarrow (E)$ $E \rightarrow E+E$ $E \rightarrow 1$ $E \rightarrow 0$

$E \Rightarrow^7_{rm} 0.(1+0)^*$ $E \Rightarrow E.E \Rightarrow E.E^* \Rightarrow E.(E)^* \Rightarrow E.(E+E)^* \Rightarrow E.(E+0)^* \Rightarrow E.(1+0)^* \Rightarrow 0.(1+0)^*$

$E \rightarrow E.E$ $E \rightarrow E^*$ $E \rightarrow (E)$ $E \rightarrow E+E$ $E \rightarrow 0$ $E \rightarrow 1$ $E \rightarrow 0$



(1) $A \Rightarrow^* w$

(2) $A \Rightarrow_{lm}{}^* w$

(3) $A \Rightarrow_{rm}{}^* w$

(4) *There is a parse tree with root* $\boldsymbol{A}$ *and yield* $\boldsymbol{w}$

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

How to obtain a derivation from a parse tree by using induction on depth of the parse tree. *(assume original depth of the tree is **n**)*

Step 1 *: Start from the root **A** and move to the children X_1 to X_k ($A \Rightarrow X_1 \ldots X_k$)*

Step 2 *: If all X_j are terminals done ; else note that each X_j is a subtree of depth at most **n-1** , hence by induction hypothesis there is a derivation $X_j \Rightarrow^* w_j \in T^*$ for each **j**. Use these derivations on any desired order on each X_j to obtain a desired derivation*

Remark *If the derivations on X_j are made from **left to right** (**right to left**) on X_j we obtain **leftmost** (**rightmost**) derivation together with the appropriate induction assumption.*

How to obtain a parse tree from a derivation by using induction on the steps of the derivation *(assume original derivation steps is **n**)*

Step 1 *: Start from variable **A** and move to the next step of the derivation where* $A \Rightarrow X_1 \ldots X_k$

Step 2 *: Set the root of the parse tree as **A** ; set each* X_j *as either an internal node if* X_j *is a variable and a leaf if* X_j *is a terminal. For each variable* X_j *the subtrees to be placed under these internal nodes follow from the induction hypothesis since their derivations have **n-1** steps or less*

HTML Example

1. *Char → a|A| …*
2. *Text → e|Char Text*
3. *Doc → e|Element Doc*
4. *Element → Text |<EM> Doc </EM> |<P> Doc |<OL> List </OL> |…*
5. *ListItem → <LI> Doc*
6. *List → e | ListItem List*

→ Subset of CFG

V= (Char, Doc, Text, Element, ListItem, List ,…) ; T = (A-z, <EM>, </EM> , <LI>, <OL>, </OL> ,<P>)

<P>
<EM> This is a warning : </EM>
<OL>
<LI> Study hard.
<LI> Do your homework.
</OL>
<P>
Else you will fail !

→ HTML Program

This is a warning :

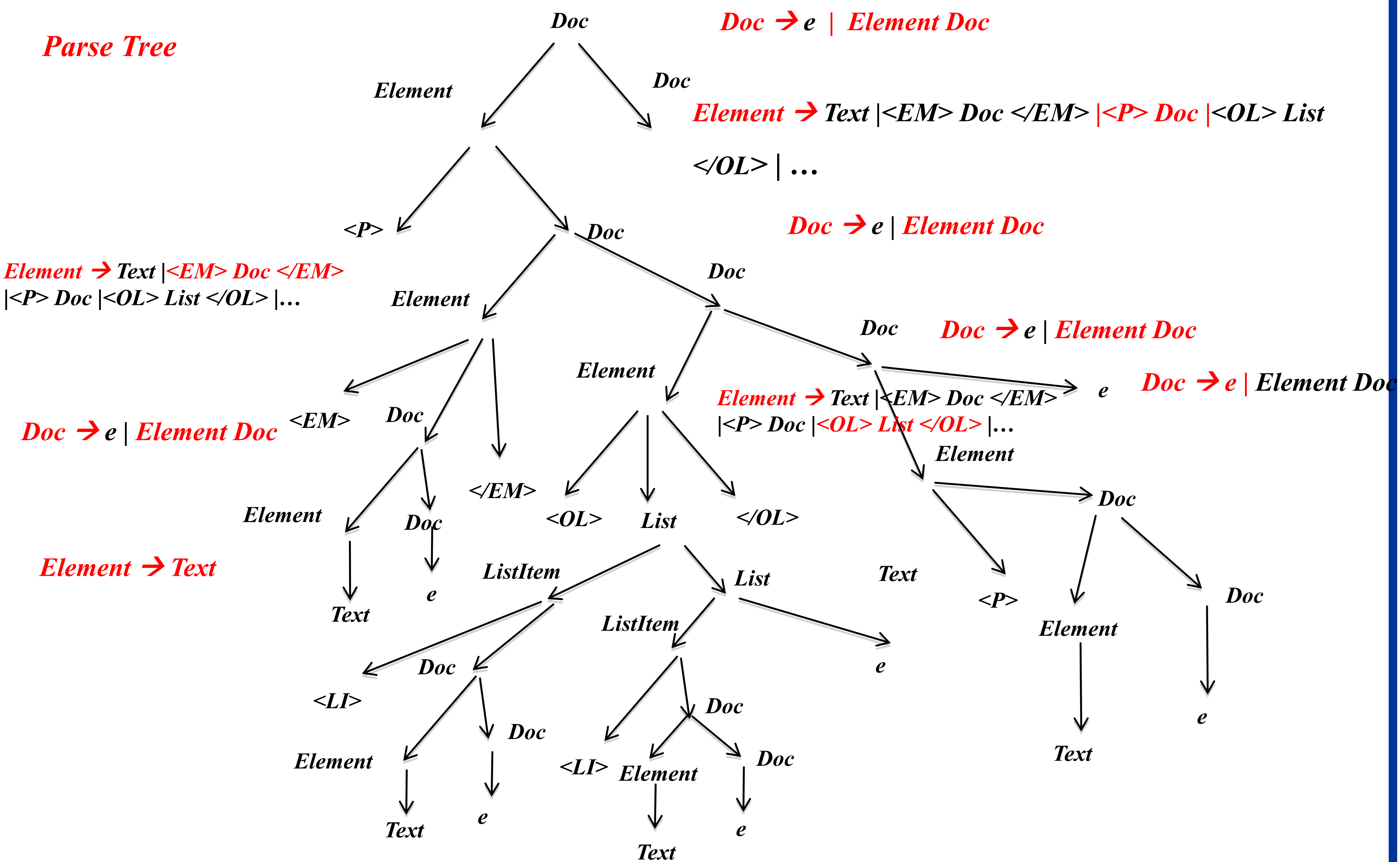
1. Study hard.

2. Do your homework.

Else you will fail !

→ Output of execution

Parse Tree


Doc
Doc → e | Element Doc
Element
Doc
Element → Text |<EM> Doc </EM> |<P> Doc |<OL> List </OL> | …
<P>
Doc
Doc → e | Element Doc
Element → Text |<EM> Doc </EM> |<P> Doc |<OL> List </OL> |…
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Doc
Doc
Doc → e | Element Doc
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Doc → e | Element Doc
Element → Text |<EM> Doc </EM> |<P> Doc |<OL> List </OL> |…
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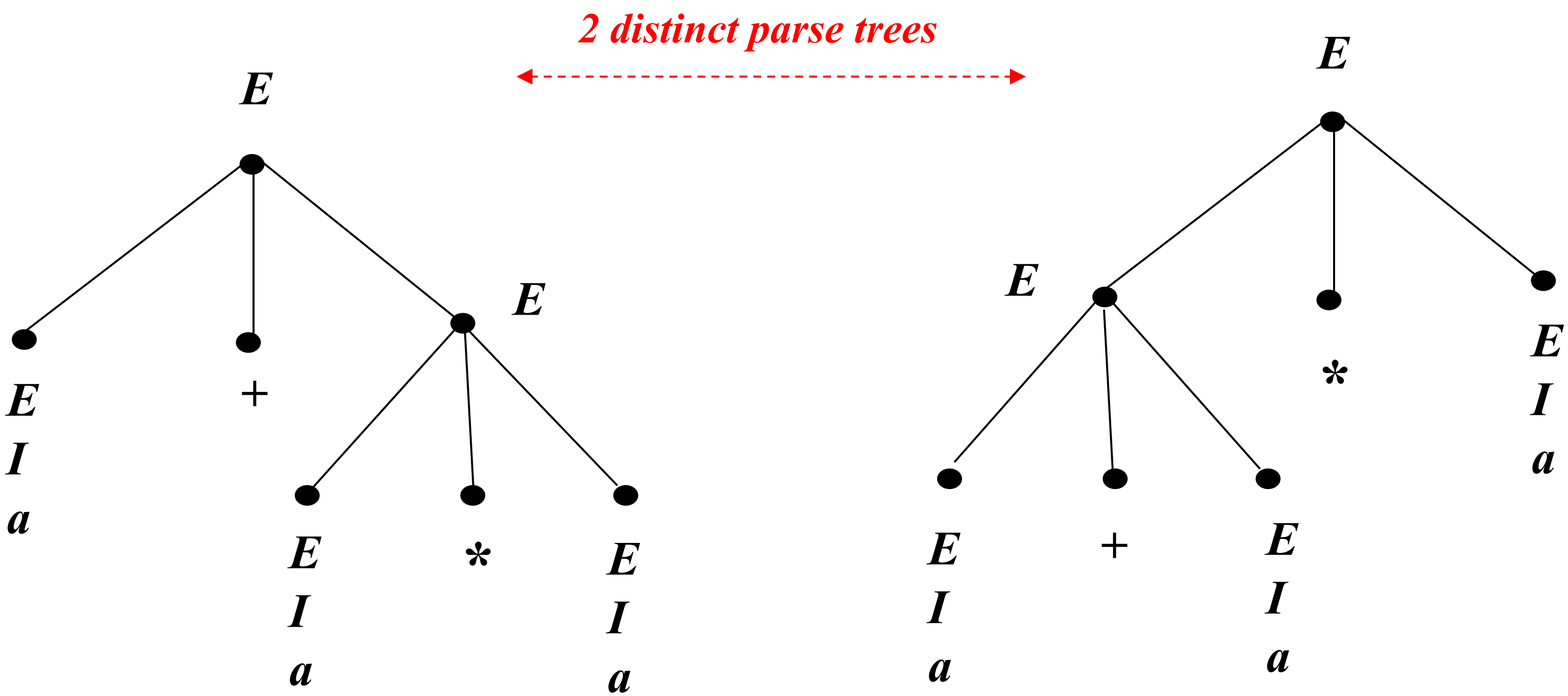
Ambiguity in Grammars and Languages

Example : a + a * a



Definition ***A Grammar G is called unambiguous if for every $w \in L_G$ there corresponds a unique parse tree. Else it is called ambiguous .***

The problem of determining whether a given grammar G is ambiguous or not is an undecidable problem !

Disambiguation = *Removing ambiguity*

Example

$E \rightarrow E+E$

$E \rightarrow E*E$

$E \rightarrow (E)$

$E \rightarrow I$

$I \rightarrow a$

$E+E*E$?

$= a+a*a$

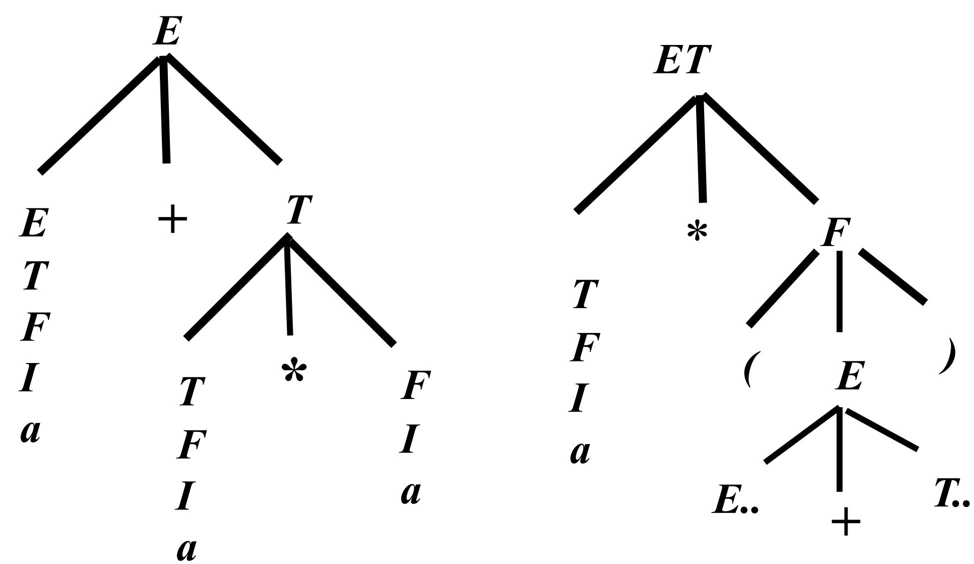
Setting priority of * over +

$E \rightarrow T \,|\, E+T$ - - - - - - - -> *E (expression) + is protected*

$T \rightarrow F \,|\, T*F$ - - - - - - - -> *T (term) * of factors*

$F \rightarrow I \,|\, (E)$ - - - - - - - -> *F (factor) is protected*

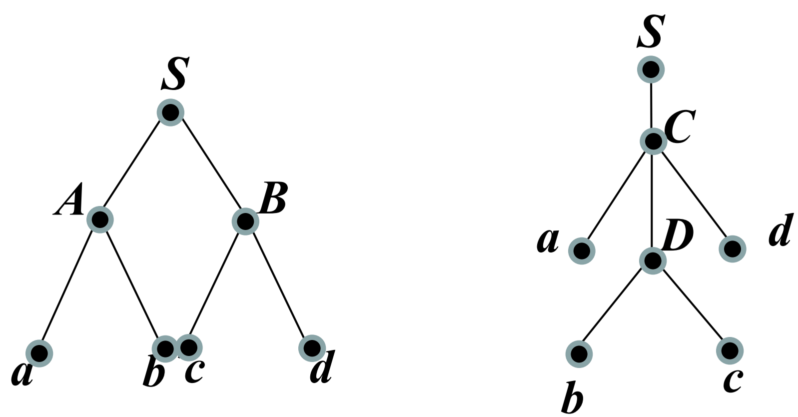
$I \rightarrow a$



Inherent Ambiguity (of CFLs)

$$L = \{ a^n b^n c^m d^m \mid n \geq 1 , m \geq 1\} \cup \{ a^n b^m c^m d^n \mid n \geq 1 , m \geq 1 \}$$

$S \rightarrow AB \mid C$
$A \rightarrow aAb \mid ab$
$B \rightarrow cBd \mid cd$
$C \rightarrow aCd \mid aD\,d$
$D \rightarrow bDc \mid bc$



Two parse trees for ***abcd***

Intuitively a string $a^k b^k c^k d^k$ will have 2 (leftmost derivations) parse trees

A nontrivial CFG Example

The CFG $\boldsymbol{G = (\{S\},\{a,b\},P,S)}$ *where* $\boldsymbol{P : S \rightarrow aSb \mid e}$ *generated the language*

$\boldsymbol{L_G=\{a^n b^n ; n \geq 0\}}$ *(which is not a regular language as proved by the PL)*

On the other hand we shall show that the CFG $\boldsymbol{G = (\{S,A,B\},\{a,b\},P,S)}$ *where*

$\boldsymbol{P : S \rightarrow aAS \mid bBS \mid e \,;\, A \rightarrow aAA \mid b \,;\, B \rightarrow bBB \mid a}$ *generates the language*

$\boldsymbol{L_G= (w \in \{a,b\}^* \mid \#a\text{'s} = \#b\text{'s})}$

(1) First observe that whenever $\boldsymbol{S \Rightarrow^* \gamma}$ *we have* $\boldsymbol{(\#a\text{'s}+\#B\text{'s})}$ *in* $\boldsymbol{\gamma=(\#b\text{'s} +\#A\text{'s})}$ *in* $\boldsymbol{\gamma}$

Check this for each production :

$\boldsymbol{S \rightarrow aAS \mid bBS}$ *; (**a(b)** and **A(B)** increase LHS (RHS) and RHS (LHS) by **1** respectively)*

$\boldsymbol{A \rightarrow aAA \mid b \,;\, B \rightarrow bBB \mid a \,;}$ *(**a(b)** and **A(B)** increase LHS and RHS by **1**)*

*+ (**b(a)** and **A(B)** are added to and subtracted from the RHS (LHS))*

A nontrivial CFG Example (cont')

$(\#a\text{'s}+\#B\text{'s})$ *in* $w=(\#b\text{'s}+\#A\text{'s})$ *in* w

Consequence of observation : at the end only terminals, **a** *and* **b** *remain in* γ

Hence if $S \Rightarrow^* w \in \{a,b\}^*$ *then* $\#a$'s = $\#b$'s *in* w

It remains to show any such sequence can be generated

This is best explained on a derivation example

$aAS \Rightarrow a^2AA\,S \Rightarrow a^3AAA\,S \Rightarrow a^4AAAAS$

Let $w = a^4\,b^3\,a^2\,b^3 \in L$

$$S \Rightarrow aAS \;\Rightarrow^3 aa^3A^4S \;\Rightarrow^3 a^4b^3\,A\,S \;\Rightarrow^2 a^4b^3\,a^2\,A^3\,S \;\Rightarrow^3 a^4b^3\,a^2\,b^3\,S \;\Rightarrow a^4b^3\,a^2\,b^3\,e$$

$S \to aAS$ $\quad A \to aAA$ $\quad A \to b$ $\quad A \to aAA$ $\quad A \to b$ $\quad S \to e$

Let $w = b^4\,a^3\,b^2\,a^3 \in L$

$$S \Rightarrow bBS \;\Rightarrow^3 bb^3B^4S \;\Rightarrow^3 b^4a^3\,B\,S \;\Rightarrow^2 b^4a^3\,b^2\,B^3\,S \;\Rightarrow^3 b^4a^3\,b^2\,a^3\,S \;\Rightarrow a^4b^3\,a^2\,b^3\,e$$

$S \to bBS$ $\quad B \to bBB$ $\quad B \to a$ $\quad B \to bBB$ $\quad B \to a$ $\quad S \to e$